Effect of Decoherence During Gates

Ben Criger

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1 Introduction

The purpose of this note is to determine the effect of biased (read: mostly dephasing) noise during the operation of certain gates. I'm going to start with the Y_{90} gate, implemented by a nice square pulse. There are two limits that can be handled succinctly, the perfectly-Markovian limit (where we can't decrease the error rate without a quantum error-correcting code), and the 'unknown constant Hamiltonian' limit, where we can apply pulse sequences, reversing the effect of the unknown Hamiltonian for $\sim 1/2$ the pulse duration. I begin with the Markovian limit, as I'm not so familiar with DD pulse sequences.

2 Helpful Math

Let's vectorize the density matrix, in the Pauli basis:

$$\rho = \frac{\hat{\mathbb{1}}}{2} + \rho_x \sigma_x + \rho_y \sigma_y + \rho_z \sigma_z \tag{1}$$

To express the equation of motion in this basis, I calculate a few commutators and dissipators.

$$\mathcal{D}\left[A\right]\left(B\right) = ABA^{\dagger} - \frac{1}{2}\left\{A^{\dagger}A,\, B\right\} \tag{2}$$

$$\mathcal{D}\left[A\right]\left(\hat{\mathbb{1}}\right) = A\hat{\mathbb{1}}A^{\dagger} - \frac{1}{2}\left\{A^{\dagger}A,\, \hat{\mathbb{1}}\right\} = \left[A,\, A^{\dagger}\right] \tag{3}$$

$$\therefore \mathcal{D}\left[\sigma_z\right]\left(\hat{\mathbb{1}}\right) = \left[\sigma_z,\, \sigma_z\right] = 0 \tag{4}$$

$$\mathcal{D}\left[\sigma_z\right]\left(\sigma_z\right) = 0 \tag{5}$$

$$\mathcal{D}\left[\sigma_z\right]\left(\sigma_{x,y}\right) = -2\sigma_{x,y} \tag{6}$$

$$\left[\sigma_y,\, \sigma_x\right] = -2i\sigma_z \quad \left[\sigma_y,\, \sigma_z\right] = 2i\sigma_x \tag{7}$$

With these in hand, we can start expressing the equation of motion for a noisy Y_{90} in this operator basis.

3 Y_{90} with Markovian Noise

A simple master equation for a Y_{90}, subject to dephasing is:

$$\dot{\rho} = -i\frac{\omega}{2}\left[\sigma_y,\, \rho\right] + \frac{\gamma}{2}\mathcal{D}\left[\sigma_z\right]\left(\rho\right), \tag{8}$$

where the factors of two are included to make the matrix description look nice, as we will see in a minute. I express the commutator and dissipator in matrix form:

$$\frac{\gamma}{2}\mathcal{D}\left[\sigma_z\right]\left(\cdot\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\gamma & 0 & 0 \\ 0 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -i\frac{\omega}{2}\left[\sigma_y,\, \cdot\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 \end{bmatrix}. \tag{9}$$

The Lindbladian is just the sum of these two terms:

$$\dot{\vec{\rho}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\gamma & 0 & \omega \\ 0 & 0 & -\gamma & 0 \\ 0 & -\omega & 0 & 0 \end{bmatrix} \vec{\rho} = \hat{L}\vec{\rho} \tag{10}$$

We take the matrix exponent $\exp(\hat{L}t)$ to get the superoperator S:

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-\frac{\gamma t}{2}}\left(\cosh(\beta t/2) - \frac{\gamma}{\beta}\sinh(\beta t/2)\right) & 0 & -\frac{2\omega}{\beta}e^{-\frac{\gamma t}{2}}\sinh(\beta t/2) \\ 0 & 0 & e^{-\frac{\gamma t}{2}} & 0 \\ 0 & \frac{2\omega}{\beta}e^{-\frac{\gamma t}{2}}\sinh(\beta t/2) & 0 & e^{-\frac{\gamma t}{2}}\left(\cosh(\beta t/2) + \frac{\gamma}{\beta}\sinh(\beta t/2)\right) \end{bmatrix} \tag{11}$$

where $\beta = \sqrt{\gamma^2 - 4\omega^2}$.

If ω is large, and γ is small (as we hope will be the case in low-noise systems), then β will be imaginary, and the hyperbolic functions will become regular trigonometric functions:

$$\beta \equiv i\nu \tag{12}$$

$$\cosh\left(\frac{\beta t}{2}\right) = \cos\left(\frac{\nu t}{2}\right) \tag{13}$$

$$\frac{\sinh(\frac{\beta t}{2})}{\beta} = \frac{\sin(\frac{\nu t}{2})}{\nu} \tag{14}$$

$$S \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-\frac{\gamma t}{2}}\left(\cos(\nu t/2) - \frac{\gamma}{\nu}\sin(\nu t/2)\right) & 0 & -\frac{2\omega}{\nu}e^{-\frac{\gamma t}{2}}\sin(\nu t/2) \\ 0 & 0 & e^{-\frac{\gamma t}{2}} & 0 \\ 0 & \frac{2\omega}{\nu}e^{-\frac{\gamma t}{2}}\sin(\nu t/2) & 0 & e^{-\frac{\gamma t}{2}}\left(\cos(\nu t/2) + \frac{\gamma}{\nu}\sin(\nu t/2)\right) \end{bmatrix} \tag{15}$$

We have control over t, and we'd like to determine how to set it in order to obtain the maximum-fidelity Y_{90}. To separate the noise from the gate we'd like to perform, we express the total superoperator as the product of a desired Y_{90} superoperator and a noise operator:

$$S = S_{\text{Noise}}S_{\text{Id}} \tag{16}$$

$$\therefore S_{\text{Noise}} = S_{\text{Id}}^{-1}S \tag{17}$$

$$S_{\text{Id}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{18}$$

$$\therefore S_{\text{Noise}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2\omega}{\nu}\exp(-\frac{\gamma t}{2})\sin(\frac{\nu t}{2}) & 0 & \exp(-\frac{\gamma t}{2})\left(\frac{\gamma}{\nu}\sin(\frac{\nu t}{2}) - \cos(\frac{\nu t}{2})\right) \\ 0 & 0 & \exp(-\gamma t) & 0 \\ 0 & \exp(-\frac{\gamma t}{2})\left(\frac{\gamma}{\nu}\sin(\frac{\nu t}{2}) + \cos(\frac{\nu t}{2})\right) & 0 & \frac{2\omega}{\nu}\exp(-\frac{\gamma t}{2})\sin(\frac{\nu t}{2}) \end{bmatrix} \tag{19}$$

To find the channel fidelity $F_\Lambda = \left\langle \Omega \middle| \Lambda \otimes \hat{\mathbb{1}}\left(|\Omega\rangle\langle\Omega|\right) \middle| \Omega \right\rangle$ (where $|\Omega\rangle$ is a Bell state), we take the trace of this superoperator and divide by 4 (I won't prove this here, but leave it as an exercise):

$$F_{S_{\text{Noise}}} = \frac{1}{4}\left[1 + \frac{4\omega}{\nu}\exp\left(-\frac{\gamma t}{2}\right)\sin\left(\frac{\nu t}{2}\right) + \exp\left(-\gamma t\right)\right] \tag{20}$$

To find out how long to leave the Hamiltonian on, we try to optimize this fidelity over t:

$$\frac{dF_{S_{\text{Noise}}}}{dt} = \frac{1}{2}\,\omega\cos\left(\frac{1}{2}\,\nu t\right)e^{\left(-\frac{1}{2}\,\gamma t\right)} - \frac{\gamma\omega e^{\left(-\frac{1}{2}\,\gamma t\right)}\sin\left(\frac{1}{2}\,\nu t\right)}{2\,\nu} - \frac{1}{4}\,\gamma e^{\left(-\gamma t\right)} = 0 \tag{21}$$

$$\therefore \sin\left(\frac{1}{2}\,\nu t\right) = \frac{\left(2\,\nu\omega\cos\left(\frac{1}{2}\,\nu t\right)e^{(\gamma t)} - \gamma\nu e^{\left(\frac{1}{2}\,\gamma t\right)}\right)e^{(-\gamma t)}}{2\,\gamma\omega} \tag{22}$$

This formula is not analytically soluble, so we just set $t = \frac{\pi}{\nu}$, to get the following (ridiculously good-looking) superoperator:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\gamma e^{\left(-\frac{\pi\gamma}{2\nu}\right)}}{\nu} & 0 & \frac{2\,\omega e^{\left(-\frac{\pi\gamma}{2\nu}\right)}}{\nu} \\ 0 & 0 & e^{\left(-\frac{\pi\gamma}{\nu}\right)} & 0 \\ 0 & -\frac{2\,\omega e^{\left(-\frac{\pi\gamma}{2\nu}\right)}}{\nu} & 0 & \frac{\gamma e^{\left(-\frac{\pi\gamma}{2\nu}\right)}}{\nu}. \end{bmatrix} \tag{23}$$

We subtract off a term $2\frac{\omega}{\nu}\exp\left(-\frac{\pi\gamma}{2\nu}\right) Y_{90}$ to obtain the remaining diagonal part of the superoperator:

$$\begin{bmatrix} -\frac{2\,\omega e^{\left(-\frac{\pi\gamma}{2\nu}\right)}}{\nu} + 1 & 0 & 0 & 0 \\ 0 & -\frac{\gamma e^{\left(-\frac{\pi\gamma}{2\nu}\right)}}{\nu} & 0 & 0 \\ 0 & 0 & -\frac{2\,\omega e^{\left(-\frac{\pi\gamma}{2\nu}\right)}}{\nu} + e^{\left(-\frac{\pi\gamma}{\nu}\right)} & 0 \\ 0 & 0 & 0 & \frac{\gamma e^{\left(-\frac{\pi\gamma}{2\nu}\right)}}{\nu} \end{bmatrix} \tag{24}$$

