Probability Theory HW5

段奎元
SID: 201821130049
dkuei@outlook.com

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1. *Page 46,13 Property 2.36*

   (a) If $f_n \xrightarrow{\text{a.e.}} f$, then every subsequence $\{f_{n_k}\}$ satisfies $f_{n_k} \xrightarrow{\text{a.e.}} f$.

   (b) If $f_n \xrightarrow{\text{a.e.}} f, f_n \xrightarrow{\text{a.e.}} f'$, then $f = f'$ a.e..

   (c) If $f_n \xrightarrow{\text{a.e.}} f, g_n = f_n$ a.e., $f = g$ a.e., then $g_n \xrightarrow{\text{a.e.}} g$.

   (d) If $f_n^{(k)} \xrightarrow{\text{a.e.}} f^{(k)}, k = 1, \cdots, m$, $g \in C(\bar{\mathbb{R}}^m)$, then
   $$g(f_n^{(1)}, \cdots, f_n^{(m)}) \xrightarrow{\text{a.e.}} g(f^{(1)}, \cdots, f^{(m)}).$$

   Solution:

   (a) Since $f_n \xrightarrow{\text{a.e.}} f$ there exists a null set N s.t. $\forall x \in \Omega \setminus N, f_n(x) \to f(x)$. Thus for every subsequence $\{f_{n_k}\}, f_{n_k}(x) \to f(x)$, i.e. $f_{n_k} \xrightarrow{\text{a.e.}} f$.

   (b) There exist two null sets N, N' such that $\forall x \in \Omega \setminus N, f_n(x) \to f(x)$, and $\forall x \in \Omega \setminus N', f_n(x) \to f'(x)$. Thus $\forall x \in \Omega \setminus (N \cup N'), f_n(x) \to f(x), f_n(x) \to f'(x)$; then $f(x) = f'(x)$ for any $x \in \Omega \setminus (N \cup N')$. Since $\mu(N \cup N') = 0$, we get $f = f'$ a.e..

   (c) There exist null sets N_1, N_2, N_3 for $f_n \xrightarrow{\text{a.e.}} f, g_n = f_n$ a.e., $f = g$ a.e. respectively. For any $x \in \Omega \setminus (\cup_{i=1}^3 N_i), f_n(x) \to f(x), g_n(x) = f_n(x), f(x) = g(x)$, so $g_n(x) \to g(x)$. Since $\mu(\cup_{i=1}^3 N_i) = 0$, we get $g_n \xrightarrow{\text{a.e.}} g$.

   (d) Denote $N_k = \{x \in \Omega^{(k)} \mid f_n^{(k)}(x) \not\to f^{(k)}(x)\}$, and we have $\mu(N_k) = 0$ by $f_n^{(k)} \xrightarrow{\text{a.e.}} f^{(k)}, k = 1, \cdots, m$. For any $(x^{(1)}, \cdots, x^{(m)}) \in \prod_{k=1}^m \Omega^{(k)} \setminus N_k$,
   $$\left(f_n^{(1)}(x^{(1)}), \cdots, f_n^{(m)}(x^{(m)})\right) \to \left(f^{(1)}(x^{(1)}), \cdots, f^{(m)}(x^{(m)})\right).$$
   Since $\mu(\prod_{k=1}^m \Omega^{(k)} \setminus N_k) = \mu(\prod_{k=1}^m \Omega^{(k)})$, the convergence keeps a.e. By continuity of g, we get
   $$g(f_n^{(1)}, \cdots, f_n^{(m)}) \xrightarrow{\text{a.e.}} g(f^{(1)}, \cdots, f^{(m)}).$$

