CS704: Lecture 2
Lambda Calculus

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[Based on notes taken by Emma Turetsky on January 22, 2010]

Abstract

This lecture introduces the lambda calculus (λ-calculus). It presents grammars for the abstract and concrete syntax of λ-calculus, and explains the notion of computation in the λ-calculus in terms of string rewriting and tree rewriting.

1 Introduction

Lambda-calculus (λ-calculus) is a model of computation invented by Alonzo Church in the 1930's. It is a stripped-down functional programming language—basically as stripped down as could possibly be: it has only *three* kinds of constructs. Nevertheless, λ-calculus is a Turing-complete language.

λ-calculus is built on two concepts:

- *abstraction*: the forming of functional expressions ("λ-terms")
- *application*: the use of a functional expression by applying it to an argument

There is one λ-calculus construct for each; the third kind of λ-calculus construct consists of variables.

Programs are functional expressions; the data domain consists of functional expressions. In other words, every program can be operated on as a piece of data, and every piece of data can be interpreted as a program.

The λ-calculus has no other kinds of data (e.g., numbers, lists, strings); however, we can performing *encodings* in which certain subsets of λ-terms can be identified as representing, e.g., the natural numbers, and other λ-terms represent the familiar functions for operating on natural numbers (e.g., plus, times). In other words, we encode an "abstract data type" by designing λ-terms that operate in a way that mimics the data and functions of the abstract data type.

2 Syntax

2.1 String View of λ-Calculus

Definition 2.1 (λ-terms) *Let V be a countably-infinite set of variables. The set of* λ-terms *are defined inductively as follows:*

1. *every variable $v \in V$ is a lambda-term*
2. *if M is a λ term, then so is the* abstraction *($\lambda x.M$)*
3. *if M and N are λ-terms, then so is the* application *(MN)*

□

Alternatively, we can express the concrete syntax of λ-calculus by means of the following grammar:

$$\begin{array}{rcl} \text{exp} & ::= & \text{var} \\ & | & (\lambda\text{var.exp}) \\ & | & (\text{exp exp}) \end{array}$$

In this grammar, the left and right parenthesis symbols are part of the subject language (i.e., they are not meta-symbols of the grammar-defining formalism).

Example 2.2

$$
\begin{array}{l}
(\lambda x.(\lambda y.(xy))) \\
(\lambda x.((\lambda y.x)y)) \\
(\lambda x.(x(\lambda y.(y(\lambda z.z))))) \\
(((\lambda x.x)(\lambda y.y))(\lambda z.z))
\end{array}
$$

□

Precedence Rules. Notation: We use small letters (such as x, y, z) for variables, and capital letters (such as M, N, P, and Q) as meta-variables standing for typical λ-terms.

We omit parentheses according to the following two precedence rules:

1. Application is left associative. For instance, $MNPQ$ stands for (((MN)P)Q). In contrast, if you want the term $(M(N(PQ)))$ you can only drop the outermost pair of parentheses: $M(N(PQ))$.
2. Application has higher precedence than abstraction. For instance, λx.yz is (λx.(yz)), not ((λx.y)z).

Example 2.3 Below on the left are some examples of λ-calculus terms in which some parentheses have been dropped. Their equivalent fully-parenthesized versions are shown on the right.

$$
\begin{array}{rcl}
\lambda x.\lambda y.xy & \equiv & (\lambda x.(\lambda y.(xy))) \\
\lambda x.(\lambda y.x)y & \equiv & (\lambda x.((\lambda y.x)y)) \\
\lambda x.x\lambda y.y\lambda z.z & \equiv & (\lambda x.(x(\lambda y.(y(\lambda z.z))))) \\
(\lambda x.x)(\lambda y.y)(\lambda z.z) & \equiv & (((\lambda x.x)(\lambda y.y))(\lambda z.z))
\end{array}
$$

□

Given a λ-term in which parentheses may have been omitted, an algorithm for fully parenthesizing the λ-term is to repeatedly perform the following steps:

1. Find the rightmost dot (".")
2. Work through the terms to the right of the dot; insert parentheses to create a left-associative list
3. Move out to the λ that corresponds to the dot, and place a left parenthesis to the left of the λ and a right parenthesis to the right of the body processed in item 2

Example 2.4

$$
\begin{array}{rcll}
\lambda x.\lambda y.\lambda x.xy\lambda z.z & \rightarrow & \lambda x.\lambda y.\lambda x.xy(\lambda z.z) & \text{items 1, 2, and 3} \\
& \rightarrow & \lambda x.\lambda y.\lambda x.((xy)(\lambda z.z)) & \text{items 1 and 2} \\
& \rightarrow & \lambda x.\lambda y.(\lambda x.((xy)(\lambda z.z))) & \text{item 3} \\
& \rightarrow & \lambda x.(\lambda y.(\lambda x.((xy)(\lambda z.z)))) & \text{items 1, 2, and 3} \\
& \rightarrow & (\lambda x.(\lambda y.(\lambda x.((xy)(\lambda z.z))))) & \text{items 1, 2, and 3}
\end{array}
$$

□

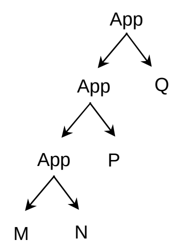
The abstract syntax of λ-calculus is defined by the following grammar:

$$
\begin{array}{rcl}
\text{term} & ::= & \text{Var(var)} \\
& | & \text{Lambda(var term)} \\
& | & \text{App(term term)}
\end{array}
$$

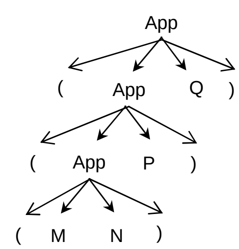
This can be considered to be a grammar that generates a language of trees. (Technically, it is called a *regular tree grammar*.) In this grammar, Var, Lambda, and App are the *operators* (or *alphabet symbols*) from which trees are constructed. (As a shorthand, in diagrams of abstract-syntax trees, we often use the symbol "λ" in place of the operator "Lambda".) The left and right parenthesis symbols are meta-symbols of the grammar-defining formalism; i.e., they are part of the meta-language, not the subject language.

2.2 Tree View of λ-Calculus

Recall that the abstract syntax gives us a tree view of λ-terms. That is, we can convert $(((MN)P)Q)$ from string form to the following tree:



The parentheses to insert (when converting back to fully-parenthesized string form) are implied by the tree structure:



When applying rewrite rules, its usually less mistake-prone to process a λ-term as follows:

$$\begin{array}{rcl} \text{string-representation} & \rightarrow & \text{tree-representation} \\ & \rightarrow & \text{normal-form in tree-representation} \\ & \rightarrow & \text{normal-form in string-representation.} \end{array}$$

3 Semantics

The intended semantics is that

- An abstraction represents a 1-argument function
- An application represents the application of a term M to input "data" N

Note that there is not a separate notion of "data" in λ-calculus. That is, all data items are themselves functional expressions: the λ-calculus consists of functional expressions that operate on functional expressions. In particular, we can look at the following λ-term:

$$(\lambda x.M)N$$

and interpret it as the application of the function $\lambda x.M$, which has formal parameter x and function body M, to actual parameter N.

3.1 β-Reduction Rule of λ-Calculus

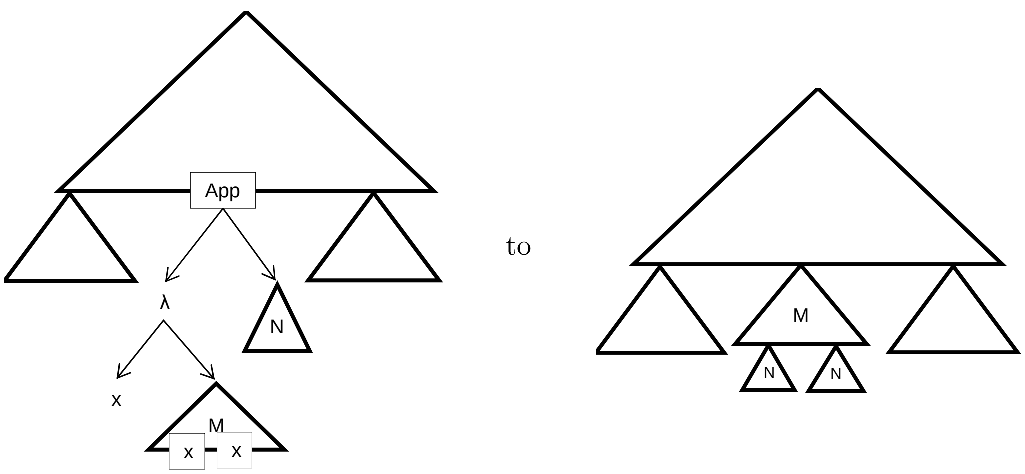
The β-reduction rule of λ-calculus allows us to resolve applications. Applications are resolved ("reduced") by replacing the formals that occur in the function body by *copies* of the actual parameter. (Thus, β-reduction is similar to the familiar programming-language notion of *call-by name*.)

