Algebra HW3

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1. *Page 111,1*
   (a) Find a normal series of D_4 consisting of 4 subgroups.
   (b) Find all composition series of the group D_4.
   (c) Do part (**??**) for the group A_4.
   (d) Do part (**??**) for the group $S_3 \times Z_2$.
   (e) Find all composition factors of S_4 and D_6.

   SOLUTION:

   (a) $D_4 > Z_4 > Z_2 > \{e\}$, it is also a composition series.
   (b) The only composition factor of D_4 is Z_2 according to the last series. By applying the Jordan-Hölder Theorem, the composition series are $D_4 > Z_4 > Z_2 > \{e\}$, $D_4 > K_4 > Z_2 > \{e\}$.
   (c) A_4
   (d) $S_3 \times Z_2 = D_6$ has Z_2, Z_3 as its simple normal subgroups.
   $S_3 \times Z_2 > S_3 > Z_3 > \{e\}$,
   $S_3 \times Z_2 > Z_6 > Z_3 > \{e\}$,
   $S_3 \times Z_2 > S_3 > Z_2 > \{e\}$,
   $S_3 \times Z_2 > Z_6 > Z_2 > \{e\}$.
   (e) The composition factors of D_6, according to the last composition series of $S_3 \times Z_2 = D_6$, are Z_2, Z_3, Z_2.
   $S_4 > A_4 >$

2. *Page 111,2* If $G = G_0 > G_1 > \cdots > G_n$ is a subnormal series of a finite group G, then $|G| = \left(\prod_{i=0}^{n-1} |G_i/G_{i+1}|\right) |G_n|$.

   SOLUTION: The conclusion is proved by induction and Lagrange Theorem.

   - If $n = 1$, $|G| = |G_0/G_1||G_1|$ follows from Lagrange Theorem.
   - It suffices to prove $|G| = \left(\prod_{t=0}^{n} |G_i/G_{i+1}|\right) |G_{n+1}|$ if $|G| = \left(\prod_{t=0}^{n-1} |G_i/G_{i+1}|\right) |G_n|$. Since $|G_n| = |G_n/G_{n+1}||G_{n+1}|$, by induction we get the conclusion.

3. *Page 111,3* If N is a simple normal subgroup of a group G and G/N has a composition series, then G has a composition series.

   <u>Solution</u>: Assuming $G/N = H_0 > H_1 > \cdots > H_n$ is the composition series, and then $G = H_0N > H_1N > \cdots > H_nN > N$ is the composition series of G.

   For any $g \in G$, there exist a unique $h_0 \in H_0$ and a unique $n \in N$ such that $g \in h_0N$ and $g = h_0n$; then we get a homorphism between G and H_0N, $G = H_0N$.

   Let $H_{n+1} = \{e\}$, then $H_{n+1} \lhd H_n$ trivially. For $i = 0, 1, \cdots, n$, $\forall h_0, h_1 \in H_{i+1}, h \in H_i, n_0, n \in N$,

$$\begin{aligned}(hN)^{-1}h_0NhN &= Nh^{-1}h_0NNh\\ &= h^{-1}h_0hN,\\ (h_1N)^{-1}h_0N &= Nh_1^{-1}h_0N\\ &= h_1^{-1}h_0hN.\end{aligned}$$

   Therefore $H_{i+1}N \lhd H_iN$ since $H_{i+1} \lhd H_i$. Because N is a simple group (with no proper normal subgroups) and $H_iN/H_{i+1}N = H_i/H_{i+1}$ is simple $G = H_0N > H_1N > \cdots > H_nN > N$ is the composition series of G.

4. *Page 112,8* If H and K are solvable subgroups of G with $H \lhd G$, then HK is a solvable subgroup of G.

   <u>Solution</u>: A group is solvable iff it has a subnoral series with every factors abelian. Assuming the series of H, K are $H = H_0 > H_1 > \cdots > H_n$, $K = K_0 > K_1 > \cdots > K_m$ respectively.

   $HK/H \cong K/(H \cap K)$ by second isomorphism Theorem.

5. *Page 112,12* Prove the Fundamental Theorem of Arithmetic by applying the Jordan-Hölder Theorem to the group Z_n.

SOLUTION: For any positive integer (except the number 1) $n \in \mathbb{Z} \setminus \{1\}$, the group Z_n is a finite group. Then Z_n must has a composition series.

Because every subgroup of Z_n is still a cyclic group, there exists a sequence $\{a_i\}$ such that $Z_n = Z_{a_0} > Z_{a_1} > \cdots > Z_{a_m}$ is a composition series. According to Problem **??**,

$$n = \left(\prod_{i=0}^{m-1} a_i/a_{i+1} \right) a_m.$$

This representation of n consist only primes since the series is composition and $Z_{a_i}/Z_{a_{i+1}} = Z_{a_i/a_{i+1}}$ is simple iff a_i/a_{i+1} is prime. By applying the Jordan-Hölder Theorem, the representation is unique.