FACULDADE DE ENGENHARIA DA UNIVERSIDADE DO PORTO

Contact Mechanics

Gonçalo Barbosa dos Santos

Computational Contact Mechanics

Supervisor: Marco Parente

April 19, 2023

Contact Mechanics

Gonçalo Barbosa dos Santos

Computational Contact Mechanics

April 19, 2023

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Understanding the behavior of solid objects in touch is the focus of the fundamental research area of contact mechanics, which falls under the umbrella of mechanics. It is a significant area of research with several applications in fields ranging from biology and geology to engineering and materials science. Understanding the interactions between two or more solid surfaces under various loading and environmental circumstances, such as friction, wear, adhesion, and deformation, is a key component of the study of contact mechanics. The purpose of this paper is to present an overview of contact mechanics' guiding concepts, methodologies, and practical applications. It will go through the fundamental ideas of contact mechanics, including contact area, pressure, and deformation, as well as several contact models and experimental methods.

Chapter 2

Contact Mechanics Methods

2.1 Lagrange Multipliers

In optimization problems, Lagrange multipliers are used to find the extremes of a function subject to one or more constraints BERTSEKAS (1982). Let $f(x)$ be the function to be optimized and let $g_i(x) = 0$ be the constraints for $i = 1, 2, ..., m$. Then the optimization problem can be formulated as:

$$\text{maximize/minimize } f(x) \quad \text{subject to } g_i(x) = 0, i = 1, 2, ..., m$$

To find the extreme subject to the constraints, we form the Lagrangian function:

$$\mathscr{L}(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

where λ_i are the Lagrange multipliers.

The Lagrange multiplier method involves finding the critical points of the Lagrangian function with respect to x and λ. This leads to a system of equations:

$$\nabla_x \mathscr{L}(x, \lambda) = 0 \quad \nabla_\lambda \mathscr{L}(x, \lambda) = 0$$

where ∇_x and ∇_λ denote the gradients with respect to x and λ, respectively.

Solving these equations simultaneously gives the optimal values of x and λ that satisfy the constraints and optimize the objective function.

The first equation $\nabla_x \mathscr{L}(x, \lambda) = 0$ gives the necessary condition for optimality:

$$\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0$$

This equation states that the gradient of the objective function $\nabla f(x)$ is proportional to the sum of the gradients of the constraints $\nabla g_i(x)$, with proportionality constants given by the Lagrange multipliers λ_i.

The second equation $\nabla_\lambda \mathscr{L}(x, \lambda) = 0$ gives the constraints:

$$g_i(x) = 0, i = 1, 2, ..., m$$

2.2 Lagrange Method in Contact Mechanics

The Lagrange multiplier method is a mathematical technique used to find the optimal values of a function subject to constraints. It is widely used in many areas of physics and engineering, including computational contact mechanics. In the context of computational contact mechanics, the Lagrange multiplier method can be used to find the contact forces between two bodies in contact .

According to Wriggers (06 October 2006) the Lagrange multiplier method is based on the idea of adding the constraints to the objective function using Lagrange multipliers. The resulting function, called the Lagrangian, is then optimized with respect to both the original variables and the Lagrange multipliers.

To explain this method in more detail, let's consider a simple optimization problem with a single constraint. The problem can be formulated as follows:

minimize f(x)

subject to g(x) = 0

where f(x) is the objective function and g(x) is the constraint function. The Lagrange multiplier method involves introducing a new variable, λ, called the Lagrange multiplier, and forming the Lagrangian as follows:

L(x, λ) = f(x) - λ g(x)

The Lagrangian is a function of both the original variables, x, and the Lagrange multiplier, λ. The optimization problem is then reformulated as follows:

minimize L(x, λ)

with respect to x and λ

The solution to this problem can be found by setting the partial derivatives of L with respect to x and λ equal to zero, and solving the resulting system of equations. This leads to a set of equations, called the Karush-Kuhn-Tucker (KKT) conditions, which must be satisfied by the optimal solution.

