Ficha nº 1
Revisões de PF

1- a) 1) $\begin{cases} f\ x = 2 * x \\ g\ x = x + 1 \end{cases}$

$(f \circ g)\ x = f\ (g\ x)$

$f\ (g\ x)$
$= f\ (x + 1)$
$= 2 * (x + 1)$
$= 2x + 1$

2) $\begin{cases} f = succ \\ g\ x = 2 * x \end{cases}$

$f\ (g\ x)$
$= f\ (2 * x)$
$= succ\ (2 * x)$

3- $\begin{cases} f = succ \\ g = length \end{cases}$

$(f \circ g)\ x$
$= (succ \circ length)\ x$
$= succ\ (length\ x)$

4- $\begin{cases} g\ (x, y) = x + y \\ f = succ \circ (2 *) \end{cases}$

$(succ \circ (2 *))\ x$
$= succ\ (2 * x)$

$f\ (g\ x)$
$= f\ (x + y)$
$= succ\ (2 * (x + y))$

b) $(f \circ g) \circ h\ x$
$= (f \circ g)(h\ x)$
$= f(g(h\ x))$
$= f((g \circ h)\ x)$
$= f \circ (g \circ h)\ x$

c) $(f \circ id)\ x$
$= f\ (id\ x)$
$= f\ x$

$(id \circ f)\ x$
$= id\ (f\ x)$
$= f\ x$

2- length :: [a] → Z
length [] = 0
length (h:t) = 1 + length t

reverse :: [a] → [a]
reverse [] = []
reverse (h:t) = (reverse t) ++ [h]

3- catMaybes :: [Maybe a] → [a]
catMaybes [] = []
catMaybes ((Nothing):t) = catMaybes t
catMaybes ((Just a):t) = (a:(catMaybes t))

4- uncurry :: (a → b → c) → (a, b) → c
uncurry f (a, b) = f a b

curry :: ((a, b) → c) → a → b → c
curry f a b = f (a, b)

flip :: (a → b → c) → b → a → c
flip f b a = f a b

5- flatten :: Ltree a -> [a]
flatten (Leaf a) = [a]
flatten (Fork (a, b)) = (flatten a) ++ (flatten b)

mirror :: Ltree a -> Ltree a
mirror (Leaf a) = Leaf a
mirror (Fork (a, b)) = Fork (mirror b, mirror a)

fmap :: (b -> a) -> Ltree b -> Ltree a
fmap f (Leaf a) = Leaf (f a)
fmap f (Fork (a, b)) = Fork (fmap f a, fmap f b)

6- a) length :: [a] -> $\mathbb{Z}$
length xs = foldr (\a -> (1+)) 0 xs

b) foldr :: (a -> b -> b) -> b -> [a] -> b
foldr g z [] = z
foldr g z (x : xs) = x 'g' foldr g z xs
$= x \text{ '(:)' } \underbrace{\text{foldr (:) [] xs}}_{f}$

Neste caso, o que a função "f" faz é concatenar a cabeça da lista passada como argumento com o resultado da aplicação da função "f" à cauda da lista. Ou seja, em efeitos práticos a função "f" devolve a mesma lista que recebeu.

$\boxed{\text{função (:)}}$

7- concat :: [[a]] -> [a]
concat [] = foldr (++) [] []
= []
concat (x : xs) = foldr (++) [] (x : xs)
= x '(++)' foldr (++) [] xs
$= x \text{ '(++)' } \underbrace{\text{foldr (++) []}}_{\text{concat}} \text{ xs}$

concat [] = []
concat (x : xs) = x ++ conc

8 - Para todo o "a" maior do que 0, retirado de "s", soma-se 1 a "a" e agrupe-se na lista a devolver

```
f :: [Z] -> [Z]
f s = foldr ((:).succ) [] (filter (\x -> (x > 0)) s)
```

9 - a)
```
m :: (a -> b) -> [a] -> [b]
m f = foldr ((:). f) []
```

b)

c)
```
k = m (\x -> [x])

k :: [a] -> [[a]]
```

Para todo elemento de uma lista, coloca esse elemento dentro de uma lista singular, formando uma lista de listas com um só elemento

d)
```
concat s
= concat (m singl "Calculo de Programas")
= concat (["C", "a", "l", "c", ...]),
= "Calculo de Programas"

concat (singl "Calculo de Programas")
= concat (["Calculo de Programas"])
= "Calculo de Programas"
```

Ficha nº2

1- a) $(f \times g)(x, y)$

$\equiv (10)$

$\langle f \circ \pi_1, g \circ \pi_2 \rangle (x, y)$

$\equiv (74), (77)$

$(f\, x, g\, y)$

b) $\pi_1 \circ (f \times g)$

$\equiv (10)$

$\pi_1 \circ \langle f \circ \pi_1, g \circ \pi_2 \rangle$

$\equiv (6)$

$f \circ \pi_1$

$$\begin{tikzcd} A \arrow[d, "f"] & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "f \times g"] & B \arrow[d, "g"] \\ C & C \times D \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & D \end{tikzcd}$$

$\pi_2 \circ (f \times g)$

$\equiv (10)$

$\pi_2 \circ \langle f \circ \pi_1, g \circ \pi_2 \rangle$

$\equiv (6)$

$g \circ \pi_2$

$id \times id = id$

$\equiv (10), (8)$

$\langle id \circ \pi_1, id \circ \pi_2 \rangle = \langle \pi_1, \pi_2 \rangle$

$\equiv (1)$

$\langle \pi_1, \pi_2 \rangle = \langle \pi_1, \pi_2 \rangle$

$$\begin{tikzcd} A \arrow[d, "id"'] & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "id"] & B \arrow[d, "id"] \\ A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \end{tikzcd}$$

3 - $(f \times g) \circ (h \times h)$

$\equiv (10)$

$(f \times g) \circ \langle h \circ \pi_1 , h \circ \pi_2 \rangle$

$\equiv (11)$

$\langle f \circ h \circ \pi_1 , g \circ h \circ \pi_2 \rangle$

$\equiv (10)$

$f \circ h \times g \circ h$

$$
\begin{tikzcd}
A \arrow[d, "h"] & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "h \times h"] & B \arrow[d] \\
C \arrow[d, "f"] & C \times D \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "f \times g"] & D \\
E & E \times F \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & F
\end{tikzcd}
$$

2 -
$$
\begin{tikzcd}
A \times B & B \times A \arrow[l, "\langle \pi_2, \pi_1 \rangle"'] & A \times B \arrow[l, "\langle \pi_2, \pi_1 \rangle"'] \arrow[ll, bend left, "id"]
\end{tikzcd}
$$

3 - $f : (A \times B) \times C \longrightarrow A \times (B \times C)$

$$
\begin{tikzcd}
A & A \times (B \times C) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \times C \\
& (A \times B) \times C \arrow[ul, "\pi_1 \circ \pi_1"] \arrow[u, "f"] \arrow[ur, "\pi_2 \times id"'] &
\end{tikzcd}
$$

$g : A \times (B \times C) \longrightarrow (A \times B) \times C$

$$
\begin{tikzcd}
A \times B & (A \times B) \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \times (B \times C) \arrow[ul] \arrow[u, "g"] \arrow[ur] &
\end{tikzcd}
$$

4- $f((a,b),c) = (a \wedge b) \oplus c$

$\hat{\wedge}(a,b) = a \wedge b$

$\hat{\oplus}(a,c) = a \oplus c$

$\equiv$ (82)

$f((a,b),c) = \hat{\wedge}(a,b) \oplus c$

$\equiv$ (82)

$f((a,b),c) = \hat{\oplus}((\hat{\wedge}(a,b)), c)$

$\equiv$ (1), (75)

$f((a,b),c) = \hat{\oplus} . (\hat{\wedge} \times id)((a,b),c)$

$\equiv$ (69)

$f = \hat{\oplus} . (\hat{\wedge} \times id)$

$$\begin{tikzcd}
\mathbb{B} \times \mathbb{B} \arrow[d, "\hat{\wedge}"] & (\mathbb{B} \times \mathbb{B}) \times \mathbb{B} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "(\hat{\wedge} \times id)"] & \mathbb{B} \arrow[d, "id"] \\
\mathbb{B} & \mathbb{B} \times \mathbb{B} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\hat{\oplus}"] & \mathbb{B} \\
 & \mathbb{B} &
\end{tikzcd}$$

$g : (\mathbb{B} \times \mathbb{B}) \times \mathbb{B} \longrightarrow (\mathbb{B} \times \mathbb{B}) \times \mathbb{B}$

5 -

$$
\begin{tikzcd}
(Atl \times Data^*)^* \\
(Atl \times Data)^* \arrow[u, "sort \,.\, (map\ (id \times sort)) \,.\, collect"'] \\
(Data \times Atl)^* \arrow[u, "converse"'] \\
(Data \times yog)^* \times (yog \times Atl)^* \arrow[u, "uncurry\ comp\ =\ \widehat{comp}"'] \\
(Data \times yog^*)^* \times (yog \times Atl^*)^* \arrow[u, "discollect \times discollect"'] \\
(Data \times yog^*)^* \quad (yog \times Atl^*)^* \arrow[u, "curry\ (id \times id)\ =\ \overline{(id \times id)}"']
\end{tikzcd}
$$

$$f = sort \,.\, (map\ (id \times sort)) \,.\, collect \,.\, converse \,.\, \widehat{comp} \,.\, (discollect \times discollect) \,.\, \overline{(id \times id)}$$

Ficha nº 3

1- a) $id = [i_1, i_2]$

$\equiv (17)$

$\begin{cases} id \circ i_1 = i_1 \\ id \circ i_2 = i_2 \end{cases}$

$\equiv (1)$

$\begin{cases} i_1 = i_1 \\ i_2 = i_2 \end{cases}$

b) $\langle k, h \rangle = \langle f, g \rangle$

$\equiv (6)$

$\begin{cases} \pi_1 \circ \langle k, h \rangle = f \\ \pi_2 \circ \langle k, h \rangle = g \end{cases}$

$\equiv (7)$

$\begin{cases} k = f \\ h = g \end{cases}$

2- $[\underline{k}, \underline{k}] = \underline{k}$

$\equiv (17)$

$\begin{cases} \underline{k} \circ i_1 = \underline{k} \\ \underline{k} \circ i_2 = \underline{k} \end{cases}$

$\equiv (13)$

$\begin{cases} \underline{k} = \underline{k} \\ \underline{k} = \underline{k} \end{cases}$

3- $id + id = id$ ((26) - Functor - id - +)

$\equiv$ (21)

$[i_1 \cdot id, i_2 \cdot id] = id$

$\equiv$ (17)

$\begin{cases} id \cdot i_1 = i_1 \cdot id \\ id \cdot i_2 = i_2 \cdot id \end{cases}$

$\equiv$ (1)

$\begin{cases} i_1 = i_1 \\ i_2 = i_2 \end{cases}$

$(f + g) \cdot i_1 = i_1 \cdot f$ ((23) - Natural - i_1)

$\equiv$ (21)

$[i_1 \cdot f, i_2 \cdot g] \cdot i_1 = i_1 \cdot f$

$\equiv$ (18)

$i_1 \cdot f = i_1 \cdot f$

$(f + g) \cdot i_2 = i_2 \cdot g$ ((24) - Natural - i_2)

$\equiv$ (21)

$[i_1 \cdot f, i_2 \cdot g] \cdot i_2 = i_2 \cdot g$

$\equiv$ (18)

