Class 2022/09/13

Useful Identities

1. $\binom{n}{r} = \binom{n}{n-r}$

2. $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ for $1 \leq r \leq n$

Combinatorial Proof

LHS: $\binom{n}{r}$ is all unordered groups of size r from n distinct objects.

RHS: $\binom{n-1}{r-1}$ is all groups of size r that contain "1"

$\binom{n-1}{r}$ is all groups of size r that don't contain "1"
They add up to be all groups of size r

- ## Binomial Theorem

$$P(n): \quad (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^n$$

for all $n \geq 1 (n \in \mathbb{R})$

- ### Example:

$$\begin{aligned}(x+y)^2 &= x^2 + 2xy + y^2 \\ &= \binom{2}{0} x^2 y^0 + \binom{2}{1} x^1 y^1 + \binom{2}{2} x^0 y^2 \\ (x+y)^3 &= \sum_{k=0}^{3} \binom{3}{k} x^{3-k} y^k \\ &= \binom{3}{0} x^3 y^0 + \binom{3}{1} x^2 y^1 + \binom{3}{2} x^1 y^2 + \binom{3}{3} x^0 y^3 \\ &= x^3 + 3x^2 y + 3xy^2 + y^3\end{aligned}$$

- ### Proof by Induction

1. Base Case: $n = 1$

$$\begin{aligned}P(1): \quad (x+y)^1 &= \sum_{k=0}^{1} \binom{1}{k} x^k y^{1-k} \\ &= \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0 \\ &= y + x\end{aligned}$$

2. Inductive Hypothesis:

   Assume $P(n-1)$ for some $n > 1, n \in \mathbb{N}$

   That is:

$$(x+y)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k}$$

   Need to show $P(n)$, that is $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

3. Induction Steps:

$$
\begin{aligned}
(x+y)^n &= (x+y)(x+y)^{n-1} \\
&= (x+y)\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k} \\
&= x\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k} + y\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k} \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{k+1} y^{n-1-k} + \sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-k}
\end{aligned}
$$

$$
\text{let } i = k+1, \qquad \text{let } j = k,
$$

$$
k = i-1,
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\binom{n-1}{i-1}x^i y^{n-i} + \sum_{i=0}^{n-1}\binom{n-1}{i}x^i y^{n-i} \\
&= \binom{n-1}{n-1}x^n + \left(\sum_{i=1}^{n-1}\binom{n-1}{i-1}x^i y^{n-i} + \sum_{i=1}^{n-1}\binom{n-1}{i}x^i y^{n-i}\right) + \binom{n-1}{0}y^n \\
&= x^n + \sum_{i=1}^{n-1}\binom{n}{i}x^i y^{n-i} + y^n \\
&= \binom{n}{n}x^n y^0 + \sum_{i=1}^{n-1}\binom{n}{i}x^i y^{n-i} + \binom{n}{0}x^0 y^n \\
&= \sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i}
\end{aligned}
$$

Q.E.D.

> First Quiz on Next Friday, Sept 23 in Recitation via Grade scope

Chapter 2

Set Theory Basics

Set

A set is a unordered collection of elements.

> Standard:
>
> Use capital letters to represent sets,
>
> lowercase to represent elements.

- Ex. $A = \{a, b, c\},\ \ B = \{b, c, d\},\ \ C = \{\{a, b, c\}, c\}$
- We say $a \in A$ if "a" is an element of A.

- $b \in A,\ b \in B,\ b \notin C$

Subset

- We say $B \subseteq A$ (B is a subset of A) if every element in B is also in A.

  $\Rightarrow B \nsubseteq A$ if there's some element in B that is not in A.

- Every set is a subset of itself.
  - $A \subseteq A$
- To conclude $A = B$, you need to show $A \subseteq B$ and $B \subseteq A$.

Empty Set

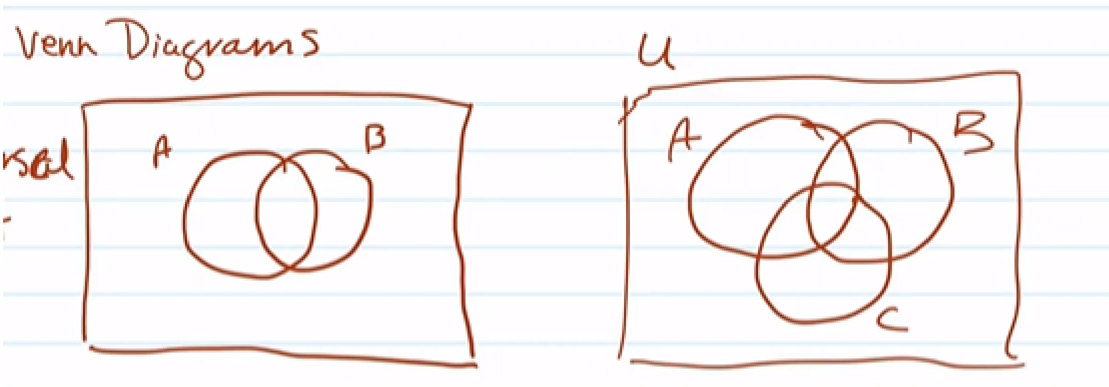
$\emptyset = \{\quad\}$ is the empty set.

- There's no element in the empty set.
  - $\emptyset \notin \emptyset$
- The empty set is a subset of any set.
  - $\emptyset \subseteq \emptyset$. (Every element in $\emptyset$ is also in $\emptyset$)

> Russell's paradox:
>
> The set of all sets that don't contain themselves.

Set Operations

Venn Diagrams



Let U be the Universal Set: U.

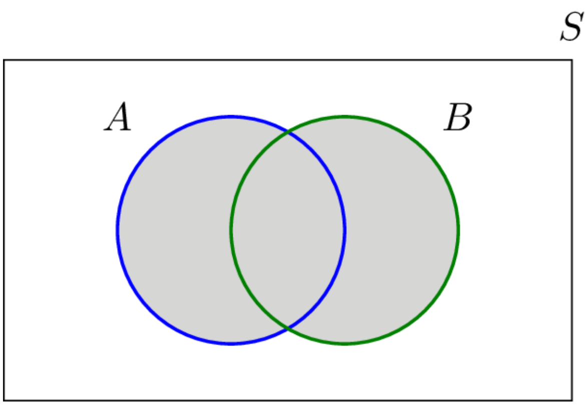
Operations:

- Union: $\cup$,
- Intersection: $\cap$,
- Complement: $A^C = A'$.

- ## Union

$$A \cup B = \{x \in U | x \in A \, or \, x \in B\}$$

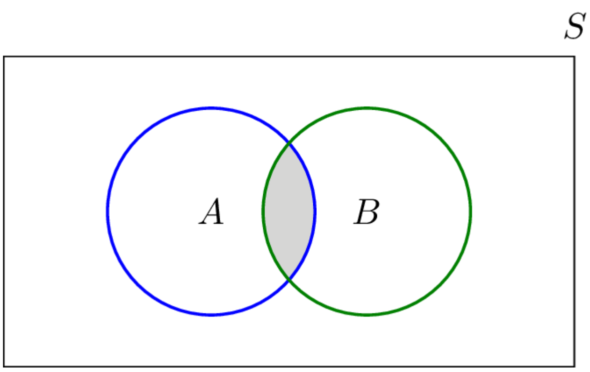
> “
>
> This is an inclusive or, meaning in A or in B or both.



- ## Intersection

$$A \cap B \, (= AB) = \{x \in U | x \in A \, and \, x \in B\}$$

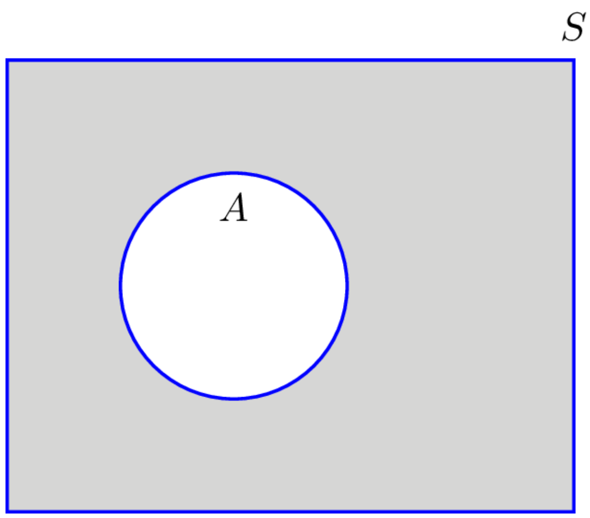
> “
>
> In our book it's usually written as AB.



- ## Complement

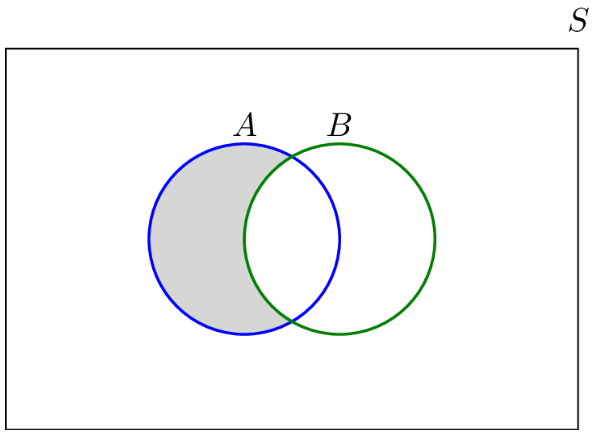
> “
>
> Only makes sense when the universe is clear.

$$A^c = \{x \in U | x \notin A\}$$



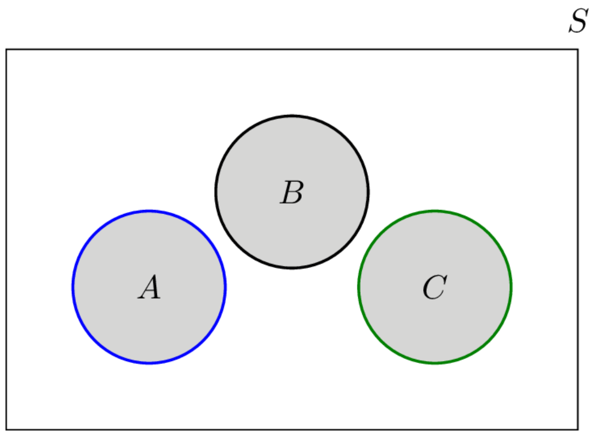
- **Difference**

$$A - B = \{x \in U | x \in A \, and \, x \notin B\} = AB^c$$



- **Disjoint**

<u>**Def**</u>: Two sets are said to be <u>Disjoint</u> if $AB = \emptyset$.



- Express $A \cup B$ as a union of disjoint sets:

$$\begin{aligned} A \cup B &= AB^c \cup AB \cup A^cB \\ &= AB^c \cup B \\ &= A \cup A^cB \end{aligned}$$