In the context of computational contact mechanics, the Lagrange multiplier method can be used to find the contact forces between two bodies in contact. The contact problem can be formulated as an optimization problem with a set of contact constraints, and the Lagrange multiplier method can be applied to find the contact forces that satisfy the contact constraints.

Let's consider the problem of finding the contact forces between two bodies in contact. The contact problem can be formulated as an optimization problem as follows:

- - The contact constrains: C(F) = 0
- - The balance of linear momentum: Ma = F

where F is the vector of contact forces, $C(F)$ is the vector of contact constraints that describe the contact conditions (e.g. non-penetration, Coulomb friction, etc.), M is the mass matrix, a is the vector of accelerations, and the balance of linear momentum represents the conservation of linear momentum.

To solve this optimization problem, the Lagrange multiplier method is used. The method involves introducing a set of Lagrange multipliers, λ, associated with the contact constraints. The Lagrangian of the problem can then be formulated as follows:

$$L(F,\lambda) = F^T Ma - \lambda^T C(F) \tag{2.1}$$

where the superscript T represents the transpose of the vector/matrix.

The optimization problem can then be reformulated as follows:

$$\text{minimize } L(F,\lambda) \quad \text{with respect to } F \text{ and } \lambda \tag{2.2}$$

The solution to this problem can be found by setting the partial derivatives of L with respect to F and λ equal to zero and solving the resulting system of equations. The KKT conditions for this problem are given by:

$$Ma + \frac{\partial C(F)}{\partial F}^T \lambda = 0 \tag{2.3}$$

$$C(F) = 0 \tag{2.4}$$

$$\lambda \geq 0 \tag{2.5}$$

$$\lambda^T C(F) = 0 \tag{2.6}$$

where $\frac{\partial C(F)}{\partial F}$ is the Jacobian matrix of the contact constraints with respect to the contact forces.

These conditions represent the necessary and sufficient conditions for the optimal solution. The first condition represents the balance of linear momentum, the second condition represents the contact constraints, and the last two conditions represent the non-negativity and complementary slackness conditions for the Lagrange multipliers.

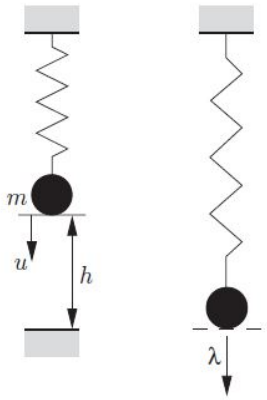

Figure 2.1: Point mass supported by a spring and free body diagram for the Lagrange multiplier method.

The Lagrange multiplier method adds to the energy of the system:

$$\Pi(u) = \frac{1}{2}k \cdot u^2 - m \cdot g \cdot u \tag{2.7}$$

a term which contains the constraint and yields:

$$\Pi(u,\lambda) = \frac{1}{2} \cdot k \cdot u^2 - m \cdot g \cdot u + \lambda \cdot c(u) \tag{2.8}$$

Lagrange multiplier λ is equivalent to the reaction force fR. The variation of 2.7 leads to two equations, since and $\delta\lambda$ can be varied independently:

$$ku\delta u - mg\delta u - \lambda\delta u = 0 \tag{2.9}$$

$$c(u)\delta\lambda = 0 \tag{2.10}$$

The first equation represents the equilibrium for the point mass including the reaction force when it touches the rigid surface 2.1, and the second equation states the fulfillment of the kinematical constraint equation for contact: u = h. Due to that, the variation is no longer restricted, and one can solve for Lagrange multiplier λ which is equivalent with the reaction force RN.

$$\lambda = kh - mg = RN \tag{2.11}$$

However, $RN \leqslant 0$ has to be checked before 2.11. If this condition is not met, and hence an adhesion force is computed, then the assumption of contact no longer holds. This means the inequality constraint is inactive and the correct solution can be computed as $u = \frac{mg}{k}$; furthermore, the reaction force or Lagrange multiplier is zero Wriggers (06 October 2006).

2.3 Penalty Method in Contact Mechanics

In computational contact mechanics, the penalty method is another commonly used technique for solving contact problems. The penalty method involves adding penalty terms to the governing equations to enforce the contact constraints.

