Quantum Fault Tolerance

One-Qubit Cliffords by Measurement

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1 Introduction

Some fault tolerant codes don't have transversal Clifford gates, which necessitates code deformation. Sometimes (like in Bravyi's 2016 QIP tutorial) these codes have weird symmetric boundary conditions that I don't understand. I'd like to compare this to a more intuitive non-unitary implementation of Cliffords, regular old lattice surgery. To accomplish this, I need to check which gates can be obtained by a protocol where:

1. An ancilla is prepared in a Pauli $+1$ eigenstate
2. a two-bit joint measurement is performed, and
3. the original ancilla qubit is measured out fault-tolerantly.

To be non-trivial, the jointly-measured Pauli has to anticommute with the prepared Pauli and the measured Pauli, there are 36 such protocols. However, consider the case in which $A = D$. In this case, the tableau evolves as follows:

$$\begin{array}{c} S_{\mathrm{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IA \\ \hline XI \\ ZI \end{array} \overset{\mathcal{M}_{BC}}{\longmapsto} \begin{array}{c} BC \\ \hline XI(XA) \\ ZI(ZA) \end{array} \overset{\mathcal{M}_{IA}}{\longmapsto} \begin{array}{c} IA \\ \hline XI \\ ZI \end{array},$$

where either of the Paulis in brackets may be present without affecting the result. Either way, the resulting unitary is the identity, we eliminate these cases, leaving 18 remaining. I test all of them in the remainder of this document, assuming all measurements produce $+1$ outcomes throughout:

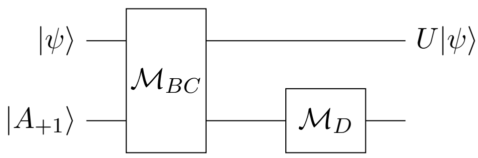
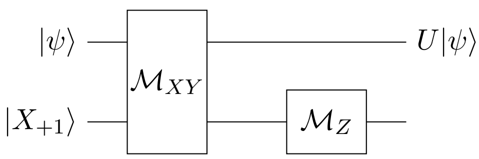
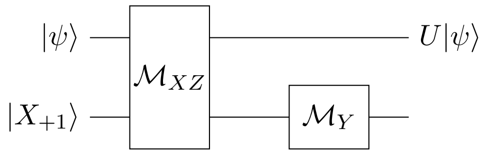


Figure 1: The basic protocol I want to analyse, with a $+1$ eigenstate prepared on the ancilla, a joint measurement of a two-qubit Pauli BC, and a final measurement of Pauli D.

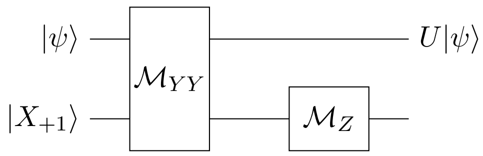
Below, I present the 18 protocols, with a little equation explaining what happens to the stabiliser and logicals throughout:



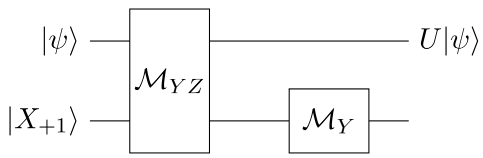
$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IX \\ \hline XI \\ ZI \end{array} \xmapsto{\mathcal{M}_{XY}} \begin{array}{c} XY \\ \hline XI \\ ZX \end{array} \xmapsto{\mathcal{M}_{IZ}} \begin{array}{c} IZ \\ \hline XI \\ YI \end{array} \therefore U \sim HPH \tag{1}$$



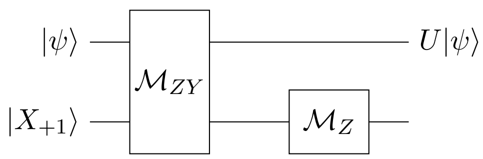
$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IX \\ \hline XI \\ ZI \end{array} \xmapsto{\mathcal{M}_{XZ}} \begin{array}{c} XZ \\ \hline XI \\ ZX \end{array} \xmapsto{\mathcal{M}_{IY}} \begin{array}{c} IY \\ \hline XI \\ YI \end{array} \therefore U \sim HPH \tag{2}$$



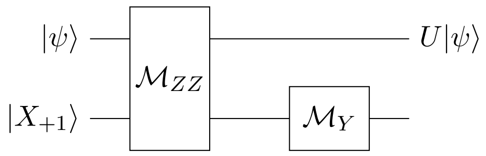
$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IX \\ \hline XI \\ ZI \end{array} \xmapsto{\mathcal{M}_{YY}} \begin{array}{c} YY \\ \hline XX \\ ZX \end{array} \xmapsto{\mathcal{M}_{IZ}} \begin{array}{c} IZ \\ \hline ZI \\ XI \end{array} \therefore U \sim H \tag{3}$$



