Theory of Probability: Class 20221004

Review: Random Variable

- A **function** that assigns a real number to each element in the sample space
- Generates a **PDF** (Probability Distribution Function)
- **DRV / Discrete Random Variable** and **Continuous Random Variable**
  - Discrete: The values that the RV can take are either finite or countably infinite
    - Gaps between values
- DRV has a **PMF**(Probability Mass Function)
  - Notation: $p : X \to [0, 1]$
  - $0 \leq p(x) \leq 1 \; \forall x \in X$
  - $\sum_{x \in X} p(x) = 1$
  - $p(a) = P(X = a)$

The Cummulative Distribution Function (CDF)

- **Notation**
  - $F(x) : \mathbb{R} \to [0, 1]$
  - $F(a) = P(X \le a) = \sum_{x \le a} p(x)$

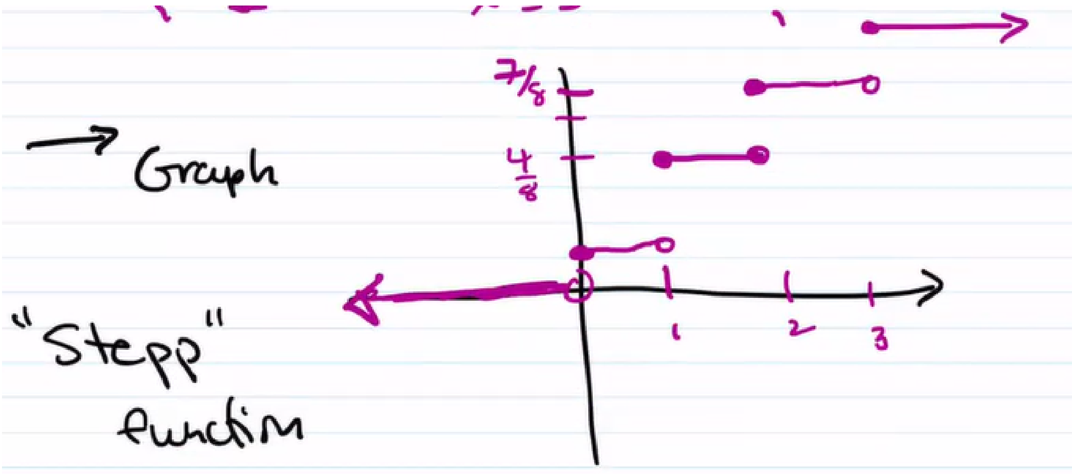
- **Examples**

1. Consider our Alpha Experiment (Flip 3 coins)
   - X is RV that counts number of heads
   - $X : S \to \{0, 1, 2, 3\}$
   -

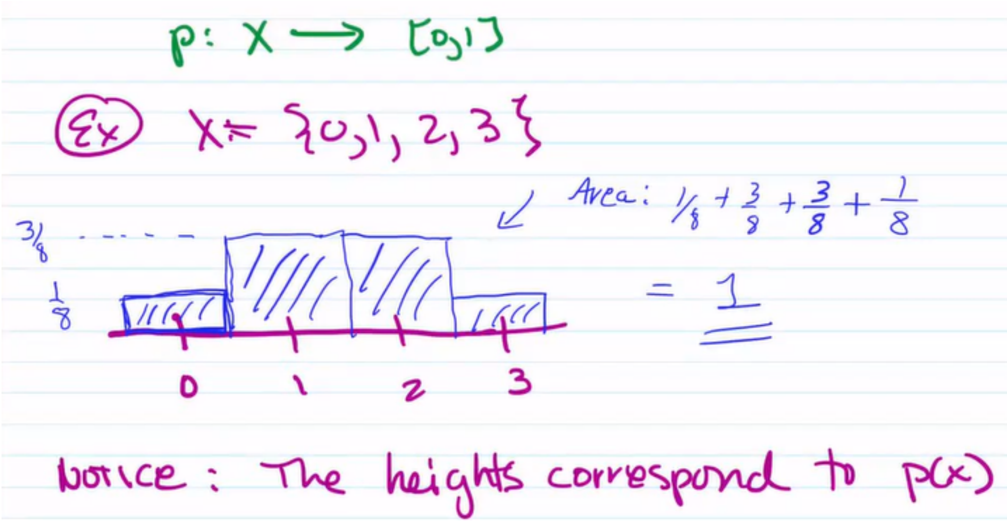
X	$P(x)$ (pmf)
0	$p(0) = P(X = 0) = 1/8$
1	$p(1) = P(X = 1) = 3/8$
2	$p(2) = P(X = 2) = 3/8$
3	$p(3) = P(X = 3) = 1/8$
	Total = 1

$$F(x) = \begin{cases} 0, & x < 0 \\ 1/8, & 0 \le x < 1 \\ 4/8, & 1 \le x < 2 \\ 7/8, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