Considering the problem before, in the penalty method, penalty terms are added to the governing equations to enforce the contact constraints. The modified optimization problem can be formulated as follows:

- minimize F
- subject to
- - The modified contact constraints: $C(F) \leq \varepsilon$
- - The balance of linear momentum: Ma = F

where ε is a penalty parameter that determines the level of enforcement of the contact constraints. The penalty terms can be formulated as follows:

$$P(F) = \frac{\beta}{2} \sum_{i=1}^{n} max(C_i(F) - \varepsilon, 0)^2$$

where β is a penalty parameter and n is the number of contact constraints.

The modified optimization problem can then be reformulated as follows:

$$\text{- The balance of linear momentum: } Ma = F + \frac{\partial P(F)}{\partial F} \tag{2.12}$$

where $\frac{\partial P(F)}{\partial F}$ is the gradient of the penalty terms with respect to the contact forces.

The penalty method can be effective in enforcing the contact constraints, but it can also lead to numerical issues such as stiffness and ill-conditioning. The choice of penalty parameters can have a significant impact on the accuracy and stability of the method.

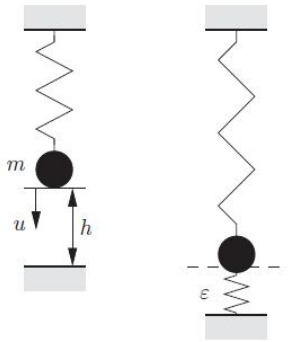

Figure 2.2: Point mass supported by a spring and a penalty spring due to the penalty term

As can be seen in Figure 2.6 the penalty parameter can be interpreted as a spring stiffness in the contact interface between point mass and rigid support. This is due to the fact that the energy of the penalty term has the same structure as the potential energy of a simple spring.

Here for an active constraint one adds a penalty term to the energy as follows:

$$\Pi(u) = \frac{1}{2}ku^2 - mgu + \frac{1}{2}\varepsilon[c(u)]^2 \quad \text{with } \varepsilon > 0 \tag{2.13}$$

The penalty parameter ε can be interpreted as a spring stiffness in the contact interface between point mass and rigid support. This is due to the fact that the energy of the penalty term has the same structure as the potential energy of a simple spring. The variation of (2.21) yields for the assumption of contact

$$ku\delta u - mg\delta u - \varepsilon c(u)\delta u = 0, \tag{2.14}$$

from which the solution

$$u = (mg + \varepsilon h)/(k + \varepsilon) \tag{2.15}$$

can be derived. The value of the constraint equation is then

$$c(u) = h - u = \frac{kh - mg}{k + \varepsilon}. \tag{2.16}$$

Since $mg \geq kh$ in the case of contact, equation 2.16 means that a penetration of the point mass into the rigid support occurs, which is physically equivalent to a compression of the spring. Note that the penetration depends upon the penalty parameter. The constraint equation is only fulfilled in the limit $\varepsilon \to \infty \Rightarrow c(u) \to 0$. Hence, in the penalty method we can distinguish two limiting cases:

1. $\varepsilon \to \infty \Longrightarrow u - h \to 0$, which means that one approaches the correct solution for very large penalty parameters. Intuitively, this is clear since that means the penalty spring stiffness is very large, and hence only very small penetration occurs. 2. $\varepsilon \to 0$ represents the unconstrained solution, and thus is only valid for inactive constraints. In the case of contact, a solution with a very small penalty parameter ε leads to a high penetration, see 2.16

The reaction force for a penalty method is computed (2.14) from $R_N = \varepsilon c(u)$. For this example, one arrives with 2.16 at

$$R_N = \lambda = \varepsilon c(u) = \frac{\varepsilon}{k + \varepsilon}(kh - mg), \tag{2.17}$$

which in the limit $\varepsilon \to \infty$ yields the correct solution obtained with tue LAGRANGE multiplier method Wriggers (06 October 2006).

2.4 Matlab Implementation

Using the previous theorems, we can plot in Matlab the Energy of the system varying the Penalty parameter, on the Penalty method, and the Lagrange parameter, on the Lagrange method.

