Two-way Analysis of Variance

Two-Way ANOVA

- Tests the equality of two or more population means when two independent variables are used: factor A and factor B (more than two factors: multi-way ANOVA).
- Each independent variables (factors) may have any number of levels.
- Same results as separate one-way ANOVA on each variable. **But interaction can be tested**.
- Saves time and effort, compared to consecutive one-way ANOVA tests.
- Assumptions are the same

Two-Way ANOVA

- Tests the equality of two or more population means when two independent variables are used: factor A and factor B (more than two factors: .
- Each independent variables (fac number of levels.
- Same results as separate one-way ANOVA on each variable. **But interaction can be tested**.
- Saves time and effort, compared to consecutive one-way ANOVA tests.
- Assumptions are the same

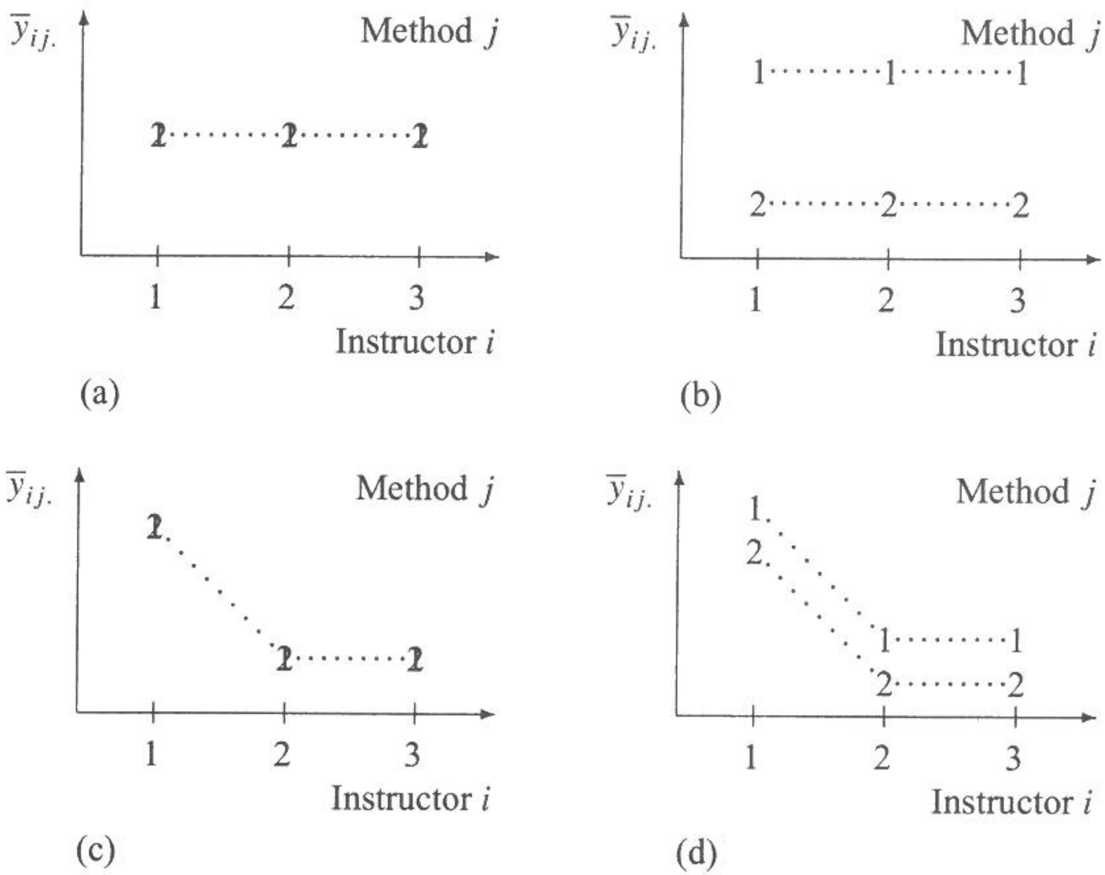
> The effect of the levels of a factor is not the same across the levels of the other factor.

Two-way ANOVA: Null Hypotheses

Tests 3 hypotheses simultaneously:

- No difference in means due to factor A
  - $H_0^A: \mu_{1.} = \mu_{2.} = \ldots = \mu_{a.}$
  - $H_1^A: \exists(i,j) : \mu_{i.} \neq \mu_{j.}$
- No difference in means due to factor B
  - $H_0^B: \mu_{.1} = \mu_{.2} = \ldots = \mu_{.b}$
  - $H_1^B: \exists(i,j) : \mu_{.i} \neq \mu_{.j}$
- No interaction of factors A and B
  - $H_0^{AB}: \forall(i,j) : \gamma_{ij} = 0$
  - $H_1^{AB}: \exists(i,j) : \gamma_{ij} \neq 0$

