Problem Set 2

Quantitative Economics, Fall 2024

November 25, 2025

This problem set consists of three problems. You have two weeks to solve them and submit your solutions (Dec. 10th 11:59 PM). You can work in teams of up to five students.

Problem 1: Portfolio choice

Consider a problem of an agent who invests her wealth W in a risk-free asset with a gross return $R_f > 1$ and a risky asset with a gross return R. The agent chooses the share ω of her wealth to invest in the risky asset. The rest, $1-\omega$, is invested in the risk-free asset. The return on the portfolio is thus

We allow ω to be an arbitrary number. There is no constraint on short selling.

$$R_p(\omega) = \omega R + (1-\omega)R_f.$$

The agent maximizes the expected utility of her wealth at the end of the period. The utility function is given by

$$u(W) = \frac{W^{1-\gamma}}{1-\gamma},$$

where $\gamma > 0$ is the coefficient of relative risk aversion. The agent's problem is thus

$$\max_{\omega} \mathbb{E}\left(u\big(WR_p(\omega)\big)\right).$$

Notice that the optimal share satisfies the first order condition

$$\mathbb{E}\left((R-R_f)u'\big(WR_p(\omega^*)\big)\right) = 0.$$

Assume that R is lognormally distributed, i.e. $\log R \sim N(\mu, \sigma^2)$. This means that R has the probability density function

$$f_R(r) = \frac{1}{r\sigma\sqrt{2\pi}} \exp\left(-\frac{(\log r - \mu)^2}{2\sigma^2}\right), \quad r > 0.$$

The first order condition can be written in integral form as

$$\int_0^{\infty} (r-R_f)u'\big(WR_p(\omega^*)\big) f_R(r)\, dr = 0,$$

where $R_p(\omega^*) = \omega^* r + (1-\omega^*)R_f$.

Your task is to write a function that takes as arguments W, R_f, γ, μ, and σ and returns the optimal share ω^*. This means that you need to find ω^* such that the above integral is zero. You can use any nonlinear equation solver you like (e.g. from the NLsolve.jl or Roots.jl packages). You will need to evaluate the integral numerically (e.g. using the QuadGK.jl package).

Steps to follow:

1. Write a function `foc_integral(`ω`, W, Rf, `γ`, `μ`, `σ`)` that numerically evaluates the integral

$$\int_0^\infty (r - R_f)\left[W(\omega r + (1-\omega)R_f)\right]^{-\gamma} f_R(r)\, dr$$

for given parameter values. Use a numerical integration package such as QuadGK.jl.

2. Verify your function works correctly by testing it with $\gamma = 0$ (linear utility, where $u'(W) = 1$). In this case, the integral simplifies to

$$\int_0^\infty (r - R_f) f_R(r)\, dr = \mathbb{E}_R - R_f = e^{\mu+\sigma^2/2} - R_f,$$

which can be computed analytically. Check that your numerical integral matches this analytical value.

3. Write a function `optimal_portfolio(W, Rf, `γ`, `μ`, `σ`)` that uses a root-finding algorithm (e.g., from NLsolve.jl or Roots.jl) to find ω^* such that `foc_integral`$(\omega^*, W, R_f, \gamma, \mu, \sigma) = 0$.

4. Compute the optimal share ω^* for the parameter values: $W = 1$, $R_f = 1.02$, $\gamma = 3$, $\mu = 0.05$, and $\sigma = 0.1$. Report the numerical value.

5. Create a plot showing how ω^* varies with risk aversion γ over the range $\gamma \in [0.1, 10]$, keeping all other parameters fixed at the values above. Add axis labels and a title to your plot.

Numerical integration approximates the integral by evaluating the integrand at many points and summing weighted values. QuadGK uses adaptive Gauss-Kronrod quadrature, which automatically adjusts the evaluation points to achieve the desired accuracy. Since the integral goes to infinity, you'll need to handle this carefully—QuadGK can handle infinite limits. The integrand decays quickly due to the lognormal density, so the integral converges. One of the objectives of this problem is to get you familiar with using numerical integration packages and understanding when numerical methods are necessary (here, because we cannot compute the expectation of a nonlinear function of a lognormal variable analytically).

