Class 2022/09/15

Axioms of Probability

- Starts with an Experiment that generates a set of all possible outcomes - Sample Space: S (The Universal set in this experiment).
- Any subset $A \subset S$ is called event.

Truths:

- For any $A \subset S$:
  - $A \cup A^c = S$
  - $AA^c = \emptyset$

Examples

- Experiment: Flip 3 coins
  - Sample Space: $S = \{HHH, HHT, HTH, THH, TTT, TTH, THT, HTT\}$
  - Size of S: $|S| = 8$
- Flip 4 Coins
  - $S = \{\dots\dots\}$
  - $|S| = 2^4 = 16$
- Roll 2 distinguishable dice
  - $S = \{\dots\}$
  - $|S| = 36$

Axioms

- Let S be the sample space for some experiment and E is any event.
- The $P(E)$ assigns a real number to E such that
  - $0 \leq P(E) \leq 1$
  - $P(S) = 1$
  - $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$ if $E_i E_j = \emptyset$ for all $i \neq j$
    - All events are pairwise disjoint
    - We say two events E_i and E_j are mutually exclusive if $E_i E_j = \emptyset$

> **Mutually Exclusive** is not the same thing as **Independent**

Truths

For any event E is a sample space S:

1. $P(E^c) + P(E) = 1$

   > How to show this?
   >
   > - $EE^c = \emptyset$ for any set E
   > - $E \cup E^c = S$
   > - $P(S) = 1$
   > - $P(E) + P(E^c) = P(E \cup E^c) = P(S) = 1$

   - Gives us $P(E) = 1 - P(E^c)$
   - It's often easier to find $P(E^c)$ than $P(E)$
2. $P(E \cup F) = P(E) + P(F) - P(EF)$

   > How to show this?
   >
   > - $E(E^c F) = \emptyset$
   > - $P(E \cup F) = P(E \cup E^c F) = P(E) + P(E^c F)$
   > - Since $F = EF \cup E^c F$, and they are disjoint,
   > - $P(F) - P(EF) = P(E^c F)$
   > - $P(E \cup F) = P(E) + P(E^c F) = P(E) + P(F) - P(EF)$

3. $IF\ E \subset F, then\ P(E) \leq P(F).$
   - Be able to prove using axioms above!

- ## Practices

1. An auto insurance company has 10000 policyholders. Each policyholder is classified as:

   (i) young or old;

   (ii) male or female;

   (iii) married or single.

   Of these policyholders, 3000 are young, 4600 are male and 7000 are married. The policyholders can also be classified as 1320 young males, 3010 married males and 1400 young married persons. Finally, 600 of the policyholders are young married males.

   Question: How many of the company's policyholders are young, female and single?

   Answer:

   Since there are 1400 young married people, 600 young married male, we have 800 young married females.

   Since there are 3000 young people and 1320 young males, we have 1680 young females.

   Since there are 1680 young females and 800 young married females, we have 880 young single females.

2. You are given $P(A \cup B) = 0.7$ and $P(A \cup B^c) = 0.9$. Determine $P(A)$.

   My Answer:

   - $P(A \cup B) = P(A \cup A^cB) = P(A) + P(A^cB)$
   - $P(A \cup B^c) = P(A) + P(A^cB^c)$
   - $A^cB^c \cup A^cB = A^c$
   - $\Rightarrow P(A^cB^c) + P(A^cB) = P(A^c)$
   - $P(A \cup B) + P(A \cup B^c) = 2P(A) + P(A^c) = P(A) + 1 = 1.6$
   - $P(A) = 0.6$

   Given Answer:

   - $P(A \cup B) = P(A) + P(B) - P(AB)$
   - $P(A \cup B^c) = P(A) + P(B^c) - P(AB^c)$
   - $P(A \cup B) + P(A \cup B^c) = 2P(A) + 1 - P(A) = P(A) + 1 = 1.6$
   - $P(A) = 0.6$

Probability of Equally Likely Outcomes

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{n(E)}{n(S)} = \frac{|E|}{|S|}$$

- ## Examples

1. Flip 3 coins. Probability of all the same flips?
   - $S = \{HHH, HHT, HTH, THH, TTT, TTH, THT, HTT\}$
   - $|S| = 8$
   - $E = \{HHH, TTT\}$
   - $|E| = 2$
   - $P(E) = \frac{2}{8} = \frac{1}{4}$
2. A drawer has 4 black socks and 8 white socks. Draw 2 at random. Probability of both are the same color?
   - $|S| = \binom{12}{2}$
   - $|E| = \binom{4}{2} + \binom{8}{2}$
   - $P(E) = \frac{\binom{4}{2} + \binom{8}{2}}{\binom{12}{2}}$
3. In 5-card poker, what is the probability of getting 3 of a kind?
   - $|S| = \binom{52}{5}$
   - $|E| = \binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}$
   - $P(E) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}}$
4. There are 7 soccer players and 6 football players. How many ways to form a group of 3 soccer and 2 football? Probability of group of 5 with 3 soccer and 2 football?
   - Ways to form the group: $|E| = \binom{7}{3} + \binom{6}{2}$
   - Form a group of 5: $|S| = \binom{13}{5}$
   - $P(E) = \frac{\binom{7}{3} + \binom{6}{2}}{\binom{13}{5}}$
5. Football Team of 20 offensive players and 20 defensive players. Pair the players into rooms. Find the probability there are no off/def pairs.
   - $|S| = \frac{40!}{2^{20} * 20!}$
   - $|E| = \frac{20!}{2^{10} * 10!} * \frac{20!}{2^{10} * 10!}$
   - $P(E) = \frac{\frac{20!}{2^{10}*10!} * \frac{20!}{2^{10}*10!}}{\frac{40!}{2^{20}*20!}}$