Definition 3.1 (Imprecise version of the β-reduction rule)

$$(\lambda x.M)N \rightarrow_{\beta} M', \text{ where } M' \text{ is } M \text{ with all occurrences of } x \text{ in } M \text{ replaced by } N.$$

□

In a tree view this converts

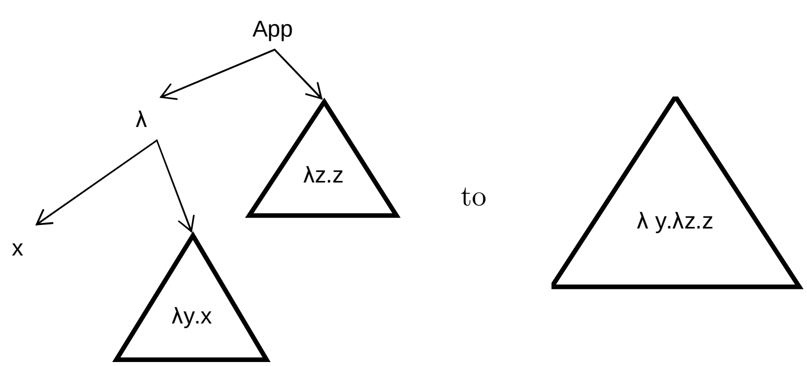


Definition 3.2 *In $(\lambda x.M)N \rightarrow M'$, an occurrence of the left-hand side pattern (i.e., $(\lambda x.M)N$) is called a* redex*; after the reduction, the corresponding occurrence of the right-hand side pattern (i.e., M') is called the* contractum. □

Example 3.3 To give a concrete example, consider the β-reduction

$$(\lambda x(\lambda y.x))(\lambda z.z) \rightarrow \lambda y.\lambda z.z$$

In tree form, the reduction takes



□

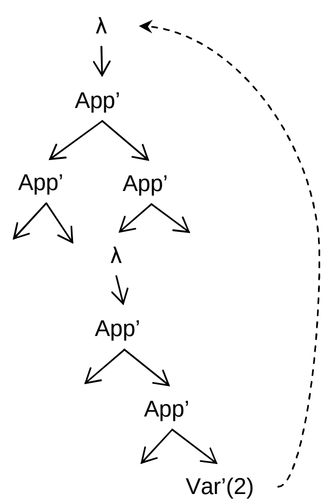
4 λ-Calculus Syntax Redux: De Bruijn Representation

De Bruijn introduced an alternative representation for λ-terms in which variables are replaced by indexes (natural numbers). There is a convention that the index represents the number of levels of enclosing λs that need to be traversed along the path to the root to find the λ with which the index (variable) is associated. The abstract syntax is changed to

$$\begin{array}{rcl} \text{term} & ::= & \text{Var}'(\text{nat}) \\ & | & \text{Lambda}'(\text{term}) \\ & | & \text{App}(\text{term term}) \end{array}$$

A tree generated by this grammar is a λ-calculus term in De Bruijn representation. The grammar generates trees of the following form, where the λ that a variable is attached to is determined by climbing up the tree towards the root, counting occurrences of λs along the way.

Example 4.1 The following figure shows (part) of a λ-term in De Bruijn representation, and in particular, shows how a sub-term that represents a variable occurrence is associated with an enclosing occurrence of a λ (i.e., Lambda) operator.



As shown by the dashed arrow, the leaf marked with the index 2 corresponds to a variable that is bound to the top λ, because that is two levels up (where counts start from 1). □