$i_2 \cdot g = i_2 \cdot g$

4-

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[dr, "i_2"'] & A+B \arrow[d, "coswap"] & B \arrow[l, "i_2"'] \arrow[dl, "i_1"] \\
 & B+A &
\end{tikzcd}$$

$coswap \cdot coswap = id$

$\equiv$ definição do coswap

$coswap \cdot [i_2, i_1] = id$

$\equiv$ (20)

$[coswap \cdot i_2, coswap \cdot i_1] = id$

$\equiv$ (17)

$$\begin{cases} id \cdot i_1 = coswap \cdot i_2 \\ id \cdot i_2 = coswap \cdot i_1 \end{cases}$$

$\equiv$ definição de coswap ; (18) ; (1)

$$\begin{cases} i_1 = i_1 \\ i_2 = i_2 \end{cases}$$

5-
$$\begin{cases} fac\ 0 = 1 \\ fac\ (n+1) = (n+1) * fac\ n \end{cases}$$

$\equiv$ definição de $\underline{0}$, $\underline{1}$, succ

$$\begin{cases} fac\ (\underline{0}\ n) = (\underline{1}\ n) \\ fac\ (succ\ n) = (succ\ n) * (fac\ n) \end{cases}$$

$\equiv$ (70) ; definição do mul

$$\begin{cases} (fac \cdot \underline{0})\ n = \underline{1}\ n \\ (fac \cdot succ)\ n = mul \cdot (succ\ n,\ fac\ n) \end{cases}$$

$\equiv$ (74), (69)

$$\begin{cases} fac \cdot \underline{0} = \underline{1} \\ fac \cdot succ = mul \cdot \langle succ, fac \rangle \end{cases}$$

$\equiv$ (27) ; (20)

$fac \cdot [\underline{0}, succ] = [\underline{1}, mul \cdot \langle succ, fac \rangle]$

6- $coassocl \circ [id + i_1, i_2 \circ i_2] = id$

$\equiv$ (20); (19)

$[coassocl \circ (id + i_1), coassocl \circ i_2 \circ i_2] = [i_1, i_2]$

$\equiv$ (27)

$$\begin{cases} coassocl \circ (id + i_1) = i_1 \\ coassocl \circ i_2 \circ i_2 = i_2 \end{cases}$$

$\equiv$ (21); (20)

$$\begin{cases} [coassocl \circ i_1 \circ id, coassocl \circ i_2 \circ i_1] = i_1 \\ coassocl \circ i_2 \circ i_2 = i_2 \end{cases}$$

$\equiv$ (17); (1)

$$\begin{cases} coassocl \circ i_1 = i_1 \circ i_1 \\ coassocl \circ i_2 \circ i_1 = i_1 \circ i_2 \\ coassocl \circ i_2 \circ i_2 = i_2 \end{cases}$$

$\equiv$ (69); (70)

$$\begin{cases} coassocl\ (i_1\ x) = i_1\ (i_1\ x) \\ coassocl\ (i_2\ (i_1\ x)) = i_1\ (i_2\ x) \\ coassocl\ (i_2\ (i_2\ x)) = i_2\ x \end{cases}$$

7-

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[dr, "\langle \underline{False}, id \rangle"'] & A + A \arrow[d, "\alpha"] & A \arrow[l, "i_2"'] \arrow[dl, "\langle \underline{True}, id \rangle"] \\ & 2 \times A & \end{tikzcd}$$

$\alpha = [\langle \underline{False}, id \rangle, \langle \underline{True}, id \rangle]$

$\equiv$ (17)

$$\begin{cases} \alpha \circ i_1 = \langle \underline{False}, id \rangle \\ \alpha \circ i_2 = \langle \underline{True}, id \rangle \end{cases}$$

$\equiv$ (69); (70); (74); (71); (4)

$\alpha\ (i_1\ x) = (False, x)$

$\alpha\ (i_2\ x) = (True, x)$

8 - $out \cdot in = id$

$\equiv$ definição de in ; (20)

$[out \cdot Leaf , out \cdot Fork] = id$

$\equiv$ (27) ; (1)

$$\begin{cases} out \cdot Leaf = i1 \\ out \cdot Fork = i2 \end{cases}$$

$\equiv$ (69) ; (70)

$out\ (Leaf\ x) = i1\ x$

$out\ (Fork\ (x,y)) = i2\ (x,y)$

Ficha nº4

1- $\underline{(b, a)}$

$\equiv$ (69) ; (72)

$(\underline{b}, \underline{a})$

$\equiv$ (72)

$(\underline{b}\ x, \underline{a}\ x)$

$\equiv$ (74) ; (69)

$\langle \underline{b}, \underline{a} \rangle$

2- a)

$$\begin{tikzcd}
1+1 & 1+1 \times A \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & A \\
 & A+A \arrow[ul, "!+!"] \arrow[u, "iso"] \arrow[ur, "{[id, id]}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A+A \arrow[r, bend left, "iso"] \arrow[r, phantom, "\cong"] & 1+1 \times A \arrow[l, bend left, "iso"]
\end{tikzcd}$$

b)

$$\begin{tikzcd}
(1+1) \times A \arrow[d, "id \times f"] & A+A \arrow[l, "iso"'] \arrow[d, "f+f"] \\
(1+1) \times A' & A'+A' \arrow[l, "iso"']
\end{tikzcd}$$

$f : A \rightarrow A'$

$(id \times f) \cdot iso = iso \cdot (f + f)$

c) $(id \times f) . iso = iso . (f + f)$

$\equiv (21) ; (20)$

$(id \times f) . iso = [iso . i1 . f , iso . i2 . f]$

$\equiv (17)$

$$\begin{cases} (id \times f) . iso . i1 = iso . i1 . f \\ (id \times f) . iso . i2 = iso . i2 . f \end{cases}$$

$\equiv (10) ; (9) ; (1)$

$$\begin{cases} \langle \pi_1 . iso . i1 , f . \pi_2 . iso . i1 \rangle = iso . i1 . f \\ \langle \pi_1 . iso . i2 , f . \pi_2 . iso . i2 \rangle = iso . i2 . f \end{cases}$$

$\equiv (6)$

$$\begin{cases} \pi_1 . iso . i1 . f = \pi_1 . iso . i1 \\ \pi_2 . iso . i1 . f = f . \pi_2 . iso . i1 \\ \pi_1 . iso . i2 . f = \pi_1 . iso . i2 \\ \pi_2 . iso . i2 . f = f . \pi_2 . iso . i2 \end{cases}$$

$\equiv$ aplicando a definição do iso ; (7)

$$\begin{cases} (! + !) . i1 . f = (! + !) . i1 \\ [id , id] . i1 . f = f . [id , id] . i1 \\ (! + !) . i2 . f = (! + !) . i2 \\ [id , id] . i2 . f = f . [id , id] . i2 \end{cases}$$

$\equiv (18) ; (23) ; (24) ;$

$$\begin{cases} i1 . ! . f = i1 . ! \\ id . f = f . id \\ i2 . ! . f = i2 . ! \\ id . f = f . id \end{cases}$$

$\equiv \quad ! . f = ! \; ; (1)$

$$\begin{cases} i1 . ! = i1 . ! \\ f = f \\ i2 . ! = i2 . ! \end{cases}$$

d) $iso = \langle ! + !, [id, id] \rangle$

$\equiv (21)$

$iso = \langle [i1 . !, i2 . !], [id, id] \rangle$

$\equiv (28), (1)$

$iso . id = [\langle i1 . !, id \rangle, \langle i2 . !, id \rangle]$

$\equiv (19), (20)$

$[iso . i1, iso . i2] = [\langle i1 . !, id \rangle, \langle i2 . !, id \rangle]$

$\equiv (27)$

$$\begin{cases} iso . i1 = \langle i1 . !, id \rangle \\ iso . i2 = \langle i2 . !, id \rangle \end{cases}$$

$\equiv (69), (70), (74)$

$$\begin{cases} iso\ (i1\ x) = (i1\ (!\ x), id\ x) \\ iso\ (i2\ x) = (i2\ (!\ x), id\ x) \end{cases}$$

$\equiv !\ x = ()\ ,\ (71)$

$$\begin{cases} iso\ (i1\ x) = (i1\ (), x) \\ iso\ (i2\ x) = (i2\ (), x) \end{cases}$$

3- $A + B \xleftarrow{id + \pi_1} A + (B \times C) \xleftarrow{i_2} B \times C \xleftarrow{\pi_2} D \times (B$

$$\begin{tikzcd} D \times (B \times C) \arrow[r, "\alpha"] \arrow[d, "d \times (b \times c)"] & A + B \arrow[d, "a + b"] \\ D' \times (B' \times C') \arrow[r, "\alpha"] & A' + B' \end{tikzcd}$$

$a : A \to A'$
$b : B \to B'$
$c : C \to C'$
$d : D \to D'$

$\alpha . (d \times (b \times c)) = (a + b) . \alpha$

4 -

$$\begin{tikzcd}
A + B \times C \arrow[r, "\alpha"] \arrow[d, "f + g \times h"'] & A + C \arrow[d, "f + h"] \\
A' + B' \times C' \arrow[r, "\alpha"'] & A' + C'
\end{tikzcd}$$

$f : A \to A'$
$g : B \to B'$
$h : C \to C'$

$\alpha = id + \pi_2$

5 -

$$\begin{tikzcd}
A \times (B + C) \arrow[r, "distr"] \arrow[d, "f \times (g + h)"'] & A \times B + A \times C \arrow[d, "f \times g + f \times h"] \\
A' \times (B' + C') \arrow[r, "distr"'] & A' \times B' + A' \times C'
\end{tikzcd}$$

$distr \cdot (f \times (g + h)) = (f \times g + f \times h) \cdot distr$

$h \cdot distr \cdot (g \times (id + f)) = h$

$\equiv$ propriedade natural do distr

$h \cdot (g \times id + g \times f) \cdot distr = h$

$\equiv$ undistr = distr° ; (F4)

$h \cdot (g \times id + g \times f) = h \cdot undistr$

6 - $\begin{cases} \nabla \cdot i_1 = id \\ \nabla \cdot i_2 = id \end{cases}$

$\equiv$ (17)

$\nabla = [id, id]$

$f \cdot \nabla$

$\equiv$ definição de ∇ ; (1)

$[f, f]$

$\equiv$ (1) ; (22)

$[id, id] \cdot (f + f)$

$\equiv$ definição de ∇

$\nabla \cdot (f + f)$

7- $[\langle f, g\rangle, \langle l, h\rangle] = \langle [f, l], [g, h]\rangle$

$\equiv (17)$

$\begin{cases} \langle [f, l], [g, h]\rangle . i1 = \langle f, g\rangle \\ \langle [f, l], [g, h]\rangle . i2 = \langle l, h\rangle \end{cases}$

$\equiv (9)$

$\begin{cases} \langle [f, l] . i1, [g, h] . i1\rangle = \langle f, g\rangle \\ \langle [f, l] . i2, [g, h] . i2\rangle = \langle l, h\rangle \end{cases}$

$\equiv (18)$

$\begin{cases} \langle f, g\rangle = \langle f, g\rangle \\ \langle l, h\rangle = \langle l, h\rangle \end{cases}$

