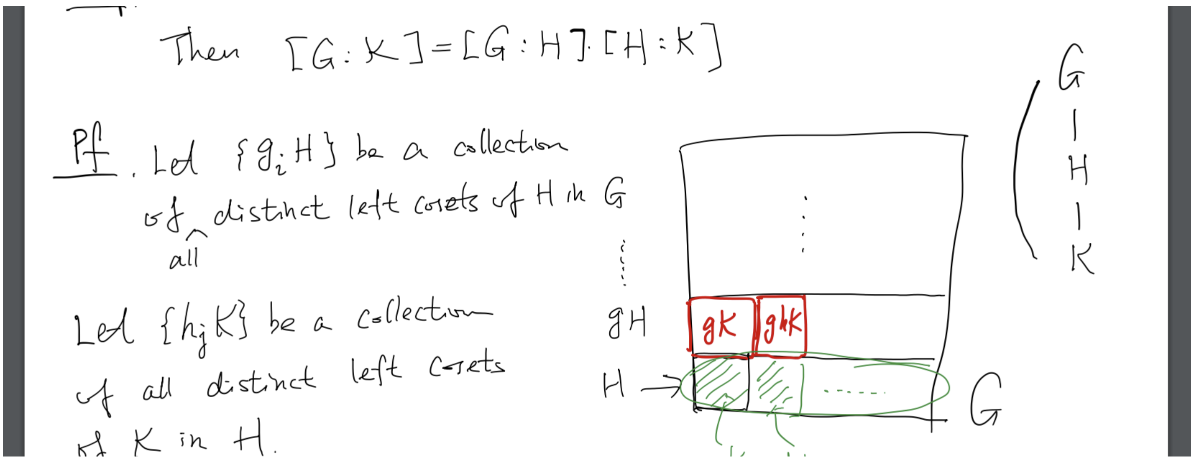
Cosets & Quotient Group 2022/10/05

Cosets

Prop

H is a subgroup of G and K is a subgroup of H. Then $[G:K]=[G:H][H:K]$



Claim

$\{g_i h_i K\}$ is a collection of all distinct left cosets of K in G.

Need to show:

- These cosets are disjoint
- Their disjoint union is G

①: Given $g_{i_1} h_{j_1} K$ and $g_{i_2} h_{j_2} K$

Observe: If $i_1 \neq i_2$. $g_{i_1} h_{j_1} K \subseteq g_{i_1} H$ ← disjoint.

$g_{i_2} h_{j_2} K \subseteq g_{i_2} H$ ←

so $g_{i_1} h_{j_1} K \cap g_{i_2} h_{j_2} K = \varnothing$.

If $i_1 = i_2$. $j_1 \neq j_2$: $h_{j_1} K \cap h_{j_2} K = \varnothing$

$\Downarrow$

$G \longrightarrow G$ is a bijection
$x \longmapsto g_{i_1} x$

$g_{i_1} h_{j_1} K \cap g_{i_1} h_{j_2} K = \varnothing$

so $g_{i_1} h_{j_1} K = g_{i_2} h_{j_2} K$ iff $i_1 = i_2$ and $j_1 = j_2$.

so these left cosets $\{g_i h_j K\}$ are all distinct.

②. We know $\bigsqcup g_i h_j K \subseteq G$, so we need to show $G \subseteq \bigsqcup g_i h_j K$.

$\forall g \in G$. $g \in g_i H$ for some i. since $G = \bigsqcup g_i H$

$g = g_i h$ for some $h \in H$. $h \in h_j K$ for some j.

since $H = \bigsqcup h_j K$.

$\Rightarrow g = g_i h \in g_i h_j K \subseteq \bigsqcup g_i h_j K$.

by ①,②

So the number of distinct left cosets of K in G ($[G:K]$)

is $\underbrace{(\text{the number of choice for } i)}_{[G:H]}$ times $\underbrace{(\text{the number of choice for } j)}_{[H:K]}$

Quotient Group

Motivation

- When will we have $gH = Hg \quad \forall g \in G$?
- Why do we need that?
  - We wish to define a group srtucture on the quotient space. (The set of left cosets)
  - $aH \cdot bH = \ ?$
  - We want $aH \cdot bH = abH$
    - If $bH = Hb$, then we can use $aH \cdot bH = abHH = abH$
- Observe:
  - $\forall g \in G,\ gH = Hg \iff \forall g \in G,\ gHg^{-1} = H$

$$\iff H \lhd G$$

- **Definition**

G is a group. N is a normal subgroup of G. We define the **quotient group** of G by N to be the set of all cosets of N in G, with composition given by $aN \cdot bN = abN$.

The quotient group is denoted by G/N.

Note : $|G/N| = [G:N]$. In particular, if $|G| < \infty$, $|G/N| = \dfrac{|G|}{|N|}$

- **Examples**

1. $K_4 = \{1, a, b, c\}$. $N = \{1, a\} =< a >$

   $K_4/N = \{N, bN\} =< bN >$. a cyclic group of order 2 (Note $N = aN$, $bN = cN$)

   Note : In G/N, the identity is N. Since $N \cdot aN = aN \cdot N = aN$

Examples. ① $K_4 = \{1, a, b, c\}$. $N = \{1, a\} = \langle a \rangle$.
(abelian)

Note: $N = aN$. $bN = cN$.

$\to K_4/N = \{N, bN\}$
$= \langle bN \rangle$. a cyclic group of order 2.