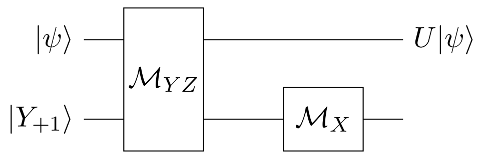
$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IX \\ \hline XI \\ ZI \end{array} \xmapsto{\mathcal{M}_{YZ}} \begin{array}{c} YZ \\ \hline XX \\ ZX \end{array} \xmapsto{\mathcal{M}_{IY}} \begin{array}{c} IY \\ \hline ZI \\ XI \end{array} \therefore U \sim H \tag{4}$$



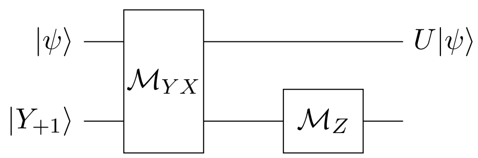
$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IX \\ \hline XI \\ ZI \end{array} \xmapsto{\mathcal{M}_{ZY}} \begin{array}{c} ZY \\ \hline XX \\ ZI \end{array} \xmapsto{\mathcal{M}_{IZ}} \begin{array}{c} IZ \\ \hline YI \\ ZI \end{array} \therefore U \sim P \tag{5}$$



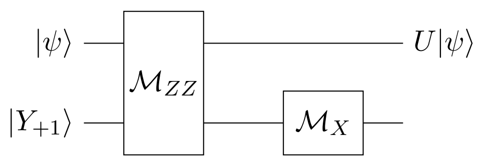
$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IX \\ \hline XI \\ ZI \end{array} \xmapsto{\mathcal{M}_{ZZ}} \begin{array}{c} ZZ \\ \hline XX \\ ZI \end{array} \xmapsto{\mathcal{M}_{IY}} \begin{array}{c} IY \\ \hline YI \\ ZI \end{array} \therefore U \sim P \tag{6}$$



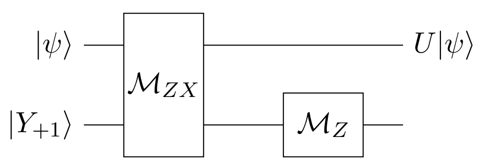
$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IY \\ \hline XI \\ ZI \end{array} \xmapsto{\mathcal{M}_{YZ}} \begin{array}{c} YZ \\ \hline XY \\ ZY \end{array} \xmapsto{\mathcal{M}_{IX}} \begin{array}{c} IX \\ \hline ZI \\ XI \end{array} \therefore U \sim H \tag{7}$$



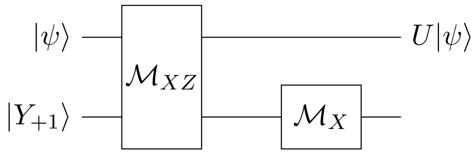
$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IY \\ \hline XI \\ ZI \end{array} \xmapsto{\mathcal{M}_{YX}} \begin{array}{c} YX \\ \hline XY \\ ZY \end{array} \xmapsto{\mathcal{M}_{IZ}} \begin{array}{c} IZ \\ \hline ZI \\ XI \end{array} \therefore U \sim H \tag{8}$$



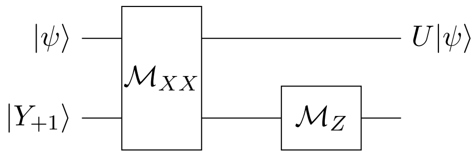
$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IY \\ \hline XI \\ ZI \end{array} \xmapsto{\mathcal{M}_{ZZ}} \begin{array}{c} ZZ \\ \hline XY \\ ZI \end{array} \xmapsto{\mathcal{M}_{IX}} \begin{array}{c} IX \\ \hline YI \\ ZI \end{array} \therefore U \sim P \tag{9}$$



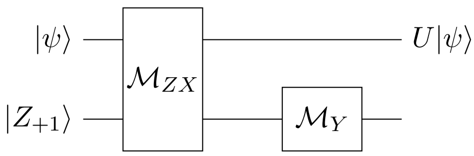
$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IY \\ \hline XI \\ ZI \end{array} \xmapsto{\mathcal{M}_{ZX}} \begin{array}{c} ZX \\ \hline XY \\ ZI \end{array} \xmapsto{\mathcal{M}_{IZ}} \begin{array}{c} IZ \\ \hline YI \\ ZI \end{array} \therefore U \sim P \tag{10}$$



