Differential Geometry HW4

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October 28, 2018

1. (M, g) is a Rienmannian manifold. For $\theta \in T^*M$, the vector $\theta^\# \in TM$ is called the dual of θ if

$$g(\theta^\#, X) = \theta(X), \forall X \in TM.$$

Define the dual metric g^*

$$\begin{aligned} g^* : \quad & \otimes^2 T^*M \to \mathbb{R} \\ & (\theta, \sigma) \mapsto g(\theta^\#, \sigma^\#). \end{aligned}$$

If $g = g_{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j$ in a local coordinate, calculate the coordinate components of g^*.

<u>Solution</u>: For coordinate dual vector $\mathrm{d}x_i$, assume $\mathrm{d}x_i^\# = A_{ik}\frac{\partial}{\partial x_k}$ where the index k appeared twice denote summation for k according to Einstein summation convention. For any coordinate vector $\frac{\partial}{\partial x_j}$,

$$\begin{aligned} g(\mathrm{d}x_i^\#, \frac{\partial}{\partial x_j}) &= g(A_{ik}\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j}) \\ &= A_{ik} g(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j}) \\ &= A_{ik} g_{kj}; \\ g(\mathrm{d}x_i^\#, \frac{\partial}{\partial x_j}) &= \mathrm{d}x_i(\frac{\partial}{\partial x_j}) \\ &= \delta_{ij}. \end{aligned}$$

Thus $A_{ik} = g^{ik}$. Because

$$\begin{aligned} g^*(\mathrm{d}x_i, \mathrm{d}x_j) &= g(\mathrm{d}x_i^\#, \mathrm{d}x_j^\#) \\ &= g(A_{ik}\frac{\partial}{\partial x_k}, A_{jl}\frac{\partial}{\partial x_l}) \\ &= A_{ik}A_{jl} g(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}) \\ &= g^{ik} g^{jl} g_{kl} \\ &= g^{ij}, \end{aligned}$$

we get $g^* = g^{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}$. □

2. The map

$$\begin{aligned} \phi : \quad & S^2 \to \mathbb{R}^4 \\ & (x, y, z) \mapsto (\frac{x^2 - y^2}{2}, xy, yz, zx) \end{aligned}$$

is an immersion. ϕ induce a Rienmannian metric from the standard metric on $\mathbb{R}^4$,

$$\begin{aligned}\phi^* \mathrm{d}s^2 &= \mathrm{d}\phi_1^2 + \mathrm{d}\phi_2^2 + \mathrm{d}\phi_3^2 + \mathrm{d}\phi_4^2 \\ &= (x\,\mathrm{d}x - y\,\mathrm{d}y)^2 + (y\,\mathrm{d}x + x\,\mathrm{d}y)^2 + (z\,\mathrm{d}y + y\,\mathrm{d}z)^2 + (x\,\mathrm{d}z + z\,\mathrm{d}x)^2 \\ &= \mathrm{d}x^2 + \mathrm{d}y^2 + (x^2 + y^2 - 2z^2)\,\mathrm{d}z^2.\end{aligned}$$

Calculate the Gauss curvature of $\phi^* \mathrm{d}s^2$.

SOLUTION: Use $x^2 + y^2 + z^2 = 1$,

$$\begin{aligned}\phi^* \mathrm{d}s^2 &= (\,\mathrm{d}\sqrt{1-(y^2+z^2)})^2 + \mathrm{d}y^2 + (1 - z^2 - 2z^2)dz^2 \\ &= (\frac{-y}{x}\,\mathrm{d}y + \frac{-z}{x}\,\mathrm{d}z)^2 + \mathrm{d}y^2 + (1-3z^2)dz^2 \\ &= \frac{y^2+x^2}{x^2}\,\mathrm{d}y^2 + \frac{yz}{x^2}\,\mathrm{d}y\,\mathrm{d}z + \frac{yz}{x^2}\,\mathrm{d}z\,\mathrm{d}y + \frac{x^2+z^2-3x^2z^2}{x^2}\,\mathrm{d}z^2.\end{aligned}$$

Caution $x \neq 0$. The dual metric $g^* = (\phi^* g)^{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}$,

$$(\phi^* g) = \frac{1}{3z^2(x^2+y^2)-1} \begin{pmatrix} 3z^2x^2 + y^2 - 1 & yz \\ yz & z^2 - 1 \end{pmatrix}.$$

