Algebra HW4

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1. *Page 106,3* If H and K are subgroups of a group G, let (H,K) be the subgroup of G generated by the elements $\{hkh^{-1}k^{-1} \mid h \in H, k \in K\}$. Show that
   (a) (H,K) is normal in $H \vee K$.
   (b) If $(H,G') = \langle e \rangle$, then $(H',G) = \langle e \rangle$.
   (c) $H \lhd G$ if and only if $(H,G) < H$.
   (d) Let $K \lhd G$ and $K \lhd H$; then $H/K < C(G/K)$ if and only if $(H,G) < K$.

   SOLUTION:

   (a) It suffices to prove $\forall a \in H, b \in K, c \in (H,K)$, $aca^{-1} \in (H,K)$ and $bcb^{-1} \in (H,K)$. We only need to prove the condition c in (H,K)'s genetor, i.e. there are $h \in H, k \in K$ such that $c = hkh^{-1}k^{-1}$. Then

$$\begin{aligned} aca^{-1} &= ahkh^{-1}k^{-1}a^{-1} \\ &= ahkh^{-1}a^{-1}k^{-1}kak^{-1}a^{-1} \\ &= (ahkh^{-1}a^{-1}k^{-1})(aka^{-1}k^{-1})^{-1} \in (H,K). \end{aligned}$$

   Similarly, $bcb^{-1} \in (H,K)$. We get $(H,K) \lhd H \vee K$.

   (b)

   (c) For all $h \in H, g \in G$,

$$\begin{aligned} H \lhd G &\Leftrightarrow ghg^{-1} \in H \\ &\Leftrightarrow gh^{-1}g^{-1} \in H \\ &\Leftrightarrow hgh^{-1}g^{-1} \in H \\ &\Leftrightarrow (H,G) < H. \end{aligned}$$

   (d)

2. *Page 106,8* If D_n is the dihedral group with generators a of order n and b of order 2, then

   (a) $a^2 \in D'_n$.

   (b) If n is odd, $D'_n \cong Z_n$.

   (c) If n is even, $D'_n \cong Z_m$, where $2m = n$.

   (d) D_n is nilpotent if and only if n is a power of 2.

   Solution:

3. *Page 106,10* S_n is solvable for $n \leq 4$, but S_3 and S_4 are not nilpotent.
   SOLUTION:

4. *Page 106,14* If $N \lhd G$ and $N \cap G' = \langle e \rangle$, then $N < C(G)$.
   SOLUTION: