Class - 2022/09/29

Conditional Probability

Independence

> Recall: Two events A, B are mutually exclusive if $A \cap B = \emptyset \to P(AB) = 0$ (They cannot occur together).

Two events A, B are independent if $P(A|B) = P(A)$ and $P(B|A) = P(B)$

That is, they have no effect on each o ther

The formula can also be written as:

$$P(A|B) = P(A) = \frac{P(AB)}{P(B)}$$
$$P(A)P(B) = P(AB)$$

Proposition

- Assume $A\&B$ are independent, show $A\&B^c$ are also independent .

Proof

- $A = AB \cup AB^c$ (They are Disjoint = Mutually Exclusive)
- $P(A) = P(AB \cup AB^c) = P(AB) + P(AB^c)$
- $P(A) = P(A)P(B) + P(AB^c)$
- $P(AB^c) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c)$
- $P(A)P(B^c) = P(AB^c)$
- Therefore, $A\&B^c$ are also independent.

Example

Suppose that chance Lannie earns an A in the course is 40% and the chance that Lannie gets an A in Analysis is 70%. If these events are independent, find the probability that Lannie:

1. Gets A in both => $P(AP) = 0.28$
2. Gets A in neither => $P(A'P') = 0.18$
3. Gets exactly one A => $P(AP') + P(A'P) = 0.42 + 0.12 = 0.54$

	P	P'	Total
A	P(AP) = 0.7*0.4 = 0.28	P(AP') = 0.7*0.6 = 0.42	0.7
A'	P(A'P) = 0.3*0.4 = 0.12	P(A'P') = 0.3*0.6 = 0.18	0.3
Total	0.4	0.6	1

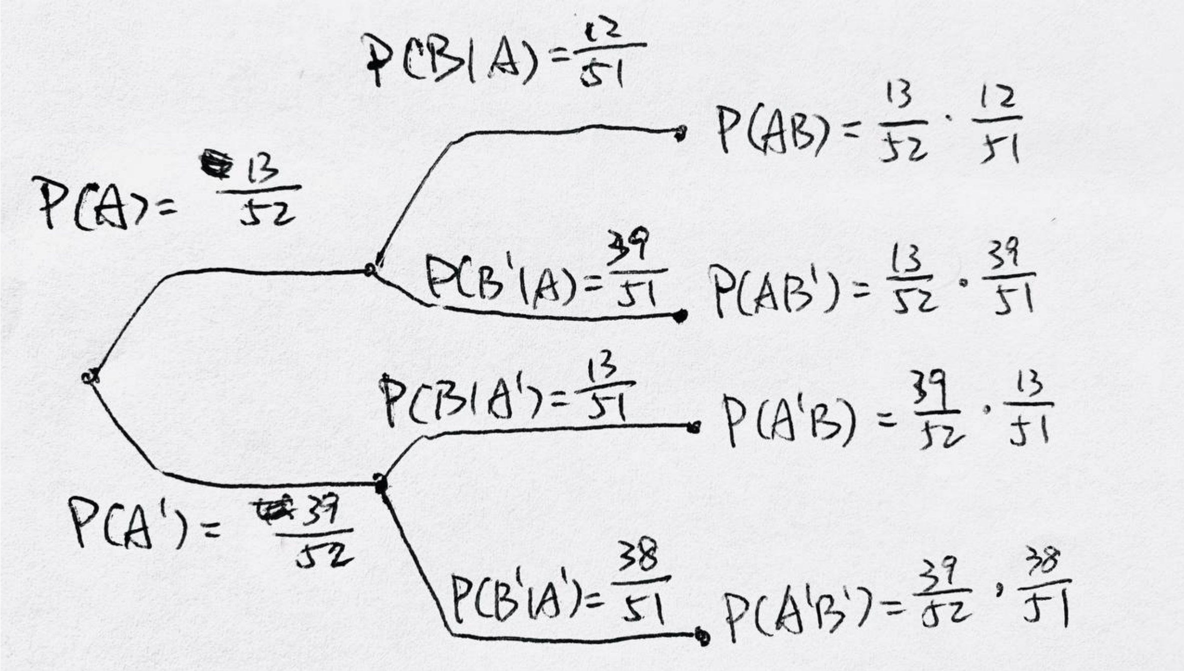
Two cards are dealt sequentially without replacement.

Suppose A is the event the first card is SPADE.

Suppose B is the event the second card is SPADE.

Find $P(B|A)$ and $P(A|B)$

Are these events independent?



$P(B|A) = \frac{12}{51} \neq \frac{1}{4} = P(B)$

$P(B) = \frac{13}{52}\frac{12}{51} + \frac{39}{52}\frac{13}{51} = \frac{1}{4}$

$A \& B$ are **NOT** independent!

Chapter 4: Random Variables

Recall: Experiment -----> Sample Space ---------> *(new)* Real number

- Ex: flip 2 coins -----> S= {HH,HT,TH,TT} ------> See example below

Definition

A random variable is a **function** that assigns a real number to each element in the sample space.

- We often use capital letters X, Y, Z to represent $R.V.$
- $X : S \to \mathbb{R}, X(s) = r$
- Often express X as its possible outcomes

Example:

- Flip 2 coins.
- Let X be the $R.V.$ that counts the number of heads.
  - $X = \{0, 1, 2\}$
  - $X : S \to \mathbb{R}$
  - $X(HH) = 2$
  - $X(HT) = 1$
  - $X(TH) = 1$
  - $X(TT) = 0$
- Let Y be the RV that subtracts number of heads from number of tails
  - $Y = \{2, 0, -2\}$

- Let Z be the RV that multiples number of heads by 3 and adds to number of tails squared
  - $Z = \{6, 4\}$
  - $Z(HH) = 6$
  - $Z(HT) = 4$
  - $Z(TH) = 4$
  - $Z(TT) = 4$
- A moral is a number RV defined on a given sample space, and it's not unique
- **However**: Once a RV is defined, then the probability distribution function(PDF) is unique
  - Examples for f_X, f_Y, f_Z
  -

	X	f_X	Y	f_Y	Z	f_Z
	0	1/4	2	1/4	6	1/4
	1	1/2	0	1/2	4	3/4
	2	1/3	-2	1/4		
Total		1		1		1

Discrete Random Variables (DRV)

A **Discrete Random Variable** on a Sample Space S, is a Random Variable that takes on a finite number of values, or countably infinite number of values.

> - Countable:
>   - finite
>   - countably infinite set: $\mathbb{N}, \mathbb{Z}, 2\mathbb{Z}, \mathbb{Q}, \ldots\ldots$ (all with equal sizes)
>   - think about it like gaps
> - Uncountable:
>   - $\mathbb{R}, (0, 1), \ldots\ldots$ or worse
>   - no gaps

Therefore, we have

Experiment ------> S --------> (Define) X --------> f_X

Example:

- Flip a coin until a heads appears. $S = \{H, TH, TTH, TTTH, \ldots..\}$
- Define X be the RV that counts the number of trials need to to get the first head.
  - $X = \{1, 2, 3, \ldots\ldots\}$
  - X is DRV

Probability Mass Function (PMF)

Our **probability distribution function** for a **DRV** is called a probability mass function (pmf)

- EX: The pmf for a DRV, X
- $p : X \to [0,1]$
  - $0 \le p(x) \le 1$ for all $x \in X$
  - $\sum_{x \in X} p(X) = 1$
  - $p(a) = P(X = a)$ (assigns probabilities to each x in X)

> For the continuous mass function, the properties would change slightly
>
> - same
> - change the sum to integral
> - not $P(X = a)$

Example

- A pmf for a DRV, I, is given by:
  - $p : I \to [0,1],\ p(i) = \frac{c\lambda^i}{i!},\ i = 0, 1, 2, \ldots (I = \{0, 1, 2, \ldots\})$
- Find the constant c.

$$\sum_{i \in I} p(i) = 1 = \sum_{i=0}^{\infty} \frac{c\lambda^i}{i!} = c\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

> Recall $: e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

$$\sum_{i=0}^{\infty} \frac{c\lambda^i}{i!} = c\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = c \cdot e^{\lambda} = 1$$
$$c = \frac{1}{e^{\lambda}} = e^{-\lambda}$$

- pmf: $p(i) = \frac{e^{-\lambda}\lambda^i}{i!}$

> PMF for an important named DRV

- $p(0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}$
- $P(I \ge 2) = 1 - P(I \le 1) = 1 - p(0) - p(1) = 1 - e^{-\lambda} - e^{-\lambda}\lambda$