Then $\theta_1 = \mathrm{d}z, \theta_2 = (1-z^2)\,\mathrm{d}y + yz\,\mathrm{d}z$ are two orthogonal dual vector on S^2 and

$$\begin{aligned}\mathrm{d}\theta_1 &= 0, \\ \mathrm{d}\theta_2 &= -2z\,\mathrm{d}z\,\mathrm{d}y + z\,\mathrm{d}y\,\mathrm{d}z \\ &= 3z\,\mathrm{d}y\,\mathrm{d}z \\ &= -\frac{3z}{1-z^2}\theta_1 \wedge \theta_2.\end{aligned}$$

Solve the structure equation

$$\begin{cases} 0 = \omega_{12} \wedge \theta_2, \\ -\dfrac{3z}{1-z^2}\theta_1 \wedge \theta_2 = -\omega_{12} \wedge \theta_1. \end{cases}$$

We get $\omega_{12} = -\frac{3z}{1-z^2}\theta_2$, and

$$\begin{aligned}\mathrm{d}\omega_{12} &= -\frac{3(1-z^2) - 3z(-2z)}{(1-z^2)^2}\,\mathrm{d}z \wedge \theta_2 \\ &= -\frac{3+3z^2}{(1-z^2)^2}\,\mathrm{d}z \wedge \theta_2 \\ &= -\frac{3+3z^2}{(1-z^2)^2}\theta_1 \wedge \theta_2.\end{aligned}$$

The Gauss curvature of $(S^2, \phi^* g)$ is $K = \frac{3+3z^2}{(1-z^2)^2}$. □

Use the **stereographic projection** $X(u,v) = \frac{1}{1+u^2+v^2}(2u, 2v, u^2+v^2-1)$ with $z \neq 1$,

$$\begin{aligned}
\phi^* \mathrm{d}s^2 =& \mathrm{d}x^2 + \mathrm{d}y^2 + (x^2+y^2-2z^2)\,\mathrm{d}z^2 \\
&= (\mathrm{d}\frac{2u}{1+u^2+v^2})^2 + (\mathrm{d}\frac{2v}{1+u^2+v^2})^2 + (1-3(\frac{u^2+v^2-1}{1+u^2+v^2})^2)(\mathrm{d}\frac{u^2+v^2-1}{1+u^2+v^2})^2 \\
&= (\frac{2}{1+u^2+v^2}\,\mathrm{d}u + \frac{-2u}{(1+u^2+v^2)^2}(2u\,\mathrm{d}u + 2v\,\mathrm{d}v))^2 \\
&+ (\frac{2}{1+u^2+v^2}\,\mathrm{d}v + \frac{-2v}{(1+u^2+v^2)^2}(2u\,\mathrm{d}u + 2v\,\mathrm{d}v))^2 \\
&+ (1-3(\frac{u^2+v^2-1}{1+u^2+v^2})^2)(\frac{2u\,\mathrm{d}u+2v\,\mathrm{d}v}{1+u^2+v^2} + \frac{u^2+v^2-1}{(1+u^2+v^2)^2}(2u\,\mathrm{d}u + 2v\,\mathrm{d}v))^2 \\
&= (\frac{2(1-u^2+v^2)}{(1+u^2+v^2)^2}\,\mathrm{d}u + \frac{-4uv}{(1+u^2+v^2)^2}\,\mathrm{d}v)^2 + (\frac{2(1+u^2-v^2)}{(1+u^2+v^2)^2}\,\mathrm{d}v + \frac{-4uv}{(1+u^2+v^2)^2}\,\mathrm{d}u)^2 \\
&+ (1-3z^2)(\frac{4u(u^2+v^2)}{(1+u^2+v^2)^2}\,\mathrm{d}u + \frac{4v(u^2+v^2)}{(1+u^2+v^2)^2}\,\mathrm{d}v)^2 \\
&= ((\frac{-2z}{1+u^2+v^2} + y^2)\,\mathrm{d}u - xy\,\mathrm{d}v)^2 + ((\frac{-2z}{1+u^2+v^2} + x^2)\,\mathrm{d}v - xy\,\mathrm{d}u)^2 \\
&+ (1-3z^2)(2x(z+\frac{1}{1+u^2+v^2})\,\mathrm{d}u + 2y(z+\frac{1}{1+u^2+v^2})\,\mathrm{d}v)^2 \\
&= ((z+z^2+y^2)\,\mathrm{d}u - xy\,\mathrm{d}v)^2 + ((z+z^2+x^2)\,\mathrm{d}v - xy\,\mathrm{d}u)^2 \\
&+ (1-3z^2)(x(z+1)\,\mathrm{d}u + y(z+1)\,\mathrm{d}v)^2 \\
&= ((z+z^2+y^2)\,\mathrm{d}u - xy\,\mathrm{d}v)^2 + ((z+z^2+x^2)\,\mathrm{d}v - xy\,\mathrm{d}u)^2 \\
&+ (1-3z^2)(x(z+1)\,\mathrm{d}u + y(z+1)\,\mathrm{d}v)^2 \\
&= (1+z-3x^2z^2)(1+z)\,\mathrm{d}u^2 - 6xyz^2(1+z)^2\,\mathrm{d}u\,\mathrm{d}v + (1+z-3y^2z^2)(1+z)\,\mathrm{d}v^2 \\
&= \begin{pmatrix} (1+z-3x^2z^2)(1+z) & -3xyz^2(1+z)^2 \\ -3xyz^2(1+z)^2 & (1+z-3y^2z^2)(1+z) \end{pmatrix} .\times \begin{pmatrix} \mathrm{d}u^2 & \mathrm{d}u\,\mathrm{d}v \\ \mathrm{d}v\,\mathrm{d}u & \mathrm{d}v^2 \end{pmatrix}.
\end{aligned}$$

