Ficha [illegible] 23/24

1. $\mu \cdot T u = id$

$\Leftrightarrow (\pi_1 \times \pi_2) \cdot (\langle id, id \rangle \times \langle id, id \rangle) = id$

$\Leftrightarrow (\pi_1 \cdot \langle id, id \rangle) \times (\pi_2 \cdot \langle id, id \rangle) = id$ $\{14\}$

$\Leftrightarrow id \times id = id$ $\{7 \ (\times 2)\}$

$\Leftrightarrow$ True $\{15\}$

$id = \mu \cdot u$

$\Leftrightarrow (\pi_1 \times \pi_2) \cdot \langle id, id \rangle = id$

$\Leftrightarrow \langle \pi_1, \pi_2 \rangle = id$ $\{11, 1 \ (\times 2)\}$

$\Leftrightarrow \pi_1 \cdot id = \pi_1 \wedge \pi_2 \cdot id = \pi_2$ $\{6\}$

$\Leftrightarrow$ True $\{1 \ (\times 2)\}$

2. F diz-se um functor se: $\begin{cases} F \ id = id \\ F(f \cdot g) = (F f) \cdot (F g) \end{cases}$

$$
\begin{aligned}
H \ id &= (F \ id) + (G \ id) && \{\text{def } H\} \\
&= id + id && \{\text{def } F, G\} \\
&= id && \{26\}
\end{aligned}
$$

$$
\begin{aligned}
H(f \cdot g) &= F(f \cdot g) + G(f \cdot g) && \{\text{def } H\} \\
&= id + (f \cdot g) && \{\text{def } F, G\} \\
&= (id \cdot id) + (f \cdot g) && \{1\} \\
&= (id + f) \cdot (id + g) && \{25\} \\
&= ((F f) + (G f)) \cdot ((F g) + (G g)) && \{\text{def } F, G \ (\times 2)\} \\
&= (H f) \cdot (H g)
\end{aligned}
$$

Logo, H é um functor.

$$
\begin{aligned}
K \ id &= (G \ id) \times (F \ id) && \{\text{def } K\} \\
&= id \times id && \{\text{def } G, F\} \\
&= id && \{15\}
\end{aligned}
$$

3

$$
\begin{aligned}
K(f\cdot g) &= G(f\cdot g)\times F(f\cdot g) && \{\text{Def } K\}\\
&= (f\cdot g)\times id && \{\text{Def } G, F\}\\
&= (f\cdot g)\times (id\cdot id) && \{1\}\\
&= (f\times id)\cdot(g\times id) && \{14\}\\
&\cancel{((G f)\times(G id))\cdot((G}\\
&= ((G f)\times(F f))\cdot((G g)\times(F g)) && \{\text{Def } G, F\ (\times 2)\}\\
&= (K f)\cdot(K g) && \{\text{Def } K\ (\times 2)\}
\end{aligned}
$$

Logo, K é um funtor.

(3.) Se F e G são funtores, então sabemos que:

$$
\begin{cases} F\ id = id \\ F(f\cdot g) = (F f)\cdot(F g) \end{cases}
\qquad
\begin{cases} G\ id = id \\ G(f\cdot g) = (G f)\cdot(G g) \end{cases}
$$

$$
\begin{aligned}
H\ id &= (F\cdot G)\ id && \{\text{Def } H\}\\
&= F\ (G\ id) && \{73\}\\
&= F\ id && \{G \text{ funtor logo } G\ id = id\}^{45}\\
&= id && \{F \text{ funtor logo } F\ id = id\}^{45}
\end{aligned}
$$

$$
\begin{aligned}
H(f\cdot g) &= (F\cdot G)(f\cdot g) && \{\text{Def } H\}\\
&= F(G(f\cdot g)) && \{73\}\\
&= F((G f)\cdot(G g)) && \{G \text{ funtor logo } G(f\cdot g) = Gf\cdot Gg\}^{44}\\
&= F(G f)\cdot F(G g) && \{F \text{ funtor logo } F(f\cdot g) = Ff\cdot Fg\}^{44}\\
&= ((F\cdot G) f)\cdot((F\cdot G) g) && \{73\ (\times 2)\}\\
&= (H f)\cdot(H g) && \{\text{Def } H\ (\times 2)\}
\end{aligned}
$$

Logo, H é um funtor.