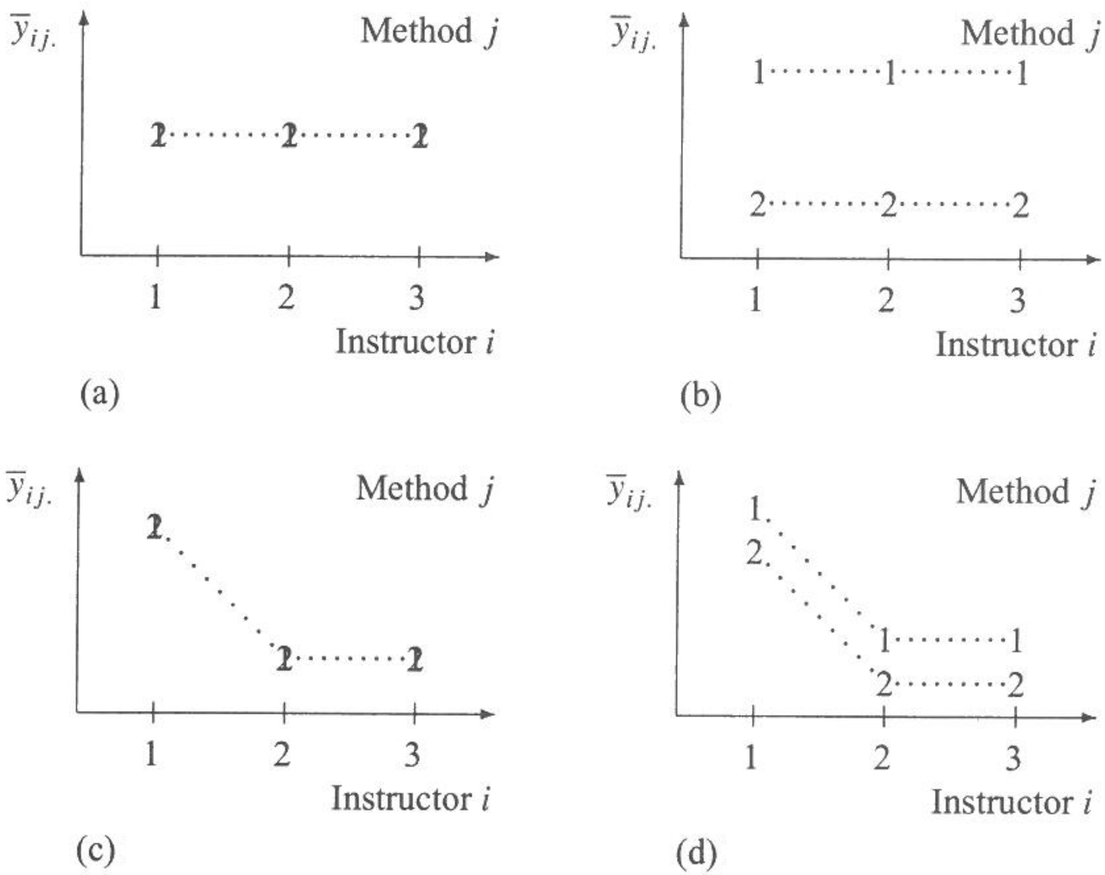
Two-way ANOVA : Interactions



Grades of students with respect to the instructor (1,2,3) and the teaching method (1,2)

Adapted from A.Dean and D.Voss, *Design and Analysis of Experiments*, Springer, 1999

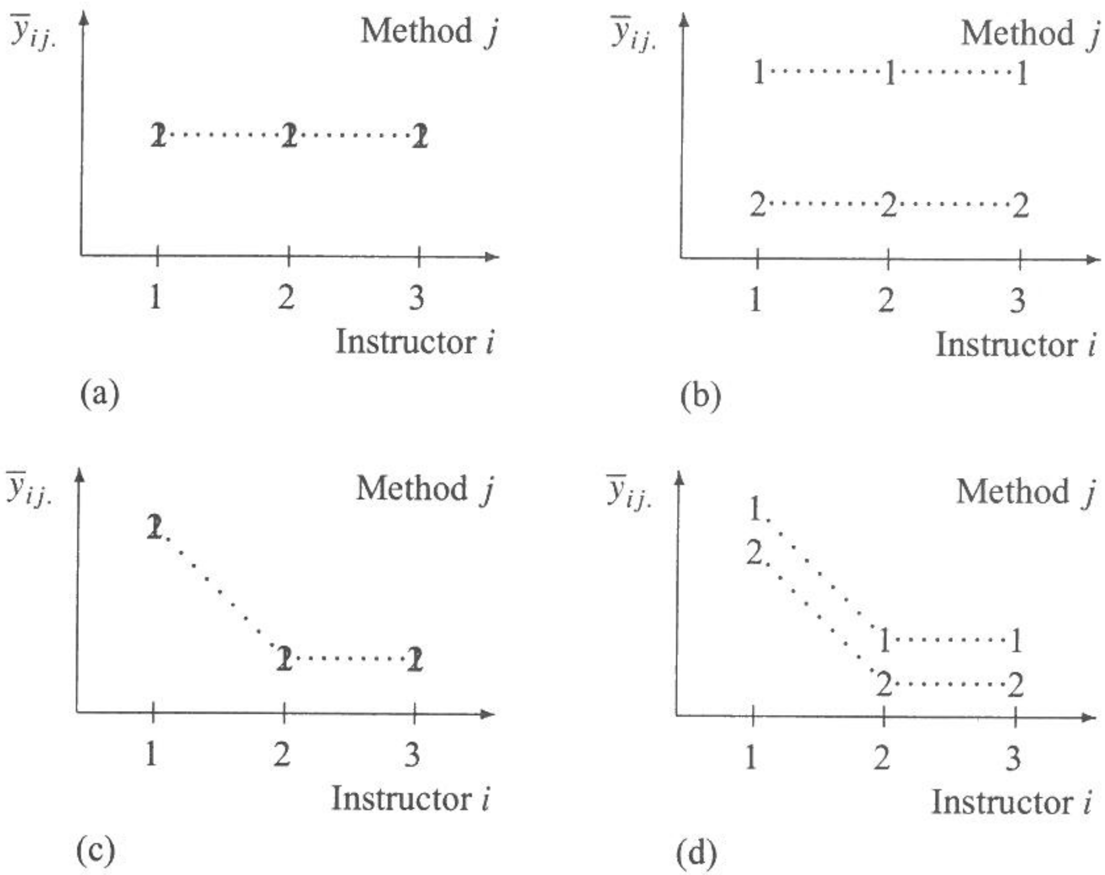
Two-way ANOVA : Interactions



a) There is no difference between instructors and methods – no interaction.