Use the other stereographic projection $Y(u,v) = \frac{1}{1+u^2+v^2}(2u, 2v, 1-u^2-v^2)$ where $(u,v) = \frac{(u,v)}{u^2+v^2}$ with $z \neq -1$, we get

$$\begin{aligned}
\phi^* \mathrm{d}s^2 &= (1+z-3x^2z^2)(1+z)(\mathrm{d}\frac{u}{u^2+v^2})^2 \\
&- 6xyz^2(1+z)^2\,\mathrm{d}\frac{u}{u^2+v^2}\,\mathrm{d}\frac{v}{u^2+v^2} + (1+z-3y^2z^2)(1+z)(\mathrm{d}\frac{v}{u^2+v^2})^2 \\
&= (1-z-3x^2z^2)(1-z)\,\mathrm{d}u^2 - 6xyz^2(1-z)^2\,\mathrm{d}u\,\mathrm{d}v + (1-z-3y^2z^2)(1-z)\,\mathrm{d}v^2 \\
&= \begin{pmatrix} (1-z-3x^2z^2)(1-z) & -3xyz^2(1-z)^2 \\ -3xyz^2(1-z)^2 & (1-z-3y^2z^2)(1-z) \end{pmatrix} .\times \begin{pmatrix} \mathrm{d}u^2 & \mathrm{d}u\,\mathrm{d}v \\ \mathrm{d}v\,\mathrm{d}u & \mathrm{d}v^2 \end{pmatrix}.
\end{aligned}$$

3. Suppose $\{v_1, \cdots, v_n\}, \{w_1, \cdots, w_n\}$ are two bases of an n-dimensional vector space V, and

$$(v_1, \cdots, v_n) = (w_1, \cdots, w_n)J.$$

Prove that

$$v_1 \wedge v_2 \wedge \cdots \wedge v_n = \det J w_1 \wedge w_2 \wedge \cdots \wedge w_n.$$

<u>Solution</u>: Use the Einstein summation convention since the summations are from 1 to n, then $v_i = w_k J_{ki}$ for $1 \leq i \leq n$.

$$\begin{aligned} v_1 \wedge v_2 \wedge \cdots \wedge v_n &= w_{k_1} J_{k_1 1} \wedge w_{k_2} J_{k_2 2} \wedge \cdots \wedge w_{k_n} J_{k_n n} \\ &= J_{k_1 1} J_{k_2 2} \cdots J_{k_n n} w_{k_1} \wedge w_{k_2} \wedge \cdots \wedge w_{k_n} \\ &= \sum_{\sigma \in S_n} J_{\sigma(1)1} J_{\sigma(2)2} \cdots J_{\sigma(n)n} w_{\sigma(1)} \wedge w_{\sigma(2)} \wedge \cdots \wedge w_{\sigma(n)} \\ &= \sum_{\sigma \in S_n} \delta(\sigma) J_{\sigma(1)1} J_{\sigma(2)2} \cdots J_{\sigma(n)n} w_1 \wedge w_2 \wedge \cdots \wedge w_n \\ &= \det J w_1 \wedge w_2 \wedge \cdots \wedge w_n. \end{aligned}$$

$\square$