Probability Theory HW6

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1. *Page 72,1* Suppose f be integrable. Prove that,

$$\int_\Omega f d\mu = \lim_{n\to\infty} \sum_{i=-\infty}^{\infty} \frac{i}{2^n}\mu(\{\frac{i}{2^n} \le f < \frac{i+1}{2^n}\}).$$

SOLUTION: $\int_\Omega f d\mu = \mu(f^+) - \mu(f^-)$, for f^+ is a nonnegative measurable function, there is the sequence of simple functions

$$f_n^+ = \sum_{i=1}^{\infty} \frac{i}{2^n} 1_{\{\frac{i}{2^n} \le f^+ < \frac{i+1}{2^n}\}} + \infty 1_{\{f^+=\infty\}}.$$

Since $f_n^+ \uparrow f^+$,

$$\begin{aligned}
\mu(f^+) &= \lim_{n\to\infty} \mu(f_n^+) \\
&= \lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{i}{2^n}\mu(\{\frac{i}{2^n} \le f^+ < \frac{i+1}{2^n}\}) + \infty\mu(\{f^+ = \infty\}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{i}{2^n}\mu(\{\frac{i}{2^n} \le f < \frac{i+1}{2^n}\}) + \infty\mu(\{f = \infty\}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mu(f^-) &= \lim_{n\to\infty} \mu(f_n^-) \\
&= \lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{i-1}{2^n}\mu(\{\frac{i-1}{2^n} < f^- \le \frac{i}{2^n}\}) + \infty\mu(\{f^- = \infty\}) \\
&= \lim_{n\to\infty} \sum_{i=-\infty}^{0} \frac{i}{2^n}\mu(\{\frac{i}{2^n} \le f < \frac{i+1}{2^n}\}) + \infty\mu(\{f = \infty\}).
\end{aligned}$$

2. *Page 72,2* Suppose f is a nonnegative measurable function. Let

$$\bar{\int}_{\Omega} f \mathrm{d}\mu = \inf\{\mu(g) \mid g \geq f, g \text{ is a simple function}\}.$$

Show that $\bar{\int}_{\Omega} f \mathrm{d}\mu$ may not equal to $\int_{\Omega} f \mathrm{d}\mu$ by an example, and explain why not define the integral by $\bar{\int}_{\Omega} f \mathrm{d}\mu$.

SOLUTION:

3. *Page 72,5* If f is a measurable complex function, then

$$|\int_{\Omega} f \mathrm{d}\mu| \leq \int_{\Omega} |f| \mathrm{d}\mu.$$

SOLUTION:

4. SOLUTION: