Contact Mechanics

Gonçalo Barbosa dos Santos

U.PORTO

FEUP FACULDADE DE ENGENHARIA
UNIVERSIDADE DO PORTO

Computational Contact Mechanics

Supervisor: Marco Parente

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Chapter 1

Introduction

Understanding the behavior of solid objects in touch is the focus of the fundamental research area of contact mechanics, which falls under the umbrella of mechanics. It is a significant area of research with several applications in fields ranging from biology and geology to engineering and materials science. Understanding the interactions between two or more solid surfaces under various loading and environmental circumstances, such as friction, wear, adhesion, and deformation, is a key component of the study of contact mechanics. The purpose of this paper is to present an overview of contact mechanics' guiding concepts, methodologies, and practical applications. It will go through the fundamental ideas of contact mechanics, including contact area, pressure, and deformation, as well as several contact models and experimental methods.

Chapter 2

Contact Mechanics Methods

2.1 Lagrange Multipliers

In optimization problems, Lagrange multipliers are used to find the extremes of a function subject to one or more constraints BERTSEKAS (1982). Let $f(x)$ be the function to be optimized and let $g_i(x) = 0$ be the constraints for $i = 1, 2, ..., m$. Then the optimization problem can be formulated as:

$$\text{maximize/minimize } f(x) \quad \text{subject to } g_i(x) = 0, i = 1, 2, ..., m$$

To find the extreme subject to the constraints, we form the Lagrangian function:

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

where λ_i are the Lagrange multipliers.

The Lagrange multiplier method involves finding the critical points of the Lagrangian function with respect to x and λ. This leads to a system of equations:

$$\nabla_x \mathcal{L}(x, \lambda) = 0 \quad \nabla_\lambda \mathcal{L}(x, \lambda) = 0$$

where ∇_x and ∇_λ denote the gradients with respect to x and λ, respectively.

Solving these equations simultaneously gives the optimal values of x and λ that satisfy the constraints and optimize the objective function.

The first equation $\nabla_x \mathcal{L}(x, \lambda) = 0$ gives the necessary condition for optimality:

$$\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0$$

This equation states that the gradient of the objective function $\nabla f(x)$ is proportional to the sum of the gradients of the constraints $\nabla g_i(x)$, with proportionality constants given by the Lagrange multipliers λ_i.

The second equation $\nabla_\lambda \mathcal{L}(x, \lambda) = 0$ gives the constraints:

$$g_i(x) = 0, i = 1, 2, ..., m$$

2.2   Lagrange Method in Contact Mechanics

The Lagrange multiplier method is a mathematical technique used to find the optimal values of a function subject to constraints. It is widely used in many areas of physics and engineering, including computational contact mechanics. In the context of computational contact mechanics, the Lagrange multiplier method can be used to find the contact forces between two bodies in contact .

According to Wriggers (06 October 2006) the Lagrange multiplier method is based on the idea of adding the constraints to the objective function using Lagrange multipliers. The resulting function, called the Lagrangian, is then optimized with respect to both the original variables and the Lagrange multipliers.

To explain this method in more detail, let's consider a simple optimization problem with a single constraint. The problem can be formulated as follows:

minimize f(x)

subject to g(x) = 0

where f(x) is the objective function and g(x) is the constraint function. The Lagrange multiplier method involves introducing a new variable, λ, called the Lagrange multiplier, and forming the Lagrangian as follows:

L(x, λ) = f(x) - λ g(x)

The Lagrangian is a function of both the original variables, x, and the Lagrange multiplier, λ. The optimization problem is then reformulated as follows:

minimize L(x, λ)

with respect to x and λ

The solution to this problem can be found by setting the partial derivatives of L with respect to x and λ equal to zero, and solving the resulting system of equations. This leads to a set of equations, called the Karush-Kuhn-Tucker (KKT) conditions, which must be satisfied by the optimal solution.

In the context of computational contact mechanics, the Lagrange multiplier method can be used to find the contact forces between two bodies in contact. The contact problem can be formulated as an optimization problem with a set of contact constraints, and the Lagrange multiplier method can be applied to find the contact forces that satisfy the contact constraints.

Let's consider the problem of finding the contact forces between two bodies in contact. The contact problem can be formulated as an optimization problem as follows:

- - The contact constrains: C(F) = 0

- - The balance of linear momentum: Ma = F

where F is the vector of contact forces, $C(F)$ is the vector of contact constraints that describe the contact conditions (e.g. non-penetration, Coulomb friction, etc.), M is the mass matrix, a is the vector of accelerations, and the balance of linear momentum represents the conservation of linear momentum.