   - **Graph**: a "Stepp" function



> - **Sketch the pmf DRV (Histogram)**



- Note that the domain of the functions are all different:

  Experiment $\rightarrow S$

  $X : S \rightarrow \mathbb{R}$

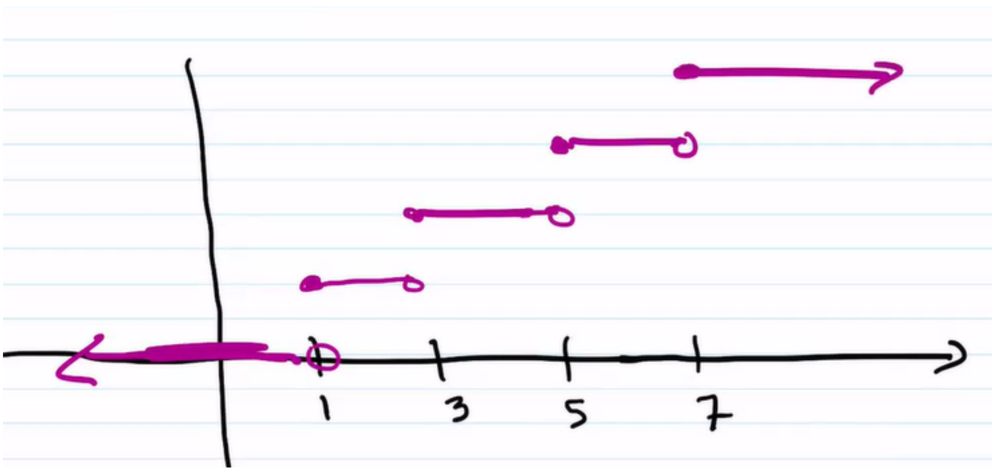
  $p(x) : \ X \rightarrow [0, 1]$

  $F(X) : \mathbb{R} \rightarrow [0, 1]$

2. T is a **DRV** which represents whole years
   - $T = \{1, 3, 5, 7\}$
   - Define **CDF**:

$$F(t) = \begin{cases} 0, & t < 1 \\ 1/4, & 1 \leq t < 3 \\ 1/2, & 3 \leq t < 5 \\ 3/4, & 5 \leq t < 7 \\ 1, & t \geq 7 \end{cases}$$

1. Sketch the cdf



2. Define the pmf

T	$P(t)$
1	$1/4$
3	$1/4$
5	$1/4$
7	$1/4$

3. $P(T=5)$
    - $=1/4$
4. $P(T>3)$
    - $=1/2$
5. $P(T\leq 5|T\geq 2)$
    - $=\dfrac{P(2<T\leq 5)}{P(T<2)}=\dfrac{P(3\ or\ 5)}{P(3\ or\ 5\ or\ 7)}=\dfrac{1/2}{3/4}=\dfrac{2}{3}$
6. $P(T=5.5)$
    - Does not make sense.
    - Not in the domain of the function
7. $F(5.5)$
    - $=3/4$

3. A shipment of 7 tvs that contains 2 defectives arrive. You purchase 3 of the tvs randomly.
    - Define X to be the RV that counts the # of defects in the purchase.

      $X=\{0,1,2\}$ <------ DRV!

    1. Find the **pmf**

    $$P(X=i)=\frac{\binom{2}{i}\binom{5}{3-i}}{\binom{7}{3}}$$

    > Hypergeometric Distribution!

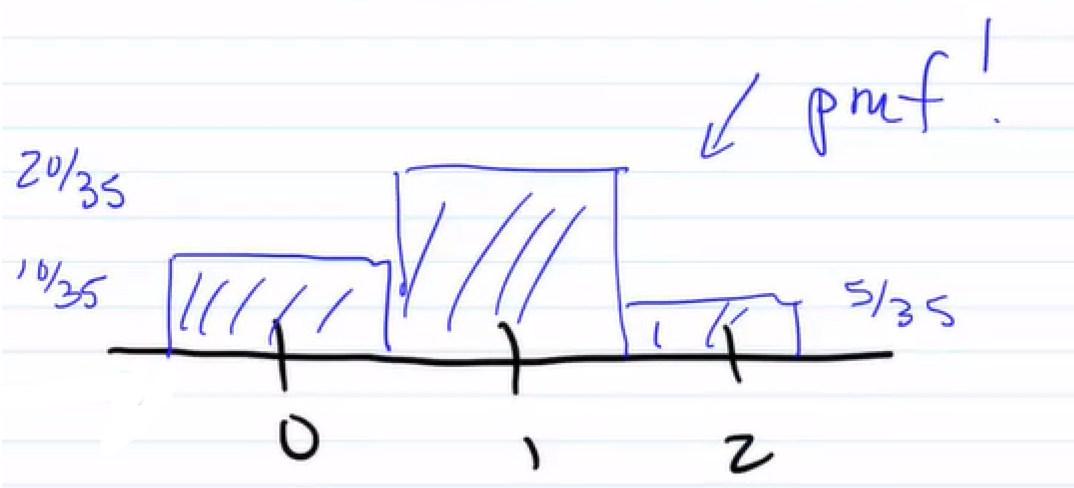
X	$P(x)$
0	$2/7$
1	$4/7$
2	$1/7$

2. Define the **cdf**

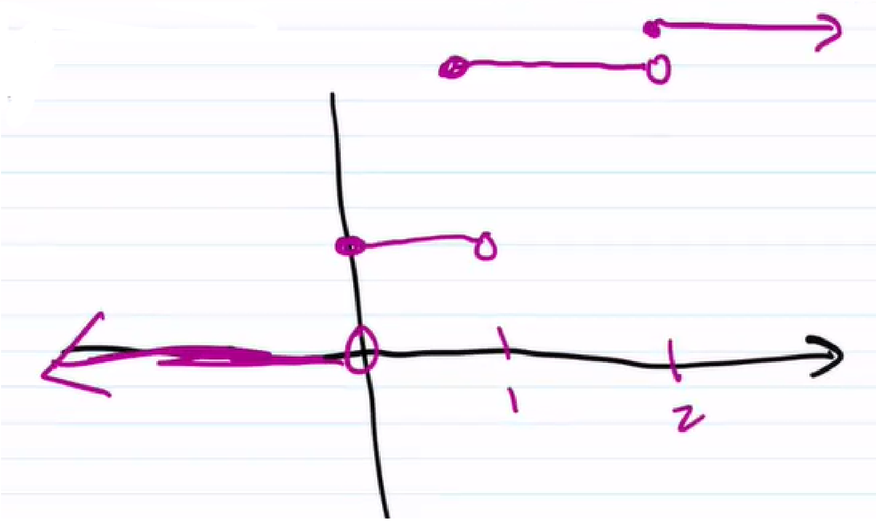
$$P(x)=\begin{cases}0, & x<0\\ 2/7, & 0\leq x<1\\ 6/7, & 1\leq x<2\\ 1, & x\geq 2\end{cases}$$

3. Sketch both

PMF:



CDF:

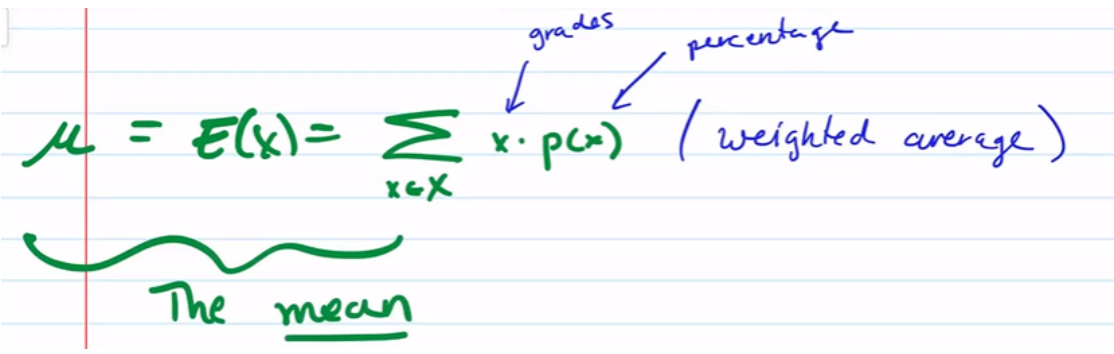


Expectation

- **Definition**

The Expectation of **DRV**, X:

$$\mu = E(X) = \sum_{x \in X} x \cdot p(x)$$



Example

1. Alpha Experiment

X	$P(x)$	$x \cdot p(x)$
0	$1/8$	0
1	$3/8$	$3/8$
2	$3/8$	$6/8$
3	$1/8$	$3/8$

$E(X) = 3/8 + 6/8 + 3/8 = 12/8 = 1.5$

> "
>
> 1.5 heads doesn't make sense
>
> What it's actually saying is that the expected value is between 1 and 2,
>
> and it's as likely to be 1 as it's likely to be 2.

2. We go to Atlantic City to play some cards

   Player pays p dollars to play game

   - Game: one card is flipped over
   - Player wins $\$3$ if the card is J or Q
   - Player wins $\$5$ if the card is A or K
   - Player wins 0 if otherwise

   Question: How much should the player pay to play for the game to be fair?

   In other words, $E(X) = 0$?

   Answer:

   Let X represent the profit earned in a given game $[Won - Paid]$

   $X = \{3 - p,\ 5 - p,\ 0 - p\}$

   $P(x) = \{2/13,\ 2/13,\ 9/13\}$

   $E(X) = \frac{2}{13}(3 - p) + \frac{2}{13}(5 - p) + \frac{9}{13}(-p) = \frac{16}{13} - p = 0$

   $\Rightarrow p = 16/13 \approx 1.23$

Law Of The Unconscious Statistician (LOTUS)

Note that we have the formula of expectation of x,

$$E(X) = \sum_x x \cdot p(x)$$

LOTUS gives us the expectation of a function of x,

$$
\begin{aligned}
E(g(X)) &= \sum_x g(x) \cdot p(x) \\
&\neq \sum_x g(x) \cdot p(g(x))
\end{aligned}
$$

x	$g(x)$	$p(x)$
x_1	$g(x_1)$	$p(x_1) \neq p(g(x_1))$
x_2	$g(x_2)$	$p(x_2)$
...	...	...

- Alpha Experiment as Example:
- $E(X) = \sum_x xp(x) = \frac{1}{8}(0) + \frac{3}{8}(1) + \frac{3}{8}(2) + \frac{1}{8}(3) = 1.5$
- $E(X^2) = \sum_x x^2 p(x) = \frac{1}{8}(0)^2 + \frac{3}{8}(1)^2 + \frac{3}{8}(2)^2 + \frac{1}{8}(3)^2 = 3$

Linear Operator

- Linear Operator:
  - $f(ax+b) = f(ax) + f(b) = af(x) + f(b)$
- Non-Linear Operator:
  - $\sin(2x) = 2 \cdot \sin(x) \cdot \cos(x) \neq 2 \cdot sin(x)$
  - $\sin(x+y) \neq \sin(x) + \sin(y)$

The Expectation is LINEAR

- $E(aX) = aE(X)$
- $E(X+b) = E(X) + b$
- $E(aX+b) = aE(X) + b$

Proof

- $E(aX) = \sum_{x \in X} (ax) \cdot p(x) = a \sum_{x \in X} x \cdot p(x) = aE(X)$
- $E(X+b) = \sum_{x \in X} (x+b)p(x) = \sum_{x \in X} xp(x) + b \sum p(x) = E(X) + b$
- $E(ax+b) = \sum_{x \in X} (ax+b)p(x) = \sum_{x \in X} axp(x) + b \sum p(x) = aE(X) + b$
- (Note that $\sum p(x) = 1$)

Remark

- Given DRVs $X_1, X_2, X_3, \ldots, X_n$
- Let DRV $Y = \sum_{i=1}^{n} X_i$
- Then $E(Y) = E(\sum_{i=1}^{n} X_i) = E(X_1) + E(X_2) + \ldots + E(X_n)$