b) There is a difference between methods but no difference between instructors – no interaction

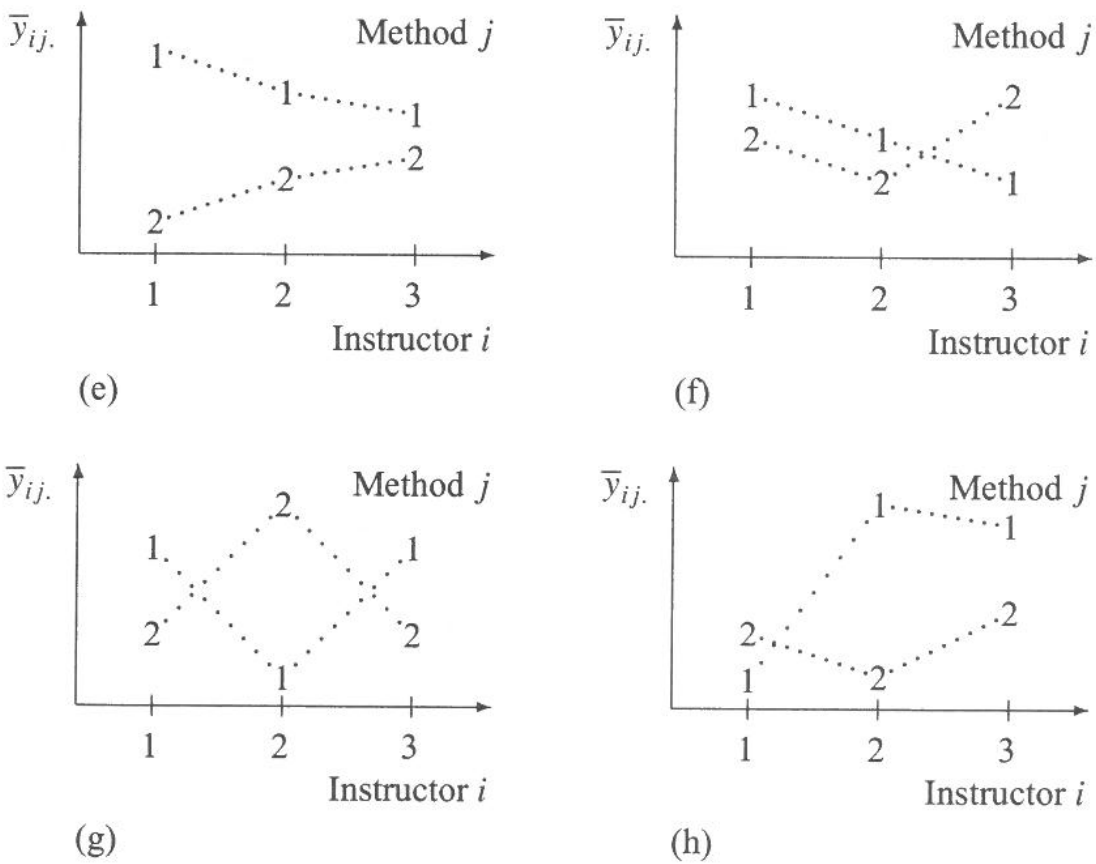
Two-way ANOVA : Interactions



c) There is a difference between instructors but no difference between methods – no interaction

d) There is a difference between instructors and methods but having the same effect – no interaction

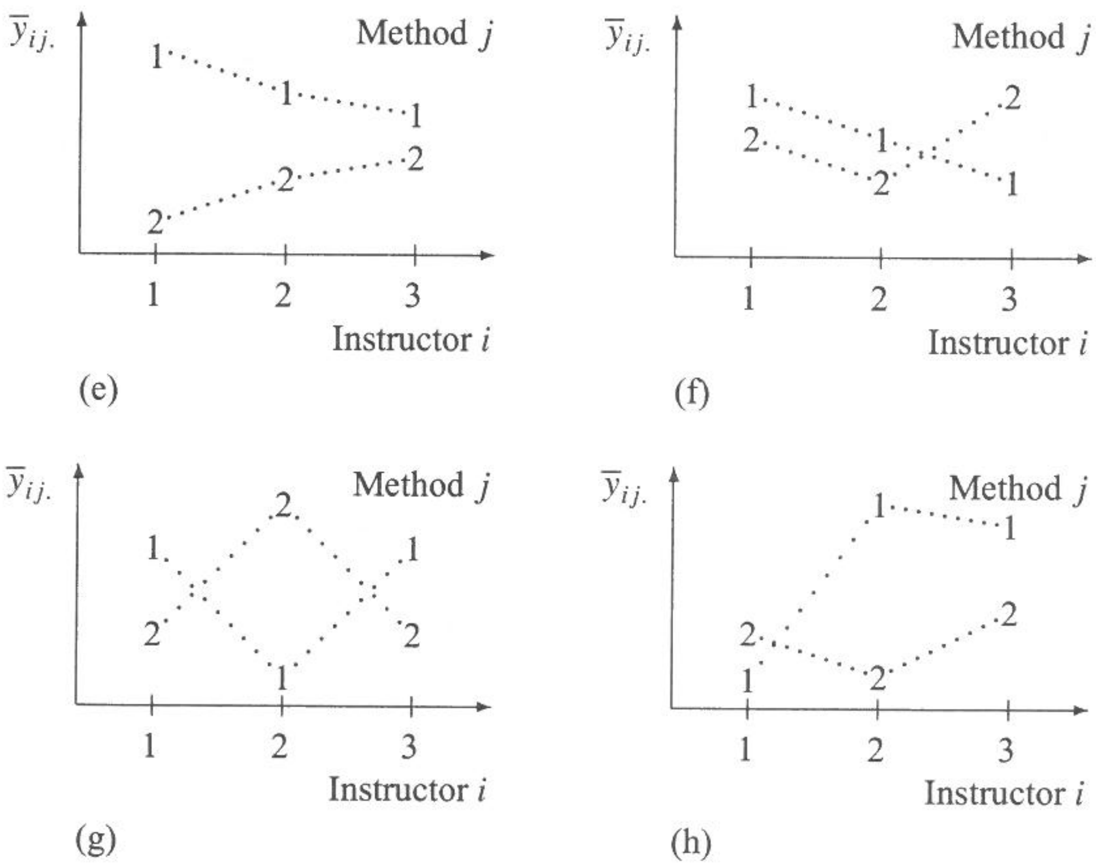
Two-way ANOVA : Interactions



e) Method 1 gives better grades but the magnitude depends of the instructor – interaction

f) Method 1 gives better grades except with instructor 3 - interaction

Two-way ANOVA : Interactions



g) Method 1 gives better grades except with instructor 2 - interaction

h) Method 1 gives better grades except with instructor 1 - interaction

Two-way ANOVA

	B						
A	1	...	j	...	b	Mean	Effect
1	y_{111}	...	y_{1j1}	...	y_{1b1}	$\bar{y}_{1..}$	α_1
	...	...	...	...	...		
	y_{11n}	...	y_{1jn}	...	y_{1bn}		
...	...	...	...	...	...	...	...
i	y_{i11}	...	y_{ij1}	...	y_{ib1}	$\bar{y}_{i..}$	α_i
	...	...	...	...	...		
	y_{i1n}	...	y_{ijn}	...	y_{ibn}		
...	...	...	...	...	...	...	...
a	y_{a11}	...	y_{aj1}	...	y_{ab1}	$\bar{y}_{a..}$	α_a
	...	...	...	...	...		
	y_{a1n}	...	y_{ajn}	...	y_{abn}		
Mean	$\bar{y}_{.1.}$	...	$\bar{y}_{.j.}$	...	$\bar{y}_{.b.}$	$\bar{y}_{...}$	
Effect	β_1	...	β_j	...	β_b		

Two-way ANOVA

	B						
A	1	...	j	...	b	Mean	Effect
1	y_{111}	...	y_{1j1}	...	y_{1b1}	$\bar{y}_{1..}$	α_1
	...	...	...	...	...		
	y_{11n}	...	y_{1jn}	...	y_{1bn}		
...	...	...	...	...	...	...	...
i	y_{i11}	...	y_{ij1}	...	y_{ib1}	$\bar{y}_{i..}$	α_i
	...	...	...	...	...		
	y_{i1n}	...	y_{ijn}	...	y_{ibn}		
...	...	...	...	...	...	...	...
a	y_{a11}	...	y_{aj1}	...	y_{ab1}	$\bar{y}_{a..}$	α_a
	...	...	...	...	...		
	y_{a1n}	...	y_{ajn}	...	y_{abn}		
Mean	$\bar{y}_{.1.}$	...	$\bar{y}_{.j.}$	...	$\bar{y}_{.b.}$	$\bar{y}_{...}$	
Effect	β_1	...	β_j	...	β_b		

Effect of interaction of level i of factor A and level j of factor B is denoted by γ_{ij} (not shown in the table)

$$\gamma_{ij} = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}$$

where $\bar{y}_{ij.}$ denotes the mean value for level i of factor A and level j of factor B

Two-way ANOVA

- Each measurement y_{ijk} can be expressed as follows

$$y_{ijk} = \bar{y}_{\ldots} + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}$$

- α_i is the main effect of the *i*-th level of factor A
- β_j is the main effect of the *j*-th level of factor B
- γ_{ij} is the effect due to the interaction between the *i*-th level factor A and the *j*-th level of factor B