8- $[i_1 \times id, i_2 \times id]$

$\equiv (10), (1)$

$[\langle i1 . \pi_1, \pi_2\rangle, \langle i2 . \pi_1, \pi_2\rangle]$

$\equiv (28)$

$\langle [i_1 . \pi_1, i_2 . \pi_1], [\pi_2, \pi_2]\rangle$

9- $(p \to f, g) . l$

$\equiv (30)$

$([f, g] . p?) . l$

$\equiv (29)$

$[f, g] . (l + l) . (p . l)?$

$\equiv (22)$

$[f . l, g . l] . (p . l)?$

$\equiv (30)$

$(p . l) \to (f . l), (g . l)$

10- $p \to \langle f, g \rangle, \langle h, i \rangle$

$\equiv (30)$

$[\langle f, g \rangle, \langle h, i \rangle] \cdot p?$

$\equiv (28)$

$\langle [f, h], [g, i] \rangle \cdot p?$

$\equiv (9), (30)$

$\langle p \to f, h ; p \to g, i \rangle$

$p \to \langle f, g \rangle, \langle f, h \rangle$

$\equiv (30)$

$[\langle f, g \rangle, \langle f, h \rangle] \cdot p?$

$\equiv (28)$

$\langle [f, f], [g, h] \rangle \cdot p?$

$\equiv (9), (30)$

$\langle p \to f, f, p \to g, h \rangle$

$\equiv (F5)$

$\langle f, p \to g, h \rangle$

$p \to (p \to a, b), (p \to c, d)$

$\equiv (30)$

$[[a, b] \cdot p?, [c, d] \cdot p?] \cdot p?$

$\equiv (22)$

$[[a, b], [c, d]] \cdot (p? + p?) \cdot p?$

$\equiv (F6)$

$[[a, b], [c, d]] \cdot (i1 + i2) \cdot p?$

$\equiv (22), (18)$

$[a, d] \cdot p?$

$\equiv (30)$

$p \to a, d$

11- for $\underset{\text{função}}{b}$ $\underset{-}{i} \longrightarrow$ constante

$$\begin{cases} \text{for } b \; i \; 0 = i \\ \text{for } b \; i \; (n+1) = b \, (\text{for } b \; i \; n) \end{cases}$$

$\equiv$ $x = \text{for } b \; i$; def do succ

$$\begin{cases} x \; 0 = i \\ x \; (\text{succ } n) = b \, (x \; n) \end{cases}$$

$\equiv$ (72), (70)

$$\begin{cases} (x \cdot \underline{0}) \, n = \underline{i} \; n \\ (x \cdot \text{succ}) \, n = (b \cdot x) \; n \end{cases}$$

$\equiv$ (69)

$$\begin{cases} x \cdot \underline{0} = \underline{i} \\ x \cdot \text{succ} = b \cdot x \end{cases}$$

$\equiv$ (27)

$[x \cdot \underline{0} \, , \, x \cdot \text{succ}] = [\underline{i} \, , \, b \cdot x]$

$\equiv$ (20); def. de in

$x \cdot \text{in} = [\underline{i} \, , \, b \cdot x]$

$\equiv$ (1); (22)

$x \cdot \text{in} = [\underline{i} \, , \, b] \cdot (id + x)$

Ficha nº 5

1- $\begin{cases} id = g \cdot f \\ f \cdot g = id \end{cases}$

$\equiv$ (69); (70)

$\begin{cases} id\ y = g\ (f\ y) \\ f\ (g\ x) = id\ x \end{cases}$

$\equiv$ (71); (F1)

$\begin{cases} y = g\ x \\ f\ y = x \end{cases}$

$\equiv$ (F1)

$\begin{cases} y = y \\ x = x \end{cases}$

2- $out \cdot in = id$

$\equiv$ (F1); (20); (19)

$[out \cdot \underline{0}, out \cdot succ] = [i1, i2]$

$\equiv$ (27)

$\begin{cases} out \cdot \underline{0} = i1 \\ out \cdot succ = i2 \end{cases}$

$\equiv$ (69); (70)

$\begin{cases} out\ (\underline{0}\ ()) = i1\ () \\ out\ (succ\ n) = i2\ n \end{cases}$

$\equiv$ def. de succ; (72)

$\begin{cases} out\ 0 = i1\ () \\ out\ (n+1) = i2\ n \end{cases}$

3- a) $(a+) = (\![\underline{a}, succ]\!)$

$\equiv (43)$

$(a+) \cdot in = [\underline{a}, succ] \cdot F(a+)$

$\equiv$ def. de in ; def. de F-naturais

$(a+) \cdot [zero, succ] = [\underline{a}, succ] \cdot (id + (a+))$

$\equiv (20) ; (22)$

$[(a+) \cdot zero, (a+) \cdot succ] = [\underline{a} \cdot id ; succ \cdot (a+)]$

$\equiv (1) ; (27)$

$$\begin{cases} (a+) \cdot zero = \underline{a} \\ (a+) \cdot succ = succ \cdot (a+) \end{cases}$$

$\equiv (69), (70)$

$$\begin{cases} (a+)(zero\ n) = \underline{a}\ n \\ (a+)(succ\ n) = succ((a+)\ n) \end{cases}$$

$\equiv$ def. de zero e succ ; (72)

$$\begin{cases} (a+)\ 0 = a \\ (a+)\ (n+1) = ((a+)\ n) + 1 \end{cases}$$

b)

$$\begin{tikzcd}
\mathbb{N}_0 \arrow[d, "(a\theta)"'] & 1 + \mathbb{N}_0 \arrow[l, "in"'] \arrow[d, "id + (a\theta)"] \\
\mathbb{N}_0 & 1 + \mathbb{N}_0 \arrow[l, "{[h, g]}"]
\end{tikzcd}$$

$\underline{h = \underline{a}}$

$g = ((a\theta)\ n) - 1$

$= sub \cdot (a\theta)$

$sub :: \mathbb{N}_0 \to \mathbb{N}_0$

$sub\ 0 = 0$

$sub\ n = n - 1$

$$\begin{tikzcd}
1 \arrow[r, "i_1"] \arrow[dr, "\underline{a}"'] & 1 + \mathbb{N}_0 \arrow[d, "{[h, g]}"] & \mathbb{N}_0 \arrow[l, "i_2"'] \arrow[dl, "sub \cdot (a\theta)"] \\
& \mathbb{N}_0 &
\end{tikzcd}$$

$$\begin{cases}(a\ominus)\ 0 = a \\ (a\ominus)\ (n+1) = ((a\ominus)\ n) - 1\end{cases}$$

$\equiv$ def. de sub e succ; (72)

$$\begin{cases}(a\ominus)(\underline{0}\ ()) = \underline{a}\ () \\ (a\ominus)(\text{succ } n) = \text{sub}\ ((a\ominus)\ n)\end{cases}$$

$\equiv$ (70); (69)

$$\begin{cases}(a\ominus) . \underline{0} = \underline{a} \\ (a\ominus) . \text{succ} = \text{sub} . (a\ominus)\end{cases}$$

$\equiv$ (27); (20); (22); (1)

$(a\ominus) . [\underline{0}, \text{succ}] = [\underline{a}, \text{sub}] . (id + (a\ominus))$

$\equiv$ (43)

$(a\ominus) = (\![\underline{a}, \text{sub}]\!)$

c) $id = \text{for succ } 0$

$\equiv$ pela definição dada

$id = (\![\text{zero}, \text{succ}]\!)$

$\equiv$ (43)

$id . in = [\text{zero}, \text{succ}] . (id + id)$

$\equiv$ definição de in; (20); (22); (1)

$[\text{zero}, \text{succ}] = [\text{zero}, \text{succ}]$

$\text{succ} = \text{for succ } 1$

$\equiv$ pela definição dada

$\text{succ} = (\![\underline{1}, \text{succ}]\!)$

$\equiv$ (43)

$\text{succ} . in = [\underline{1}, \text{succ}] . (id + \text{succ})$

$\equiv$ definição de in; (20); (22)

$[\text{succ} . \underline{0}, \text{succ} . \text{succ}] = [\underline{1}, \text{succ} . \text{succ}]$

$\equiv$ (27)

$$\begin{cases}\text{succ} . \underline{0} = \underline{1} \\ \text{succ} . \text{succ} = \text{succ} . \text{succ}\end{cases}$$

$\equiv$ (69); (70); (72)

$$\begin{cases} succ\ 0 = 1 \\ succ \cdot succ = succ \cdot succ \end{cases}$$

$\equiv$ def. do succ

$$\begin{cases} 0 + 1 = 1 \\ succ \cdot succ = succ \cdot succ \end{cases}$$

d) $id = (\![nil, cons]\!)$

$\equiv$ pela definição dada

$id \cdot in = [nil, cons] \cdot (id + id \times id)$

$\equiv$ (15); (26); (1)

$in = [nil, cons]$

e) $f = for\ id\ i$

$\equiv$ pela definição dada

$f = (\![\underline{i}, id]\!)$

$\equiv$ (43)

$f \cdot in = [\underline{i}, id] \cdot (id + f)$

$\equiv$ def. de in; (20); (22); (1)

$[f \cdot zero, f \cdot succ] = [\underline{i}, f]$

$\equiv$ (27)

$$\begin{cases} f \cdot zero = \underline{i} \\ f \cdot succ = f \end{cases}$$

$\equiv$ (69); (70); def. de zero e succ; (72)

$$\begin{cases} f\ 0 = i \\ f\ (n+1) = f\ n \end{cases}$$

$g = f_0\ 1\ 1$

$\equiv$ pela definição dada

$g = ([1, 1])$

$\equiv$ (43)

$g \cdot in = [1, 1] \cdot (id + g)$

$\equiv$ def. de in; (20); (21); (1)

$[g \cdot zero, g \cdot succ] = [1, 1 \cdot g]$

$\equiv$ (3); (27)

$$\begin{cases} g \cdot zero = 1 \\ g \cdot succ = 1 \end{cases}$$

$\equiv$ (69); (70);

$$\begin{cases} g(zero \cdot ()) = 1\ () \\ g(succ\ n) = 1\ n \end{cases}$$

$\equiv$ def. de zero e succ; (72)

$$\begin{cases} g\ 0 = 1 \\ g\ (n+1) = 1 \end{cases}$$

Ambas as funções representam a função constante "1"

4 - a)

$$\begin{tikzcd} A^* \arrow[d, "k"'] & 1 + A \times A^* \arrow[l, "in"'] \arrow[d, "id + id \times k"] \\ A & 1 + A \times A \arrow[l, "g"] \end{tikzcd}$$

$$g = [\underline{1}, (\hat{*})]$$

$$h = (\!|[\underline{1}, (\hat{*})]|\!)$$

b)

$$\begin{tikzcd} A^* \arrow[d, "h"'] & 1 + A \times A^* \arrow[l, "in"'] \arrow[d, "id + id \times h"] \\ A^* & 1 + A \times A^* \arrow[l, "g"] \end{tikzcd}$$

$$g = [nil, (\hat{++}) \cdot swap \cdot (singl \times id)]$$

$h = (\!|[nil, (\hat{++}) \cdot swap \cdot (singl \times id)]|\!)$

$\equiv$ (43)

$h \cdot in = [nil, (\hat{++}) \cdot swap \cdot (singl \times id)] \cdot (id + id \times h)$

$\equiv$ def. de in ; (20) ; (22) ; (1)

$[h \cdot nil, h \cdot cons] = [nil, (\hat{++}) \cdot swap \cdot (singl \times id) \cdot (id \times h)$

$\equiv$ (14) ; (1)

$[h \cdot nil, h \cdot cons] = [nil, (\hat{++}) \cdot swap \cdot (singl \times h)]$

$\equiv$ (27)

$$\begin{cases} h \cdot nil = nil \\ h \cdot cons = (\hat{++}) \cdot swap \cdot (singl \times h) \end{cases}$$

$\equiv$ (69) ; (70)

$$\begin{cases} h\ [\,] = [\,] \\ h\ (h : t) = (\hat{++})\ (swap\ (singl\ h,\ h\ t)) \end{cases}$$

c)

$$\begin{tikzcd}
A^* \arrow[d, "h"'] & 1 + A \times A^* \arrow[l, "in"'] \arrow[d, "id + id \times h"] \\
B^* & 1 + A \times B^* \arrow[l, "g"]
\end{tikzcd}$$

$$g = [nil, cons \cdot (f \times id)]$$

d)

$$\begin{tikzcd}
A^* \arrow[d, "h"'] & 1 + A \times A^* \arrow[l, "in"'] \arrow[d, "id + id \times h"] \\
A & 1 + A \times A \arrow[l, "g"]
\end{tikzcd}$$

$$g = [\underline{0}, \widehat{max}]$$

e)

$$\begin{cases} filter\ p\ [\ ] = [\ ] \\ filter\ p\ (h:t) = x \mathbin{++} filter\ p\ t \\ \qquad where\ x = if\ (p\ h)\ then\ [h]\ else\ [\ ] \end{cases}$$

$\equiv$ def. de h, singl, nil

$$\begin{cases} h\ [\ ] = [\ ] \\ h\ (h:t) = x \mathbin{++} h\ t \\ \qquad where\ x = if\ (p\ h)\ then\ (singl\ h)\ else\ (nil\ h) \end{cases}$$

$\equiv$ (76)

$$\begin{cases} h\ [\ ] = [\ ] \\ h\ (h:t) = x \mathbin{++} h\ t \\ \qquad where\ x = (p \rightarrow singl, nil)\ h \end{cases}$$

$\equiv$

$$\begin{cases} h\ [\ ] = [\ ] \\ h\ (h:t) = (p \rightarrow singl, nil)\ h \mathbin{++} h\ t \end{cases}$$

$\equiv$ def. de nil, cons e (++) ; (82)

$$\begin{cases} h(\text{nil}()) = \text{nil}() \\ h(\text{cons}(h,t)) = (\hat{++})((p \to \text{singl}, \text{nil})\, h,\ h\ t) \end{cases}$$

$\equiv$ (75) ; (70)

$$\begin{cases} (h \cdot \text{nil})\,() = \text{nil}\,() \\ (h \cdot \text{cons})(h,t) = ((\hat{++}) \cdot (p \to \text{singl}, \text{nil} \times h))\,(h,t) \end{cases}$$

$\equiv$ (69) ; (27)

$$[h \cdot \text{nil},\ h \cdot \text{cons}] = [\text{nil},\ (\hat{++}) \cdot (p \to \text{singl}, \text{nil} \times h)]$$

$\equiv$ (20) ; (1) ; (14)

$$h \cdot [\text{nil}, \text{cons}] = [\text{nil},\ (\hat{++}) \cdot (p \to \text{singl}, \text{nil} \times id) \cdot (id \times h$$

$\equiv$ (1) ; (22)

$$h \cdot [\text{nil}, \text{cons}] = [\text{nil},\ (\hat{++}) \cdot (p \to \text{singl}, \text{nil} \times id)] \cdot (id + id \times h)$$

$\equiv$ (43)

$$h = (\!|[\text{nil},\ (\hat{++}) \cdot (p \to \text{singl}, \text{nil} \times id)]|\!)$$

5 - $\begin{cases} \text{fac}\ 0 = 1 \\ \text{fac}\ (n+1) = (n+1) * \text{fac}\ n \end{cases}$

$\equiv$ (82) ; (72) ; def. de succ

$$\begin{cases} \text{fac}\ (\underline{0}\ n) = \underline{1}\ n \\ \text{fac}\ (\text{succ}\ n) = (\hat{*})\ (\text{succ}\ n,\ \text{fac}\ n) \end{cases}$$

$\equiv$ (74) ; (70)

$$\begin{cases} (\text{fac} \cdot \underline{0})\ n = \underline{1}\ n \\ (\text{fac} \cdot \text{succ})\ n = ((\hat{*}) \cdot \langle \text{succ}, \text{fac} \rangle)\ n \end{cases}$$

$\equiv$ (69) ; (27)

$$[\text{fac} \cdot \underline{0},\ \text{fac} \cdot \text{succ}] = [\underline{1},\ (\hat{*}) \cdot \langle \text{succ}, \text{fac} \rangle]$$

$\equiv$ (20) ; (22) ; (1)

$$\text{fac} \cdot [\underline{0}, \text{succ}] = [\underline{1},\ (\hat{*})] \cdot (id + \langle \text{succ}, \text{fac} \rangle)$$

A função não pode ser vista diretamente como um catamorfismo de números naturais

6- $(a+) = ([\underline{a}, succ])$

$\equiv$

$(a+) =$ for succ a

```
int aPlus(int n){
    int r = a;
    int j;
    for(j=1; j<n+1; j++)
        r = succ(r);
    return r;
}
```

f = for id i

```
int f(int n){
    int r = i;
    int j;
    for(j=1; j<n+1; j++)
        r = id(r);
    return r;
}
```

f = for $\underline{i}$ i $\qquad\qquad$ $\underline{i}$ x = const(i, x)

```
int g(int n){
    int r = i;
    int j;
    for(j=1; j<n+1; j++)
        r = const(i, r);
    return r;
}
```

Ficha nº 6

1- 1)

$$\begin{tikzcd} \mathbb{N}_0^* \arrow[d, "positivos"'] & 1 + \mathbb{N}_0 \times \mathbb{N}_0^* \arrow[l, "in"'] \arrow[d, "id + id \times positivos"] \\ \mathbb{N}_0^* & 1 + \mathbb{N}_0 \times \mathbb{N}_0^* \arrow[l, "g"] \end{tikzcd}$$

$$g = [nil, (+^\wedge+) \cdot ((>=0) \to singl, nil \times id)$$

2)

$$\begin{tikzcd} (A^*)^* \arrow[d, "concat"'] & 1 + A^* \times (A^*)^* \arrow[l, "in"'] \arrow[d, "id + id \times concat"] \\ A^* & 1 + A^* \times A^* \arrow[l, "g"] \end{tikzcd}$$

$$g = [nil, (+^\wedge+)]$$

3)

$$\begin{tikzcd} LTree\ A \arrow[d, "zeros"'] & A + LTree\ A \times LTree\ A \arrow[l, "in"'] \arrow[d, "id + zeros^2"] \\ LTree\ \mathbb{N}_0 & A + LTree\ \mathbb{N}_0 \times LTree\ \mathbb{N}_0 \arrow[l, "g"] \end{tikzcd}$$

$$g = [Leaf \cdot \underline{0}, Fork]$$

4)

$$\begin{tikzcd} BTree\ A \arrow[d, "conta"'] & 1 + A \times (BTree\ A)^2 \arrow[l] \arrow[d, "id + id \times conta^2"] \\ \mathbb{N}_0 & 1 + A \times \mathbb{N}_0^2 \arrow[l, "g"] \end{tikzcd}$$

$$g = [\underline{0}, succ \cdot (\hat{+}) \cdot \pi_2]$$

5)

$$\begin{tikzcd}
\mathrm{LTree}\ A \arrow[d] & A + (\mathrm{LTree}\ A)^2 \arrow[l, "in"'] \arrow[d, "id + mirror^2"] \\
\mathrm{LTree}\ A & A + (\mathrm{LTree}\ A)^2 \arrow[l, "g"]
\end{tikzcd}$$

$$g = [Leaf, Fork \cdot swap]$$

6)

$$\begin{tikzcd}
\mathrm{FTree}\ B\ A \arrow[d, "converte"'] & B + A \times (\mathrm{FTree}\ B\ A)^2 \arrow[l, "in"'] \arrow[d, "id + id \times converte^2"] \\
\mathrm{BTree}\ A & B + A \times (\mathrm{BTree}\ A)^2 \arrow[l, "g"]
\end{tikzcd}$$

$$g = in_{BTree} = [\underline{Empty}, Node]$$

7)

$$\begin{tikzcd}
\mathrm{Expr}\ V\ O \arrow[d, "vars"'] & V + O \times (\mathrm{Expr}\ V\ O)^* \arrow[l, "in"'] \arrow[d, "id + id \times (map\ vars)"] \\
V^* & V + O \times (V^*)^* \arrow[l, "g"]
\end{tikzcd}$$

$$g = [singl, concat \cdot \pi_2]$$

2- $vars = (\![singl, concat \cdot \pi_2]\!)$

$\equiv$ (43); def. de in

$vars \cdot [Var, Op] = [singl, concat \cdot \pi_2] \cdot (id + id \times (map\ vars))$

$\equiv$ (20); (22); (1)

$[vars \cdot Var, vars \cdot Op] = [singl, concat \cdot \pi_2 \cdot (id \times (map\ vars$

$\equiv$ (13); (27)

$$\begin{cases} vars \cdot Var = singl \\ vars \cdot Op = concat \cdot (map\ vars) \cdot \pi_2 \end{cases}$$

$\equiv$ (69); (70)

$$\begin{cases} vars\ (Var\ v) = singl\ v \\ vars\ (Op\ (o, l)) = concat\ (map\ vars\ (\pi_2\ (o, l))) \end{cases}$$

$\equiv$ def. de singl; (77)

$$\begin{cases} vars\ (Var\ v) = [v] \\ vars\ (Op\ (o, l)) = concat\ (map\ vars\ l) \end{cases}$$

3- $f \cdot (\!|g|\!) = (\!|h|\!)$

$\equiv$ (43) - Universal-Cata

$f \cdot (\!|g|\!) \cdot in = h \cdot F\,(f \cdot (\!|g|\!))$

$\equiv$ (41) - Functor-F

$f \cdot (\!|g|\!) \cdot in = h \cdot F f \cdot F (\!|g|\!)$

$\equiv$ (44) - Cancelamento-Cata

$f \cdot g \cdot F (\!|g|\!) = h \cdot F f \cdot F (\!|g|\!)$

$\Leftarrow$ (5) - Leibniz

$f \cdot g = h \cdot F f$

4 - $sumprod\ a = (a*) \cdot sum$

$\equiv$ (F1) ; def. de sum

$$(a*) \cdot (\![zero, add]\!) = (\![zero, add \cdot ((a*) \times id)]\!)$$

$\Leftrightarrow$

$$(a*) \cdot [zero, add] = [zero, add \cdot ((a*) \times id)] \cdot (id + id \times (a*))$$

$\equiv$ (20) ; (22) ; (1)

$$[(a*) \cdot zero, (a*) \cdot add] = [zero, add \cdot ((a*) \times id) \cdot (id \times (a*))]$$

$\equiv$ (27) ; (14) ; (1)

$$\begin{cases} (a*) \cdot zero = zero \\ (a*) \cdot add = add \cdot ((a*) \times (a*)) \end{cases}$$

$\equiv$ (69) ; (70)

$$\begin{cases} (a*)(zero()) = zero() \\ (a*)(add(x, y)) = add((a*)x, (a*)y) \end{cases}$$

$\equiv$ def. de zero, add e $(a*)$

$$\begin{cases} a * 0 = 0 \\ a * (x + y) = a * x + a * y \end{cases}$$

No primeiro caso, temos presente o elemento absorvente da multiplicação (0), ou seja, a afirmação é verdadeira. No segundo caso temos presente a propriedade distributiva da multiplicação

Ficha nº 7

1- $f \cdot (\text{for } f\ i) = \text{for } f\ (f\ i)$

$\equiv$ def. de for

$f \cdot (\!|[\underline{i}, f]|\!) = (\!|[\underline{(f\ i)}, f]|\!)$

$\equiv$ (46)

$f \cdot [\underline{i}, f] = [\underline{(f\ i)}, f] \cdot (id + f)$

$\equiv$ (20); (22)

$[f \cdot \underline{i}, f \cdot f] = [\underline{(f\ i)} \cdot id, f \cdot f]$

$\equiv$ (4); (3)

$[\underline{(f\ i)}, f \cdot f] = [\underline{(f\ i)}, f \cdot f]$

2- $\langle f, \langle g, j \rangle\rangle = (\!|\langle h, \langle k, l \rangle\rangle|\!)$

$\equiv$ (50)

$$\begin{cases} f \cdot in = h \cdot F\langle f, \langle g, j\rangle\rangle \\ \langle g, j\rangle \cdot in = \langle k, l\rangle \cdot F\langle f, \langle g, j\rangle\rangle \end{cases}$$

$\equiv$ (9)

$$\begin{cases} f \cdot in = h \cdot F\langle f, \langle g, j\rangle\rangle \\ \langle g \cdot in, j \cdot in\rangle = \langle k \cdot F\langle f, \langle g, j\rangle\rangle, l \cdot F\langle f, \langle g, j\rangle\rangle\rangle \end{cases}$$

$\equiv$ (96)

$$\begin{cases} f \cdot in = h \cdot F\langle f, \langle g, j\rangle\rangle \\ g \cdot in = k \cdot F\langle f, \langle g, j\rangle\rangle \\ j \cdot in = l \cdot F\langle f, \langle g, j\rangle\rangle \end{cases}$$

$\equiv$ (F2)

3- $\begin{cases} impar\ 0 = false \\ impar\ (n+1) = par\ n \\ par\ 0 = true \\ par\ (n+1) = impar\ n \end{cases}$

$\equiv$ (82); def. de succ

$\begin{cases} impar\ (\underline{0}\ n) = \underline{false}\ n \\ impar\ (succ\ n) = par\ n \\ par\ (\underline{0}\ n) = \underline{true}\ n \\ par\ (succ\ n) = impar\ n \end{cases}$

$\equiv$ (69); (70)

$\begin{cases} impar \cdot \underline{0} = \underline{false} \\ impar \cdot succ = par \\ par \cdot \underline{0} = \underline{true} \\ par \cdot succ = impar \end{cases}$

$\equiv$ (27)

$\begin{cases} [impar \cdot \underline{0},\ impar \cdot succ] = [\underline{false},\ par] \\ [par \cdot \underline{0},\ impar \cdot succ] = [\underline{true},\ impar] \end{cases}$

$\equiv$ (20); (7)

$\begin{cases} impar \cdot [\underline{0},\ succ] = [\underline{false},\ \pi_2 \cdot \langle impar, par \rangle] \\ par \cdot [\underline{0},\ succ] = [\underline{true},\ \pi_1 \cdot \langle impar, par \rangle] \end{cases}$

$\equiv$ (1); (22)

$\begin{cases} impar \cdot in = [\underline{false},\ \pi_2] \cdot (id + \langle impar, par \rangle) \\ par \cdot in = [\underline{true},\ \pi_1] \cdot (id + \langle impar, par \rangle) \end{cases}$

Concluímos, assim, que $h = [\underline{false}, \pi_2]$ e $k = [\underline{true}, \pi_1]$

$$\begin{cases} impar \cdot in = [\underline{false}, \pi_2] \cdot F\langle impar, par\rangle \\ par \cdot in = [\underline{true}, \pi_1] \cdot F\langle impar, par\rangle \end{cases}$$

$\equiv$ (50)

$\langle impar, par\rangle = (|\langle [\underline{false}, \pi_2], [\underline{true}, \pi_1]\rangle|)$

$\equiv$ (28)

$\langle impar, par\rangle = (|[\langle \underline{false}, \underline{true}\rangle, \langle \pi_2, \pi_1\rangle]|)$

$\equiv$ def. de $swap = \langle \pi_2, \pi_1\rangle$

$\langle impar, par\rangle = (|[\langle \underline{false}, \underline{true}\rangle, swap]|)$

$\equiv$ def. de ciclo for

$\langle impar, par\rangle = for\ swap\ (false, true)$

4 - $fac \cdot in = [\underline{1}, mul \cdot \langle succ, fac\rangle]$

$\equiv$ (1); (22)

$fac \cdot in = [\underline{1}, mul] \cdot (id + \langle succ, fac\rangle)$

$$\begin{cases} succ\ 0 = 1 \\ succ\ (n+1) = succ\ n + 1 \end{cases}$$

$\equiv$ (72); def. de succ;

$$\begin{cases} succ\ (\underline{0}\ n) = \underline{1}\ n \\ succ\ (succ\ n) = succ\ (succ\ n) \end{cases}$$

$\equiv$ (74); (70); (69)

$$\begin{cases} succ \cdot \underline{0} = \underline{1} \\ succ \cdot succ = succ \cdot succ \end{cases}$$

$\equiv$ (27); (20)

$succ \cdot [\underline{0}, succ] = [\underline{1}, succ \cdot succ]$

$\equiv$ (7)

$succ \cdot [\underline{0}, succ] = [\underline{1}, succ \cdot \pi_1 \cdot \langle succ, fac\rangle]$

$\equiv$ (1); (22); def. de in

$succ \cdot in = [\underline{1}, succ \cdot \pi_1] \cdot (id + \langle succ, fac\rangle)$

$$\begin{cases} fac \cdot in = [\underline{1}, mul] \cdot (id + \langle succ, fac \rangle) \\ succ \cdot in = [\underline{1}, succ \cdot \pi_1] \cdot (id + \langle succ, fac \rangle) \end{cases}$$

$\equiv$ (50)

$\langle succ, fac \rangle = (\!| \langle [\underline{1}, succ \cdot \pi_1], [\underline{1}, mul] \rangle |\!)$

$\equiv$ (28)

$\langle succ, fac \rangle = (\!| [\langle \underline{1}, \underline{1} \rangle, \langle succ \cdot \pi_1, mul \rangle] |\!)$

$\equiv$ (6)

$fac = \pi_2 \cdot (\!| [\langle \underline{1}, \underline{1} \rangle, \langle succ \cdot \pi_1, mul \rangle] |\!)$

$\equiv$ def. do ciclo for

$fac = \pi_2 \cdot (for \langle succ \cdot \pi_1, mul \rangle (1, 1))$

$fac = \pi_2 \cdot (for \langle succ \cdot \pi_1, mul \rangle (1, 1))$

$\equiv$ revertendo o que se fez acima

$$\begin{cases} fac \cdot in = [\underline{1}, mul] \cdot (id + \langle succ, fac \rangle) \\ succ \cdot in = [\underline{1}, succ \cdot \pi_1] \cdot (id + \langle succ, fac \rangle) \end{cases}$$

$\equiv$ def. de in; (20); (22)

$$\begin{cases} [fac \cdot \underline{0}, fac \cdot succ] = [\underline{1} \cdot id, mul \cdot \langle succ, fac \rangle] \\ [succ \cdot \underline{0}, succ \cdot succ] = [\underline{1} \cdot id, succ \cdot \pi_1 \cdot \langle succ, fac \rangle] \end{cases}$$

$\equiv$ (1), (7), (27)

$$\begin{cases} fac \cdot \underline{0} = \underline{1} \\ fac \cdot succ = mul \cdot \langle succ, fac \rangle \\ succ \cdot \underline{0} = \underline{1} \\ succ \cdot succ = succ \cdot succ \end{cases}$$

$\equiv$ (69); (72); (74); (70); def. de succ

$$\begin{cases} fac\ 0 = 1 \\ fac\ (n+1) = mul(succ\ n, fac\ n) \\ succ\ 0 = 1 \\ succ\ (n+1) = succ(succ\ n) \end{cases}$$

5 -

$$\begin{cases} f_1\ [\,] = [\,] \\ f_1\ (h:t) = h : (f_2\ t) \\ f_2\ [\,] = [\,] \\ f_2\ (h:t) = f_1\ t \end{cases}$$

$\equiv$ Def. de nil e cons ; (77)

$$\begin{cases} f_1\ (nil\ ()) = nil\ () \\ f_1\ (cons\ (h,t)) = cons\ (h, f_2\ t) \\ f_2\ (nil\ ()) = nil\ () \\ f_2\ (cons\ (h,t)) = f_1\ (\pi_2\ (h,t)) \end{cases}$$

$\equiv$ (70); (69); (75); (1)

$$\begin{cases} f_1 \cdot nil = nil \\ f_1 \cdot cons = cons \cdot (id \times f_2) \\ f_2 \cdot nil = nil \\ f_2 \cdot cons = f_1 \cdot \pi_2 \end{cases}$$

$\equiv$ (27); (20)

$$\begin{cases} f_1 \cdot [nil, cons] = [nil, cons \cdot (id \times f_2)] \\ f_2 \cdot [nil, cons] = [nil, f_1 \cdot \pi_2] \end{cases}$$

$\equiv$ (13); (7)

$$\begin{cases} f_1 \cdot [nil, cons] = [nil, cons \cdot (id \times \pi_2 \cdot \langle f_1, f_2 \rangle)] \\ f_2 \cdot [nil, cons] = [nil, \pi_2 \cdot (id \times \pi_1 \cdot \langle f_1, f_2 \rangle)] \end{cases}$$

$\equiv$ (1); (14)

$$\begin{cases} f_1 \cdot [nil, cons] = [nil, cons \cdot (id \times \pi_2) \cdot (id \times \langle f_1, f_2 \rangle)] \\ f_2 \cdot [nil, cons] = [nil, \pi_2 \cdot (id \times \pi_1) \cdot (id \times \langle f_1, f_2 \rangle)] \end{cases}$$

$\equiv$ (13); (22); (1)

$$\begin{cases} f_1 \cdot [nil, cons] = [nil, cons \cdot (id \times \pi_2)] \cdot (id + id \times \langle f_1, f_2 \rangle) \\ f_2 \cdot [nil, cons] = [nil, \pi_1 \cdot \pi_2] \cdot (id + id \times \langle f_1, f_2 \rangle) \end{cases}$$

$\equiv$ (50)

$$\langle f_1, f_2 \rangle = (\!|\langle [nil, cons \cdot (id \times \pi_2)], [nil, \pi_1 \cdot \pi_2] \rangle|\!)$$

$$
\begin{tikzcd}
A^* \arrow[d, "\langle f_1, f_2 \rangle"'] & 1 + A \times A^* \arrow[l, "in"'] \arrow[d, "id + id \times \langle f_1, f_2 \rangle"] \\
A^* \times A^* & 1 + A \times (A^* \times A^*) \arrow[l, "g"]
\end{tikzcd}
$$

$$g = \langle [nil, cons \cdot (id \times \pi_2)], [nil, \pi_1 \cdot \pi_2] \rangle$$

$$[0,1,2,3] \xrightarrow{f_1} [0,2]$$

$$[0,1,2,3] \xrightarrow{f_2} [1,3]$$

A função f_1 retorna da lista dada todos os elementos presentes em posições impares.
A função f_2 retorna da lista dada todos os elementos presentes em posições pares.

