Effect of Decoherence During Gates

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1 Introduction

The purpose of this note is to determine the effect of biased (read: mostly dephasing) noise during the operation of certain gates. I'm going to start with the Y_{90} gate, implemented by a nice square pulse. There are two limits that can be handled succinctly, the perfectly-Markovian limit (where we can't decrease the error rate without a quantum error-correcting code), and the 'unknown constant Hamiltonian' limit, where we can apply pulse sequences, reversing the effect of the unknown Hamiltonian for $\sim {}^{1}/_{2}$ the pulse duration. I begin with the Markovian limit, as I'm not so familiar with DD pulse sequences.

2 Helpful Math

Let's vectorize the density matrix, in the Pauli basis:

$$\rho = \frac{\hat{\mathbb{1}}}{2} + \rho_x\sigma_x + \rho_y\sigma_y + \rho_z\sigma_z \tag{1}$$

To express the equation of motion in this basis, I calculate a few commutators and dissipators.

$$\begin{aligned}
&\mathcal{D}\left[A\right](B) = ABA^\dagger - \frac{1}{2}\left\{A^\dagger A,\, B\right\} &(2)\\
&\mathcal{D}\left[A\right]\left(\hat{\mathbb{1}}\right) = A\hat{\mathbb{1}}A^\dagger - \frac{1}{2}\left\{A^\dagger A,\, \hat{\mathbb{1}}\right\} = \left[A,\, A^\dagger\right] &(3)\\
&\therefore \mathcal{D}\left[\sigma_z\right]\left(\hat{\mathbb{1}}\right) = \left[\sigma_z,\, \sigma_z\right] = 0 &(4)\\
&\mathcal{D}\left[\sigma_z\right]\left(\sigma_z\right) = 0 &(5)\\
&\mathcal{D}\left[\sigma_z\right]\left(\sigma_{x,y}\right) = -2\sigma_{x,y} &(6)\\
&\left[\sigma_y,\, \sigma_x\right] = -2i\sigma_z \quad \left[\sigma_y,\, \sigma_z\right] = 2i\sigma_x &(7)
\end{aligned}$$

With these in hand, we can start expressing the equation of motion for a noisy Y_{90} in this operator basis.

3 Y_{90} with Markovian Noise

A simple master equation for a Y_{90}, subject to dephasing is:

$$\dot{\rho} = -i\frac{\omega}{2}\left[\sigma_y,\, \rho\right] + \frac{\gamma}{2}\mathcal{D}\left[\sigma_z\right](\rho)\,, \tag{8}$$

where the factors of two are included to make the matrix description look nice, as we will see in a minute. I express the commutator and dissipator in matrix form:

$$\frac{\gamma}{2}\mathcal{D}\left[\sigma_z\right](\cdot) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\gamma & 0 & 0 \\ 0 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -i\frac{\omega}{2}\left[\sigma_y,\, \cdot\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 \end{bmatrix}. \tag{9}$$

The Lindbladian is just the sum of these two terms:

$$\dot{\vec{\rho}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\gamma & 0 & \omega \\ 0 & 0 & -\gamma & 0 \\ 0 & -\omega & 0 & 0 \end{bmatrix} \vec{\rho} = \hat{L}\vec{\rho} \tag{10}$$

We take the matrix exponent $\exp(\hat{L}t)$ to get the superoperator S:

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-\frac{\gamma t}{2}}\left(\cosh(\beta t/2) - \frac{\gamma}{\beta}\sinh(\beta t/2)\right) & 0 & -\frac{2\omega}{\beta}e^{-\frac{\gamma t}{2}}\sinh(\beta t/2) \\ 0 & 0 & e^{-\frac{\gamma t}{2}} & 0 \\ 0 & \frac{2\omega}{\beta}e^{-\frac{\gamma t}{2}}\sinh(\beta t/2) & 0 & e^{-\frac{\gamma t}{2}}\left(\cosh(\beta t/2) + \frac{\gamma}{\beta}\sinh(\beta t/2)\right) \end{bmatrix} \tag{11}$$

where $\beta = \sqrt{\gamma^2 - 4\omega^2}$.

If ω is large, and γ is small (as we hope will be the case in low-noise systems), then β will be imaginary, and the hyperbolic functions will become regular trigonometric functions:

$$\beta \equiv i\nu \tag{12}$$

$$\cosh\left(\frac{\beta t}{2}\right) = \cos\left(\frac{\nu t}{2}\right) \tag{13}$$

$$\frac{\sinh(\frac{\beta t}{2})}{\beta} = \frac{\sin(\frac{\nu t}{2})}{\nu} \tag{14}$$

$$S \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-\frac{\gamma t}{2}}\left(\cos(\nu t/2) - \frac{\gamma}{\nu}\sin(\nu t/2)\right) & 0 & -\frac{2\omega}{\nu}e^{-\frac{\gamma t}{2}}\sin(\nu t/2) \\ 0 & 0 & e^{-\frac{\gamma t}{2}} & 0 \\ 0 & \frac{2\omega}{\nu}e^{-\frac{\gamma t}{2}}\sin(\nu t/2) & 0 & e^{-\frac{\gamma t}{2}}\left(\cos(\nu t/2) + \frac{\gamma}{\nu}\sin(\nu t/2)\right) \end{bmatrix} \tag{15}$$

