Algebra HW6

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1. *Page 133,3* If R is a ring and $a \in R$, then $J = \{r \in R \mid ra = 0\}$ is a left ideal and $K = \{r \in R \mid ar = 0\}$ is a right ideal in R.

   SOLUTION: For any $r \in R$, $i, j \in J$, $k, l \in K$,

$$\begin{aligned} (ri)a &= r(ia) = r0 = 0, \\ (i-j)a &= ia - ja = 0 - 0 = 0; \\ a(kr) &= (ak)r = 0r = 0, \\ a(k-l) &= ak - al = 0 - 0 = 0. \end{aligned}$$

   Thus $ri \in J, i - j \in J$, J is a left ideal; $kr \in K, k - l \in K$, K is a right ideal. □

2. *Page 133,8* Let R be a ring with identity and S the ring of all $n \times n$ matrices over R. J is an ideal of S if and only if J is the ring of all $n \times n$ matrices over I for some ideal I in R.

   [Hint: Given J, let I be the set of all those elements of R that appear as the row 1-column 1 entry of some matrix in J. Use the matrices $E_{r,s}$, where $1 \le r, s \le n$, and $E_{r,s}$ has 1_R as the row r-column s entry and 0 elsewhere. Observe that for a matrix $A = (a_{ij}), E_{p,r}AE_{s,q}$ is the matrix with a_{rs} the row p-column q entry and 0 elsewhere.]

   SOLUTION:

   "⇒": Consider the set $I_{rs} = \{a \in R \mid aE_{rs} \in J\}$. Since the elementary matrix switching the i-th and j-th row or column $P_{ij} \in R$, $I_{rs} = \{a \in R \mid aE_{rs} \in J\} = \{a \in R \mid P_{1s}aE_{rs}P_{1r} \in J\} = I_{11}$. Thus denote $I = I_{rs}$ for all $1 \le r, s \le n$. I is an ideal since for any $a, b \in I, x \in R$,

$$\begin{aligned} (a-b)E_{11} &= aE_{11} - bE_{11} \in J, \\ (xa)E_{11} &= xE_{11}aE_{11} \in J, \\ (ax)E_{11} &= aE_{11}xE_{11} \in J. \end{aligned}$$

   In addition, J is the ring of all $n \times n$ matrices over I since for any $1 \le r, s \le n$, $(a_{ij}) \in J$, $a_{rs}E_{rs} = (a_{ij})E_{rs} \in J$ infers $a_{rs} \in I_{rs} = I$.

   "⇐": For any $(a_{ij}), (b_{ij}) \in J$ where $a_{ij}, b_{ij} \in I$ for $1 \le i, j \le n$. For any $(x_{ij}) \in R$,

$$\begin{aligned} (a_{ij}) - (b_{ij}) &= ((a-b)_{ij}) \in J, \\ (a_{ij})(x_{ij}) &= (\sum_{k=1}^{n} a_{ik}x_{kj}) \in J, \\ (x_{ij})(a_{ij}) &= (\sum_{k=1}^{n} x_{ik}a_{kj}) \in J. \end{aligned}$$

   Thus J is an ideal of S.

□

3. *Page 133,10*

   (a) Show that Z is a principal ideal ring [see **Theorem I.3.1**].

   (b) Every homomorphic image of a principal ideal ring is also a principal ideal ring.

   (c) Z_m is a principal ideal ring for every $m > 0$.

   SOLUTION:

   (a) By **Theorem I.3.1**, every subgroup of Z has the form $\langle m \rangle$ for some $m \in Z$. Since every ideal of Z must be subgroup of Z, it must have the form $\langle m \rangle$ and is a principal ideal. In addition, $\forall n \in Z$, $km \in \langle m \rangle$, $nkm = (nk)m \in \langle m \rangle$. Since Z is communitative, $\langle m \rangle$ is an ideal of Z. Thus every ideal of Z is of the form $\langle m \rangle$.

   (b) Assume P is a principal ideal ring, R is a ring, ϕ is a homomorphism from P to R; then $\phi(P)$ is a ring. For any ideal I in $\phi(P)$, $\forall a, b \in \phi^{-1}(I), x \in P$,

$$\begin{aligned} \phi(a-b) &= \phi(a) - \phi(b) \in I, \\ \phi(xa) &= \phi(x)\phi(a) \in I, \\ \phi(ax) &= \phi(a)\phi(x) \in I. \end{aligned}$$

   Thus $\phi^{-1}(I)$ is an ideal in P. For any ideal J in P, $\forall a, b \in J, x \in P$,

$$\begin{aligned} \phi(a) - \phi(b) &= \phi(a-b) \in \phi(J), \\ \phi(x)\phi(a) &= \phi(xa) \in \phi(J), \\ \phi(a)\phi(x) &= \phi(ax) \in \phi(J). \end{aligned}$$

   Thus $\phi(J)$ is an ideal in $\phi(P)$.
   Suppose $\phi^{-1}(I) = \langle m \rangle$ for some $m \in P$, then $\phi^{-1}(\langle \phi(m) \rangle)$ is an ideal in P and $m \in \phi^{-1}(\langle \phi(m) \rangle)$. Since $\phi^{-1}(I)\langle m \rangle = \phi^{-1}(\langle \phi(m) \rangle)$, we get $I = \langle \phi(m) \rangle$ is a principal ideal and $\phi(P)$ is a principal ideal ring.