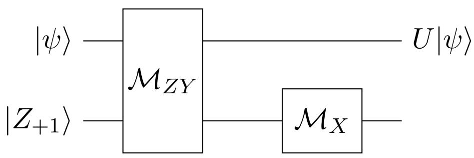
$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IY \\ \hline XI \\ ZI \end{array} \overset{\mathcal{M}_{XZ}}{\longmapsto} \begin{array}{c} XZ \\ \hline XI \\ ZY \end{array} \overset{\mathcal{M}_{IX}}{\longmapsto} \begin{array}{c} IX \\ \hline XI \\ YI \end{array} \therefore U \sim HPH \tag{11}$$



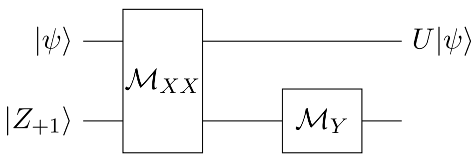
$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IY \\ \hline XI \\ ZI \end{array} \overset{\mathcal{M}_{XX}}{\longmapsto} \begin{array}{c} XX \\ \hline XI \\ ZY \end{array} \overset{\mathcal{M}_{IZ}}{\longmapsto} \begin{array}{c} IZ \\ \hline XI \\ YI \end{array} \therefore U \sim HPH \tag{12}$$



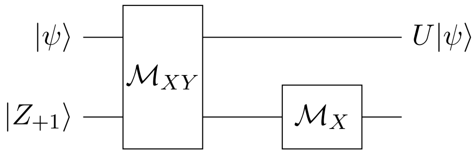
$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IZ \\ \hline XI \\ ZI \end{array} \overset{\mathcal{M}_{ZX}}{\longmapsto} \begin{array}{c} ZX \\ \hline XZ \\ ZI \end{array} \overset{\mathcal{M}_{IY}}{\longmapsto} \begin{array}{c} IY \\ \hline YI \\ ZI \end{array} \therefore U \sim P \tag{13}$$



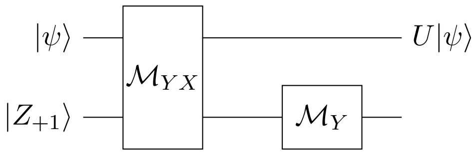
$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IZ \\ \hline XI \\ ZI \end{array} \overset{\mathcal{M}_{ZY}}{\longmapsto} \begin{array}{c} ZY \\ \hline XZ \\ ZI \end{array} \overset{\mathcal{M}_{IX}}{\longmapsto} \begin{array}{c} IX \\ \hline YI \\ ZI \end{array} \therefore U \sim P \tag{14}$$



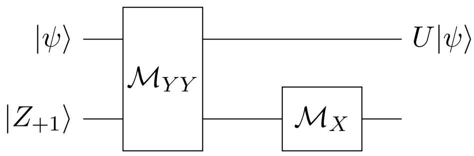
$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IZ \\ \hline XI \\ ZI \end{array} \overset{\mathcal{M}_{XX}}{\longmapsto} \begin{array}{c} XX \\ \hline XI \\ ZZ \end{array} \overset{\mathcal{M}_{IY}}{\longmapsto} \begin{array}{c} IY \\ \hline XI \\ YI \end{array} \therefore U \sim HPH \tag{15}$$



$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IZ \\ \hline XI \\ ZI \end{array} \xmapsto{\mathcal{M}_{XY}} \begin{array}{c} XY \\ \hline XI \\ ZZ \end{array} \xmapsto{\mathcal{M}_{IX}} \begin{array}{c} IX \\ \hline XI \\ YI \end{array} \therefore U \sim HPH \tag{16}$$



$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IZ \\ \hline XI \\ ZI \end{array} \xmapsto{\mathcal{M}_{YX}} \begin{array}{c} YX \\ \hline XZ \\ ZZ \end{array} \xmapsto{\mathcal{M}_{IY}} \begin{array}{c} IY \\ \hline ZI \\ XI \end{array} \therefore U \sim H \tag{17}$$



$$\begin{array}{c} S_{\text{init}} \\ \hline \overline{X} \\ \overline{Z} \end{array} = \begin{array}{c} IZ \\ \hline XI \\ ZI \end{array} \xmapsto{\mathcal{M}_{YY}} \begin{array}{c} YY \\ \hline XZ \\ ZZ \end{array} \xmapsto{\mathcal{M}_{IX}} \begin{array}{c} IX \\ \hline ZI \\ XI \end{array} \therefore U \sim H \tag{18}$$

2 Comments

It looks like each code involved in the gate must be distance 3. This is because one logical operator from each of the initial codes remains in the logical group of the intermediate code. If using a destructive measurement at the end, you could settle for having a single weight-2 logical, but there are no codes with two large logicals and one small (at least small ones with $k = 1$).

3 Two-Qubit Cliffords

It would be interesting to generalise this to two-qubit Cliffords (CNOTs, CZs, and shallow compiled gates), but there are a too many possible circuits for me to spam them all.