TOP - 20221018 - Poiss

> This is from a lecture recording from Fall 2020

Approximating Poisson with Binomial

Let $X \sim Bin(n,\ p)$

- pmf: $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$
- $E(X) = np$
- $Var(X) = np(1-p)$

Claim

Claim: If n is "large" enough and p "small" enought such that np is "moderate", then using $\lambda = np$, $X \sim Pois(\lambda)$, that is $X \sim Pois(np)$.

Rules

Rule of Thumb

- $n \geq 20\ \ \&\ \ p \leq 0.05$

  or

- $n \geq 100\ \ \&\ \ np \leq 10$

- **Proof**

  - Given $X \sim Bin(n,\ p)$
  - Let $\lambda = np \quad \Rightarrow\ p = \lambda/n$
  - pmf:

$$
\begin{aligned}
P(X=k) &= \binom{n}{k} p^k (1-p)^{n-k} \\
&= \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \frac{n!}{(n-k)!\,k!} \cdot \frac{\lambda^k}{n^k} \cdot \frac{(1-\lambda/n)^n}{(1-\lambda/n)^k} \\
&= \frac{n!}{(n-k)!\,n^k} \cdot \left(\frac{\lambda^k}{k!}\right) \cdot \frac{(1-\lambda/n)^n}{(1-\lambda/n)^k} \\
&= \frac{n(n-1)\ldots(n-(n-k)+1)}{n \ \cdot \ n \quad \ldots . \quad n} \cdot \frac{\lambda^k}{k!} \cdot \frac{(1-\lambda/n)^n}{(1-\lambda/n)^k}
\end{aligned}
$$

    - Note that $\dfrac{n!}{(n-k)!} = n \cdot (n-1) \cdot (n-2) \ldots (n-(n-k)+1)$
      - has k items
    - Thus $\dfrac{n!}{(n-k)!n^k} = \dfrac{n(n-1)(n-2)\ldots(n-(n-k)+1)}{n \cdot \quad n \ \cdot \quad n \quad \ldots . \quad n}$

  - Now,

$$
\mathrm{Pmf} = \frac{n(n-1)\ldots(n-(n-k)+1)}{n \ \cdot \ n \quad \ldots . \quad n} \cdot \left(\frac{\lambda^k}{k!}\right) \cdot \frac{(1-\lambda/n)^n}{(1-\lambda/n)^k}
$$

  - As $n \to \infty$,
    - $\dfrac{n}{n} = \dfrac{n-1}{n} = \ldots = \dfrac{n-(n-k)+1}{n} = 1$
    - $(1-\lambda/n)^n \ = e^{-\lambda}$

      > “
      >
      > - **Important Limit Expansions of e^x**
      >
      > 1. $$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$
      > 2. $$\lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n = e^{-x}$$

    - $(1-\lambda/n)^k = 1 \quad (\text{as } \lambda/n \to 0)$

  - Therefore,

$$
\begin{aligned}
\mathrm{Pmf}_{n\to\infty} &= 1 \cdot \left(\frac{\lambda^k}{k!}\right) \cdot \frac{e^{-\lambda}}{1} \\
&= \frac{e^{-\lambda} \cdot \lambda^k}{k!} \\
&= P(X=k) \quad \text{ for } X \sim Pois(\lambda)
\end{aligned}
$$

Recap

Let $X \sim Bin(n,\ p)$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \approx \frac{e^{-\lambda}\lambda^k}{k!} = P(X=k) \quad \text{for } X \sim Pois(\lambda = np)$$

- When n is large