Class 2022/10/20

Matching Revisited

Recap

Recall the matching problem:

- N men with hats through them into center and then host randomly gave each man a hat

What's the probability no man got matched with his hat?

- We used inclusion / Exclusion principle
- Let E_i = event the i^{th} man got matched with his hat
- then $E_1 \cup E_2 \ldots \cup E_N$ represent the event at least one man gets his hat
- $E_1 E_2 E_3 \ldots E_N$ represent everyone gets matched with hat
- Then,

$$\begin{aligned} P(none) &= 1 - P(E_1 \cup E_2 \ldots \cup E_N) = \\ &= 1 - \left[\sum_{i=1}^{N} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \cdots + (-1)^{n+1} P(E_{i_1} E_{i_2} \ldots E_{i_n}) \right] \\ &= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} \ldots\ldots \end{aligned}$$

- Finally, we concluded as $N \to \infty, \quad P(none) \approx e^{-1}$

- **When $N \to \infty$**

$$P(E_i) = \frac{(N-1)!}{N!} = \frac{1}{N}$$
$$P(E_iE_j) = \frac{(N-2)!}{N!} = \frac{1}{N(N-1)}$$
$$P(E_i|E_j) = \frac{P(E_iE_j)}{P(E_i)} = \frac{\frac{1}{N(N-1)}}{\frac{1}{N}} = \frac{1}{N-1}$$

- Note that when N is very large,
  - $P(E_i|E_j) = \dfrac{1}{N-1} \approx \dfrac{1}{N} = P(E_i)$

    $\Rightarrow P(E_i|E_j) \approx P(E_i)$
  - independent?
- Weakly Dependent

- **Using Poisson**

So, consider the following

Let X be the RV that cound the number of matches (success = match)

- 2 outcomes S/F
- Fixed number of trials (N)
- "kinda" independent (as $N \to \infty$)

$\Rightarrow X$ is "kinda" Binomial

- $X \sim Bin(N,\ p = 1/N)$

And Since $N \to \infty$,

- $X \sim Pois(\lambda = Np = N\frac{1}{N} = 1)$

  $\sim Pois(1)$

Therefore, using Poisson,

$$P(none) = P(X=0) = \frac{e^{-1}\lambda^0}{0!} = e^{-1}$$

> Useful #44 in HW6

Chapter 5: Continous Random Variables

Definitions for CRV

Let X be a Countinous Random Variable

X can take on uncountably many values (similar to $\mathbb{R}$ or (a, b), no gaps!)

- Ex. time, length, distance, height

Probability Density Function (pdf)

- the Probability distribution function for the CRV is called a Probability Density Function (pdf)
- $f : X \to [0, 1]$
- (extend--->) $f : \mathbb{R} \to [0, 1] = \ldots (x \in X), and\ 0(x = else)$

Properties

1. $\displaystyle\int_{-\infty}^{\infty} f(x)dx = 1$
2. **<u>UNLIKE</u> the DRV** (whose $p(x) = P(X = x)$)

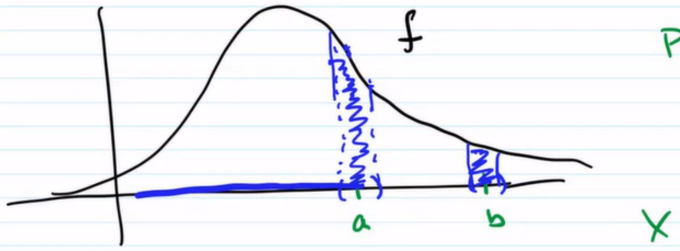
   $f(x) \neq P(X = x)$

   In fact $P(X = x) = 0,\ \forall x$

   For CRV,

   - $P(x_1 < X < x_2) = \displaystyle\int_{x_1}^{x_2} f(x)dx$
   - $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$
   - $P(X = a) = 0$
     - $P(x \leq X \leq x) = \int_x^x f(t)dt = 0$

Ex a pdf, f below



$P(X=a) = P(X=b) = 0$

$P(\text{within } \varepsilon \text{ of } a) > P(\text{within } \varepsilon \text{ of } b)$

$f(a) > f(b)$

$P(a-\varepsilon \leq X < a+\varepsilon) > P(b-\varepsilon \leq X < b+\varepsilon)$

$$\int_{a-\varepsilon}^{a+\varepsilon} f(x)dx > \int_{b-\varepsilon}^{b+\varepsilon} f(x)dx$$

CDF

- $F(x) = P(X \leq x)$
- $F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$

Things to Know:

$$P(X \leq a) = F(a) = \int_{-\infty}^{a} f(x)dx$$
$$P(X < a) = 1 - F(a) = \int_{a}^{\infty} f(x)dx$$

Sometimes, it's useful to find the closed form of $F(X)$

PDF

CRV, X and $f(x)$ pmf, $f(x) \geq 0$ for all $x \in \mathbb{R}$

1. $\int_{-\infty}^{\infty} f(x)dx = 1$
2. $F(x) = \int_{-\infty}^{x} f(t)dt = P(X \leq x)$

   $F'(x) = f(x)$ (**FTC**!)
3. $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$

   $E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx$
4. $Var(X) = E(X^2) - E(X)^2 = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx - \left[\int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx\right]^2$

> In DRV, we count
>
> In CRV, we integrate

- Example

1. Ex Consider a CRV, X and its pdf

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

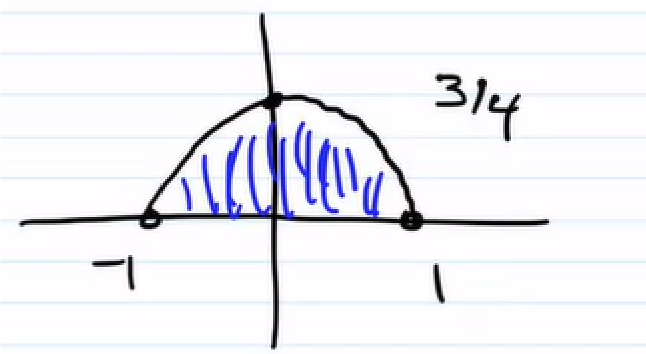
a) Find E(X)
b) If possible, find closed form of F(x)
c) $P(X \le 1/2 \mid X > 1/3)$

   - Find c

$$\int_{-1}^{1} f(x)dx = 1$$
$$\int_{-1}^{1} c(1-x^2)dx = 1$$
$$c\int_{-1}^{1} (1-x^2)dx = 1$$
$$c\left(x - \frac{x^3}{3}\Big|_{-1}^{1}\right) = 1$$
$$\frac{4}{3}c = 1$$
$$c = \frac{3}{4}$$

Then the pdf is:

$$f(x) = \begin{cases} \frac{3}{4}(1-x^2), & -1 < x < 1 \\ 0, & else \end{cases}$$

Graph: 

- Expectation:

$$E(X) = \int_{-1}^{1} x \frac{3}{4}(1-x^2)dx = \frac{3}{4}\int_{-1}^{1} x - x^3 \, dx = 0$$

- Find closed form:

$$P(X < 1/2) = F(1/2) = \int_{-1}^{1/2} f(x)\,dx$$

$$P(X < 1/4) = F(1/4) = \int_{-1}^{1/4} f(x)\,dx$$

$$P(X < -1/4) = F(-1/4) = \int_{-1}^{-1/4} f(x)\,dx$$

  Instead of calculating $P(X < a)$ each time when we have a new a, we want to find an answer for all $P(X < a)$, which is called a closed form.

$$\begin{aligned} P(X < a) = F(a) &= \int_{-1}^{a} f(x)dx \\ &= \int_{-1}^{a} \frac{3}{4}(1-x^2)dx \\ &= \frac{3}{4}\int_{-1}^{a}(1-x^2)dx \\ &= \frac{3}{4}\Big(x - \frac{x^3}{3}\Big|_{-1}^{a}\Big) \\ &= \frac{3}{4}\Big(x - \frac{x^3}{3}\Big|_{-1}^{a}\Big) \\ &= \frac{3}{4}\Big(a - \frac{a^3}{3} + 1 - \frac{1}{3}\Big) \end{aligned}$$

  Therefore our closed form is

$$F(a) = \frac{3}{4}\Big(a - \frac{a^3}{3} + \frac{2}{3}\Big)$$

- $P(X \leq 1/2 \mid X > 1/3)$

  We can use the closed form to solve this

$$\begin{aligned} P(X \leq \frac{1}{2} \mid X > \frac{1}{3}) &= \frac{P(\frac{1}{3} < X \leq \frac{1}{2})}{P(X > \frac{1}{3})} \\ &= \frac{\int_{1/3}^{1/2} f(x)dx}{\int_{1/3}^{1} f(x)dx} \\ &= \frac{F(\frac{1}{2}) - F(\frac{1}{3})}{1 - F(\frac{1}{3})} \end{aligned}$$

2. X is CRV with pdf:

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & \text{—} \end{cases}$$

And further, suppose $E(X) = 3/5$

Your task, find a & b

$a = 3/5, \; b = 6/5$

3. Let X be a CRV, with pdf $f_X(x)$ and cdf $F_X(x)$

   Find the pdf for $Y = 2X$, that is to find $f_Y(y)$

   <u>**Process:**</u>

   Start with the CDF of Y,

   $F_Y(a) = P(Y \leq a) = P(2X \leq a) = P(X \leq a/2) = F_x(a/2)$

   Take derivative,

$$\begin{aligned} F_Y(a) &= F_X(a/2) \\ \frac{d}{da} F_Y(a) &= \frac{d}{da} F_X(a/2) \\ f_Y(a) &= f_x(a/2) \cdot 1/2 \end{aligned}$$

> We'll revisit these a lot