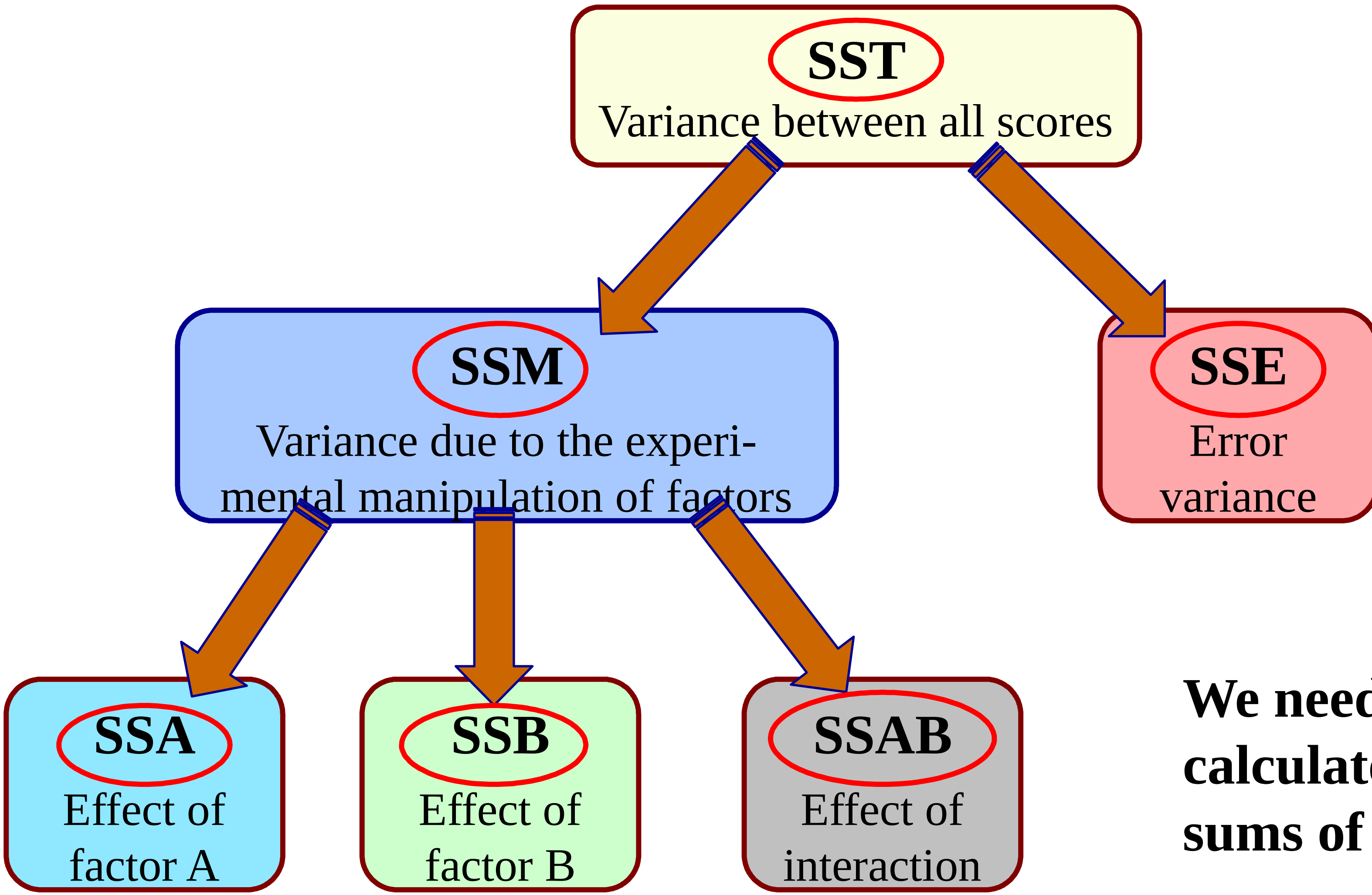
Two-way ANOVA

- Each measurement y_{ijk} can be expressed as follows

$$y_{ijk} = \bar{y}_{...} + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}$$

- In this model, we split the total variation in all measurements into two components:
  1. Variation due to the effects of the levels of each factor
  2. Variation due to the interaction of factors
  3. Variation due to the errors

Two-way ANOVA sums of squares (SS)



We need to calculate all these sums of squares

(adapted from Andy Field slides, SAGE)

Two-way ANOVA

- Each measurement y_{ijk} can be expressed as follows

$$y_{ijk} = \bar{y}_{...} + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}$$

- Variation due to the effects of A: SSA = $bn\sum_{i=1}^{a}(\bar{y}_{i..} - \bar{y}_{...})^2$
- Variation due to the effects of B: SSB = $an\sum_{j=1}^{b}(\bar{y}_{.j.} - \bar{y}_{...})^2$
- Variation due to the interactions: SSAB

$$= n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2$$

- Variation due to errors: SSE = $\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{ij.})^2$
- Total Variation: SST = SSA + SSB+SSAB+SSE

Two-way ANOVA

- Use the F statistic for each factor and interaction

$$F_a = \frac{SSA/(a-1)}{SSE/(ab(n-1))}$$

$$F_b = \frac{SSB/(b-1)}{SSE/(ab(n-1))}$$

$$F_{ab} = \frac{SSAB/((a-1)(b-1))}{SSE/(ab(n-1))}$$

Two-way ANOVA table

	SS	df	MS	F	P
Factor A	SSA	a-1	MSA = SSA/(a-1)	MSA/MSE	Tail area above F
Factor B	SSB	b-1	MSB = SSB/(b-1)	MSB/MSE	Tail area above F
Interaction	SSAB	(a-1)(b-1)	MSAB=SSAB/(a-1)(b-1))	MSAB/MSE	Tail area above F
Within	SSE	ab(n-1)	MSE = SSE/(ab(n-1))		
Total	SST	abn-1			

If the interaction is significant, then, it does not make sense to study the factors separately.