Problem 2: Simulated method of moments

Consider a log-income process that follows an AR(1) specification with stochastic volatility:

$$\log y_{t+1} = \rho \log y_t + \varepsilon_{t+1},$$

where the innovation has a mixture distribution:

$$\varepsilon_{t+1} \sim \begin{cases} N(0, \sigma_L^2) & \text{with probability } p \\ N(0, \sigma_H^2) & \text{with probability } 1-p \end{cases}$$

With probability p, the shock has low volatility σ_L; with probability $1-p$, it has high volatility $\sigma_H > \sigma_L$. This captures periods of calm versus turbulent economic conditions.

Because of the mixture distribution, the unconditional moments of this process cannot be derived in closed form and must be computed via simulation.

Your task is to estimate the parameters $\theta = (\rho, p)$ using the *Simulated Method of Moments (SMM)*, treating σ_L and σ_H as known.

Steps to follow:

1. Generate "observed" data by simulating the model for $T = 500$ periods with true parameters $\rho = 0.90$, $p = 0.80$, $\sigma_L = 0.10$, and $\sigma_H = 0.30$. Start with $\log y_1 = 0$. **Set the random seed to 2024** to ensure reproducibility. After simulating, **discard the first 100 observations** as a burn-in period to reduce dependence on the initial condition.

2. From your observed data, compute the following sample moments:

$$\widehat{m}_1 = \mathrm{std}(\log y_t), \quad \text{(standard deviation)}$$
$$\widehat{m}_2 = \mathrm{corr}(\log y_t, \log y_{t-1}), \quad \text{(lag-1 autocorrelation)}$$
$$\widehat{m}_3 = \mathrm{kurtosis}(\Delta \log y_t), \quad \text{(excess kurtosis of changes)}$$

   where $\Delta \log y_t = \log y_t - \log y_{t-1}$.

3. Write a function `simulate_model(`θ`, T, `σ_L`, `σ_H`)` that takes parameters $\theta = (\rho, p)$, simulates the model for T periods, and returns the simulated time series $\{\log y_t^{\mathrm{sim}}\}_{t=1}^T$. The function should also discard the first 100 observations as burn-in.

4. Write a function `smm_objective(`θ`, observed_data, `σ_L`, `σ_H`, S)` that:

   - Computes the observed moments $(\widehat{m}_1, \widehat{m}_2, \widehat{m}_3)$ from the data
   - For each simulation $s = 1, \ldots, S$, calls `simulate_model` to generate data, then computes the three moments
   - Averages the simulated moments across S simulations: $(\bar{m}_1^{\mathrm{sim}}, \bar{m}_2^{\mathrm{sim}}, \bar{m}_3^{\mathrm{sim}})$
   - Returns the SMM objective:

$$Q(\theta) = (\widehat{m}_1 - \bar{m}_1^{\mathrm{sim}})^2 + (\widehat{m}_2 - \bar{m}_2^{\mathrm{sim}})^2 + (\widehat{m}_3 - \bar{m}_3^{\mathrm{sim}})^2$$

   Use $S = 100$ simulations to reduce simulation noise. **Important:** Set a random seed at the beginning of this function (based on the parameter values) so that the objective function is deterministic for a given θ.

5. Use an optimization algorithm (e.g., from Optim.jl) to find $\widehat{\theta} = (\widehat{\rho}, \widehat{p})$ that minimizes $Q(\theta)$. Use initial guess $\theta_0 = (0.85, 0.70)$. Impose constraints: $\rho \in (0.5, 0.99)$ and $p \in (0.5, 0.95)$. Report your estimates and compare them to the true values $\rho = 0.90$ and $p = 0.80$.