Note: In G/N. the identity is N. since $\begin{cases} N.aN = (1a)N = aN \\ aN.N = (a1)N = aN. \end{cases}$

② $S_3 = \{id, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$.

$H = \langle (1\ 2\ 3) \rangle = \{id, (1\ 2\ 3), (1\ 3\ 2)\}$.

Exercise. If $[G:H]=2$, then H is a normal subgroup of G.

$S_3/H = \{\underline{H}, \underline{(1\,2)H}\}$. $\quad H=(1\,2\,3)H=(1\,3\,2)H$

$(1\,2)H=(1\,3)H=(2\,3)H$.

$= \langle (1\,2)H \rangle$

cyclic group of order 2.

③. $(\mathbb{R},+)$. Consider the subgroup $\mathbb{Z}$.

In $\mathbb{R}/\mathbb{Z}$, $a+\mathbb{Z} = b+\mathbb{Z} \Leftrightarrow a-b \in \mathbb{Z}$

so: all the elements in $\mathbb{R}/\mathbb{Z}$ are uniquely represented on $[0,1]$, except $0+\mathbb{Z}=1+\mathbb{Z}$.

so: all the elements in $\mathbb{R}/\mathbb{Z}$ are uniquely represented on $[0,1]$, except $0+\mathbb{Z}=1+\mathbb{Z}$.

$\Downarrow$

$\mathbb{R}/\mathbb{Z} =$  $\leftarrow$ glue 0 and 1 to form a circle.

(since 0 and 1 represent the same coset).

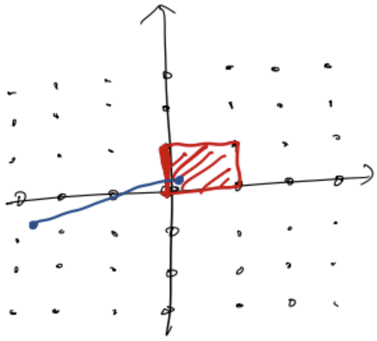
Algebraically, it can be described as the multiplicative group of complex numbers of norm 1.

(It's the unit circle on the complex plane).

$$\underline{e^{i\theta_1}} \cdot \underline{e^{i\theta_2}} = e^{i(\theta_1+\theta_2)}$$

Generalisation. $\mathbb{R}^2$: $(x_1, y_1)+(x_2, y_2)=(x_1+x_2, y_1+y_2)$.
is a group.

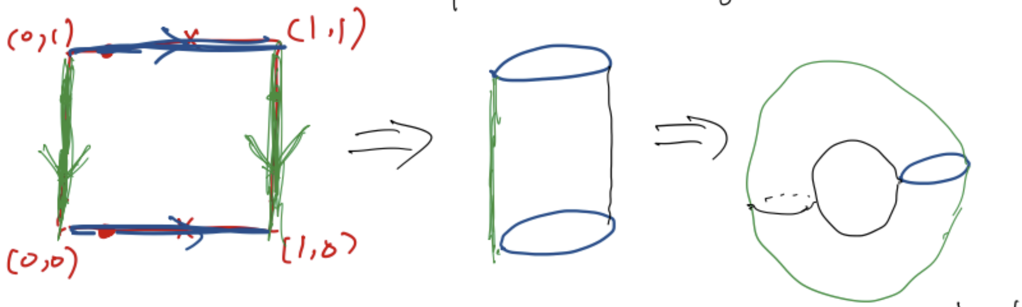
$\mathbb{Z}^2 = \{(x,y)\in\mathbb{R}^2 \mid x\in\mathbb{Z}, y\in\mathbb{Z}\}$.



$\mathbb{R}^2/\mathbb{Z}^2$. $\quad (x,y)+\mathbb{Z}^2=(x',y')+\mathbb{Z}^2$

$\iff \begin{cases} x-x'\in\mathbb{Z}. \\ y-y'\in\mathbb{Z}. \end{cases}$

Every coset has a unique representative in the unit square $[0,1]\times[0,1]$.
except the boundary of the square



called a "solid torus"

TORUS = $S^1\times S^1$

donut is called a "solid torus".

TORUS = $S^1\times S^1$
(surface of a donut)

Klein Bottle :

This bottle doesn't have inside or outside.

Topology

Remark. • A topological space with a group structure on it is called a topological group.

• If a smooth surface (manifold) has a group structure satisfying certain conditions, it's called a Lie group.

- **Group of $\mathbb{Z}/n\mathbb{Z}$**

Next example will be the study of $\mathbb{Z}/n\mathbb{Z}$. $(n \geq 2)$.

$n=0$. $\mathbb{Z}/\{0\} = \mathbb{Z}$.

$n=1$ $\mathbb{Z}/\mathbb{Z}$ is a tricial group.

Denote $\bar{k} = k + n\mathbb{Z}$. Then in $\mathbb{Z}/n\mathbb{Z}$, the compositim is

$$\bar{k} + \bar{l} = \overline{k+l}.$$

The identity element is $\bar{0}$.

$\mathbb{Z}/n\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{n-1}\}$. since $\bar{a} = \bar{b} \Leftrightarrow a-b \in n\mathbb{Z}$
$\Leftrightarrow n \mid a-b$

$|\mathbb{Z}/n\mathbb{Z}| = n$.

$= \langle \bar{1} \rangle$

Cyclic group of order n.