6- $H f = F f + G f$
$= id + f$

$K f = G f \times F f$
$= f \times id$

8- $\mu \cdot T\mu$
$\equiv$ def. de μ, $\mu \circ Tf$
$(\pi_1 \times \pi_2) \cdot (\langle id, id \rangle \times \langle id, id \rangle)$
$\equiv (14)$
$(\pi_1 \cdot \langle id, id \rangle) \times (\pi_2 \cdot \langle id, id \rangle)$
$\equiv (7)$
$id_A \times id_B$
$\equiv (15)$
$id_{A \times B}$
$\equiv (8)$
$\langle \pi_1, \pi_2 \rangle$
$\equiv (1)$
$\langle \pi_1 \cdot id, \pi_2 \cdot id \rangle$
$\equiv (11)$
$(\pi_1 \times \pi_2) \cdot \langle id, id \rangle$
$\equiv$ def. de μ e u
$\mu \cdot u$

9-
$$\begin{tikzcd}
\mathrm{LTree}\ A \arrow[d, "mirror"'] & A + (\mathrm{LTree}\ A)^2 \arrow[l, "in"'] \arrow[d, "id + mirror^2"] \\
\mathrm{LTree}\ A & A + (\mathrm{LTree}\ A)^2 \arrow[l, "g"]
\end{tikzcd}$$

$g = [Leaf, Fork \cdot swap]$

$\equiv (1), (25)$

$g = [Leaf, Fork] \cdot (id + swap)$

$mirror \cdot mirror = id$

$\equiv (45); (F3)$

$mirror \cdot (\!| in \cdot (id + swap) |\!) = (\!| in |\!)$

$\Leftarrow (46)$

$mirror \cdot in \cdot (id + swap) = in \cdot (id + mirror^2)$

$\equiv$ def. de in; (20); (22)

$[mirror \cdot Leaf \cdot id, mirror \cdot Fork \cdot swap] = [Leaf \cdot id, Fork \cdot mirror^2]$

$\equiv (1); (27)$

$\begin{cases} mirror \cdot Leaf = Leaf \\ mirror \cdot Fork \cdot swap = Fork \cdot mirror^2 \end{cases}$

$\equiv (1)$, $swap \cdot swap = id$

$\begin{cases} mirror \cdot Leaf = Leaf \\ mirror \cdot Fork \cdot swap = Fork \cdot mirror^2 \cdot swap \cdot swap \end{cases}$

$\Leftarrow (5)$

$\begin{cases} mirror \cdot Leaf = Leaf \\ mirror \cdot Fork = Fork \cdot mirror^2 \cdot swap \end{cases}$

(69); def. do $swap$; (70)

$\begin{cases} mirror\ (Leaf\ a) = Leaf\ a \\ mirror\ (Fork\ (x, y)) = Fork\ ((mirror \times mirror)\ (y, x)) \end{cases}$

$\equiv (75)$

$mirror\ (Leaf\ a) = Leaf\ a$

$mirror\ (Fork\ (x, y)) = Fork\ (mirror\ y, mirror\ x)$

Ficha nº 8

1- $\langle (|i|), (|j|) \rangle = (|i \times j|) . \langle F\pi_1, F\pi_2 \rangle$

$\equiv (11)$

$\langle (|i|), (|j|) \rangle = (| \langle i . F\pi_1, j . F\pi_2 \rangle |)$

$\equiv (50)$

$\begin{cases} (|i|) . in = i . F\pi_1 . F\langle (|i|), (|j|) \rangle \\ (|j|) . in = j . F\pi_2 . F\langle (|i|), (|j|) \rangle \end{cases}$

$\equiv (41)$

$\begin{cases} (|i|) . in = i . F(\pi_1 . \langle (|i|), (|j|) \rangle) \\ (|j|) . in = j . F(\pi_2 . \langle (|i|), (|j|) \rangle) \end{cases}$

$\equiv (7), (43)$

true

2- $aux = \langle sum, length \rangle$

$\equiv$ def. de sum e length

$aux = \langle (|[zero, add]|), (|[zero, succ . \pi_2]|) \rangle$

$\equiv (51), (11)$

$aux = (| \langle [zero, add] . F\pi_1 , [zero, succ . \pi_2] . F\pi_2 \rangle |)$

$\equiv (43)$

$aux . [nil, cons] = \langle [zero, add] . F\pi_1 , [zero, succ . \pi_2] . F\pi_2 \rangle$

$\equiv (20), (9), (41)$

$[aux . nil, aux . cons] = \langle [zero, add] . F(\pi_1 . aux), [zero, succ . \pi_2$

$\equiv Ff = id + id \times f ; (22); (1)$

$[aux . nil, aux . cons] = \langle [zero, add . (id \times \pi_1 . aux)], [zero, succ .$

$\equiv (13); (28)$

$[aux . nil, aux . cons] = [\langle zero, zero \rangle, \langle add . (id \times \pi_1 . aux), succ$

$\equiv (27)$

$\begin{cases} aux . nil = \langle zero, zero \rangle \\ aux . cons = \langle add . (id \times \pi_1 . aux), succ . \pi_2 . aux . \pi_2 \rangle \end{cases}$

$\equiv (69); (70)$

$\begin{cases} aux(nil()) = \langle zero, zero \rangle () \\ aux(cons(h, t)) = \langle add . (id \times \pi_1 . aux), succ . \pi_2 . aux . \pi_2 \end{cases}$

$\equiv$ def. de nil e cons; (74); (72)

$$\begin{cases} aux\ [\,] = (0,0) \\ aux\ (h : t) = (add \cdot (id \times \pi_1 \cdot aux)(h,t),\ succ \cdot \pi_2 \cdot aux \cdot \pi_2 (h,t)) \end{cases}$$

$\equiv$ (75); (71); (77); (70)

$$\begin{cases} aux\ [\,] = (0,0) \\ aux\ (h : t) = (add\ (h, \pi_1 (aux\ t)),\ succ\ (\pi_2 (aux\ t))) \end{cases}$$

$\equiv$ aux t = (x, y); (77); def. de succ e add

$$\begin{cases} aux\ [\,] = (0,0) \\ aux\ (h : t) = (h + x,\ 1 + y) \\ \qquad \text{where } (x, y) = aux\ t \end{cases}$$

average = ratio . <sum, length>

$\equiv$ def. de aux = <sum, length>

average = ratio . aux

$\equiv$ (69); (70)

average l = ratio (aux l)

$\equiv$ aux l = (n, d); def. de ratio

```
average l = n / d
    where (n, d) = aux l
          aux [ ] = (0,0)
          aux (h : t) = (h + x, 1 + y)
              where (x, y) = aux t
```

3-

$$\begin{tikzcd} \mathrm{LTree}\,\mathbb{N}_0 \arrow[d, "sum"'] & \mathbb{N}_0 + (\mathrm{LTree}\,\mathbb{N}_0)^2 \arrow[l, "in"'] \arrow[d, "id + sum^2"] \\ \mathbb{N}_0 & \mathbb{N}_0 + (\mathbb{N}_0)^2 \arrow[l, "g"] \end{tikzcd}$$

$$g = [id, (\hat{+})]$$

$$\begin{tikzcd} \mathrm{LTree}\,\mathbb{N}_0 \arrow[d, "length"'] & \mathbb{N}_0 + (\mathrm{LTree}\,\mathbb{N}_0)^2 \arrow[l] \arrow[d, "id + length^2"] \\ \mathbb{N}_0 & \mathbb{N}_0 + \mathbb{N}_0^2 \arrow[l, "g"] \end{tikzcd}$$

$$g = [\underline{1}, (\hat{+})]$$

$aux = \langle sum, length \rangle$

$\equiv$ def. de sum e length

$aux = \langle (\![[id, (\hat{+})]]\!), (\![[\underline{1}, (\hat{+})]]\!) \rangle$

$\equiv$ pela demonstração anterior

$[aux \cdot Leaf, aux \cdot Fork] = \langle [id, (\hat{+})] \cdot F(\pi_1 \cdot aux), [\underline{1}, (\hat{+})] \cdot F(\pi$

$\equiv F f = id + f^2$; (22); (11)

$[aux \cdot Leaf, aux \cdot Fork] = \langle [id, (\hat{+}) \cdot (\pi_1 \cdot aux)^2], [\underline{1}, (\hat{+}) \cdot (\pi_2 \cdot aux$

$\equiv$ (28); (27)

$$\begin{cases} aux \cdot Leaf = \langle id, \underline{1} \rangle \\ aux \cdot Fork = \langle (\hat{+}) \cdot (\pi_1 \cdot aux \times \pi_1 \cdot aux), (\hat{+}) \cdot (\pi_2 \cdot aux \times \pi_2 \cdot a \end{cases}$$

$\equiv$ (69); (70); (74); (75)

$$\begin{cases} aux\,(Leaf\ a) = (id\ a, \underline{1}\ a) \\ aux\,(Fork\,(e, d)) = ((\hat{+})\ (\pi_1(aux\ e), \pi_1(aux\ d)), (\hat{+})\ (\pi_2(aux \end{cases}$$

$\equiv$ (71); (72); (82); (77)

$$\begin{cases} aux\,(Leaf\ a) = (a, 1) \\ aux\,(Fork\,(e, d)) = (x + a, y + b) \\ \qquad \text{where } (x, y) = aux\ e \\ \qquad\qquad (a, b) = aux\ d \end{cases}$$

average = ratio ∘ ⟨sum, length⟩
≡ def. de aux ; (69) ; (70)
average t = ratio (aux t)
≡ def. do ratio
average t = s / l
where (s, l) = aux t

4- $\boxed{B(X, Y) = X \times Y}$

$B(id, id) = id \times id$
≡ (15)
$B(id, id) = id$

$B(f \circ g, k \circ h) = f \circ g \times k \circ h$
≡ (14)
$B(f \circ g, k \circ h) = f \times k \circ g \times h$
≡ def. de B(X, Y)
$B(f \circ g, k \circ h) = B(f, k) \circ B(g, h)$

$\boxed{B(X, Y) = X + Y}$

$B(id, id) = id + id$
≡ (26)
$B(id, id) = id$

$B(f \circ g, k \circ h) = f \circ g + k \circ h$
≡ (25)
1/2 (aux d))) $B(f \circ g, k \circ h) = f + k \circ g + h$
≡ def. do B(X, Y)
$B(f \circ g, k \circ h) = B(f, k) \circ B(g, h)$

$$\boxed{B(X,Y) = X + Y \times Y}$$

$B(id, id) = id + id \times id$

$\equiv (15)$

$B(id, id) = id + id$

$\equiv (26)$

$B(id, id) = id$

$B(f \cdot g, h \cdot k) = f \cdot g + h \cdot k \times h \cdot k$

$\equiv (14)$

$B(f \cdot g, h \cdot k) = f \cdot g + h \times h \cdot k \times k$

$\equiv (25)$

$B(f \cdot g, h \cdot k) = f + h \times h \cdot g + k \times k$

$\equiv$ def. de $B(X,Y)$

$B(f \cdot g, h \cdot k) = B(f, h) \cdot B(g, k)$

5- $f^* = T f$

$\equiv (48)$

$f^* = (\!|\ in \cdot B(f, id)\ |\!)$

$\equiv$ def. de in e $B(X,Y)$

$f^* = (\!|\ [nil, cons] \cdot (id + f \times id)\ |\!)$

$\equiv (22)$

$f^* = (\!|\ [nil \cdot id, cons \cdot (f \times id)]\ |\!)$

$\equiv (43); (1)$

$f^* \cdot [nil, cons] = [nil, cons \cdot (f \times id)] \cdot (id + id \times f^*)$