To solve this optimization problem, the Lagrange multiplier method is used. The method involves introducing a set of Lagrange multipliers, λ, associated with the contact constraints. The Lagrangian of the problem can then be formulated as follows:

$$L(F, \lambda) = F^T M a - \lambda^T C(F) \tag{2.1}$$

where the superscript T represents the transpose of the vector/matrix.

The optimization problem can then be reformulated as follows:

$$\text{minimize } L(F, \lambda) \quad \text{with respect to } F \text{ and } \lambda \tag{2.2}$$

The solution to this problem can be found by setting the partial derivatives of L with respect to F and λ equal to zero and solving the resulting system of equations. The KKT conditions for this problem are given by:

$$Ma + \frac{\partial C(F)}{\partial F}^T \lambda = 0 \tag{2.3}$$

$$C(F) = 0 \tag{2.4}$$

$$\lambda \geq 0 \tag{2.5}$$

$$\lambda^T C(F) = 0 \tag{2.6}$$

where $\frac{\partial C(F)}{\partial F}$ is the Jacobian matrix of the contact constraints with respect to the contact forces.

These conditions represent the necessary and sufficient conditions for the optimal solution. The first condition represents the balance of linear momentum, the second condition represents the contact constraints, and the last two conditions represent the non-negativity and complementary slackness conditions for the Lagrange multipliers.
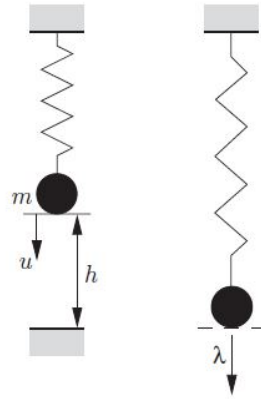
Figure 2.1: Point mass supported by a spring and free body diagram for the Lagrange multiplier method.

The Lagrange multiplier method adds to the energy of the system:

$$\Pi(u) = \frac{1}{2} k \cdot u^2 - m \cdot g \cdot u \tag{2.7}$$

a term which contains the constraint and yields:

$$\Pi(u, \lambda) = \frac{1}{2} \cdot k \cdot u^2 - m \cdot g \cdot u + \lambda \cdot c(u) \tag{2.8}$$

Lagrange multiplier λ is equivalent to the reaction force fR. The variation of 2.7 leads to two equations, since and $\delta\lambda$ can be varied independently:

$$ku\delta u - mg\delta u - \lambda\delta u = 0 \tag{2.9}$$

$$c(u)\delta\lambda = 0 \tag{2.10}$$

The first equation represents the equilibrium for the point mass including the reaction force when it touches the rigid surface 2.1, and the second equation states the fulfillment of the kinematical constraint equation for contact: u = h. Due to that, the variation is no longer restricted, and one can solve for Lagrange multiplier λ which is equivalent with the reaction force RN.

$$\lambda = kh - mg = RN \tag{2.11}$$

However, $RN \leqslant 0$ has to be checked before 2.11. If this condition is not met, and hence an adhesion force is computed, then the assumption of contact no longer holds. This means the inequality constraint is inactive and the correct solution can be computed as $u = \frac{mg}{k}$; furthermore, the reaction force or Lagrange multiplier is zero Wriggers (06 October 2006).

2.3   Penalty Method in Contact Mechanics

In computational contact mechanics, the penalty method is another commonly used technique for solving contact problems. The penalty method involves adding penalty terms to the governing equations to enforce the contact constraints.

Considering the problem before, in the penalty method, penalty terms are added to the governing equations to enforce the contact constraints. The modified optimization problem can be formulated as follows:

- minimize F

- subject to

- - The modified contact constraints: $C(F) \leq \varepsilon$

- - The balance of linear momentum: Ma = F

where ε is a penalty parameter that determines the level of enforcement of the contact constraints. The penalty terms can be formulated as follows:

$$P(F) = \frac{\beta}{2} \sum_{i=1}^{n} max(C_i(F) - \varepsilon, 0)^2$$

where β is a penalty parameter and n is the number of contact constraints.
The modified optimization problem can then be reformulated as follows:

$$- \text{The balance of linear momentum: } Ma = F + \frac{\partial P(F)}{\partial F} \tag{2.12}$$

where $\frac{\partial P(F)}{\partial F}$ is the gradient of the penalty terms with respect to the contact forces.