We have control over t, and we'd like to determine how to set it in order to obtain the maximum-fidelity Y_{90}. To separate the noise from the gate we'd like to perform, we express the total superoperator as the product of a desired Y_{90} superoperator and a noise operator:

$$S = S_{\text{Noise}} S_{\text{Id}} \tag{16}$$

$$\therefore S_{\text{Noise}} = S_{\text{Id}}^{-1} S \tag{17}$$

$$S_{\text{Id}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{18}$$

$$\therefore S_{\text{Noise}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2\omega}{\nu}\exp(-\frac{\gamma t}{2})\sin(\frac{\nu t}{2}) & 0 & \exp(-\frac{\gamma t}{2})\left(\frac{\gamma}{\nu}\sin(\frac{\nu t}{2}) - \cos(\frac{\nu t}{2})\right) \\ 0 & 0 & \exp(-\gamma t) & 0 \\ 0 & \exp(-\frac{\gamma t}{2})\left(\frac{\gamma}{\nu}\sin(\frac{\nu t}{2}) + \cos(\frac{\nu t}{2})\right) & 0 & \frac{2\omega}{\nu}\exp(-\frac{\gamma t}{2})\sin(\frac{\nu t}{2}) \end{bmatrix} \tag{19}$$

To find the channel fidelity $F_\Lambda = \left\langle \Omega \middle| \Lambda \otimes \hat{\mathbb{1}}\left(|\Omega\rangle\langle\Omega|\right) \middle| \Omega \right\rangle$ (where $|\Omega\rangle$ is a Bell state), we take the trace of this superoperator and divide by 4 (I won't prove this here, but leave it as an exercise):

$$F_{S_{\text{Noise}}} = \frac{1}{4}\left[1 + \frac{4\omega}{\nu}\exp\left(-\frac{\gamma t}{2}\right)\sin\left(\frac{\nu t}{2}\right) + \exp\left(-\gamma t\right)\right] \tag{20}$$

To find out how long to leave the Hamiltonian on, we try to optimize this fidelity over t:

$$\frac{dF_{S_{\text{Noise}}}}{dt} = \frac{1}{2}\,\omega\cos\left(\frac{1}{2}\,\nu t\right)e^{\left(-\frac{1}{2}\,\gamma t\right)} - \frac{\gamma\omega e^{\left(-\frac{1}{2}\,\gamma t\right)}\sin\left(\frac{1}{2}\,\nu t\right)}{2\,\nu} - \frac{1}{4}\,\gamma e^{(-\gamma t)} = 0 \tag{21}$$

$$\therefore \sin\left(\frac{1}{2}\,\nu t\right) = \frac{\left(2\,\nu\omega\cos\left(\frac{1}{2}\,\nu t\right)e^{(\gamma t)} - \gamma\nu e^{\left(\frac{1}{2}\,\gamma t\right)}\right)e^{(-\gamma t)}}{2\,\gamma\omega} \tag{22}$$

This formula is not analytically soluble, so we just set $t = \frac{\pi}{\nu}$, to get the following (ridiculously good-looking) superoperator:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\gamma e^{\left(-\frac{\pi\gamma}{2\,\nu}\right)}}{\nu} & 0 & \frac{2\,\omega e^{\left(-\frac{\pi\gamma}{2\,\nu}\right)}}{\nu} \\ 0 & 0 & e^{\left(-\frac{\pi\gamma}{\nu}\right)} & 0 \\ 0 & -\frac{2\,\omega e^{\left(-\frac{\pi\gamma}{2\,\nu}\right)}}{\nu} & 0 & \frac{\gamma e^{\left(-\frac{\pi\gamma}{2\,\nu}\right)}}{\nu}. \end{bmatrix} \tag{23}$$

We subtract off a term $2\frac{\omega}{\nu}\exp\left(-\frac{\pi\gamma}{2\nu}\right)Y_{90}$ to obtain the remaining diagonal part of the superoperator:

$$\begin{bmatrix} -\frac{2\,\omega e^{\left(-\frac{\pi\gamma}{2\,\nu}\right)}}{\nu} + 1 & 0 & 0 & 0 \\ 0 & -\frac{\gamma e^{\left(-\frac{\pi\gamma}{2\,\nu}\right)}}{\nu} & 0 & 0 \\ 0 & 0 & -\frac{2\,\omega e^{\left(-\frac{\pi\gamma}{2\,\nu}\right)}}{\nu} + e^{\left(-\frac{\pi\gamma}{\nu}\right)} & 0 \\ 0 & 0 & 0 & \frac{\gamma e^{\left(-\frac{\pi\gamma}{2\,\nu}\right)}}{\nu} \end{bmatrix} \tag{24}$$

This is equivalent to a Pauli map with the following probabilities:

$$\begin{aligned} p_I &= -\frac{\omega e^{\left(-\frac{\pi\gamma}{2\,\nu}\right)}}{\nu} + \frac{1}{4}e^{\left(-\frac{\pi\gamma}{\nu}\right)} + \frac{1}{4} & p_X &= -\frac{\gamma e^{\left(-\frac{\pi\gamma}{2\,\nu}\right)}}{2\,\nu} - \frac{1}{4}e^{\left(-\frac{\pi\gamma}{\nu}\right)} + \frac{1}{4} \\ p_Y &= -\frac{\omega e^{\left(-\frac{\pi\gamma}{2\,\nu}\right)}}{\nu} + \frac{1}{4}e^{\left(-\frac{\pi\gamma}{\nu}\right)} + \frac{1}{4} & p_Z &= \frac{\gamma e^{\left(-\frac{\pi\gamma}{2\,\nu}\right)}}{2\,\nu} - \frac{1}{4}e^{\left(-\frac{\pi\gamma}{\nu}\right)} + \frac{1}{4} \end{aligned} \tag{25}$$

These probabilities approach 0 in the small γ limit and approach $\frac{1}{4}$ in the large γ limit. I'm willing to bet that they are always between 0 and 1.

4 Questions

1. Show that "shorting" the gate time optimally (maximizing the channel fidelity) doesn't appreciably raise the fidelity over just setting $\omega t = \frac{\pi}{\nu}$.
2. How does all this change when we add amplitude damping?