Class: 2022/10/24

Recall: define $sgn: S_n \longrightarrow \{\pm 1\}$

$\forall \sigma \in S_n$. define

$$sgn(\sigma) = \det\begin{bmatrix} e_{\sigma(1)} \cdots\cdots e_{\sigma(n)} \end{bmatrix}.$$

we've verified sgn is a homomorphism.

Note that if $\sigma = (i\ j)$ is a 2-cycle, then $sgn(\sigma) = -1$. since the matrix is obtained from identity matrix by switching i-th & j-th column.

when $n \geq 2$. S_n contains 2-cycles, so sgn is surjective

Denote $A_n = \ker(sgn)$. called the alternating group.

By the First Isomorphism Theorem.

$$S_n / A_n \cong \{\pm 1\}.$$

In particular. $\frac{|S_n|}{|A_n|} = 2$. $|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$

A_n is a normal subgroup of S_n. (it's the kernel of sgn)

$$A_n = \{\sigma \in S_n \mid sgn(\sigma) = +1\}$$

- If $sgn(\sigma) = +1$, we call it an even permutation.
- If $sgn(\sigma) = -1$, we call it an odd permutation

How to compute $sgn(\sigma)$?

* We have calculated the sgn. of a 2-cycle, which is -1.
* sgn. is a homomorphism.
* $(a_1\ a_2\ \dots\ a_k) = (a_1\ a_k)(a_1\ a_{k-1})\dots(a_1\ a_2)$
* Each $\sigma \in S_n$ is a product of disjoint cycles.

$$sgn(a_1\ a_2\ \dots\ a_k) = sgn(a_1\ a_k)\cdot sgn(a_1\ a_{k-1})\cdots sgn(a_1\ a_2)$$
$$= (-1)^{k-1}$$

Then, $\sigma = \tau_1\tau_2\cdots\tau_m$, with $\tau_1,\dots,\tau_m$ disjoint cycles.
τ_i is a l_i-cycle

$$sgn(\sigma) = sgn(\tau_1)\, sgn(\tau_2)\cdots sgn(\tau_m)$$
$$= (-1)^{l_1-1}\cdot(-1)^{l_2-1}\cdots(-1)^{l_m-1}$$

Example. $\sigma = (1\ 3\ 5)(2\ 4)(6\ 8\ 9) \in S_{10}$.

$$sgn(\sigma) = (-1)^{3-1}\cdot(-1)^{2-1}\cdot(-1)^{3-1} = (+1)\cdot(-1)\cdot(+1) = -1.$$

Sgn(σ) is also called the parity of σ:

σ is a product of 2-cycles.

sgn(σ). tells us, if we write σ as product of 2-cycles.

how many 2-cycles are there in the expression up to odd or even.

If there're even numbers of 2-cycles. $sgn(\sigma) = +1$.

If there're odd numbers of 2-cycles. $sgn(\sigma) = -1$.

Alternating Group $A_n = \{\sigma \in S_n \mid sgn(\sigma) = +1\}$.

- $A_1 = S_1 = \{id\}$.
- when $n \geq 2$, we have discussed that $|A_n| = \frac{|S_n|}{2}$.

$A_2 = \{id\}$. $S_2 = \{id, (1\ 2)\}$.

- $A_3 = \{id, (1\ 2\ 3), (1\ 3\ 2)\}$.
- $A_4 = \{\boxed{id, (12)(34), (1\ 3)(2\ 4), (14)(23)}, (1\ 2\ 3), (1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 2), (1\ 3\ 4), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3)\}$

is a normal subgroup of A_4, which is isomorphic to K_4.

$\forall\ \sigma \in A_4$. $\sigma(1\ 2)(3\ 4)\sigma^{-1} = (\sigma(1)\ \sigma(2))(\sigma(3)\ \sigma(4))$.

[recall: $\sigma(a_1 \ldots a_k)\sigma^{-1} = (\sigma(a_1) \ldots \sigma(a_k))$]

Example 2.5. There are four conjugacy classes in A_4:

$$\{(1)\}, \quad \{(12)(34), (13)(24), (14)(23)\},$$

$$\{(123), (243), (134), (142)\}, \quad \{(132), (234), (143), (124)\}.$$

Notice the 3-cycles (123) and (132) are *not* conjugate in A_4. All 3-cycles in A_4 are conjugate in the larger group S_4, *e.g.*, $(132) = (23)(123)(23)^{-1}$ and the conjugating permutation (23) is not in A_4.

- $\underline{A_5}$:

$\underline{\text{Def}}$. A group is $\underline{\text{simple}}$ if it has no "$\underline{\text{proper}}$" normal subgroups (a subgroup H of G is proper if $H \neq \{1\}$, $H \neq G$)

i.e., A group G is simple if its only normal subgroups are $\{1\}$, G.

By this definition, A_4 is $\underline{\text{not}}$ simple.

$\mathbb{Z}/p\mathbb{Z}$ is simple for p prime.

<u>Thm</u>. A_5 is simple.

<u>pf</u>. If we compute the <u>conjugacy classes</u> of A_5 :

Note: elements of same cycle type in A_5 may not be conjugates.

More generally, H is a subgroup of G.
x and y are conjugates in G doesn't imply they're conjugates in H.

conjugacy classes in A_5:

id	$(1\ 2\ 3\ 4\ 5)$	$(2\ 1\ 3\ 4\ 5)$	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$
$a_1 = 1$	$a_2 = 12$	$a_3 = 12$	$a_4 = 15$	$a_5 = 20$

$60 = 1 + 12 + 12 + 15 + 20$

if H is a normal subgroup of A_5. then
$|H|$ should be a sum of some of $a_1, \dots, a_5$. ($a_1 = 1$ has to be in the sum.)
but the only way to make $|H|$ a divisor of $|A_5| = 60$ is $|H| = 1$ or 60. so H is not proper.
we conclude A_5 is simple.

<u>Thm</u>. A_n is simple for $n \geq 5$.

Pf. (Dummit & Foote). By Induction.

- we've proved A_5 is simple.
- Assume A_{n-1} is simple. Consider A_n. $(n \geq 6)$

Suppose H is a proper normal subgroup of A_n.

For each $1 \leq i \leq n$, let $G_i = \{\sigma \in A_n \mid \sigma(i) = i\} \cong A_{n-1}$.
which is simple by induction.

$H \cap G_i$ is a normal subgroup of G_i.

so: $\underline{H \cap G_i = G_i}$ or $\underline{H \cap G_i = \{id\}}$. (because G_i is simple)

If $H \cap G_i = G_i$ for some $1 \le i \le n$. it means $H \ge G_i$.

Observe : $\forall\, \sigma \in A_n.\ \sigma G_i \sigma^{-1} = G_{\sigma(i)}$

H is normal in A_n. so $G_{\sigma(i)} = \sigma G_i \sigma^{-1} \le H$

For any $\tau \in A_n$, we can write $\tau = \tau_1 \tau_2 \cdots \tau_{k-1} \tau_k$

as a product of even numbers of 2-cycles.

$\tau_1 = (a_1\ a_2)$, $\tau_2 = (a_3\ a_4)$. Note $\tau_1\tau_2 = (a_1\ a_2)(a_3\ a_4)$

so $\tau_1\tau_2$ fixes $a \notin \{a_1, a_2, a_3\ a_4\}$. so $\tau_1\tau_2 \in G_a$

similarly, $\tau_3\tau_4 \in G_i$ for some i.

$\tau_5\tau_6 \in G_i$ for some i, ...

we get. τ is a product of elements from $\{G_i\}$.

each $G_i \le H$. so τ is a product of elements in H.

it means $A_n \le H$. contradict to H proper.

so $H \cap G_i = G_i$ is impossible. it has to be $H \cap G_i = \{id\}$ for all $1 \le i \le n$.

$H \cap G_i = \{id\}$ means id is the only element of H that fixes i.

so for any $\tau \in H$, $\tau \ne id$. $\tau(i) \ne i$ $\forall\ 1 \le i \le n$.

this implies $\boxed{\tau_1 \ne \tau_2 \text{ in } H,\ \tau_1(i) \ne \tau_2(i),\ \forall\ 1 \le i \le n}$. (*)

(otherwise. $\tau_1(i) = \tau_2(i)$ for some $i \Rightarrow \tau_2^{-1} \cdot \tau_1(i) = i \Rightarrow \tau_2^{-1} \cdot \tau_1 = id$ $\Rightarrow \tau_1 = \tau_2$.)

Now for $\sigma \in H$, $\sigma \ne id$. write σ as a product of disjoint cycles.

① If the product contains a cycle of length at least 3:

$\sigma = (a_1\ a_2\ a_3\ \dots)(\dots)(\dots)$

we can find $\eta \in A_n$. $\eta(a_1) = a_1$. $\eta(a_2) = a_2$. $\eta(a_3) \neq a_3$.

$\sigma' = \eta\sigma\eta' = (\eta(a_1)\ \eta(a_2)\ \eta(a_3)\ \dots)(\dots)(\dots)$

$= (a_1\ a_2\ \eta(a_3)\ \dots)(\dots)(\dots)$.

σ, σ' are different, since $\sigma(a_2) = a_3$. $\sigma'(a_2) = \eta(a_3)$.

But $\sigma(a_1) = a_2 = \sigma'(a_1)$, contradict to (*)

② If the product consists of only 2-cycles.

$\sigma = (a_1\ a_2)(a_3\ a_4)(a_5\ a_6)\dots$

we can let $\eta = (a_1\ a_2)(a_3\ a_5)$.

$\sigma' = \eta\sigma\eta' = (a_1\ a_2)(a_3\ a_6)(a_4\ a_5)$.

$\sigma(a_3) = a_4$. $\sigma'(a_3) = a_6$. so $\sigma \neq \sigma'$.

But. $\sigma(a_1) = a_2 = \sigma'(a_1)$. contradict to (*).

So we cannot find a non-identity σ in H. contradiction.

We conclude H cannot be a proper normal subgroup of A_n.

In the lecture notes, there's a more computational proof.

Idea is: show for $n \geq 5$
- A_n is generated by 3-cycles
- all 3-cycles are conjugates in A_n.
- If H is a normal subgroup of A_n. with $H \neq \{id\}$ then H contains at least one 3-cycle.