For each period, draw a Bernoulli random variable to determine whether the innovation is low or high volatility, then draw the innovation from the corresponding normal distribution.

In Julia, use `Random.seed!(2024)` at the beginning of your code. The burn-in period allows the process to move away from the arbitrary starting value $\log y_1 = 0$ toward its long-run distribution. Use observations $t = 101, \ldots, 500$ for estimation.

Use *excess kurtosis* (the fourth standardized moment minus 3). For a normal distribution, excess kurtosis equals zero. The mixture distribution produces positive excess kurtosis, capturing the fat tails. This moment is informative about the probability p. In Julia, you can compute this using the `kurtosis` function from the Statistics package, which returns excess kurtosis by default.

Simulate for $T + 100$ periods total, then return only the last T observations. This ensures the simulated moments aren't biased by the initial condition.

6. Compare your observed data to data simulated using your estimated parameters (using a single simulation with $T = 500$). Create plots showing:

   - Time series of observed vs. simulated log income (first 200 periods)
   - Histogram of $\Delta \log y_t$: observed vs. simulated (to visualize the mixture distribution)

   Do the estimated parameters reproduce the key features of the observed data?

7. **(Bonus)** To study the finite-sample properties of the SMM estimator, repeat the following procedure for $M = 100$ Monte Carlo replications:

   - Generate a new "observed" dataset with $T = 500$ periods (after burn-in) using the true parameters, with a different random seed for each replication
   - Estimate $\widehat{\theta} = (\widehat{\rho}, \widehat{p})$ using SMM
   - Store the estimates

   Then compute and report:

   - The mean and standard deviation of $\widehat{\rho}$ and $\widehat{p}$ across replications
   - The bias: $\text{mean}(\widehat{\rho}) - \rho$ and $\text{mean}(\widehat{p}) - p$
   - Create scatter plot of $(\widehat{\rho}, \widehat{p})$ across replications, with the true values marked
   - Create histograms of $\widehat{\rho}$ and $\widehat{p}$ with vertical lines at the true values

   Is the estimator approximately unbiased? How does the number of simulations S affect the results? (Try $S = 50$ vs $S = 200$.)

This explores how the estimator behaves across repeated samples. In each replication, you generate new data from the true model, estimate the parameters, and record the estimates. This reveals the estimator's bias, variance, and sampling distribution.

Problem 3: Transition dynamics in a neoclassical growth model

Consider a discrete-time neoclassical growth model with a representative household that has preferences over consumption sequences:

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad u(c) = \frac{c^{1-\gamma}}{1-\gamma},$$

where $\beta \in (0, 1)$ is the discount factor and $\gamma > 0$ is the coefficient of relative risk aversion.

The production function is Cobb-Douglas:

$$y_t = A k_t^{\alpha},$$

The capital stock depreciates at rate $\delta \in (0, 1)$ each period. The resource constraint states that output can be consumed or invested (net of depreciation).

where $A > 0$ is total factor productivity and $\alpha \in (0,1)$ is the capital share.

The capital accumulation equation is:

$$k_{t+1} = (1-\delta)k_t + i_t,$$

where $\delta \in (0,1)$ is the depreciation rate and i_t is investment.

The resource constraint is:

$$c_t + i_t = y_t.$$

The first-order conditions for the household's problem yield the Euler equation:

$$u'(c_t) = \beta u'(c_{t+1}) \left[\alpha A k_{t+1}^{\alpha-1} + (1-\delta) \right].$$

This equation states that the marginal utility of consumption today must equal the discounted marginal utility of consumption tomorrow times the gross return on capital.

The steady state $(\bar{k}, \bar{c})$ satisfies:

$$\frac{1}{\beta} = \alpha A \bar{k}^{\alpha-1} + (1-\delta),$$
$$\bar{c} = A\bar{k}^{\alpha} - \delta\bar{k}.$$

Your task: Suppose the economy starts at an initial capital stock $k_0 < \bar{k}$ and converges to the steady state. Compute the deterministic transition path $\{k_t, c_t\}_{t=0}^{T}$ by solving a system of nonlinear equations.

