Algebra HW5

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1. *Page 120,3* A ring R such that $a^2 = a$ for all $a \in R$ is called a **Boolean ring**. Prove that every Boolean ring R is commutative and $a + a = 0$ for all $a \in R$.

   SOLUTION: For all $a \in R$, $a + a = a + a^2 = a + (-a)^2 = a + (-a) = 0$. For any $a, b \in R$,

$$\begin{aligned} ab - ba &= ab + ba \\ &= (a+b)^2 - (a^2 + b^2) \\ &= (a+b) - (a+b) = 0. \end{aligned}$$

   The Boolean ring R is commutative. □

2. *Page 120,8* Let R be the set of all 2×2 matrices over complex field $\mathbb{C}$ of the form

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}.$$

   Then R is a division ring that is isomorphic to the division ring K of real quaternions. *Hint:* The fundamental quaternion units $1, i, j, k$ of K map to the matrices respectively,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

   SOLUTION: For any two elements of R,

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} - \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} = \begin{pmatrix} z - x & w - y \\ -\bar{w} + \bar{y} & \bar{z} - \bar{x} \end{pmatrix} \in R,$$

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} = \begin{pmatrix} xz - w\bar{y} & yz + w\bar{x} \\ -\bar{y}\bar{z} - \bar{w}x & \bar{x}\bar{z} - \bar{w}y \end{pmatrix} \in R.$$

   Therefore R is a ring under addition and multiplication of 2×2 matrices. Because

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \frac{1}{z\bar{z} + w\bar{w}} \begin{pmatrix} \bar{z} & -w \\ \bar{w} & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

   R is a division ring.

   Construct ϕ mapping the fundamental quaternion units $1, i, j, k$ of K to the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

respectively. It suffices to verify the multiplication between the matrices is the same as the real quaternions'. $\phi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity of multiplication.

$$\phi(i)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \phi(-1), \phi(j)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \phi(-1), \phi(k)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \phi(-1)$$

$$\begin{aligned}
\phi(i)\phi(j) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \phi(k), \quad \phi(j)\phi(i) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -\phi(k) = \phi(-k), \\
\phi(j)\phi(k) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \phi(i), \quad \phi(k)\phi(j) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -\phi(i) = \phi(-i), \\
\phi(k)\phi(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \phi(j), \quad \phi(i)\phi(k) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\phi(j) = \phi(-j).
\end{aligned}$$

Thus ϕ is a homeomorphism. And for any element of R,

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} = a\phi(1) + b\phi(i) + c\phi(j) + d\phi(k),$$

it is generated by the four matrices. Thus ϕ is an isomorphism from R to the division ring K. $\square$

3. *Page 120,11(The Freshman's Dream).* Let R be a commutative ring with identity of prime characteristic p. If $a, b \in R$, then $(a \pm b)^{p^n} = a^{p^n} \pm b^{p^n}$ for all integers $n \geq 0$. [Note that $b = -b$ if $p = 2$.]

   <u>Solution</u>: Use the *Binomial Theorem* and $\binom{p^n}{k}$ divisible by p for $1 \leq k \leq p^n - 1$,

$$\begin{aligned}
(a+b)^{p^n} &= \sum_{k=0}^{p^n} \binom{p^n}{k} a^k b^{p^n-k} \\
&= \sum_{k=1}^{p^n-1} \binom{p^n}{k} a^k b^{p^n-k} + a^0 b^{p^n} + a^{p^n} b^0 \\
&= a^{p^n} + b^{p^n}.
\end{aligned}$$

   Substitude b by $-b$, we get $(a-b)^{p^n} = a^{p^n} + (-b)^{p^n} = a^{p^n} + (-1)^{p^n} b^{p^n}$. If $p \neq 2$, $(-1)^{p^n} = -1$; if $p = 2$, $(-b)^{p^n} = b^{p^n-1}(-b) = -b^{p^n}$. In both cases, $(a-b)^{p^n} = a^{p^n} - b^{p^n}$. $\square$

4. *Page 120,13* In a ring R the following conditions are equivalent.

   (a) R has no nonzero nilpotent elements.

   (b) If $a \in R$ and $a^2 = 0$, then $a = 0$.

   <u>Solution</u>:

   (a) If $a \in R$ and $a^2 = 0$, a is a nilpotent element. Since R has no nonzero nilpotent elements, $a = 0$.

   (b) If $a \in R$ is a nilpotent element and $a^n = 0$; If $n = 1, 2, a = 0$. If $n > 2$, $a^{2(n-1)} = a^{n-2}a^n = 0$, then $a^{n-1} = 0$. Recursively, we get $a^2 = 0$ and then $a = 0$. So any nilpotent element of R is zero.

   $\square$