Mathematical Statistics: HW 1

Due Feb 2, 2023

1. Chapter 1, Exercise 5

   Let A and B be arbitrary events. Let C be the event that either A occurs or B occurs, but not both. Express C in terms of A and B using any of the basic operations of union, intersection, and complement.

   Answer:

   Since C means A or B occurs, but not both, we have

   $$C = A \cup B - A \cap B = (A \cap B^c) \cup (B \cap A^c)$$

2. Chapter 1, Exercise 11

   The first three digits of a university telephone exchange are 452. If all the sequences of the remaining four digits are equally likely, what is the probability that a randomly selected university phone number contains seven distinct digits?

   Answer:

   Since there are 3 digits already used, for the forth digit, we have 7 different choice, and 6 choice for the next one, and so on. Also, if we don't consider the repetition, we have 10 cases for each digit. Therefore, the result is:

   $$\frac{7 * 6 * 5 * 4}{10^4} = 0.084$$

3. Chapter 1, Exercise 46

   Urn A has three red balls and two white balls, and urn B has two red balls and five white balls. A fair coin is tossed. If it lands heads up, a ball is drawn from urn A; otherwise, a ball is drawn from urn B.

   a. What is the probability that a red ball is drawn?

   b. If a red ball is drawn, what is the probability that the coin landed heads up?

   Answer:

Let A be the event that the coin lands heads up, B be the event that the coin land tails up. R be the event that the red ball is drawn, W be the event that the white ball is drawn.

a. $P(A) = P(B) = 0.5$

$P(R) = P(R|A)P(A) + P(R|B)P(B) = \frac{3/5+2/7}{2} \approx 0.443$

b. $P(A|R) = \dfrac{P(AR)}{P(R)} = \dfrac{P(R|A)P(A)}{P(R|A)P(A) + P(R|B)P(B)} = \dfrac{3/5}{3/5 + 2/7} \approx 0.677$

4. Chapter 1, Exercise 54

This problem introduces a simple meteorological model, more complicated versions of which have been proposed in the meteorological literature. Consider a sequence of days and let R_i denote the event that it rains on day i. Suppose that $P(R_i|R_{i-1}) = \alpha$ and $P(R_i^c|R_{i-1}^c) = \beta$. Suppose further that only today's weather is relevant to predicting tomorrow's; that is, $P(R_i|R_{i-1} \cap R_{i-2} \cap \cdots \cap R_0) = P(R_i|R_i - 1)$.

a. If the probability of rain today is p, what is the probability of rain tomorrow?

b. What is the probability of rain the day after tomorrow?

c. What is the probability of rain n days from now? What happens as n approaches infinity?

<u>Answer:</u>

a.

$$\begin{aligned} P(R_{i+1}) &= P(R_{i+1}|R_i)P(R_i) + P(R_{i+1}|R_i^c)P(R_i^c) \\ &= \alpha p + (1-\beta)(1-p) \\ &= \alpha p + 1 - \beta - p + \beta p \\ &= 1 - \beta + \alpha p - p + \beta p \\ &= 1 - \beta + p(\alpha + \beta - 1) \\ &= 1 - \beta + P(R_i)(\alpha + \beta - 1) \end{aligned}$$

b.

$$\begin{aligned} P(R_{i+2}) &= 1 - \beta + P(R_{i+1})(\alpha + \beta - 1) \\ &= 1 - \beta + (1 - \beta + p(\alpha + \beta - 1))(\alpha + \beta - 1) \\ &= 1 - \beta + (1 - \beta + \alpha p - p + \beta p)(\alpha + \beta - 1) \\ &= (\alpha + \beta - 1)^2 p + (1 - \beta)(\alpha + 1) - (1 - \beta)^2 \end{aligned}$$

c.

$$\begin{aligned} P(R_{i+n}) &= P(R_{i+n}|R_{i+n-1}) \cdot P(R_{i+n-1}) + P(R_{i+n}|R_{i+n-1}^c)P(R_{i+n-1}^c) \\ &= (\alpha + \beta - 1) \cdot P(R_{i+n-1}) + 1 - \beta \\ &= (\alpha + \beta - 1)^2 \cdot P(R_{i+n2}) + (1 - \beta)(\alpha + \beta - 1 + 1) \\ &= (\alpha + \beta - 1)^n \cdot p + (1 - \beta) \sum_{k=0}^{n-1} (\alpha + \beta - 1)^k \\ &= (\alpha + \beta - 1)^n \cdot p + (1 - \beta) \cdot \frac{(\alpha + \beta - 1)^n - 1}{\alpha + \beta - 2} \end{aligned}$$