$\equiv (20); (22); (1)$

$[f^* \cdot nil, f^* \cdot cons] = [nil, cons \cdot (f \times id) \cdot (id \times f^*)]$

$\equiv (27); (14); (1)$

$$\begin{cases} f^* \cdot nil = nil \\ f^* \cdot cons = cons \cdot (f \times f^*) \end{cases}$$

$\equiv (69); (70); (75);$ def. de nil e $cons$

$$\begin{cases} f^* [\,] = [\,] \\ f^* (h : t) = f\ h : f^*\ t \end{cases}$$

$\text{Rose } f = T f$

$\equiv (48)$

$\text{Rose } f = (\!|\, \text{in} \cdot B(f, id) \,|\!)$

$\equiv$ def. do $\text{in} \cdot B(f, id)$

$\text{Rose } f = (\!|\, \text{Ros} \cdot (f \times id^*) \,|\!)$

$\equiv (43)$

$\text{Rose } f \cdot \text{in} = \text{Ros} \cdot (f \times id^*) \cdot (id \times (\text{Rose } f)^*)$

$\equiv$ def. do in ; (44) ; (1)

$\text{Rose } f \cdot \text{Ros} = \text{Ros} \cdot (f \times \text{Rose } f)$

$\equiv (69) ; (70) ; (76)$

$\text{Rose } f\, (\text{Ros}\,(x, xs)) = \text{Ros}\,(f\, x, (\text{Rose } f)^*\, xs)$

6-

$$\begin{tikzcd}
\text{BTree } A \arrow[d, "f"'] & 1 + A \times (\text{BTree } A)^2 \arrow[l, "\text{in}"'] \arrow[d, "id + id \times f^2"] \\
C & 1 + A \times C^2 \arrow[l, "g"]
\end{tikzcd}$$

$B(A, \text{BTree } A) = 1 + A \times (\text{BTree } A)^2$

$B(f, g) = id + f \times g^2$

$$\begin{tikzcd}
\text{NEList } A \arrow[d, "f"'] & A + A \times \text{NEList } A \arrow[l] \arrow[d, "id + id \times f"] \\
C & A + A \times C \arrow[l, "g"]
\end{tikzcd}$$

$B(A, \text{NEList } A) = A + A \times \text{NEList } A$

$B(f, g) = id + f \times g$

Ficha nº 9

1- T in B(X,Y) B(f,g) F f ~~BT~~ T f

$\mathbb{N}_0$ $[\underline{0}, succ]$ — — $id + f$ —

A^* $[nil, cons]$ $1 + X \times Y$ $id + f \times g$ $id + id \times f$ $(\![nil, co$

$BTree\ A$ $[Empty, Node]$ $1 + X \times Y^2$ $id + f \times g^2$ $id + id \times f^2$ $(\![Empty,$

$LTree\ A$ $[Leaf, Fork]$ $X + Y^2$ $f + g^2$ $id + f^2$ $(\![Leaf . f,$

2- $length = (\![\underline{0}, succ . \pi_2]\!)$

$sum = (\![\underline{0}, (\hat{+})]\!)$

$map\ f = T\ f = (\![nil, cons . (f \times id)]\!)$

$length = sum . (map\ \underline{1})$

$\equiv$ def. de sum e map

$length = (\![\underline{0}, (\hat{+})]\!) . T\ \underline{1}$

$\equiv$ (49)

$length = (\![\underline{0}, (\hat{+})] . B(\underline{1}, id)]\!)$

$\equiv$ def. de $B(X,Y)$; (22); (1)

$length = (\![\underline{0}, (\hat{+}) . (\underline{1} \times id)]\!)$

$\equiv$ (43)

$length . [nil, cons] = [\underline{0}, (\hat{+}) . (\underline{1} \times id)] . (id + id \times length)$

$\equiv$ (20); (22); (1)

$[length . nil, length . cons] = [\underline{0}, (\hat{+}) . (\underline{1} \times id) . (id \times length$

$\equiv$ (27); (14); (1)

$\begin{cases} length . nil = \underline{0} \\ length . cons = (\hat{+}) . (\underline{1} \times length) \end{cases}$

$\equiv$ (69); (70); (75); (72); (82); def. de nil e cons

$\begin{cases} length\ [\ ] = 0 \\ length\ (h : t) = 1 + length\ t \end{cases}$

$length = length \cdot (\mathsf{map}\ f)$

$\equiv$ def. de length e map

$length = (\![\underline{0}, succ \cdot \pi_2]\!) \cdot \mathsf{T} f$

$\equiv$ (49); def. de $B(X, Y)$

$length = (\![\underline{0}, succ \cdot \pi_2] \cdot (id + f \times id)]\!)$

$\equiv$ (22); (1); (43)

$length \cdot [nil, cons] = [\underline{0}, succ \cdot \pi_2 \cdot (f \times id)] \cdot (id + id \times length)$

$\equiv$ (20); (22); (1); (12)

$[length \cdot nil, length \cdot cons] = [\underline{0}, succ \cdot id \cdot \pi_2 \cdot (id \times length)]$

$\equiv$ (27); (1); (13)

$$\begin{cases} length \cdot nil = \underline{0} \\ length \cdot cons = succ \cdot length \cdot \pi_2 \end{cases}$$

$\equiv$ (69); (70); def. do nil e cons; (72)

$$\begin{cases} length\ [\,] = 0 \\ length\ (h : t) = succ\ (length\ (\pi_2\ (h, t))) \end{cases}$$

$\equiv$ (77); def. do succ

$length\ [\,] = 0$

$length\ (h : t) = 1 + length\ t$

3- $length \cdot concat = sum \cdot map\ length$

$\equiv$ def. de sum e map; (49); def. de $B(X, Y)$

$length \cdot concat = (\![0, add]\!) \cdot (id + length \times id)$

$\equiv$ (F4); (22); (1)

$length \cdot (\![nil, conc]\!) = (\![0, add \cdot (length \times id)]\!)$

$\Leftarrow$ (46)

$length \cdot [nil, conc] = [0, add \cdot (length \times id)] \cdot (id + id \times length)$

$\equiv$ (20); (22); (1); (27)

$\begin{cases} length \cdot nil = \underline{0} \\ length \cdot conc = add \cdot (length \times id) \cdot (id \times length) \end{cases}$

$\equiv$ $length\ [\,] = 0 \Leftrightarrow true$; (14); (1)

$length \cdot conc = add \cdot (length \times length)$

$\equiv$ ...

true

4- $count = count \cdot BTree\ f$

$\equiv$ def. de count

$count = (\![zero, succ \cdot add \cdot \pi_2]\!) \cdot BTree\ f$

$\equiv$ (49)

$count = (\![zero, succ \cdot add \cdot \pi_2] \cdot B(f, id)]\!)$

$\equiv$ def. de $B(f, g)$; (22); (1)

$count = (\![zero, succ \cdot add \cdot \pi_2 \cdot (f \times (id \times id))]\!)$

$\equiv$ (13); (13)

$count = (\![zero, succ \cdot add \cdot id \cdot \pi_2]\!)$

$\equiv$ (1)

$count = (\![zero, succ \cdot add \cdot \pi_2]\!)$

$\equiv$ def. de count

$count = count$

5- $(Tf) \cdot mirror = mirror \cdot (Tf)$

$\equiv$ def. do mirror

$(Tf) \cdot (\!|in \cdot (id + swap)|\!) = (\!|in \cdot (id + swap)|\!) \cdot (Tf)$

$\equiv (49)$

$(Tf) \cdot (\!|in \cdot (id + swap)|\!) = (\!|in \cdot (id + swap) \cdot B(f, id)|\!)$

$\Leftarrow (46)$

$(Tf) \cdot in \cdot (id + swap) = in \cdot (id + swap) \cdot B(f, id) \cdot (id + Tf \times Tf)$

$\equiv$ def. do $B(X,Y)$; (25); (1)

$(Tf) \cdot in \cdot (id + swap) = in \cdot (id + swap) \cdot (f + Tf \times Tf)$

$\equiv (48)$; (25); (1)

$(\!|in \cdot B(f, id)|\!) \cdot in \cdot (id + swap) = in \cdot (f + swap \cdot (Tf \times Tf))$

$\equiv (44)$; (48)

$in \cdot B(f, id) \cdot (id + Tf \times Tf) \cdot (id + swap) = in \cdot (f + swap \cdot (Tf \times Tf))$

$\equiv (23)$; (1)

$in \cdot B(f, id) \cdot (id + (Tf \times Tf) \cdot swap) = in \cdot (f + swap \cdot (Tf \times Tf))$

$\equiv$ def. do $B(X,Y)$; (25); (1)

$in \cdot (f + (Tf \times Tf) \cdot swap) = in \cdot (f + swap \cdot (Tf \times Tf))$

$\equiv$ def. de $swap = \langle \pi_2, \pi_1 \rangle$; (9); (11)

$in \cdot (f + \langle Tf \cdot \pi_2, Tf \cdot \pi_1 \rangle) = in \cdot (f + \langle \pi_2 \cdot (Tf \times Tf), \pi_1 \cdot (Tf \times Tf) \rangle)$

$\equiv (12)$; (13)

$in \cdot (f + \langle Tf \cdot \pi_2, Tf \cdot \pi_1 \rangle) = in \cdot (f + \langle Tf \cdot \pi_2, Tf \cdot \pi_1 \rangle)$

6- $h = [\![(id + \langle f, id \rangle) \cdot out_{\mathbb{N}_0}]\!]$

$\equiv (52);$

$out_{List} \cdot h = (id + id \times h) \cdot (id + \langle f, id \rangle) \cdot out_{\mathbb{N}_0}$

$\equiv out \cdot in = id$ (isomorfismo)

$h \cdot in_{\mathbb{N}_0} = in_{List} \cdot (id + id \times h) \cdot (id + \langle f, id \rangle)$

$\equiv (25); (1)$

$h \cdot in_{\mathbb{N}_0} = in_{List} \cdot (id + (id \times h) \cdot \langle f, id \rangle)$

$\equiv (11);$ def. de $in_{\mathbb{N}_0}$ e in_{List}; (1)

$h \cdot [\underline{0}, succ] = [nil, cons] \cdot (id + \langle f, h \rangle)$

$\equiv (20); (22); (1); (27)$

$$\begin{cases} h \cdot \underline{0} = nil \\ h \cdot succ = cons \cdot \langle f, h \rangle \end{cases}$$

$\equiv (69); (70); (74)$

$$\begin{cases} h(\underline{0}()) = nil() \\ h(succ\ n) = cons(f\ n, h\ n) \end{cases}$$