   (c) $Z_m \cong Z/mZ$, construct a map

$$\begin{aligned} \phi : \quad Z &\to Z_m \\ z &\mapsto [z]_m \end{aligned}$$

   For any $a, b \in Z$,

$$\begin{aligned} \phi(a) + \phi(b) &= [a]_m + [b]_m = [a+b]_m, \\ \phi(a)\phi(b) &= [a]_m[b]_m = [ab]_m, \end{aligned}$$

   thus ϕ is an epimorphism. By conclusions in the above two, $Z_m = \phi(Z)$ is a principal ideal ring.

□

4. *Page 133,23* An element e in a ring R is said to be idempotent if $e^2 = e$. An element of the center of the ring R is said to be central. If e is a central idempotent in a ring R with identity, then

   (a) $1_R - e$ is a central idempotent;

   (b) eR and $(1_R - e)R$ are ideals in R such that $R = eR \times (1_R - e)R$.

   Solution:

   (a) For any $a \in R$,

   $$\begin{aligned} a(1_R - e) &= a - ae = a - ea = (1_R - e)a; \\ (1_R - e)^2 &= 1_R - e - e + e^2 = 1_R - e. \end{aligned}$$

   Thus $1_R - e$ is a central idempotent.

   (b) For any central idempotent $f \in R$, $fa, fb \in fR$ and $x \in R$,

   $$\begin{aligned} fa - fb &= f(a - b) \in fR, \\ x(fa) &= (xf)a = (fx)a = f(xa) \in fR, \\ (fa)x &= f(ax) \in fR. \end{aligned}$$

   Thus fR is an ideal and so do $eR, (1_R - e)R$.
   For any $r \in R$, $r = er + (1_R - e)r \in eR + (1_R - e)R$ thus $R = eR + (1_R - e)R$. For $a \in eR \cap (1_R - e)R$, suppose $a = ex = (1_R - e)y$ for some $x, y \in R$.

   $$\begin{aligned} a &= ea + (1_R - e)a \\ &= e(1_R - e)y + (1_R - e)ex \\ &= (e - e^2)y + (e - e^2)x \\ &= 0. \end{aligned}$$

   Therefore, $R = eR \times (1_R - e)R$.

   $\square$

5. *Page 133,24* Idempotent elements $e_1, \cdots, e_n$ in a ring R [see Exercise 23] are said to be orthogonal if $e_ie_j = 0$ for $i \neq j$. If $R, R_1, \cdots, R_n$ are rings with identity, then the following conditions are equivalent:

   (a) $R \cong R_1 \times \cdots \times R_n$.

   (b) R contains a set of orthogonal central idempotents [Exercise 23] $\{e_1, \cdots, e_n\}$ such that $e_l + e_2 + \cdots + e_n = 1_R$ and $e_iR \cong R_i$ for each i.

   (c) R is the internal direct product $R = A_1 \times \cdots \times A_n$ where each A_i is an ideal of R such that $A_i \cong R_i$.
   [Hint: (a)→(b): The elements $f_1 = (1_{R_1}, 0, \cdots, 0), f_2 = (0, 1_{R_2}, \cdots, 0),\ , f_n = (0, \cdots, 0, 1_{R_n})$ arf orthogonal cfntral idfmpotfnts in $S = R_1 \times \cdots \times R_n$ such that $f_1 + \cdots + f_n = 1_s$ and $f_iS \cong R_i$. (b) → (c) Note that $A_k = e_kR$ is the principal ideal $\langle e_k \rangle$ in R and that e_kR is itself a ring with identity e_k.]

   Solution:

(a)⇒(b): Let $S = R_1 \times \cdots \times R_n$, the elements $e_1 = (1_{R_1}, 0, \cdots, 0), e_2 = (0, 1_{R_2}, \cdots, 0), \cdots, e_n = (0, \cdots, 0, 1_{R_n})$ are orthogonal since $e_ie_j \in R_i \cap R_j = \{0\}$ for $i \neq j$. $1_S = \sum_{k=1}^n e_k$. For any $1 \leq i \leq n$, $a = \sum_{k=1}^n e_ka_k \in S$,

$$\begin{aligned} e_ia &= e_ia_i = a_ie_i = ae_i, \\ e_i^2 &= (0, \cdots, 1_{R_i}^2, \cdots, 0) = e_i. \end{aligned}$$

Thus e_i is a central idempotent. Since $R \cong S$, R also contains a set of orthogonal central idempotents satisfies the conditions.

(b)$\Rightarrow$(c): Let $A_i = e_i R \cong R_i$, then $R = A_1 \times \cdots \times A_n$.

(c)$\Rightarrow$(a): It suffices to show $R_1 \times R_2 \cong A_1 \times A_2$ where the products are both external products by **Theorem 2.24**. Suppose the isomorphism $\phi_1 : R_1 \to A_1$ and $\phi_2 : R_2 \to A_2$, and define

$$\begin{aligned} \phi : \quad & R_1 \times R_2 \to A_1 \times A_2 \\ & (r_1, r_2) \mapsto (\phi_1(r_1), \phi_2(r_2)). \end{aligned}$$

ϕ is an isomorphism between groups. For any $(r_1, r_2), (s_1, s_2) \in R_1 \times R_2$,

$$\begin{aligned} (\phi_1(r_1), \phi_2(r_2))(\phi_1(s_1), \phi_2(s_2)) &= (\phi_1(r_1)\phi_1(s_1), \phi_2(r_2)\phi_2(s_2)) \\ &= (\phi_1(r_1 s_1), \phi_2(r_2 s_2)) \\ &= \phi((r_1 s_1, r_2 s_2)), \end{aligned}$$

so ϕ is an isomorphism between rings. $R = A_1 \times \cdots \times A_n \cong R_1 \times \cdots \times R_n$.

$\square$