Differential Geometry HW3

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1. A tensor is **decomposable** if it is the tensor product of several vectors. Is every tensor decomposable? Prove or give a counter example.

SOLUTION: No. Assume that V^* is a 2-dimensional dual vector space over $\mathbb{R}$ with a basis $\{e_1, e_2\}$. For a tensor $v \otimes w \in V^* \otimes V^*$, there exist $x \in \mathbb{R}^4$ such that

$$v \otimes w = (e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2) \cdot x.$$

If $v \otimes w$ is decomposable, $\exists a_1 e_1 + a_2 e_2, b_1 e_1 + b_2 e_2 \in V^*$ such that

$$\begin{aligned} v \otimes w &= (a_1 e_1 + a_2 e_2) \otimes (b_1 e_1 + b_2 e_2) \\ &= a_1 b_1 e_1 \otimes e_1 + a_1 b_2 e_1 \otimes e_2 + a_2 b_1 e_2 \otimes e_1 + a_2 b_2 e_2 \otimes e_2 \\ &= (e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2) \cdot (a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)^T. \end{aligned}$$

Then $x = (a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)^T$. Every tensor is decomposable means every $x \in \mathbb{R}^4$ has the later form; this is not true. For example, $x = (0, 1, 1, 1)$, then $a_1 b_1 = 0$ infers $a_1 = 0$ or $b_1 = 0$; hence $a_1 b_2 = 0$ or $a_2 b_1 = 0$, but $x_2 = x_3 = 1 \neq 0$. Contradiction! □

2. Calculate the induce metric of an ellipsoid surface,

$$S_2^n = \{x \in \mathbb{R}^{n+1} \mid 4x_0^2 + \sum_{i=1}^n x_i^2 = 1\} \hookrightarrow \mathbb{R}^{n+1}.$$

SOLUTION: Denote $N = (1/2, 0, \cdots, 0), S = (-1/2, 0, \cdots, 0) \in S_2^n$, $V_1 = S_2^n \setminus \{N\}, V_2 = S_2^n \setminus \{S\}$. $\{V_1, V_2\}$ is an open cover of S_2^n. Define the stereographic projection $X : \mathbb{R}^n \to V_1, Y : \mathbb{R}^n \to V_2$,

$$\begin{aligned} X(\vec{x}) &= \frac{1}{1+\vec{x}^2}(\frac{\vec{x}^2 - 1}{2}, 2\vec{x}), \\ Y(\vec{x}) &= \frac{1}{1+\vec{y}^2}(\frac{1-\vec{y}^2}{2}, 2\vec{y}) \end{aligned}$$

where $\vec{x} \in \mathbb{R}^n, (a, 2\vec{x}) \in \mathbb{R}^{n+1}$ for any $a \in \mathbb{R}$, $\vec{y} = \frac{\vec{x}}{\vec{x}^2}$.

Denote the induce metrics of $(X, \mathbb{R}^n), (Y, \mathbb{R}^n)$ as g^+, g^- respectively, the tagent vector $\frac{\partial}{\partial x_\alpha} \in T\mathbb{R}^{n+1}$ as $e_\alpha = (0, \cdots, 1, \cdots, 0)$ (the α-th component is 1 and all other components are 0) for $0 \leq \alpha \leq n$. Then in $V_1 = X(\mathbb{R}^n)$,

$$\begin{aligned} \mathrm{d}X(\frac{\partial}{\partial x_i}) &= \frac{\partial X}{\partial x_i} \\ &= \frac{1}{(\vec{x}^2+1)^2}(2x_i, 2\delta_{1i}(\vec{x}^2+1) - 4x_1 x_i, \cdots, 2\delta_{ni}(\vec{x}^2+1) - 4x_n x_i) \end{aligned}$$

for $1 \leq i \leq n$. Then

$$
\begin{aligned}
g_{ij}^+ &= \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_X = \langle \mathrm{d}X(\frac{\partial}{\partial x_i}), \mathrm{d}X(\frac{\partial}{\partial x_j}) \rangle_{\mathbb{R}^{n+1}} = \langle \frac{\partial X}{\partial x_i}, \frac{\partial X}{\partial x_i} \rangle \\
&= \frac{4x_i x_j}{(\vec{x}^2+1)^4} + \sum_{k=1}^{n} \frac{(2\delta_{ki}(\vec{x}^2+1) - 4x_k x_i)(2\delta_{kj}(\vec{x}^2+1) - 4x_k x_j)}{(\vec{x}^2+1)^4} \\
&= \frac{4\delta_{ij}(\vec{x}^2+1)^2 - 12x_i x_j}{(\vec{x}^2+1)^4}.
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
g^+ &= g_{ij}^+ \mathrm{d}x_i \mathrm{d}x_j \\
&= \frac{4\delta_{ij}(\vec{x}^2+1)^2 - 12x_i x_j}{(\vec{x}^2+1)^4} \mathrm{d}x_i \mathrm{d}x_j.
\end{aligned}
$$

Similarly,

$$
g^- = \frac{4\delta_{ij}(\vec{y}^2+1)^2 - 12y_i y_j}{(\vec{y}^2+1)^4} \mathrm{d}y_i \mathrm{d}y_j.
$$

On $V_1 \cap V_2$,

$$
\begin{aligned}
g^- &= \frac{4\delta_{ij}(\vec{y}^2+1)^2 - 12y_i y_j}{(\vec{y}^2+1)^4} \mathrm{d}y_i \mathrm{d}y_j \\
&= \frac{4\delta_{ij}(1/\vec{x}^2+1)^2 - 12x_i x_j/\vec{x}^4}{(1/\vec{x}^2+1)^4 \vec{x}^4} \mathrm{d}x_i \mathrm{d}x_j \\
&= \frac{4\delta_{ij}(\vec{x}^2+1)^2 - 12x_i x_j}{(\vec{x}^2+1)^4} \mathrm{d}x_i \mathrm{d}x_j
\end{aligned}
$$

□