2. *Page 47,14 Theorem 2.38(2)*

   Suppose $f, f_n, n \geq 1$ are finite measurable functions. Then $f_n - f_m \xrightarrow{\text{a.e.}} 0$ if and only if
   $$\forall \varepsilon > 0, \mu(\cap_{n=1}^\infty \cup_{v=1}^\infty \{|f_{n+v} - f_n| \geq \varepsilon\}) = 0.$$
   Specially when μ is finite, $f_n - f_m \xrightarrow{\text{a.e.}} 0$ if and only if
   $$\forall \varepsilon > 0, \mu(\cup_{v=1}^\infty \{|f_{n+v} - f_n| \geq \varepsilon\}) \to 0 (n \to \infty).$$

   Solution: $f_n - f_m \xrightarrow{\text{a.e.}} 0$ if and only if $|f_{n+v} - f_n| \xrightarrow{\text{a.e.}} 0$ by letting $n = \min\{m, n\}, v = |m - n|$. Notice $\{|f_{n+v} - f_n| \geq \varepsilon\} = \{x : |f_{n+v} - f_n| \geq \varepsilon\}$.

"⇒": There exists a null set A such that $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}_+$ s.t. $\forall n \geq N, v \in \mathbb{N}_+, x \in \Omega \setminus A, |f_{n+v}(x) - f_n(x)| < \varepsilon$. Then $\mu(\cup_{n=N}^{\infty} \cup_{v=1}^{\infty} \{x : |f_{n+v}(x) - f_n(x)| \geq \varepsilon\}) = 0$, and apparently

$$\mu(\cap_{n=1}^{\infty} \cup_{v=1}^{\infty} \{|f_{n+v} - f_n| \geq \varepsilon\}) = 0, \mu(\cup_{v=1}^{\infty}\{|f_{n+v} - f_n| \geq \varepsilon\}) \to 0(n \to \infty).$$

"⇐": When μ is finite, $\mu(\cup_{v=1}^{\infty}\{|f_{n+v} - f_n| \geq \varepsilon\}) \to 0(n \to \infty)$ means $\forall \phi > 0 \exists N$ s.t. $\mu(\cup_{v=1}^{\infty}\{|f_{n+v} - f_n| \geq \varepsilon\}) < \phi$ for all $n \geq N$. Then $\mu(\cap_{n=1}^{\infty} \cup_{v=1}^{\infty} \{|f_{n+v} - f_n| \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$.

3. *Page 47,16* Let $\xi_n = 1_{A_n}$, then $\xi_n \xrightarrow{\mathbb{P}} 0$ if and only if $\mathbb{P}(A_n) \to 0$.

<u>Solution</u>: For any $\varepsilon > 0$, $\{|\xi_n| \geq \varepsilon\} = \{A_n\}$.

$$\begin{aligned} \xi_n \xrightarrow{\mathbb{P}} 0 &\Leftrightarrow \forall \varepsilon > 0, \mathbb{P}(|\xi_n| \geq \varepsilon) \to 0(n \to \infty) \\ &\Leftrightarrow \forall \varepsilon > 0, \mathbb{P}(A_n) \to 0(n \to \infty) \end{aligned}$$

4. *Page 47,22* For any random variable sequence ξ_n, there is a positive integer sequence a_n s.t. $a_n \xi_n \xrightarrow{\mathbb{P}} 0$.

<u>Solution</u>: $\xi_n \xrightarrow{\mathbb{P}} 0 \Leftrightarrow \forall \varepsilon, \phi > 0 \exists N > 0, \mathbb{P}(|\xi_n| \geq \varepsilon) < \phi, \forall n \geq N$

5. *Page 47,24* Prove two theorems 2.49 and 2.50.

Theorem 2.49 If $\xi_n - \xi_n' \xrightarrow{\mathbb{P}} 0$ and $\xi_n' \xrightarrow{d} \xi$, then $\xi_n \xrightarrow{d} \xi$.

Theorem 2.50 If $\xi_n \xrightarrow{d} \xi, \eta_n \xrightarrow{d} a(\text{const})$, then $\xi_n + \eta_n \xrightarrow{d} \xi + a$.

<u>Solution</u>: