Class 20220920

Review

Axiom of Probability

- Let S be the sample space for some experiment and E is any event
- The $P(E)$ assigns a real number to E such that
  - $0 \leq P(E) \leq 1$
  - $P(S) = 1$
  - $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$ if $E_iE_j = \emptyset$ for all $i \neq j$ (Pairwise Disjoint)

Facts

1. $P(E^c) + P(E) = 1$
2. $P(E \cup F) = P(E) + P(F) - P(EF)$
3. $\text{If } E \subset F, \text{ then } P(E) \leq P(F)$

   Proof:

$$
\begin{aligned}
F &= E \cup E^cF && \text{(disjoint sets)} \\
P(F) &= P(E) + P(E^cF) && \text{(Axiom 3)} \\
P(E^cF) &\geq 0 && \text{(Axiom 1)} \\
P(F) &= P(E) + P(E^cF) \\
&\geq P(E)
\end{aligned}
$$

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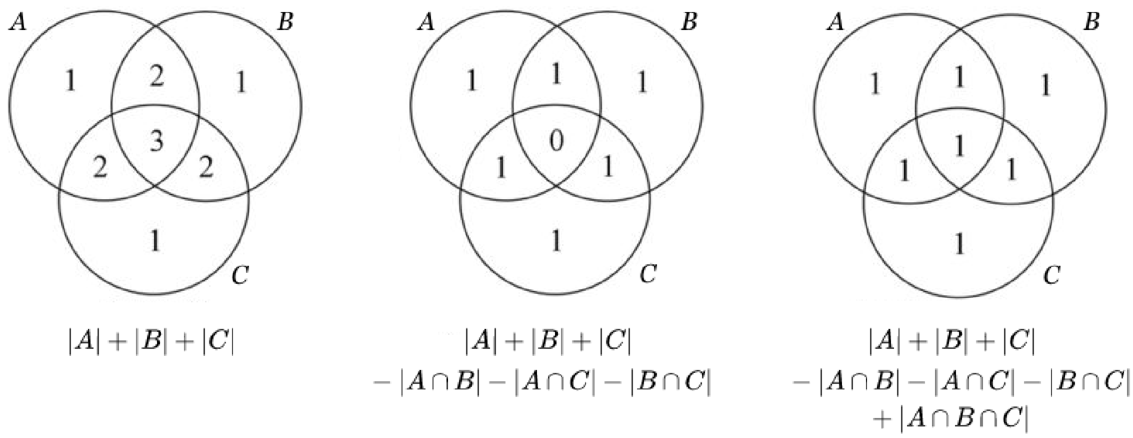
Inclusion / Exclusion

We already know: $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1E_2)$

For three sets:

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1E_2) - P(E_2E_3) - P(E_1E_3) + P(E_1E_2E_3)$$



Again, the number in each subset represents how many times that subset is counted. Here, the overlap of all three subsets must be added back in due to it being removed after the second step.

For 4 sets:

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3 \cup E_4) = \quad & P(E_1) + P(E_2) + P(E_3) + P(E_4) \\ & - P(E_1E_2) - P(E_1E_3) - P(E_1E_4) - P(E_2E_3) - P(E_2E_4) - P(E_3E_4) \\ & + P(E_1E_2E_3) + P(E_1E_3E_4) + P(E_2E_3E_4) \\ & - P(E_1E_2E_3E_4) \end{aligned}$$

- ## Generalization

$$\begin{aligned} P\Big(\bigcup_{i=1}^{n} E_i\Big) = \quad & \sum_{i=1}^{n} P(E_i) - \sum_{i_i<i_2} P(E_{i_1}E_{i_2}) + \sum_{i_i<i_2<i_3} P(E_{i_1}E_{i_2}E_{i_3}) - \dots \\ & + (-1)^{n+1} P(E_{i_1}E_{i_2}\dots E_{i_n}) \\ (\text{Note: } n \geq 1,\ n \in \mathbb{N}) \end{aligned}$$

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>
> We can PROVE it by induction

