Probability Theory HW3

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October 16, 2018

1. Construct a counter example to show that if μ is not σ finite, its extension from a semialgebra to the minimal σ-algebra may not be unique.

   Solution: The semialgebra $\mathscr{S} = \{(a, b] : a \le b\}$, $\mu((a, b]) = 0$ if $(a, b] \cap (0, 1] = \varnothing$ and $\mu((a, b]) = \infty$ if $(a, b] \cap (0, 1] \neq \varnothing$. Then the extension $\mu(\{1\})$ can be 0 or ∞.

2. $\mathscr{S}$ is a semialgebra, μ is a finite measure on $\mathscr{S}$, the triple $(\Omega, \mathscr{A}^*, \mu^*)$ is the completion of extension of μ on $\sigma(\mathscr{S})$. Let

$$\begin{aligned}\mu_*(A) &= \sup\{\textstyle\sum_n \mu(A_n) : A_n \in \mathscr{S} \text{ pairwise disjoint}, \ \sum_n A_n \subset A\},\\ \mathscr{A}_* &= \{A \subset \Omega : \mu^*(A) = \mu_*(A)\}.\end{aligned}$$

   Prove that $\mathscr{A}^* \supset \mathscr{A}_*$.

   Solution: $\forall A \in \mathscr{A}_*, D \subset \Omega$,

$$\begin{aligned}\mu^*(A \cap D) &= \inf\{\textstyle\sum_{n=1}^\infty \mu(B_n) : B_n \in \mathscr{S}, A \cap D \subset \bigcup_{n=1}^\infty B_n\}\\ \mu^*(A^c \cap D) &= \inf\{\textstyle\sum_{n=1}^\infty \mu(C_n) : C_n \in \mathscr{S}, A^c \cap D \subset \bigcup_{n=1}^\infty C_n\}\\ &= \inf\{\textstyle\sum_{n=1}^\infty \mu(C_n) : C_n \in \mathscr{S}, \bigcap_{n=1}^\infty C_n^c \subset A \cup D^c\}\\ &=\end{aligned}$$

3. Let $\mathscr{C} = \{C_{a,b} = [-b, a) \cup (a, b] : 0 < a < b\}$, and define the measure $\mu(C_{a,b}) = b - a$. Prove that μ can extend to a measure on $\sigma(\mathscr{C})$. Is $[1, 2]$ a μ^*-measurable set?

   Solution: Let $\mathscr{S} = \mathscr{C} \cup \{\mathbb{R}, \varnothing\}$,

   (a) $\{\mathbb{R}, \varnothing\} \subset \mathscr{S}$

   (b) $\forall A, B \in \mathscr{S}$, if $A = C_{a,b}, B = C_{c,d} \in \mathscr{C}$, $C_{a,b} \cap C_{c,d} = C_{\max a,c,\min b,d} \in \mathscr{C} \subset \mathscr{S}$. In the case either of $A, B \in \{\mathbb{R}, \varnothing\}$, $A \cap B \in \mathscr{S}$ is trivial.

   (c) $\forall A, A_1 \in \mathscr{S}, A_1 \subset A$, the case $A, A_1 \in \{\mathbb{R}, \varnothing\}$ is trivial. If $A = C_{a,b}, A_1 = C_{a_1,b_1}$, then $A \setminus A_1 = C_{a,a_1} + C_{b_1,b}$.

   so $\mathscr{S}$ is a semialgebra. Assign μ on $\mathbb{R}, \varnothing$ to $\infty, 0$ respectively, and then μ is a measure on $\mathscr{S}$ analogue of the semialgebra on $\mathbb{R}$. By **Theorem 1.42** μ has a unique extension.

   For $C_{1,2} \subset \mathbb{R}$, $\mu^*(C_{1,2}) = 1$ and $\mu^*([1, 2]) = \mu^*([-2, -1]) = 1$, i.e.$\mu^*(C_{1,2}) \neq \mu^*([1, 2] \cap C_{1,2}) + \mu^*([-2, -1]^c \cap C_{1,2})$. $[1, 2]$ is not a μ^*-measurable set.

4. Let $f : \mathbb{R} \ni x \mapsto \frac{x}{3} \in \mathbb{R}, A_0 = [0, 1]$. Then the sequence of $A_{n+1} = f(A_n) \cup (\frac{2}{3} + f(A_n))(n \geq 0)$ is monotone decreasing. The limitation of A_n is called Cantor set, $C = \bigcap_n A_n$. Prove the Lebesgue measure of Cantor set is 0.

   Solution: It suffices to prove the outer measure of C is 0. $\forall n \geq 0$, $f(A_n) \cap (\frac{2}{3} + f(A_n)) = \varnothing$ and A_n is sum of some intervals since $f(A_n) \subset f(A_0) = [0, \frac{1}{3}]$. Then

$$\begin{aligned}
\mu(A_{n+1}) &= \mu(f(A_n)) + \mu((\frac{2}{3} + f(A_n))) && \text{finite additivity} \\
&= \frac{2}{3}\mu(A_n) && \text{length of interval} \\
&= (\frac{2}{3})^{n+1} && \text{inductively.}
\end{aligned}$$

   This follows

$$\mu(C) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} (\frac{2}{3})^n = 0,$$

   by the continuity of measure.