(4.)

$$\langle\langle f, g\rangle, j\rangle = (\!|\langle\langle h, k\rangle, l\rangle|\!)$$

$$\Leftrightarrow \langle\langle f, g\rangle, j\rangle\cdot in = \langle\langle h, k\rangle, l\rangle\cdot F\langle\langle f, g\rangle, j\rangle \qquad \{46\}$$

$$\Leftrightarrow \begin{cases} \langle f, g\rangle\cdot in = \langle h, k\rangle\cdot F\langle\langle f, g\rangle, j\rangle \\ j\cdot in = l\cdot F\langle\langle f, g\rangle, j\rangle \end{cases} \qquad \{9\ (\times 2), 16\}$$

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$$\Leftrightarrow \begin{cases} g \cdot in = k \cdot F \langle\langle f, g\rangle, j\rangle \\ j \cdot in = l \cdot F \langle\langle f, g\rangle, j\rangle \end{cases}$$ {g(x2), 16}

(5.)

$$loop\ (a, b) = (2 + b,\ 2 - a)$$

$$= ((2+)\ b,\ (2+)\ (neg\ a))$$ {Notação Prefixa $neg\ x = -x$}

$$= ((2+)\ b,\ ((2+) \cdot neg)\ a)$$ {73}

$$= ((2+) \times ((2+) \cdot neg))\ (b, a)$$ {78}

$$= ((2+) \times ((2+) \cdot neg))\ (swap\ (a, b))$$ {2f swap}

$$= ((2+) \times ((2+) \cdot neg)) \cdot swap\ (a, b)$$ {73}

Logo, $\boxed{loop = ((2+) \times ((2+) \cdot neg)) \cdot swap}$ {tirando variáveis (72)}

$aux = for\ loop\ (4, -2)$ {Como queremos aplicar a lei de recursividade mútua, o gene do cata tem de ser um split}

{2f for} $= (\![\ (\underline{4, -2}),\ ((2+) \times ((2+) \cdot neg)) \cdot swap\ ]\!)$

{10, 9, $(a,b) = \langle \underline{a}, \underline{b}\rangle$} $= (\![\ \langle \underline{4}, \underline{(-2)}\rangle,\ \langle (2+) \cdot \pi_1 \cdot swap,\ ((2+) \cdot neg) \cdot \pi_2 \cdot swap\rangle\ ]\!)$

{28} $= (\!| \langle [\underline{4},\ (2+) \cdot \pi_1 \cdot swap],\ [\underline{(-2)},\ ((2+) \cdot neg) \cdot \pi_2 \cdot swap] \rangle |\!)$

Logo,

$$\langle f, g\rangle = (\!| \langle [\underline{4},\ (2+) \cdot \pi_1 \cdot swap],\ [\underline{(-2)},\ ((2+) \cdot neg) \cdot \pi_2 \cdot swap] \rangle |\!)$$

$$\Leftrightarrow \begin{cases} f \cdot in = [\underline{4},\ (2+) \cdot \pi_1 \cdot swap] \cdot F \langle f, g\rangle \\ g \cdot in = [\underline{(-2)},\ ((2+) \cdot neg) \cdot \pi_2 \cdot swap] \cdot F \langle f, g\rangle \end{cases}$$ {53}

$$\Rightarrow \begin{cases} f \cdot [\underline{0}, succ] = [\underline{4},\ (2+) \cdot \pi_1 \cdot swap] \cdot (id + \langle f, g\rangle) \\ g \cdot [\underline{0}, succ] = [\underline{(-2)},\ ((2+) \cdot neg) \cdot \pi_2 \cdot swap] \cdot (id + \langle f, g\rangle) \end{cases}$$ {2f in, F (x2)}

$$\Leftrightarrow \begin{cases} \begin{cases} f \cdot \underline{0} = \underline{4} \\ f \cdot succ = (2+) \cdot \pi_1 \cdot swap \cdot \langle f, g\rangle \end{cases} \\ \begin{cases} g \cdot \underline{0} = \underline{(-2)} \\ g \cdot succ = (2+) \cdot neg \cdot \pi_2 \cdot swap \cdot \langle f, g\rangle \end{cases} \end{cases}$$ {20, 22, 27, 1} (x2)

$$\Rightarrow \begin{cases} \begin{cases} f\ 0 = 4 \\ f\ (n+1) = 2 + g\ n \end{cases} \\ \begin{cases} g\ 0 = -2 \\ g\ (n+1) = 2 - f\ n \end{cases} \end{cases}$$ {72, 73, 75, 77, 79} {72, 73, 75, 77, 79}

6.

$$\begin{cases} impar\ 0 = false \\ impar\ (n+1) = par\ n \end{cases}$$

$$\Rightarrow \begin{cases} impar\ (\underline{0}\ n) = \underline{false}\ n \\ impar\ (succ\ n) = par\ n \end{cases} \qquad \{75(\times 2)\}$$

$$\Rightarrow \begin{cases} impar \cdot \underline{0} = \underline{false} \\ impar \cdot succ = par \end{cases} \qquad \{73(\times 2), 72(\times 2)\}$$

$$\Leftrightarrow impar \cdot in = [\underline{false}, par] \qquad \{18(\times 2), 17\}$$

$$\Rightarrow impar \cdot in = [\underline{false} \cdot id, \pi_2 \cdot \langle impar, par\rangle] \qquad \{1, 7\}$$

$$\Rightarrow impar \cdot in = [\underline{false}, \pi_2] \cdot (id + \langle impar, par\rangle) \qquad \{22\}$$

$$\Leftrightarrow impar \cdot in = [\underline{false}, \pi_2] \cdot F\ \langle impar, par\rangle \qquad \{def\ F\}$$

Pela mesma lógica, chegamos a:

$$par \cdot in = [\underline{True}, \pi_1] \cdot F\ \langle impar, par\rangle$$

$$\begin{cases} impar \cdot in = [\underline{false}, \pi_2] \cdot F\ \langle impar, par\rangle \\ par \cdot in = [\underline{True}, \pi_1] \cdot F\ \langle impar, par\rangle \end{cases}$$

$$\Leftrightarrow \langle impar, par\rangle = (\!| \langle [\underline{false}, \pi_2], [\underline{True}, \pi_1] \rangle |\!) \qquad \{53\}$$

$$\Leftrightarrow \langle impar, par\rangle = (\!| [\langle \underline{false}, \underline{True}\rangle, \langle \pi_2, \pi_1\rangle] |\!) \qquad \{28\}$$

$$\Leftrightarrow \langle impar, par\rangle = (\!| [\underline{(false, True)}, swap] |\!) \qquad \{\langle \underline{a}, \underline{b}\rangle = \underline{(a,b)}\}$$

$$\Leftrightarrow \langle impar, par\rangle = for\ swap\ (false, True) \qquad \{def\ for\}$$

7.