- ## Example

1. $$\begin{aligned} P\Big(\bigcup_{i=1}^{6} E_i\Big) = \quad & \sum_{i=1}^{n} P(E_i) - \sum_{i_i<i_2} P(E_{i_1}E_{i_2}) + \sum_{i_i<i_2<i_3} P(E_{i_1}E_{i_2}E_{i_3}) \\ & - \sum_{i_i<i_2<i_3<i_4} P(E_{i_1}E_{i_2}E_{i_3}E_{i_4}) + \sum_{i_i<i_2<i_3<i_4<i_5} P(E_{i_1}E_{i_2}E_{i_3}E_{i_4}E_{i_5}) \\ & + P(E_{i_1}E_{i_2}E_{i_3}E_{i_4}E_{i_5}E_{i_6}) \end{aligned}$$
2. Deal a 13 card hand from a deck of 52 cards.

Calculate the probability that it contains the ace and king of at least one suit.

$Let\ A_1 = \spadesuit AK, A_2 = \diamondsuit AK, A_3 = \heartsuit AK, A_4 = \clubsuit AK$

At least one of these events must occur ⇒ Use Union:

$P(A_1 \cup A_2 \cup A_3 \cup A_4)$

$$
\begin{aligned}
P(A_i) &= \frac{\binom{50}{11}}{\binom{52}{13}} \\
P(A_{i_1}A_{i_2}) &= \frac{\binom{48}{9}}{\binom{52}{13}} \\
P(A_{i_1}A_{i_2}A_{i_3}) &= \frac{\binom{46}{7}}{\binom{52}{13}} \\
P(A_{i_1}A_{i_2}A_{i_3}A_{i_4}) &= \frac{\binom{48}{9}}{\binom{52}{13}}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
P(A_1 \cup A_2 \cup A_3 \cup A_4) = \quad & 4P(A_i) - \binom{4}{2} P(A_{i_1}A_{i_2}) \\
& + \binom{4}{3} P(A_{i_1}A_{i_2}A_{i_3}) - P(A_{i_1}A_{i_2}A_{i_3}A_{i_4})
\end{aligned}
$$

Matching Problem

Set-up

N men walk into a room and throw their hats into center. The host mixes them all up and randomly distribute the hats to the men.

Questions:

1. What is the probability no one gets their own hat?

   $P(\text{none}) = 1 - P(\text{none})^c$

2. What is the probability at least one gets his own hat?

   $P(\text{at least one}) = P(\text{none})^c = 1 - P(\text{none})$

They are compliment to each other

Let: $E_i = \text{man } i \text{ gets his own hat}$

Then,

- $P(\text{none}) = 1 - P(E_1 \cup .... \cup E_N)$
- $P(\text{none})^c = P(E_1 \cup .... \cup E_N)$

Also,

$$
\begin{aligned}
&P(E_1) = \frac{|E_i|}{|S|} = \frac{(N-1)!}{N!} = \frac{1}{N} \\
&P(E_1E_2) = \frac{(N-2)!}{N!} \\
&\ldots
\end{aligned}
$$

Therefore, $P(\text{none})^c =$

$$
\begin{aligned}
P(\bigcup_{i=1}^{N} E_i) &= \sum_{i=1}^{N} P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) + \cdots + (-1)^{n+1} P(E_{i_1}E_{i_2}\ldots E_{i_n}) \\
&= N(\frac{1}{N}) - \binom{N}{2}\frac{(N-2)!}{N!} + \binom{N}{3}\frac{(N-3)!}{N!}\ldots \\
&= 1 - \frac{1}{2!} + \frac{1}{3!}\ldots\ldots + (-1)^{n+1}\frac{1}{N!}
\end{aligned}
$$

$$
\begin{aligned}
P(\text{none}) &= 1 - [1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots + (-1)^{n+1}\frac{1}{N!}] \\
&= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^{n}\frac{1}{N!}
\end{aligned}
$$

- ## Expansion of e^x

You should remember:

$$
\begin{aligned}
e^x &= \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2!} + \ldots. + \frac{x^n}{n!} + \ldots. \\
e &= \sum_{i=0}^{\infty} \frac{1}{i!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots\ldots. \\
e^{-1} &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!}\ldots\ldots
\end{aligned}
$$

This is vary important in a lot of probability

- $P(\text{none})$ is **NOT** equal to e^{-1}
- **BUT**, as $N \to \infty, \quad P(\text{none}) \to e^{-1}$