Class 4: Subgroups, Additive Integer Group and Its Subgroup - 2022/09/19

Group Cont.

Order

Definition: The order of a group G is the number of element G has, denoted by $|G|$.

- If $|G| < \infty$, we say G is a finite group.
- Otherwise G is an infinite group

Notation of composition in groups

If we write the composition of a group is a multiplicative way, we have:

$$
\begin{aligned}
k > 0, \qquad & g^k = \underbrace{g * g * \ldots g}_{\text{k-copies}}, \\
& g^{-k} = \underbrace{g^{-1} * g^{-1} \ldots g^{-1}}_{\text{k-copies}} \\
k = 0, \qquad & g^0 = 1
\end{aligned}
$$

Note:

$$g^k \cdot g^l = g^{k+l}$$

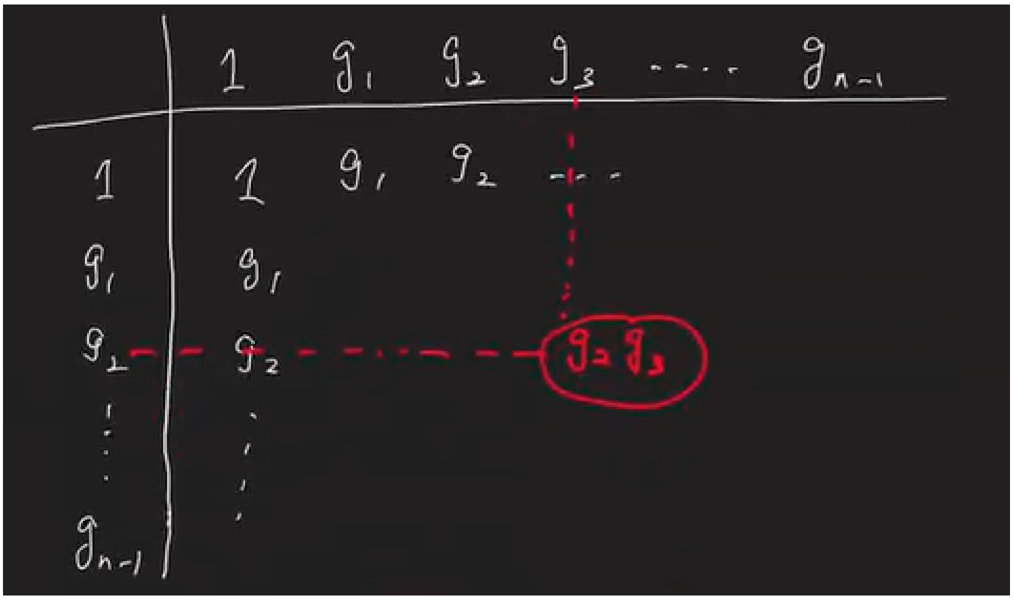
- **Examples of Groups**

1. $G = \{\pm 1\}$, composition is multiplication.

   $|G| = 2$. Finite group.

$$
\begin{aligned}
(+1) \cdot (+1) &= +1 \\
(+1) \cdot (-1) &= -1 \\
(-1) \cdot (+1) &= -1 \\
(-1) \cdot (-1) &= +1
\end{aligned}
$$

   - Since it's a finite group, we can use the multiplication table to show the results.

> ## Multiplication Table
>
> - "Multiplication Table" for a finite group is a way to express the result of all the compositions for this group.
> - Example: G is a finite group, $G = \{1, g_1, g_2, \ldots, g_{n-1}\}$
>
> 
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>
> - In particular, for $G = \{\pm 1\}$ we just defined,
>
> | | +1 | -1 |
> |---|---|---|
> | +1 | +1 | -1 |
> | -1 | -1 | +1 |

> - The trivial group,
>
> | | 1 |
> |---|---|
> | 1 | 1 |

2. <u>Klein</u> Four Group

> Named after the mathematician Klein

- $K_4 = \{1, a, b, c\}$
- Law of composition is given by the table:

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

- Note:
  - $|K_4| = 4$
  - $\forall x \in K_4,\ \ x^{-1} = x$
- Associativity: It was checked that $(xy)z = x(yz),\ \forall x, y, z \in K_4$.
  - $4^3 = 64$ cases
- K_4 is an abelian group.

  > A finite group G is Abelian $\iff$ its multiplication table is symmetric along diagonal

3. Symmetric Groups S_n

> First group in math history

- <u>Recall:</u>
  - $X = \{1, 2, 3, \ldots, n\}$.
  - S_n is the groups of all bijections (permutations) on X.
  - with function composition
- Problem: We don't have a good way to describe the elements (functions)
  - We can use cycle

Cycle

- ## Definition

  - A <u>cycle</u> $(a_1\ a_2\ \dots\ a_k)$, where $a_1, a_2, \dots a_k$ are distinct elements in $\{1, 2, \dots, n\}$, is an element in S_n that sends $a_1 \mapsto a_2, a_2 \mapsto a_3$, ... , $a_{k-1} \mapsto a_k,\ a_k \mapsto a_{k+1}$, while fixing the remaining numbers.
  - A cycle $(a_1\ a_2\ \dots\ a_k)$ is called a <u>k-cycle</u>

- ## Example

  - $\sigma = (1 \quad 2 \quad 3) \in S_5$
  - $\sigma : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$
  - $\sigma(1) = 2,\ \sigma(2) = 3,\ \sigma(3) = 1$

> <u>NOTE</u>: $(1 \quad 2 \quad 3) = (2 \quad 3 \quad 1) = (3 \quad 1 \quad 2)$

Not every elements are cycles as the permutations do not always work in a cyclic way

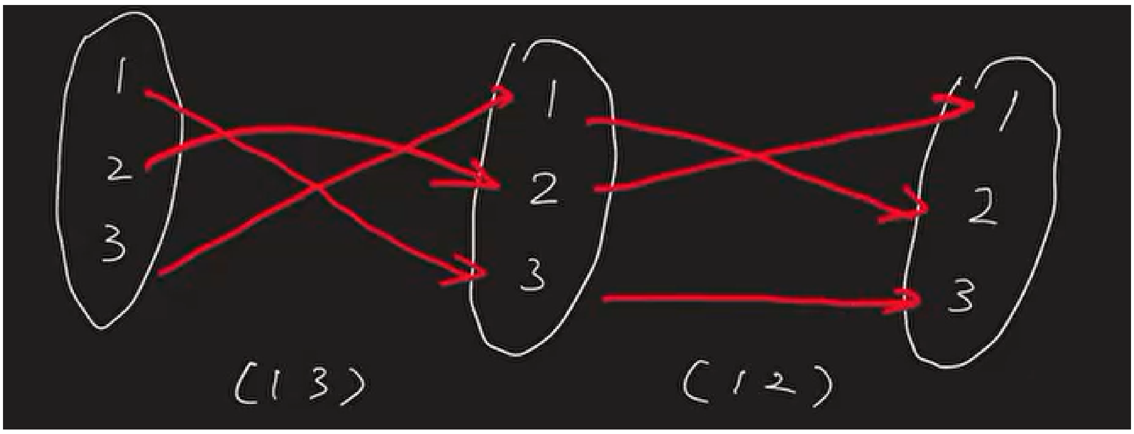
But cycle is enough to describe small groups

- $S_1 = \{id\}$
- $S_2 = \{id, (1\ 2)\}$
- $S_3 = \{id, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$
- $|S_n| = n!$

- ## Compute

  - $(1\ 2)(1\ 3)\ =?$

    > Each of the cycles is a bijective function. Here, we are composing a pair of functions
    >
    > $f_1 \circ f_2(x) = f_1(f_2(x))$, so $(1\ 3)$ should be calculated first

    $f(1) = 3, f(2) = 1, f(3) = 2$

    Therefore, $(1\ 2)(1\ 3) = (1\ 3\ 2)$

  - Fun facts

    - $(1\ 2)^2 = (1\ 3)^2 = (2\ 3)^2 = id$
    - $(1\ 2\ 3)(1\ 3\ 2) = id$

- S_3 is non-Abelian, and it's the smallest non-Abelian group
- Note: not all permutations are cycles.
  - For example, in S_4:
  - $\sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 4, \sigma(4) = 3,$
  - We can write it as the composition of two cycles
  - $\sigma = (1\ 2)(3\ 4)$

Propositions

1. (Will prove later) Each element of S_n can be written as a product of disjoint cycles, in a unique way up to reordering of these cycles
   - Disjoint cycles: $(a_1 \dots a_k)(b_1 \dots b_m)$ are disjoint if $a_1 \dots a_k, b_1 \dots b_m$ are all distinct numbers
2. Two disjoint cycles commute
   - $(a_1 \dots a_k)(b_1 \dots b_m) = (b_1 \dots b_m)(a_1 \dots a_k)$

Exercise

- List all the 24 elements in S_4, ad product of disjoint cycles

Subgroups

Wish

- Given a group G, we want to study a subset of G that is also a group with the same composition as that of G.

Definition

- A subset of a group G, $H \subseteq G$, is a **subgroup** of G if:
  1. Closure: $\forall h_1, h_2 \in H,\ \ h_1 h_2 \in H$
  2. Identity: $1 \in H$
     - It's the same identity as in G
  3. Inverse: $\forall h \in H, h^{-1} \in H$
- In other words, it's a smaller subset lives inside of G and use the same composition, and it's still a group

Examples

- For any G, it has two "uninteresting" subgroups,
  - G
  - $\{1\}$
  - They coincide when $G = \{1\}$

- $G = (\mathbb{R}, +)$, then $H = \mathbb{Z}$ is a subgroup
    - $\forall k_1, k_2 \in \mathbb{Z}, k_1 + k_2 \in \mathbb{Z}$
    - $0 \in \mathbb{Z}$
    - $\forall k \in \mathbb{Z}, -k \in \mathbb{Z}$
- For $S_n, \quad H = \{\sigma \in S_n | \sigma(n) = n\}$
    - H consists of all the functions that fixes the last element n
    - Then H is a subgroup of S_n
    - Actually, H can be regarded as a copy of S_{n-1}

Proposition

- H is a nonempty subset of a group G satisfying that $\forall a, b \in H, a^{-1}b \in H.$ Then H is a subgroup of G.

Proof:

- $H \neq \emptyset, \exists h \in H.$ then $1 = h^{-1}h \in H \ (a = h, b = h)$
- $\forall h \in H, h^{-1} = h^{-1}1 \in H \ (a = h, b = 1)$
- $\forall h_1, h_2 \in H, h_1h_2 = (h_1^{-1})^{-1}h_2 \in H \ (a = h_1^{-1}, b = h_2)$

Example

- Recall: We defined $GL_n(\mathbb{R})$ - General Linear Group
    - $GL_n(\mathbb{R})$ consists of all $n \times n$ invertible real-matrices.
    - Composition is multiplication
- Let $SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) | \ det(A) = 1\}$
- Then $SL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$ called Special Linear Group
- Proof:
    - $\forall A, b \in SL_n(\mathbb{R}),$
    - $det(A^{-1}B) = det(A^{-1})det(B) = det(A)^{-1}det(B) = 1$

      $\Rightarrow A^{-1}B \in SL_n(\mathbb{R})$
    - So by the prop., $SL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$