Since $0 < \alpha < 1, 0 < \beta < 1$. So $|\alpha + \beta - 1| < 1$, so $\lim_{n \to \infty} (\alpha + \beta - 1)^n = 0$

So $\lim\limits_{n \to \infty} P(R_{i+n}) = \lim\limits_{n \to \infty} (\beta - 1) \cdot \dfrac{1}{\alpha + \beta - 2} = \dfrac{\beta - 1}{\alpha + \beta - 2}$

5. Chapter 1, Exercise 71

   Show that if A, B, and C are mutually independent, then $A \cap B$ and C are independent and $A \cup B$ and C are independent.

   Answer:

   If A, B, C mutually independent, we have:

   $P(A \cap B \cap C) = P(A)P(B)P(C)$

   $P(A \cap B) = P(A)P(B)$

   $P(C \cap B) = P(C)P(B)$

   $P(A \cap C) = P(A)P(C)$

   Therefore, $P((A \cap B) \cap C) = P(A \cap B \cap C) = P(A)P(B) \cdot P(C) = P(A \cap B)P(C)$

   Thus $A \cap B$ and C are independent

   Also,

$$\begin{aligned} P((A \cup B) \cap C) &= P((A \cap C) \cup (B \cap C)) \\ &= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C) \\ &= P(A)P(C) + P(B)P(C) - P(A)P(B)P(C) \\ &= \big(P(A) + P(B) - P(A \cap B)\big)P(C) \\ &= P(A \cup B)P(C) \end{aligned}$$

   Thus $A \cup B$ and C are independent

6. Chapter 2, Exercise 3

   The following table shows the cumulative distribution function of a discrete random variable. Find the frequency function.

k	$F(k)$
0	0
1	.1
2	.3
3	.7
4	.8
5	1.0

Answer:

$$\begin{aligned}p(1) &= 0.1 - 0 = 0.1 \\ P(2) &= 0.3 - 0.1 = 0.2 \\ p(3) &= 0.7 - 0.3 = 0.4 \\ p(4) &= 0.8 - 0.7 = 0.1 \\ p(5) &= 1.0 - 0.8 = 0.2\end{aligned}$$

k	$f(k)$
1	0.1
2	0.2
3	0.4
4	0.1
5	0.2

7. Chapter 2, Exercise 40

Suppose that X has the density function $f(x) = cx^2$ for $0 \le x \le 1$ and $f(x) = 0$ otherwise.

a. Find c. **b.** Find the cdf. **c.** What is $P(.1 \le X < .5)$?

Answer:

a.

$$\int_0^1 cx^2 = 1 = \frac{cx^3}{3}\Big|_0^1 = \frac{c}{3}$$

Therefore $c = 3$

b.

$$F_X(x) = \int_0^x 3x^2 = x^3\Big|_0^x = x^3$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^3, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

c.

$$P(0.1 \le X < 0.5) = F_X(0.5) - F_X(0.1) = 0.5^3 - 0.1^3 = 0.124$$

8. Chapter 2, Exercise 46

   T is an exponential random variable, and $P(T < 1) = .05$. What is λ?

   Answer:

   Let T be an exponential random variable. Therefore, $T \sim Exp(\lambda$.

   Thus we have $F_T(t) = 1 - e^{-\lambda t}$

   $P(T < 1) = F_T(1) = 1 - e^{-\lambda} = 0.05$

   Thus $e^{-\lambda} = 0.95$

   $\lambda = -\ln 0.95 \approx 0.051$

9. Chapter 2, Exercise 62

   Show that if X has a density function f_X and $Y = aX + b$, then

   $$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

   Answer:

   $$Y = aX + b$$

   1. if $a < 0$

      $$F_Y(y) = P(Y \leq y) = P(aX + b \leq y) = P(X \geq \frac{y-b}{a}) = 1 - F_X(\frac{y-b}{a})$$

      Take derivative of both sides:

      $$f_Y(y) = -\frac{1}{a} f_X(\frac{y-b}{a})$$

   2. if $a > 0$

      $$F_Y(y) = P(Y \leq y) = P(aX + b \leq y) = P(X \leq \frac{y-b}{a}) = F_X(\frac{y-b}{a})$$

      Take derivative of both sides:

      $$f_Y(y) = \frac{1}{a} f_X(\frac{y-b}{a})$$

   Therefore, we have

   $$f_Y(y) = \frac{1}{|a|} f_X(\frac{y-b}{a})$$