Problem Set 3

Quantitative Economics, Fall 2025

December 18, 2025

This problem set consists of four problems. Submit your solutions until Jan. 14th 11:59 PM. You can work in teams of up to five students.

Problem 1: Joint Markov processes

Consider a stochastic process with two state variables: Z_t (exogenous) and X_t (endogenous).

$Z_t \in \{z_1, z_2, z_3\}$ follows a Markov process with transition matrix:

$$P = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}$$

$X_t \in \{0, 1, 2, 3, 4, 5\}$ evolves according to the policy function:

$$X_{t+1} = \sigma(X_t, Z_t) = \begin{cases} 0 & \text{if } Z_t = z_1 \\ X_t & \text{if } Z_t = z_2 \\ \min(X_t + 1, 5) & \text{if } Z_t = z_3, X_t \leq 4 \\ 3 & \text{if } Z_t = z_3, X_t = 5 \end{cases}$$

Write code that:

1. Constructs the joint transition matrix Q for (X_t, Z_t) of size 18×18.

Use this ordering: $(0, z_1), (0, z_2), (0, z_3), (1, z_1), \ldots, (5, z_3)$.

2. Computes the stationary distribution ψ^* satisfying $\psi^* = Q^\top \psi^*$ and derives:
   - Marginal distributions: $\psi^*_X(x) = \sum_z \psi^*(x, z)$ and $\psi^*_Z(z) = \sum_x \psi^*(x, z)$
   - Mean of X: $\mathbb{E}[X] = \sum_x x \cdot \psi^*_X(x)$
   - Conditional means: $\mathbb{E}[X \mid Z = z]$ for each $z \in \{z_1, z_2, z_3\}$
3. Creates plots showing: (i) marginal distribution of X, (ii) marginal distribution of Z, (iii) conditional mean $\mathbb{E}[X \mid Z]$.

Problem 2: Basil's Orchid Anxiety

After purchasing a rare orchid from the festival for price p, Basil brings it home. The orchid is currently in bud form and perhaps will bloom soon. Basil doesn't know when the bloom will occur - it's random. He really hopes to take a photo of the orchid at its bloom before he goes away on a trip in 20 days ($T = 20$).

Each day $t = 1, 2, ..., T - 1$, he must decide whether to:

- Keep waiting for the natural bloom (generates anxiety)
- Use a special fertilizer that increases bloom probability but risks damage

The fertilizer can be applied multiple times, but overfertilization kills the orchid. Each application of fertilizer boosts bloom probability but also increases "stress" on the plant. If stress gets too high, the orchid dies permanently.

The orchid can be in one of three states:

- Not yet bloomed $O_t = 0$
- In bloom $O_t = 1$
- Dead $O_t = -1$

There are $S_{max} + 1$ possible stress levels for the orchid: $S_t \in \{0, 1, 2, \ldots, S_{max}\}$. If stress reaches S_{max} and Basil applies fertilizer again, the orchid dies ($O_t = -1$).

At each day t, if the orchid has not yet bloomed, Basil chooses an action $a_t \in \{W, A\}$:

- W = Wait (do nothing)
- A = Apply fertilizer

If the orchid is dead or in bloom, Basil cannot take any action and the state remains unchanged.

Transition: Wait ($a_t = W$). If Basil waits, the orchid blooms next day with probability λ. Stress remains unchanged.

The two possible transitions from state $(0, S)$ are:

$$
\begin{aligned}
(0, S) &\to (1, S) && \text{w.p. } \lambda \\
(0, S) &\to (0, S) && \text{w.p. } 1 - \lambda
\end{aligned}
$$

Transition: Apply fertilizer ($a_t = A$). Fertilizer temporarily boosts bloom probability to $\lambda + \delta$. The stress increment ΔS follows a Poisson distribution with rate ρ, i.e., $\Delta S \sim \text{Poisson}(\rho)$. Let $\pi_k = \frac{e^{-\rho}\rho^k}{k!}$ denote the probability that stress increases by k units.