Steps to follow:

1. Set the following parameter values: $\beta = 0.96$, $\alpha = 0.33$, $A = 1$, $\delta = 0.1$, and initial capital $k_0 = 0.5\bar{k}$ (half of steady state). Compute the steady state capital $\bar{k}$ and consumption $\bar{c}$ analytically using the steady state conditions above.

The initial capital k_0 is a given initial condition, not an unknown. It represents where the economy starts before transitioning to steady state.

2. To compute the transition path, we solve for the sequence $\{c_t, k_t\}_{t=0}^{T}$ using the following system of equations:

The system has $2T+1$ unknowns: $\{c_t\}_{t=0}^{T}$ (T+1 unknowns) and $\{k_t\}_{t=1}^{T}$ (T unknowns). We have T Euler equations (for $t = 0, \ldots, T-1$), T capital accumulation equations (for $t = 0, \ldots, T-1$), and the terminal condition $c_T = \bar{c}$ (steady state consumption), giving us $2T+1$ equations total.

   - Euler equations for $t = 0, \ldots, T-1$:

$$c_t^{-\gamma} = \beta c_{t+1}^{-\gamma} \left[\alpha A k_{t+1}^{\alpha-1} + (1-\delta) \right]$$

   - Capital accumulation for $t = 0, \ldots, T-1$:

$$k_{t+1} = (1-\delta)k_t + Ak_t^{\alpha} - c_t$$

   - Terminal condition: $c_T = \bar{c}$

   Write a function `transition_equations(x, params)` that takes a vector x of length $2T+1$ (where T is a parameter) containing $[c_0, \ldots, c_T, k_1, \ldots, k_T]$ and returns a vector of residuals corresponding to the Euler equations, capital accumulation equations, and terminal condition. The function should return zero when the transition path satisfies all equilibrium conditions.

3. Use a nonlinear equation solver (e.g., NLsolve.jl with the `nlsolve` function) to solve the system. Do it for two values of γ: 0.5 and 2.0. **Choosing T:** Start with $T = 100$ periods. Check whether k_T is sufficiently close to $\bar{k}$ (say, within 0.1%). If not, increase T (try $T = 150$ or $T = 200$) until the terminal condition is essentially satisfied. For the initial guess, you can use:

$$c_t^{(0)} = \bar{c} \quad \text{(constant at steady state)},$$
$$k_t^{(0)} = k_0 + \frac{t}{T}(\bar{k} - k_0) \quad \text{(linear interpolation)}.$$

Report whether the solver converged and the final norm of the residuals.

The economy converges to steady state asymptotically, so you need T large enough that $k_T \approx \bar{k}$. Higher γ (lower IES) typically means slower convergence and requires larger T. Check convergence by comparing k_T to $\bar{k}$ or by plotting the capital path to see if it flattens out.
A good initial guess is important for convergence. Linear interpolation between the initial and steady state values usually works well for growth model transitions.

4. Create three plots comparing the transition dynamics for the two values of γ (0.5 and 2.0):

- Plot 1: Capital stock k_t over time, with a horizontal line showing the steady state $\bar{k}$.
- Plot 2: Consumption rate c_t/y_t over time, with a horizontal line showing the steady state consumption rate $\bar{c}/\bar{y}$ where $\bar{y} = A\bar{k}^{\alpha}$.
- Plot 3: Investment rate $i_t/y_t = (y_t - c_t)/y_t$ over time, with a horizontal line showing the steady state investment rate $\delta\bar{k}/(A\bar{k}^{\alpha})$.

Add appropriate axis labels and titles.

Describe the economic intuition: what are the effects of different intertemporal elasticities of substitution on the speed of convergence and the consumption/investment dynamics?