This is equivalent to a Pauli map with the following probabilities:

$$\begin{aligned} p_I &= -\frac{\omega e^{\left(-\frac{\pi\gamma}{2\nu}\right)}}{\nu} + \frac{1}{4}e^{\left(-\frac{\pi\gamma}{\nu}\right)} + \frac{1}{4} \quad & p_X &= -\frac{\gamma e^{\left(-\frac{\pi\gamma}{2\nu}\right)}}{2\,\nu} - \frac{1}{4}e^{\left(-\frac{\pi\gamma}{\nu}\right)} + \frac{1}{4} \\ p_Y &= -\frac{\omega e^{\left(-\frac{\pi\gamma}{2\nu}\right)}}{\nu} + \frac{1}{4}e^{\left(-\frac{\pi\gamma}{\nu}\right)} + \frac{1}{4} \quad & p_Z &= \frac{\gamma e^{\left(-\frac{\pi\gamma}{2\nu}\right)}}{2\,\nu} - \frac{1}{4}e^{\left(-\frac{\pi\gamma}{\nu}\right)} + \frac{1}{4} \end{aligned} \tag{25}$$

These probabilities approach 0 in the small γ limit and approach $\frac{1}{4}$ in the large γ limit. I'm willing to bet that they are always between 0 and 1.

4 Amplitude Damping

We'd also like, if possible, to consider the effect of amplitude damping (or T_1 noise) on the Y_{90} gate. It will likely not be possible to express this noise as a mixed-Clifford channel, given that it is non-unital. There are two approaches to coping with this; either we obtain a mixed-Clifford channel which optimally approximates the gate, or we fold the noisy Y_{90} into the nearest state-preparation or measurement location, modeling the effect of T_1 on a mixture of $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ rather than superpositions. This second approach ought to be easier (analytically soluble, rather than requiring an SDP), and apply most of the time (since syndrome extraction from a CSS code usually only uses H/Y_{90} gates to prepare superposition states and rotate measurement bases), so I'll try that first.

We calculate the effect of a σ_- dissipator in the Pauli basis:

$$\mathcal{D}\left[\sigma_-\right]\left(\hat{\mathbb{1}}\right) = \left[\sigma_-,\, \sigma_+\right] = \sigma_z \tag{26}$$

$$\mathcal{D}\left[\sigma_-\right]\left(\sigma_x\right) = \sigma_-\sigma_x\sigma_+ - \frac{1}{2}\left\{\sigma_+\sigma_-,\, \sigma_x\right\} = |0\rangle\langle 1|\sigma_x|1\rangle\langle 0| - \frac{1}{2}\left\{|1\rangle\langle 1|,\, \sigma_x\right\} = -\frac{1}{2}\sigma_x \tag{27}$$

$$\mathcal{D}\left[\sigma_-\right]\left(\sigma_y\right) = -\frac{1}{2}\sigma_y \tag{28}$$

$$\mathcal{D}\left[\sigma_-\right]\left(\sigma_z\right) = |0\rangle\langle 1|\sigma_z|1\rangle\langle 0| - \frac{1}{2}\left\{|1\rangle\langle 1|,\, \sigma_z\right\} = -\sigma_z \tag{29}$$

This allows us to express the new Lindbladian in matrix form (note that I mess with coefficient definitions to try to get the matrix to look nice, your coefficients may vary):

$$\dot{\rho} = -i\frac{\omega}{2}\left[\sigma_y,\, \rho\right] + \frac{\gamma_\phi}{2}\mathcal{D}\left[\sigma_z\right](\rho) + \gamma_-\mathcal{D}\left[\sigma_-\right](\rho) \tag{30}$$

$$\dot{\vec{\rho}} = \begin{bmatrix} 0 & 0 & 0 & -\gamma_- \\ 0 & -\gamma_\phi - \frac{1}{2}\gamma_- & 0 & \omega \\ 0 & 0 & -\gamma_\phi - \frac{1}{2}\gamma_- & 0 \\ \gamma_- & -\omega & 0 & 0 \end{bmatrix}\vec{\rho} = \hat{L}\vec{\rho} \tag{31}$$

5 Questions

1. Show that "shorting" the gate time optimally (maximizing the channel fidelity) doesn't appreciably raise the fidelity over just setting $\omega t = \frac{\pi}{\nu}$.

2. How does all this change when we add amplitude damping?