The penalty method can be effective in enforcing the contact constraints, but it can also lead to numerical issues such as stiffness and ill-conditioning. The choice of penalty parameters can have a significant impact on the accuracy and stability of the method.
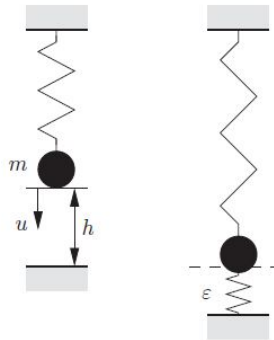

Figure 2.2: Point mass supported by a spring and a penalty spring due to the penalty term

As can be seen in Figure 2.6 the penalty parameter can be interpreted as a spring stiffness in the contact interface between point mass and rigid support. This is due to the fact that the energy of the penalty term has the same structure as the potential energy of a simple spring.

Here for an active constraint one adds a penalty term to the energy as follows:

$$\Pi(u) = \frac{1}{2}ku^2 - mgu + \frac{1}{2}\varepsilon[c(u)]^2 \quad \text{with } \varepsilon > 0 \tag{2.13}$$

The penalty parameter ε can be interpreted as a spring stiffness in the contact interface between point mass and rigid support. This is due to the fact that the energy of the penalty term has the same structure as the potential energy of a simple spring. The variation of (2.21) yields for the assumption of contact

$$ku\delta u - mg\delta u - \varepsilon c(u)\delta u = 0, \tag{2.14}$$

from which the solution

$$u = (mg + \varepsilon h)/(k + \varepsilon) \tag{2.15}$$

can be derived. The value of the constraint equation is then

$$c(u) = h - u = \frac{kh - mg}{k + \varepsilon}. \tag{2.16}$$

Since $mg \geq kh$ in the case of contact, equation 2.16 means that a penetration of the point mass into the rigid support occurs, which is physically equivalent to a compression of the spring. Note that the penetration depends upon the penalty parameter. The constraint equation is only fulfilled in the limit $\varepsilon \to \infty \Rightarrow c(u) \to 0$. Hence, in the penalty method we can distinguish two limiting cases:

1. $\varepsilon \to \infty \implies u - h \to 0$, which means that one approaches the correct solution for very large penalty parameters. Intuitively, this is clear since that means the penalty spring stiffness is very large, and hence only very small penetration occurs. 2. $\varepsilon \to 0$ represents the unconstrained solution, and thus is only valid for inactive constraints. In the case of contact, a solution with a very small penalty parameter ε leads to a high penetration, see 2.16

The reaction force for a penalty method is computed (2.14) from $R_N = \varepsilon c(u)$. For this example, one arrives with 2.16 at

$$R_N = \lambda = \varepsilon c(u) = \frac{\varepsilon}{k + \varepsilon}(kh - mg), \tag{2.17}$$

which in the limit $\varepsilon \to \infty$ yields the correct solution obtained with tue LAGRANGE multiplier method Wriggers (06 October 2006).

2.4   Matlab Implementation

Using the previous theorems, we can plot in Matlab the Energy of the system varying the
Penalty parameter, on the Penalty method, and the Lagrange parameter, on the Lagrange method.
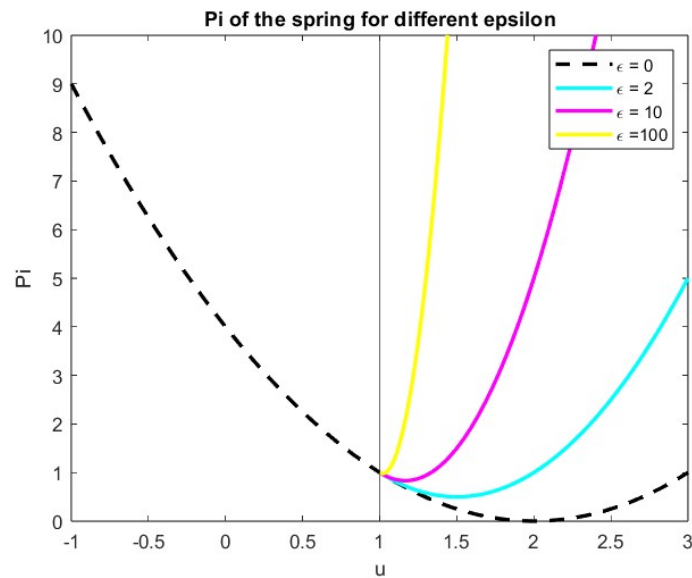
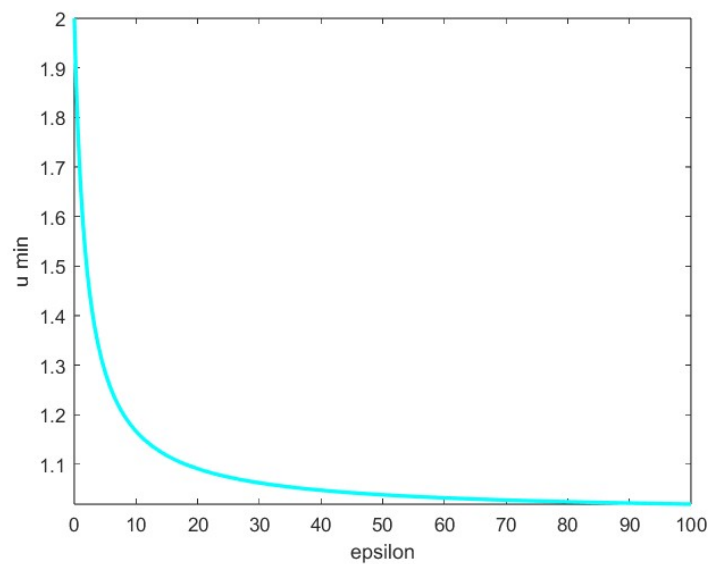Figure 2.3: Plot of the Energy - Penalty method