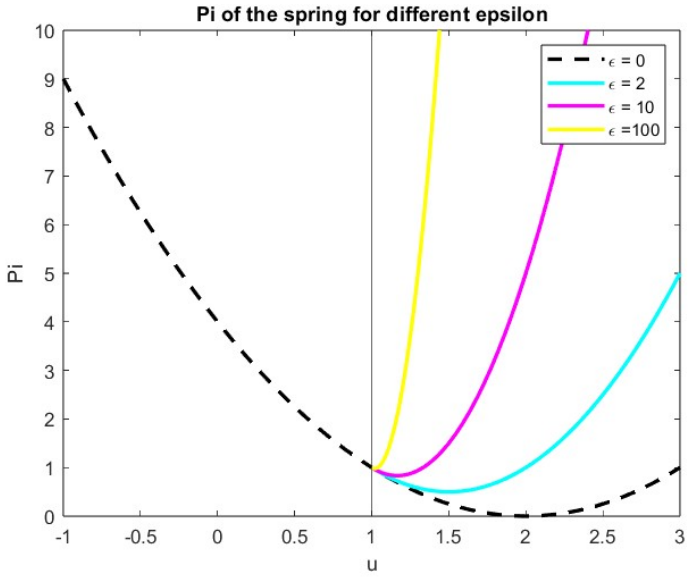

Figure 2.3: Plot of the Energy - Penalty method

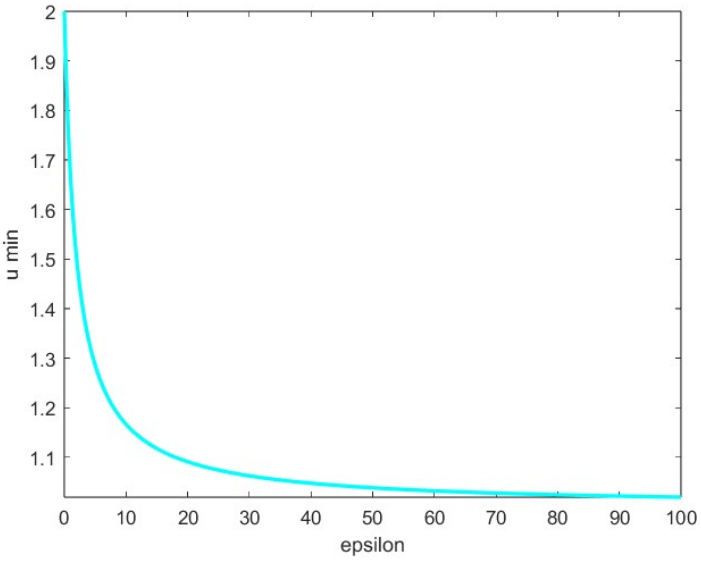

Figure 2.4: Plot of the displacement - Penalty method

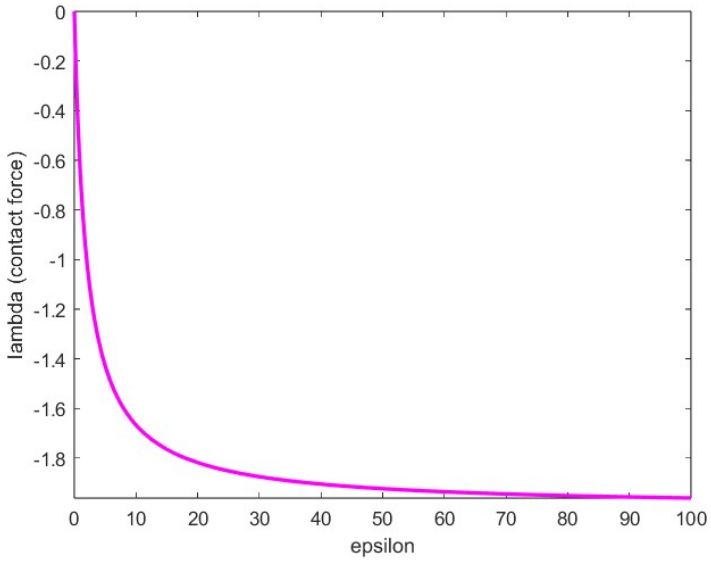

Figure 2.5: Plot of the contact force - Penalty method

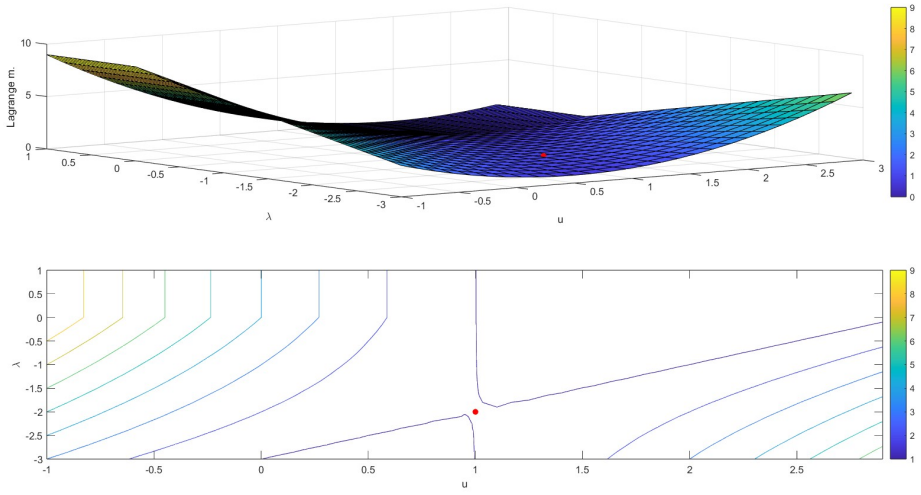

Figure 2.6: Plot of the Energy - Lagrange method

As we can see on the Lagrange method the solution to the problem: u = 1, λ = -2: the contact is established.

2.5 Augmented Lagrangian

The Augmented Lagrangian is a function used in constrained optimization problems. It is defined as:

$$Lc(x,\lambda) = f(x) + \lambda^T h(x) + \frac{c}{2}||h(x)||^2 \tag{2.18}$$

Where f(x) is the objective function, h(x) are the constraints, λ are the Lagrange multipliers, and c is a penalty parameter. This function is used to convert constrained optimization problems into unconstrained ones that can be solved using methods like the Quadratic Penalty Approach or the Method of Multipliers.

The Quadratic Penalty Approach, Barrier method, and Interior Point method are all techniques used to solve constrained optimization problems. The main difference between these methods is the way they handle the constraints.

The Barrier method adds a barrier function to the objective function that penalizes points outside of the feasible region. This method iteratively solves a sequence of unconstrained problems with decreasing values of the barrier parameter until a solution is found.