Example Two-way ANOVA table

	SS	df	MS	F	P (α=0.05)
Factor A	3.3714	3	MSA = 1.1238	460.2	
Factor B	0.5152	2	MSB = 0.2576	105.5	
Interaction	0.4317	6	MSAB=0.072	29.5	
Within	0.0293	12	MSE = 0.0024		
Total	4.3476	23			

Example Two-way ANOVA table

	SS	df	MS	F	P (α=0.05)
Factor A	3.3714	3	MSA = 1.1238	460.2	
Factor B	0.5152	2	MSB = 0.2576	105.5	
Interaction	0.4317	[illegible]	[illegible]	29.5	
Within	0.0293	12	MSE = 0.0024		
Total	4.3476	23			

This factor has a large impact

Example Two-way ANOVA table

	SS	df	MS	F	P (α=0.05)
Factor A	3.3714	3	MSA = 1.1238	460.2	
Factor B	0.5152	2	MSB = 0.2576	105.5	
Interaction	0.4317	6	MSAB=0.072	29.5	
Within	0.0293	12	MSE = 0.00		
Total	4.3476				

But this one also has some impact

Example Two-way ANOVA table

	SS	df	MS	F	P (α=0.05)
Factor A	3.3714	3	MSA = 1.1238	460.2	
Factor B	0.5152	2	MSB = 0.2576	105.5	
Interaction	0.4317	6	MSAB=0.072	29.5	
Within	0.0293	12	MSE = 0.0024		
Total	4.3476				

And the interaction between both might have some impact as well.

Example Two-way ANOVA table

	SS	df	MS	F	P (α=0.05)
Factor A	3.3714	3	MSA = 1.1238	460.2	~0
Factor B	0.5152	2	MSB = 0.2576	105.5	~0
Interaction	0.4317	6	MSAB=0.072	29.5	~0
Within	0.0293	12	MSE = 0.0024		
Total	4.3476	23			

The **H_0 for main effects and interaction are rejected!**
Therefore, start searching for differences in the interaction.

Example: Two-Way ANOVA in R

Measurements	System A		System B		System C	
	Prog A	Prog B	Prog A	Prog B	Prog A	Prog B
1	0.0952	0.1432	0.1382	0.1082	0.0966	0.1066
2	0.0871	0.1343	0.1332	0.1032	0.1200	0.1100
3	0.0969	0.1314	0.1482	0.1182	0.1152	0.1252
4	0.1054	0.1443	0.1430	0.1130	0.1375	0.1275
5	0.0812	0.1312	0.1483	0.1083	0.1298	0.1398

Measurements of the time (in microseconds) required to perform a subroutine on three different systems programmed by two different programmers.

Is there a difference between systems ?
Is there a difference between programmers?
Is there any interaction between systems and programmers?

Example: Two-Way ANOVA in R

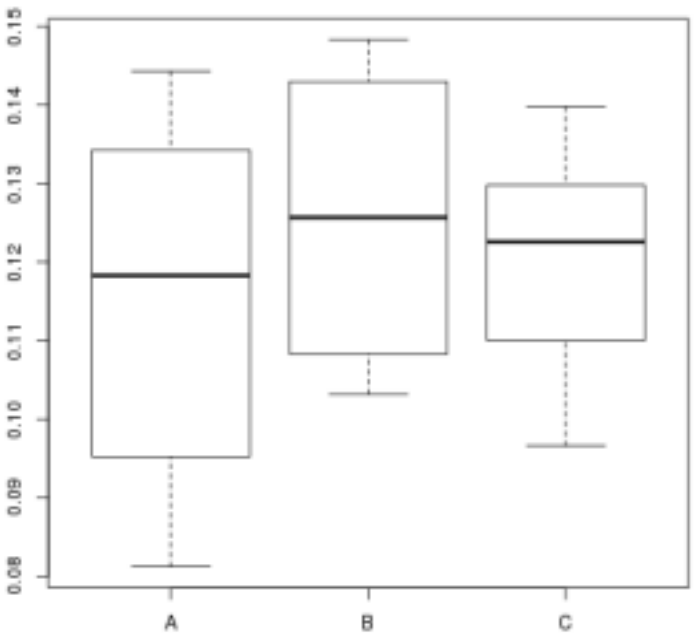
Step 1- State the hypotheses

Hypothesis

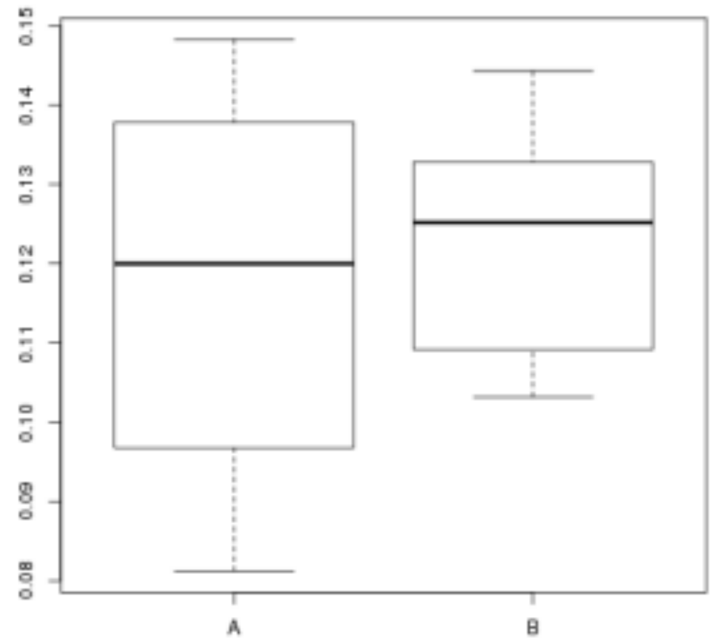
- $H_0^S : \mu_{A.} = \mu_{B.} = \mu_{C.}$ → All systems have equal means
- $H_0^P : \mu_{.A} = \mu_{.B}$ → All programmers have equal means
- $H_0^{SP} : \forall(i,j)\ \gamma_{ij} = 0$ → There is no interaction

Example: Two-Way ANOVA in R

Exploratory data analysis




```
boxplot( Time ~ System, data=D)
```

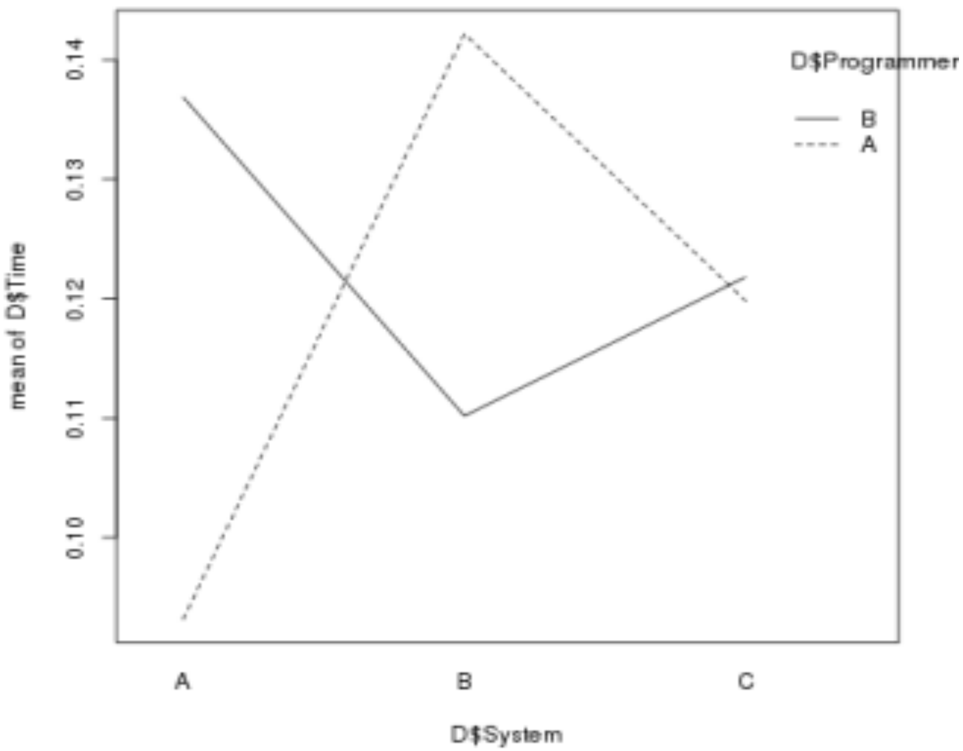



```
boxplot( Time ~ Programmer, data=D)
```

Almost no difference between systems and between programmers?

Example: Two-Way ANOVA in R

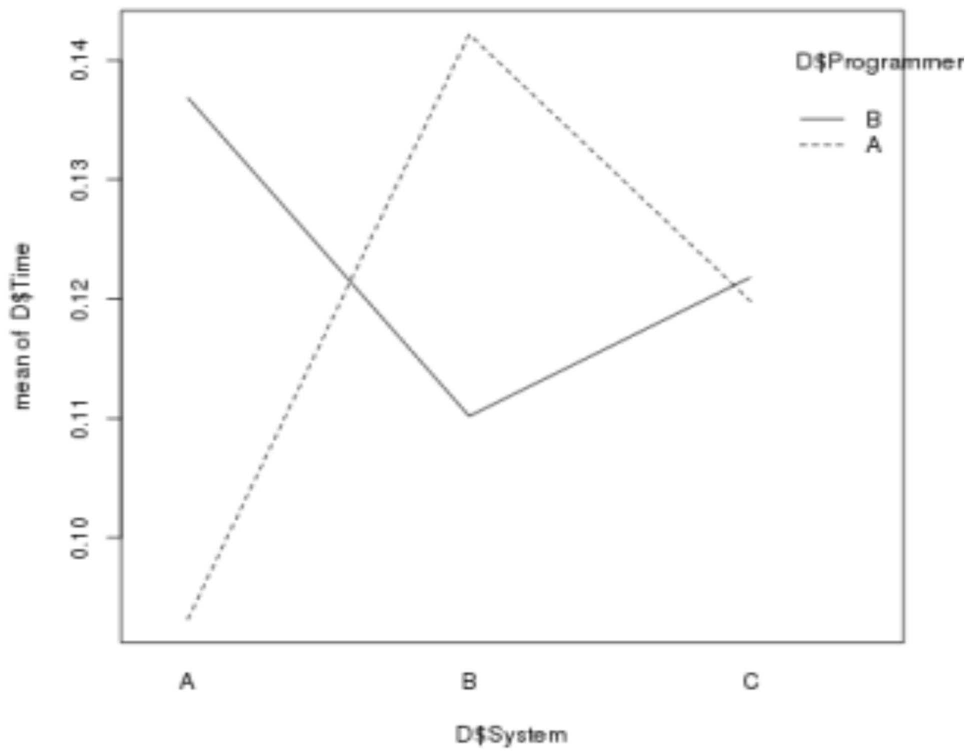
Exploratory data analysis




```
interaction.plot(D$System, D$Programmer,D$Time)
```

Example: Two-Way ANOVA in R

Exploratory data analysis



But there is a strong interaction between systems and programmers!