Figure 2.4: Plot of the displacement - Penalty method
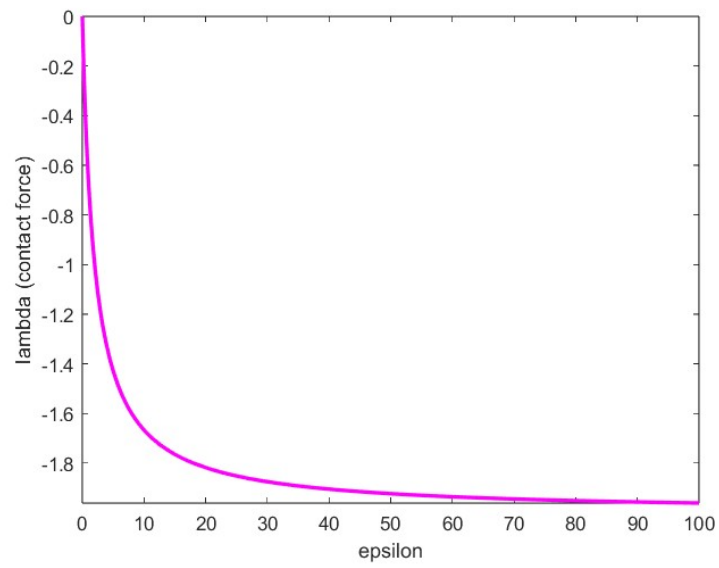
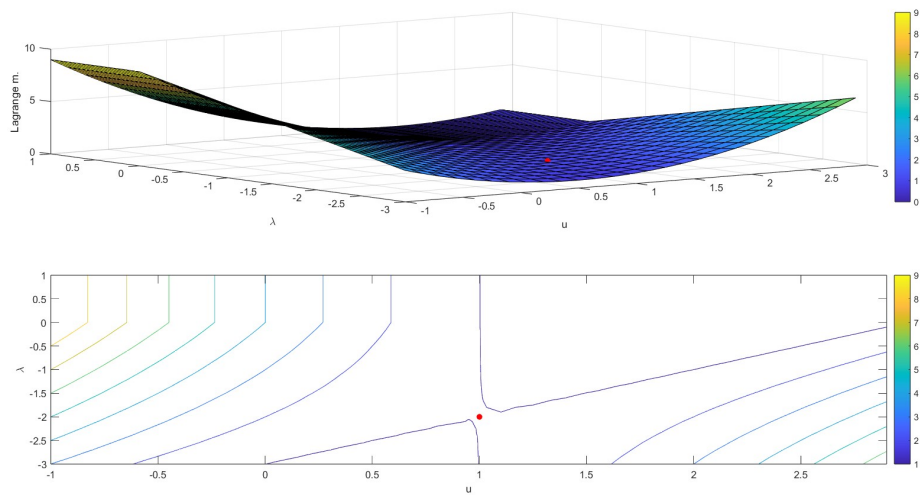Figure 2.5: Plot of the contact force - Penalty method

Figure 2.6: Plot of the Energy - Lagrange method

As we can see on the Lagrange method the solution to the problem: u = 1, λ = -2: the contact is established.

References

DIMITRI P. BERTSEKAS. Chapter 4 - exact penalty methods and lagrangian methods. In DIMITRI P. BERTSEKAS, editor, *Constrained Optimization and Lagrange Multiplier Methods*, pages 179–301. Academic Press, 1982. ISBN 978-0-12-093480-5. doi: https://doi.org/10.1016/B978-0-12-093480-5.50008-8. URL https://www.sciencedirect.com/science/article/pii/B9780120934805500088.

Peter Wriggers. *Computational Contact Mechanics Second Edition*. Springer edition, 06 October 2006. ISBN 978-3-540-32609-0. doi: https://doi.org/10.1007/978-3-540-32609-0.