The Interior Point method, on the other hand, uses a logarithmic barrier function to transform the constrained optimization problem into an unconstrained one. It then solves this problem using Newton's method or other similar techniques.

The Quadratic Penalty Approach adds a quadratic penalty term to the objective function that penalizes points outside of the feasible region. This approach iteratively solves unconstrained minimization problems with increasing values of the penalty parameter until a solution is found.

The Method of Multipliers is a technique used to solve constrained optimization problems. Here is an example problem that can be solved using this method:

Minimize f(x) = $x1^2$ + $x2^2$ subject to the constraint h(x) = x1 + x2 - 1 = 0.

To solve this problem using the Method of Multipliers, we first form the Lagrangian function:

$$L(x,\lambda) = f(x) + \lambda h(x), \tag{2.19}$$

Where λ is the Lagrange multiplier. We then use the Method of Multipliers to solve the following unconstrained minimization problem in 2.18 where c is a penalty parameter. We iteratively solve this problem with increasing values of c until a solution is found. Singh

2.6 Contact Search

The search for contact between solids is a significant challenge in computational contact mechanics. It involves two phases: spatial search and contact detection. In the spatial search, there are two problem classes:

The first involves small deformations where the topology of the system is fixed, and all possible connections and neighboring pairs are known. In this case, only contact detection is necessary, which itself can be evaluated efficiently.