$\equiv$ def. de nil, succ, cons e f; (72)

$$\begin{cases} h\ 0 = [\,] \\ h(n+1) = (2n+1) : h\ n \end{cases}$$

Ficha nº10

1- depth = depth . LTree f

≡ (F1) ; def. de LTree

depth = ([one, succ . umax]) . T f

≡ (49)

depth = ([one, succ . umax] . B(f, id))

≡ def. de B(X,Y)

depth = ([one, succ . umax] . (f + id × id))

≡ def. de one ; (22)

depth = ([1 . f, succ . umax . (id × id)])

≡ (3) ; (15) ; (1) ; def. de one

depth = ([one, succ . umax])

2- T f = (in . B(f, id))

≡ (43)

T f . in = in . B(f, id) . F(T f)

≡ (47) ;

T f . in = in . B(f, id) . B(id, T f)

≡ B(f, g) . B(h, k) = B(f . h, g . k)

T f . in = in . B(f . id, id . T f)

≡ (1) ; B(f, g) . B(h, k) = B(f . h, g . k)

T f . in = in . B(id, T f) . B(f, id)

≡ out é a função inversa de in, tal que in . out = id

out . T f = B(id, T f) . B(f, id) . out

≡ (52) ; (47)

T f = [(B(f, id) . out)]

2- $length = (\!|[zero, succ \cdot \pi_2]|\!)$

$\equiv$ (43)

$length \cdot in_{List} = [zero, succ \cdot \pi_2] \cdot F\, length$

$\equiv$ (1); (22)

$length \cdot in_{List} = [zero, succ] \cdot (id + \pi_2) \cdot F\, length$

$\equiv$ def. de in_{N_0}

$length \cdot in_{List} = in_{N_0} \cdot (id + \pi_2) \cdot F\, length$

$\equiv$ out é a função inversa de in, tal que in . out são isomorfismos

$out_{N_0} \cdot length = (id + \pi_2) \cdot F\, length \cdot out_{List}$

$\equiv$ def. de F f (listas)

$out_{N_0} \cdot length = (id + \pi_2) \cdot (id + id \times length) \cdot out_{List}$

$\equiv$ (25);

$out_{N_0} \cdot length = ((id \cdot id) + (\pi_2 \cdot (id \times length))) \cdot out_{List}$

$\equiv$ (13), (25)

$out_{N_0} \cdot length = (id + length) \cdot (id + \pi_2) \cdot out_{List}$

$\equiv$ def. de F f (N_0)

$out_{N_0} \cdot length = F\, length \cdot (id + \pi_2) \cdot out_{List}$

$\equiv$ (52)

$length = [\!((id + \pi_2) \cdot out_{List})\!]$

4- $mirror = (\!| in \cdot (id + swap) |\!)$

$\equiv (43)$

$mirror \cdot in = in \cdot (id + swap) \cdot F\, mirror$

$\equiv$ { out é a função inversa de in, tal que in e out são isomorfismos }

$out \cdot mirror = (id + swap) \cdot F\, mirror \cdot out$

$\equiv$ { def. de F f }

$out \cdot mirror = (id + swap) \cdot (id + mirror \times mirror) \cdot out$

$\equiv$ { def. de $swap = \langle \pi_2, \pi_1 \rangle$; (25); (9) }

$out \cdot mirror = ((id \cdot id) + (\langle \pi_2 \cdot (mirror \times mirror), \pi_1 \cdot (mirror \times mirror) \rangle)) \cdot out$

$\equiv (12); (13); (11)$

$out \cdot mirror = ((id \cdot id) + ((mirror \times mirror) \cdot \langle \pi_2, \pi_1 \rangle)) \cdot out$

$\equiv$ (25); def. de F f e swap

$out \cdot mirror = F\, mirror \cdot (id + swap) \cdot out$

$\equiv (52)$

$mirror = [\!((id + swap) \cdot out)\!]$

$mirror \cdot mirror = id$

$\equiv$ (54); def. de mirror

$[\!((id + swap) \cdot out)\!] \cdot mirror = [\!(out)\!]$

$\Leftarrow (55)$

$(id + swap) \cdot out \cdot mirror = F\, mirror \cdot out$

$\equiv$ def. de F f

$(id + swap) \cdot out \cdot mirror = (id + mirror \times mirror) \cdot out$

$\equiv$ def. de mirror; (53)

$(id + swap) \cdot F\, mirror \cdot (id + swap) \cdot out = (id + mirror \times mirror) \cdot out$

$\Leftarrow$ (5); def. de F f

$(id + swap) \cdot (id + mirror \times mirror) \cdot (id + swap) = (id + mirror \times mirror)$

$\equiv (25); (1)$

$(id + swap) \cdot (id + (mirror \times mirror) \cdot swap) = (id + mirror \times mirror)$

$\equiv$ def. de $swap = \langle \pi_2, \pi_1 \rangle$; (11); (25); (1)

$(id + swap \cdot \langle mirror \cdot \pi_2, mirror \cdot \pi_1 \rangle) = (id + mirror \times mirror)$

$\equiv$ (12); (13); (9); def. de swap

$(id + swap \cdot swap \cdot (mirror \times mirror)) = (id + mirror \times mirror)$

$\equiv$ $swap \cdot swap = id$; (1)

$(id + mirror \times mirror) = (id + mirror \times mirror)$

5- $\text{repeat} = (\!| i_2 . \langle id, id \rangle |\!)$

$\equiv (52)$

$out . \text{repeat} = F\,\text{repeat} . i_2 . \langle id, id \rangle$

$\equiv$ def. de $F f$; (24)

$out . \text{repeat} = i_2 . (id \times \text{repeat}) . \langle id, id \rangle$

$\equiv$ in é a função inversa de out, tal que in e out são isomorfismos;

$\text{repeat} = in . i_2 . \langle id, \text{repeat} \rangle$

$\equiv$ def. de $in = [nil, cons]$; (18)

$\text{repeat} = cons . \langle id, \text{repeat} \rangle$

$\equiv$ (69); (74); (70); def. de cons; (71)

$\text{repeat}\ a = a : \text{repeat}\ a$

$(map\ f) . \text{repeat} = \text{repeat} . f$

$\equiv$ def. de map e repeat

$T f . (\!| i_2 . \langle id, id \rangle |\!) = \text{repeat} . f$

$\equiv (58)$

$(\!| B(f, id) . i_2 . \langle id, id \rangle |\!) = \text{repeat} . f$

$\equiv$ def. de $B(X,Y)$

$(\!| (id + f \times id) . i_2 . \langle id, id \rangle |\!) = \text{repeat} . f$

$\equiv (24)$

$(\!| i_2 . (f \times id) . \langle id, id \rangle |\!) = \text{repeat} . f$

$\Leftarrow$ def. de repeat ; (55)

$F f . i_2 . (f \times id) . \langle id, id \rangle = i_2 . \langle id, id \rangle . f$

$\equiv$ def. de $F f$; (24)

$i_2 . (id \times f) . (f \times id) . \langle id, id \rangle = i_2 . \langle id, id \rangle . f$

$\equiv (14) ; (1)$

$i_2 . (f \times f) . \langle id, id \rangle = i_2 . \langle id, id \rangle . f$

$\equiv (11) ; (9) ; (1)$

$i_2 . \langle f, f \rangle = i_2 . \langle f, f \rangle$

6- $unzip\ xs = (map\ \pi_1\ xs,\ map\ \pi_2\ xs)$

$\equiv (74); (69)$

$unzip = \langle map\ \pi_1,\ map\ \pi_2 \rangle$

$\equiv$ def. de $map\ f = T f$; (48)

$unzip = \langle (\!| in \cdot B(\pi_1, id) |\!),\ (\!| in \cdot B(\pi_2, id) |\!) \rangle$

$\equiv (51), (11)$

$unzip = (\!| \langle in \cdot B(\pi_1, id) \cdot F\pi_1,\ in \cdot B(\pi_2, id) \cdot F\pi_2 \rangle |\!)$

$\equiv (47),\ B(f,g) \cdot B(h,k) = B(f \cdot h, g \cdot k); (1)$

$unzip = (\!| \langle in \cdot B(\pi_1, \pi_1),\ in \cdot B(\pi_2, \pi_2) \rangle |\!)$

$\equiv$ def. de $B(f,g)$ e in; (22); (1); (43)

$unzip \cdot in = \langle [nil,\ cons \cdot (\pi_1 \times \pi_1)],\ [nil,\ cons \cdot (\pi_2 \times \pi_2)] \rangle \cdot F\,unzip$

$\equiv (28)$

$unzip \cdot in = [\langle nil, nil \rangle,\ \langle cons \cdot (\pi_1 \times \pi_1),\ cons \cdot (\pi_2 \times \pi_2) \rangle] \cdot F\,unzip$

$\equiv$ def. de in e $F f$; (20); (22); (1); (27)

$$\begin{cases} unzip \cdot nil = \langle nil, nil \rangle \\ unzip \cdot cons = \langle cons \cdot (\pi_1 \times \pi_1),\ cons \cdot (\pi_2 \times \pi_2) \rangle \cdot (id \times unzip) \end{cases}$$

$\equiv (9); (14); (1)$

$$\begin{cases} unzip \cdot nil = \langle nil, nil \rangle \\ unzip \cdot cons = \langle cons \cdot (\pi_1 \times \pi_1 \cdot unzip),\ cons \cdot (\pi_2 \times \pi_2 \cdot unzip) \rangle \end{cases}$$

$\equiv (69); (70); (74); (75); (77)$

$$\begin{cases} unzip\ (nil\ ()) = (nil\ (),\ nil\ ()) \\ unzip\ (cons\ ((a, b), xs)) = (cons\ (a, \pi_1 (unzip\ xs)),\ cons\ (b, \pi_2 (unzip\ xs))) \end{cases}$$

$\equiv$ def. de nil e $cons$; $(as, bs) = unzip\ xs$; (77)

$$\begin{cases} unzip\ [\,] = ([\,], [\,]) \\ unzip\ ((a, b) : xs) = (a : as,\ b : bs) \\ \qquad \text{where } (as, bs) = unzip\ xs \end{cases}$$

Ficha nº 11

1- $solve = (\!|conquer|\!) \cdot [\!(divide)\!]$

$\equiv$ (44), (53); in e out são isomorfismos

$solve = conquer \cdot F(\!|conquer|\!) \cdot out \cdot in \cdot F[\!(divide)\!] \cdot divide$

$\equiv out \cdot in = id$

$solve = conquer \cdot F(\!|conquer|\!) \cdot F[\!(divide)\!] \cdot divide$

$\equiv$ (41)

$solve = conquer \cdot F((\!|conquer|\!) \cdot [\!(divide)\!]) \cdot divide$

$\equiv$ (F2)

$solve = conquer \cdot F\,solve \cdot divide$

2-

$$\begin{tikzcd}
BTree\ A \arrow[d, "inorder"'] & 1 + A \times (BTree\ A)^2 \arrow[l, "in"'] \arrow[d, "id + id \times inorder^2"] \\
A^* & 1 + A \times (A^*)^2 \arrow[l, "g"]
\end{tikzcd}$$

Através da análise da função inorder, mais particularmente da função f, concluímos que o que esta função faz é concatenar a sublista da esquerda com o elemento da raiz e a sublista da direita, de forma a obter uma lista ordenada de forma crescente.
Por forma a obter a lista ordenada de forma decrescente, apenas temos de "trocar" a árvore da direita com a da esquerda, e vice-versa. Sendo tal, temos que α é igual a:

$\alpha = (id + id \times swap)$