Algebra HW2

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1. *Page 63,4* Give an example to show that the weak direct product is not a coproduct in the category of all groups. (Hint: it suffices to consider the case of two factors $G \times H$)

   SOLUTION: Consider the weak direct product $\mathbb{Z}_2 \times \mathbb{Z}_2 = D_2 = \{1, a, b, ab\}$ where $a^2 = b^2 = 1$. $\mathbb{Z}_2 \cong \{1, a\}$. The morphism from $\{1, a\}$ to D_2 is not unique because it can be $f : 1 \mapsto 1, a \mapsto b$, or the inclusion map $\mathrm{id} : 1 \mapsto 1, a \mapsto a$. Hence the (weak) direct product is not a coproduct in the category of all groups.

2. *Page 63,7* Let H, K, N be nontrivial normal subgroups of a group G and suppose $G = H \times K$. Prove that N is in the center of G or N intersects one of H, K nontrivially. Give examples to show that both possibilities can actually occur when G is nonabelian.

   SOLUTION: `NOTATION: It is necessary that` $N \neq H, K$ `(otherwise,` $N = H$ `intersects both` H, K `trivially and may not in the center of` G`).`

   It suffices to prove $N \subset C(G)$ when $N \cap H = N \cap K = \{e\}$. For any $h \in H, n \in N$, $hnh^{-1} \in N$ since $N \trianglelefteq G$, then $hnh^{-1}n^{-1} \in N$ since N is a subgroup. Meanwhile $hnh^{-1}n^{-1} = h(nh^{-1}n^{-1}) \in H$, so $hnh^{-1}n^{-1} \in H \cap N = \{e\}$ i.e. $hnh^{-1}n^{-1} = e$. We get $hn = nh$ and for any $k \in K$ $kn = nk$ by the same impication.
   For any element in G, it equals to hk for some $h \in H, k \in K$ since $G = H \times K$. $\forall n \in N$,

   $$(hk)n = h(kn) = h(nk) = (hn)k = (nh)k = n(hk)$$

   shows that $N \subset C(G)$.

   (a) The group $G = \mathbb{Z}_2 \times S_3 =$

   $$\{e = (1,1), (1,132), (1,231), (1,213), (1,312), (1,321),$$
   $$(21,1), (21,132), (21,231), (21,213), (21,312), (21,321)\}$$

has $\mathbb{Z}_2, \mathbb{Z}_2 \times A_3, \mathbb{Z}_2 \times S_3, \mathbb{Z}_3, A_3$ and S_3 as the nontrivial normal subgroup. The center of group G $C(G) = C(\mathbb{Z}_2) \times C(S_3) = \mathbb{Z}_2$ because

$$\begin{aligned} C(G) = C(H \times K) &= \{(h_0, k_0) \mid \forall (h,k) \in G, (hh_0, kk_0) = (h_0 h, k_0 k)\} \\ &= \{h_0 \mid \forall h \in H, hh_0 = h_0 h\} \times \{k_0 \mid \forall k \in K, kk_0 = k_0 k\} = C(H) \times C(K). \end{aligned}$$

(b) The group $G = \mathbb{Z}_2 \times (\mathbb{Z}_2 \times S_3)$ has normal subgroups $\{(1,1,1),(21,21,1)\}$ in the center of G with intersects both $\mathbb{Z}_2$ and $D_6 = \mathbb{Z}_2 \times S_3$ both trivially.

3. *Page 68,9* The group defined by the generator b and the relation $b^m = e(m \in \mathbb{N}^*)$ is the cyclic group $\mathbb{Z}_m$.

<u>SOLUTION</u>: If F is the free group on $\{b\}$, consider the morphism $f : F \to \mathbb{Z}_m$ mapping b^n to $[n]_m$ for any $n \in \mathbb{Z}$. $\forall l, n \in \mathbb{Z}$,

$$f(b^l b^n) = f(b^{l+n}) = [l+n]_m = [l]_m + [n]_m = f(b^l) + f(b^n),$$

and $[n]_m = f(b^n)$, so f is a group epimorphism.
$\mathrm{Ker}(f) = \{b^n \mid f(b^n) = [n]_m = [0]_m, \text{ i.e. } m|n\}$, so $F/\mathrm{Ker}(f) \cong \mathbb{Z}_m$. $F/\mathrm{Ker}(f)$ is the group defined by the generator b and the relation $b^m = e(m \in \mathbb{N}^*)$.