Olhando para as definições de insg, as únicas diferenças estão em $(n+1)$ que é substituído por $(fsuc\ n)$. Logo, temos de provar que:

$$n+1 = fsuc\ n \Leftrightarrow \boxed{fsuc = succ}$$

$$\begin{cases} fsuc\ 0 = 1 \\ fsuc\ (n+1) = (fsuc\ n)+1 \end{cases} \Leftrightarrow \begin{cases} fsuc \cdot \underline{0} = \underline{1} \\ fsuc \cdot succ = succ \cdot fsuc \end{cases} \qquad \{75(\times 2), 73(\times 3), 72(\times 2)\}$$

$$\Leftrightarrow fsuc \cdot in = [\underline{1}, succ \cdot fsuc] \qquad \{18(\times 2), 17\}$$

$$\Leftrightarrow \cancel{fsuc = k}\ fsuc \cdot in = [\underline{1}, succ] \cdot id + fsuc \qquad \{1, 22\}$$

$$\Leftrightarrow \boxed{fsuc = (\!| [\underline{1}, succ] |\!)}$$

$\boxed{fsuc = succ}$

$\Leftrightarrow succ = (\![[\underline{1}, succ]]\!)$ {def fsuc}

$\Leftrightarrow succ \cdot in = [\underline{1}, succ] \cdot F\ succ$ {46}

$\Leftrightarrow succ \cdot in = [\underline{1}, succ] \cdot (id + succ)$ {Def F N_0}

$\Leftrightarrow \begin{cases} succ \cdot \underline{0} = \underline{1} \\ succ \cdot succ = succ \cdot succ \end{cases}$ {20, 22, 1, 27}

$\Leftrightarrow 1 = 1$ {72, 73, 75}

$\Leftrightarrow$ True

Logo, fsuc = succ e, por isso, as duas definições de insg são válidas.

$$\begin{cases} \begin{cases} insg\ 0 = [\,] \\ insg\ (n+1) = (fsuc\ n) : insg\ n \end{cases} \\ \begin{cases} fsuc\ 0 = 1 \\ fsuc\ (n+1) = (fsuc\ n) + 1 \end{cases} \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{cases} insg \cdot \underline{0} = nil \\ insg \cdot succ = cons \cdot \langle fsuc, insg \rangle \end{cases} & \{75(\times 2), 73(\times 3), 77, 72(\times 2)\} \\ \begin{cases} fsuc \cdot \underline{0} = \underline{1} \\ fsuc \cdot succ = succ \cdot fsuc \end{cases} & \{75(\times 2), 73(\times 3), 72(\times 2)\} \end{cases}$$

$\Leftrightarrow \begin{cases} insg \cdot in = [nil, cons \cdot \langle fsuc, insg \rangle] \\ fsuc \cdot in = [\underline{1}, succ \cdot fsuc] \end{cases}$ {18($\times$4), 17($\times$2)}

$\Leftrightarrow \begin{cases} insg \cdot in = in_2 \cdot F \langle fsuc, insg \rangle & \{1, 22\} \\ fsuc \cdot in = [\underline{1}, succ \cdot \pi_1] \cdot F \langle fsuc, insg \rangle & \{7, 1, 22\} \end{cases}$

~~insg~~

$\Leftrightarrow \langle fsuc, insg \rangle = (\![\langle [\underline{1}, succ \cdot \pi_1], in_2 \rangle]\!)$ {53}

$\Leftrightarrow \langle fsuc, insg \rangle = (\![[\langle \underline{1}, \underline{[\,]} \rangle, \langle succ \cdot \pi_1, cons \rangle]]\!)$ {28}

$\Leftrightarrow \langle fsuc, insg \rangle = (\![[\underline{(1, [\,])}, \langle succ \cdot \pi_1, cons \rangle]]\!)$ {$\langle \underline{a}, \underline{b} \rangle = \underline{(a, b)}$}

$\Leftrightarrow \langle fsuc, insg \rangle = for\ \langle (+1) \cdot \pi_1, cons \rangle\ (1, [\,])$ {Def for}

$\Leftrightarrow insg = \pi_2 \cdot (for\ \langle (+1) \cdot \pi_1, cons \rangle\ (1, [\,]))$ {6}

(8.)

$$\left\{\begin{array}{l}\left\{\begin{array}{l} f_1\ [\,] = [\,] \\ f_1\ (h:t) = h : (f_2\ t)\end{array}\right. \\ \left\{\begin{array}{l} f_2\ [\,] = [\,] \\ f_2\ (h:t) = f_1\ t\end{array}\right.\end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l}\left\{\begin{array}{l} f_1 \cdot nil = nil \\ f_1 \cdot cons = cons \cdot (id \times f_2)\end{array}\right. \\ \left\{\begin{array}{l} f_2 \cdot nil = nil \\ f_2 \cdot cons = f_1 \cdot \pi_2\end{array}\right.\end{array}\right.$$

{75(x2), 73(x3), 78, 72(x2)}

{75(x2), 73(x2), 79, 72(x2)}

$$\Leftrightarrow \left\{\begin{array}{l} f_1 \cdot in = [nil, cons \cdot (id \times f_2)] \\ f_2 \cdot in = [nil, f_1 \cdot \pi_2]\end{array}\right.$$

{18(x2), 17(x2)}

$$\Leftrightarrow \left\{\begin{array}{l} f_1 \cdot in = in \cdot F\ f_2 \\ f_2 \cdot in = [nil, \pi_2 \cdot (id \times f_1)]\end{array}\right.$$

{1, 13, 22}

$$\Leftrightarrow \left\{\begin{array}{l} f_1 \cdot in = in \cdot F\ f_2 \\ f_2 \cdot in = [nil, \pi_2] \cdot F\ f_1\end{array}\right.$$

{1, 22}

$$\Leftrightarrow \left\{\begin{array}{l} f_1 \cdot in = in \cdot F\ (\pi_2 \cdot \langle f_1, f_2\rangle) \\ f_2 \cdot in = [nil, \pi_2] \cdot F\ (\pi_1 \cdot \langle f_1, f_2\rangle)\end{array}\right.$$

{7(x2)}

$$\Leftrightarrow \left\{\begin{array}{l} f_1 \cdot in = in \cdot F\pi_2 \cdot F\langle f_1, f_2\rangle \\ f_2 \cdot in = [nil, \pi_2] \cdot F\pi_1 \cdot F\langle f_1, f_2\rangle\end{array}\right.$$

{42(x2)}

$$\Leftrightarrow \left\{\begin{array}{l} f_1 \cdot in = [nil, cons \cdot (id \times \pi_2)] \cdot F\langle f_1, f_2\rangle \\ f_2 \cdot in = [nil, \overset{\pi_2}{\cancel{cons}} \cdot (id \times \pi_1)] \cdot F\langle f_1, f_2\rangle\end{array}\right.$$

{22(x2), 1(x2)}

{53}

$$\Leftrightarrow \langle f_1, f_2\rangle = (\!|\langle [nil, cons \cdot (id \times \pi_2)], [nil, \overset{\pi_2}{\cancel{cons}} \cdot (id \times \pi_1)]\rangle|\!)$$

$$\Leftrightarrow \boxed{\langle f_1, f_2\rangle = (\!|[\langle nil, nil\rangle, \langle cons \cdot (id \times \pi_2), \pi_1 \cdot \pi_2\rangle]|\!)}$$

{28, 13}

$$\begin{tikzcd} A^* \arrow[d, "\langle f_1, f_2\rangle"'] \arrow[r, shift left, "out"'] & 1 + A \times A^* \arrow[l, shift left, "in"'] \arrow[d, "F\langle f_1, f_2\rangle"] \\ A^* \times A^* & 1 + A \times (A^* \times A^*) \arrow[l, "{[\langle nil, nil\rangle, \langle cons \cdot (id \times \pi_2), \pi_1 \cdot \pi_2\rangle]}"] \end{tikzcd}$$

f_1 devolve apenas os elementos em indices pares da lista, enquanto que f_2 faz o mesmo para os impares.