The systems A and B can have different performance depending of who is the programmer

```
interaction.plot(D$System, D$Programmer,D$Time)
```

Example: Two-Way ANOVA in R

Step 2 – Calculations

Test no interactions, only *main effects*

```
> aov.out = aov(Time~System+Programmer, data=D)
> summary(aov.out)

            Df Sum Sq    Mean Sq   F value Pr(>F)
System       2 0.000623 0.0003115   0.834  0.446
Programmer   1 0.000157 0.0001569   0.420  0.523
Residuals   26 0.009714 0.0003736
```

No rejection of H_0^S or H_0^P at significance level of 5%.
Systems take similar times.
The time is not affect by programmers
But we are not testing interactions..

Example: Two-Way ANOVA in R

Step 2 – Calculations

Test *interactions* and *main effects*

```
> aov.out = aov(Time~System*Programmer, data=D)
> summary(aov.out)

 Df    Sum Sq   Mean Sq F value    Pr(>F)
System              2 0.000623 0.000312   2.965   0.0707 .
Programmer          1 0.000157 0.000157   1.493   0.2337
System:Programmer   2 0.007192 0.003596  34.219 9.38e-08 ***
Residuals          24 0.002522 0.000105
```

Rejection of H_0^{SP} at significance level 5% which means that there is an interaction between systems and programmers.
We need to check assumptions.

Example: Two-Way ANOVA in R
Check assumptions

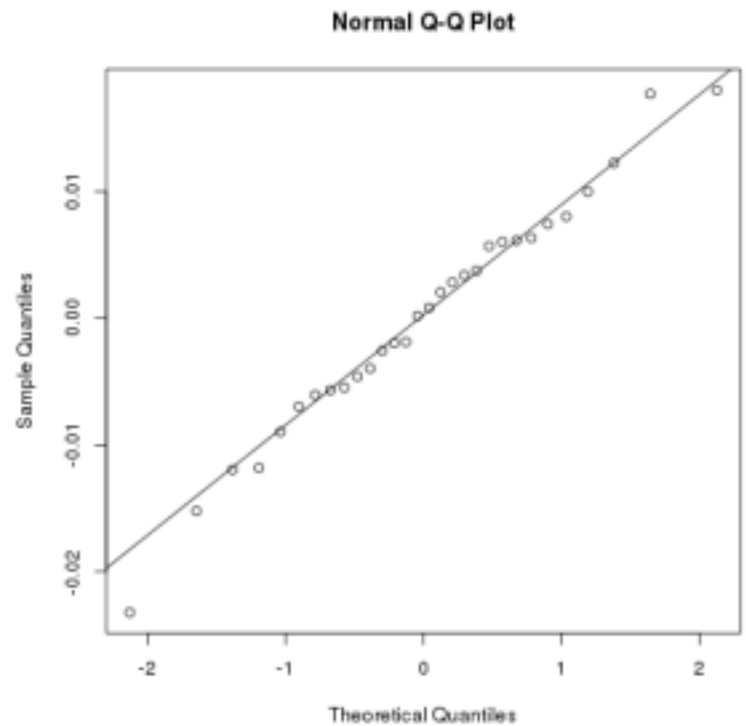
Test normality of the residuals

```
> qqnorm(aov.out$res)
> qqline(aov.out$res)

> shapiro.test(aov.out$res)

  Shapiro-Wilk normality test

W = 0.986, p-value = 0.9523
```



- We assume normality

Example: Two-Way ANOVA in R

Check assumptions

Test homogeneity of variances

```
> bartlett.test(Time~interaction(System,Programmer), data=D)

	Bartlett test of homogeneity of variances

data:  Time by System by Programmer
Bartlett's K-squared = 6.6152, df = 5, p-value = 0.2509
```

- We assume that variances are similar.

Example: Two-Way ANOVA in R

Post-hoc analysis

Tukey HSD

```
> t = TukeyHSD(aov.out,alternative="two.sided")
> print(t)
```

```
$`System:Programmer`
                diff          lwr           upr     p adj
B:A-A:A  0.04902  0.028973919  0.0690660806 0.0000012
C:A-A:A  0.02666  0.006613919  0.0467060806 0.0047273
A:B-A:A  0.04372  0.023673919  0.0637660806 0.0000077
B:B-A:A  0.01702 -0.003026081  0.0370660806 0.1295242
C:B-A:A  0.02866  0.008613919  0.0487060806 0.0022271
C:A-B:A -0.02236 -0.042406081 -0.0023139194 0.0226329
A:B-B:A -0.00530 -0.025346081  0.0147460806 0.9614023
B:B-B:A -0.03200 -0.052046081 -0.0119539194 0.0006248
C:B-B:A -0.02036 -0.040406081 -0.0003139194 0.0450286
A:B-C:A  0.01706 -0.002986081  0.0371060806 0.1280049
B:B-C:A -0.00964 -0.029686081  0.0104060806 0.6754123
C:B-C:A  0.00200 -0.018046081  0.0220460806 0.9995743
B:B-A:B -0.02670 -0.046746081 -0.0066539194 0.0046571
C:B-A:B -0.01506 -0.035106081  0.0049860806 0.2238190
C:B-B:B  0.01164 -0.008406081  0.0316860806 0.4870652
```