The second problem class involves large deformations and arbitrary movements of contact points over adjacent elements of the contacting body. In this case, the spatial search must be able to locate a slave segment far from the neighboring area.

In both cases, different search algorithms have been developed to improve the efficiency of the process, such as exhaustive all-to-all search and grid cell algorithms, which are based on a subdivision of the simulation space into uniform cells or adaptive grids. The objective is to find the possible contact partners and determine the pairs of objects or finite elements which actually intersect and are in contact. This process directly affects considerably the processing time and quality of the results obtained by the contact algorithm. Doca (2010)

2.7 The contact detection algorithm

The Table II - The contact detection algorithm presented in the research paper "Algorithms for the Analysis of 3D Finite Strain Contact Problems" focuses on the detection of contact between deformable solids. The algorithm has several steps, which are described below in detail.

The first step (Step 1) is to determine the nodal average normal vectors for the slave nodes. This is necessary for the approximate projection of nodes onto the target faces.

The second step (Step 2) is to define the bucket indexes of global detection, which involves dividing the 3D space into small cubes, or buckets, to enable efficient searching for nodes and faces.

The third step (Step 3) involves iterating over each group of elements (master and slave) and finding candidate faces for each incident node.

In step 4, for each group, the algorithm iterates over all incident nodes.

Step 5 determines the bucket(s) to search for the given node.

In step 6, the slave node's radius of influence is determined.

The algorithm then iterates over all candidate faces (Step 7). If the face does not belong to one of the 27 neighborhood buckets of the given node, the algorithm cycles (Step 8).

Similarly, if the candidate face contains any node coincident with the given node, the algorithm cycles (Step 9).

In step 10, the algorithm determines the radius of influence of the given face.

If the distance between the center of the candidate face and the given node (drs) is greater than the sum of the radius of influence of the node (Rr) and the radius of influence of the face (Rs), the algorithm cycles (Step 11).

In step 12, the algorithm checks if the given node can be associated with the candidate face by verifying the violation of local co-ordinates.

Step 13 calculates, by a ranking strategy, the "cost" of forming the element Nkr $\cup$ Fk, which is a function of the gap estimate and step 12.

Step 14 checks if the cost of forming the contact element is less than the previous minimum. If yes, the cost is updated, and the candidate face is marked for association with the given node.

In step 15, the algorithm iterates over all candidate faces for the given node. If the node has some face associated, the algorithm defines the element Nkr $\cup$ Fks forming the new connectivities in step 16.

If the node does not have any associated face, then the element is deleted from the connectivities if it existed previously (Step 17).

Step 18 is completed when all nodes in each group are processed.

If there were any changes in the connectivities (i.e., new contacts added), then the algorithm recycles the code's pointers and reorder degrees of freedom to minimize fill-in (Step 20).

In summary, the algorithm's purpose is to search for potential contact pairs between master and slave surfaces while avoiding false contact during the numerical analysis of the finite strain problem. It accomplishes this by iteratively checking each node in the slave surface for proximity to a master target face and determining whether contact is present based on certain criteria, including the violation of local co-ordinates and ranking strategies. The algorithm is validated and applied to several illustrative examples discussed in the research paper.

References

DIMITRI P. BERTSEKAS. Chapter 4 - exact penalty methods and lagrangian methods. In DIMITRI P. BERTSEKAS, editor, *Constrained Optimization and Lagrange Multiplier Methods*, pages 179–301. Academic Press, 1982. ISBN 978-0-12-093480-5. doi: https://doi.org/10.1016/B978-0-12-093480-5.50008-8. URL https://www.sciencedirect.com/science/article/pii/B9780120934805500088.

Thiago Doca. Prodem seminar on computational contact mechanics, 2010.

Aarti Singh. Augmented lagrangian the method of multiplier. Disponível em https://www.cs.cmu.edu/~pradeepr/convexopt/Lecture_Slides/Augmented-lagrangian.pdf.

Peter Wriggers. *Computational Contact Mechanics Second Edition*. Springer edition, 06 October 2006. ISBN 978-3-540-32609-0. doi: https://doi.org/10.1007/978-3-540-32609-0.