Differential Geometry HW2

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1. The transition function between two axis of the vector bundle is smooth.

<u>Solution</u>: For any $i = 1, 2, \cdots, n$, $\frac{\partial}{\partial y^i} = A_{ij} \frac{\partial}{\partial x^j}$ since $\frac{\partial}{\partial x^j}$ is a basis of T_pM. For any axis function x^k, make the vector acts on it,

$$\begin{aligned} \frac{\partial x^k}{\partial y^i} &= A_{ij} \frac{\partial x^k}{\partial x^j} \\ &= A_{ij} \delta_{jk} \\ &= A_{ik} \end{aligned}$$

Then $\frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}$, the Jacob matrix is $\frac{\partial x^j}{\partial y^i}$.

2. M is a differential manifold. For $X, Y \in TM$, verify that $X \circ Y$ is not in TM and $[X, Y] \in TM$.

<u>Solution</u>: For any $f, g \in C^\infty(M)$,

$$\begin{aligned} X \circ Y(fg) &= X(fY(g) + Y(f)g) \\ &= X(f)Y(g) + fX \circ Y(g) + X \circ Y(f)g + Y(f)X(g); \\ [X, Y](fg) &= X \circ Y(fg) - Y \circ X(fg) \\ &= X(f)Y(g) + fX \circ Y(g) + X \circ Y(f)g + Y(f)X(g) \\ &- (Y(f)X(g) + fY \circ X(g) + Y \circ X(f)g + X(f)Y(g)) \\ &= f[X, Y](g) + [X, Y](f)g. \end{aligned}$$

$X \circ Y$ does not satisfy Leibniz formula, is not in TM.

For any $\alpha, \beta \in \mathbb{R}$, there are constent functions in $C^\infty(M)$ which equal to α, β respectively. We denote the constent functions by α, β themselves.

$$\begin{aligned} [X, Y](f + g) &= X(Y(f + g)) - Y(X(f + g)) \\ &= X(Y(f) + Y(g)) - Y(X(f) + Y(g)) \\ &= X \circ Y(f) + X \circ Y(g) - Y \circ X(f) - Y \circ X(g) \\ &= [X, Y]f + [X, Y]g; \\ [X, Y](\alpha f + \beta g) &= [X, Y](\alpha f) + [X, Y](\beta g) \\ &= \alpha[X, Y](f) + [X, Y](\alpha)f + \beta[X, Y](g) + [X, Y](g)\beta \\ &= \alpha[X, Y](f) + \beta[X, Y](g). \end{aligned}$$

$[X, Y]$ satisfies Leibniz formula and has linear property. $[X, Y] \in TM$.

3. The communator $[X, Y]$ can expand by axis vectors in a coordinate neighbourhood.

SOLUTION: Suppose $X = X_i \frac{\partial}{\partial x^i}$, $Y = Y_i \frac{\partial}{\partial x^i}$. For an axis function $x_k \in C^\infty(M)$,

$$\begin{aligned}
[X, Y](x_k) &= X(Yx_k) - Y(Xx_k) \\
&= X(Y_i \frac{\partial x_k}{\partial x^i}) - Y(X_i \frac{\partial x_k}{\partial x^i}) \\
&= X(Y_k) - Y(X_k) \\
&= X_i \frac{\partial Y_k}{\partial x^i} - Y_i \frac{\partial X_k}{\partial x^i}.
\end{aligned}$$

Since we have known $[X, Y] \in TM$, thus $[X, Y]$ can be represented by $\frac{\partial}{\partial x^i}$,

$$[X, Y] = (X_i \frac{\partial Y_k}{\partial x^i} - Y_i \frac{\partial X_k}{\partial x^i}) \frac{\partial}{\partial x^k}.$$