Quantum Error Correction

Transversal Cliffords on $[\![n,\, n-2,\, 2]\!]$ Codes

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August 31, 2016

1 Introduction

It has been noted that there exists an $[\![8,\, 3,\, 2]\!]$ code in which a transversal $T(= R_{z,\pi/8})$ gate produces a doubly-controlled Z on the logical qubits. It would be nice to generalize this result to non-cube structures, or even larger square tilings of the sphere. To begin, I'm going to recap the equivalent result in 2D, then generalize it to non-squares.

2 Transversal Phase Gate in the $[\![4,\, 2,\, 2]\!]$ Code

The stabilizers and logical operators of the $[\![4,\, 2,\, 2]\!]$ code should be old hat by now:

$$S = \langle S_X = XXXX, S_Z = ZZZZ \rangle\,, \quad L = \left\langle \overline{X}_0 = XXII,\, \overline{X}_1 = XIXI,\, \overline{Z}_0 = IZIZ,\, \overline{Z}_1 = IIZZ \right\rangle \tag{1}$$

The action of a bipartite phase gate $P \otimes P^\dagger \otimes P^\dagger \otimes P$ (where $P = R_{z,\pi/4}$) preserves the stabilizer, since $XXXX$ can be regenerated by multiplying $YYYY$ and $ZZZZ$. The effect on the logical X operators is to map them to $-YYII$ and $-YIYI$. Multiplying these new operators by S_Z takes them to $XXZZ = \overline{X}_0\overline{Z}_1$ and $XZXZ = \overline{X}_1\overline{Z}_0$, performing a CPHASE gate on the logical qubits. In the following section, we generalize this to codes on even-weight polygons.

3 Polygonal Phase Gate

We begin again with two stabilizers, $S_Z = Z^{\otimes n}$ and $S_X = X^{\otimes n}$, producing an $[\![n,\, n-2,\, 2]\!]$ code. If $n = 0 \mod 4$, we can perform a similar phase gate $P \otimes P^{\dagger \otimes n-2} \otimes P$, and the stabilizer will be preserved, since it is preserved under $P^{\otimes 2} \otimes P^{\dagger \otimes 2}$ and $P^{\dagger \otimes 4}$ separately. If $n = 2 \mod 4$, and we applied the same gate, we would end up with $-X^{\otimes n}$ in the stabilizer. In this case, we apply the gate $P \otimes P^{\dagger \otimes n-1}$ instead.

The $n-2$ pairs of logical operators can be written as $\overline{X}_j = X_0 X_j$, $\overline{Z}_j = Z_j Z_{n-1}$ for j between 1 and $n-2$. As before, $X_0 X_j \mapsto -Y_0 Y_j$. If we multiply by S_Z, the result is $X_0 X_j \cdot \bigotimes_{k \neq 0,j} Z_k$. It is important to note that the set $\{k \,|\, k \neq 0, j\}$ is even-size, so that the set $\{k \,|\, k \neq 0, j, n-1\}$ is odd-size. This implies that, if we take the product of $\overline{Z_k}, k \neq j$, the qubit $n-1$ will appear in the argument an odd number of times, and therefore be in the support of the product. This implies that the transversal gate described above takes $\overline{X}_j$ to $\overline{X}_j \cdot \bigotimes_{k \neq j} \overline{Z}_j$, performing a complete graph's worth of CPHASE gates in a single timestep.

4 Future Work

- Hadamards
- Polyhedra
- Replace each face of the cube with four squares.