If $S + k > S_{\max}$ for some increment k, the orchid dies:

$$
(0, S) \to (-1, S_{\max}) \quad \text{w.p.} \sum_{k: S+k > S_{\max}} \pi_k
$$

Otherwise, the orchid may bloom or remain unbloomed:

$$\begin{aligned}(0,S) &\to (1,S+k) \quad \text{w.p. } (\lambda+\delta)\pi_k, \quad \forall k : S+k \le S_{\max} \\ (0,S) &\to (0,S+k) \quad \text{w.p. } (1-\lambda-\delta)\pi_k, \quad \forall k : S+k \le S_{\max}\end{aligned}$$

Absorbing states. If the orchid is already in bloom ($O_t = 1$) or dead ($O_t = -1$), it remains in that state with probability 1. The level of stress no longer matters.

Per-period rewards. The flow payoff in periods $t = 1, 2, ..., T$ depends on the orchid status:

$$r(O_t, t) = \begin{cases} -(c_0 + c_1 t + c_2 t^2) & \text{if } O_t = 0 \\ \alpha & \text{if } O_t = 1 \\ -(c_0 + c_1 t + c_2 t^2) - \psi & \text{if } O_t = -1 \end{cases}$$

Terminal payoff. In addition, at $T = 20$, there is a terminal payoff (in addition to $r(O_T, T)$):

$$\Phi(O_T, p) = \begin{cases} -\kappa p & \text{if } O_T = 0 \\ \theta p & \text{if } O_T = 1 \\ -\omega p & \text{if } O_T = -1 \end{cases}$$

where $\omega > \kappa > 0$ and $\theta > 0$.

Basil maximizes expected total discounted utility:

$$\max_{\{a_t\}_{t=1}^T} \mathbb{E}\left[\sum_{t=1}^{T} r(O_t, t) + \Phi(O_T, p)\right].$$

Steps to follow:

1. Write a Bellman equation that represents Basil's problem. Let $V_t(O_t, S_t)$ be the value function at day t when the orchid is in state O_t with stress level S_t.

2. Write a function that solves the Bellman equation using backward induction. Start from period T and work backwards to period 1.

   The function should return:

   (a) the value function $V_t(O_t, S_t)$ for all $t \in \{1, \ldots, T\}$, $O_t \in \{0, 1, -1\}$, and $S_t \in \{0, \ldots, S_{max}\}$;

   (b) the optimal policy $a_t^*(O_t, S_t)$ indicating whether to Wait or Apply fertilizer at each state and time.

To handle the Poisson distribution, you need to truncate it at some large value.

3. Assume $T = 20$, $S_{max} = 10$, $\lambda = 0.05$, $\delta = 0.25$, $\rho = 5$, $c_0 = 0.5$, $c_1 = 0.5$, $c_2 = 0.1$, $\alpha = 5$, $\psi = 10$, $p = 100$, $\kappa = 0.5$, $\theta = 2$, $\omega = 2.5$, and answer the following questions:

Submit clearly commented code and plots. No need to write separate functions for each question.

4. (a) What is the optimal action at $t = 1$ when $O_1 = 0$ and $S_1 = 0$ (orchid just purchased, no stress)?
   (b) Create a plot showing the optimal policy as a function of stress level S for $t \in \{1, 5, 10, 15, 19\}$ when $O_t = 0$. Does Basil become more or less willing to use fertilizer as the deadline approaches?
   (c) What is the expected total utility starting from $(t = 1, O_1 = 0, S_1 = 0)$?
   (d) Simulate 1000 paths following the optimal policy starting from $(t = 1, O_1 = 0, S_1 = 0)$. What fraction of paths result in: (i) successful bloom, (ii) orchid never blooming, (iii) orchid dying?
   (e) Using the same 1000 simulations, calculate the average number of times Basil applies fertilizer per path. How does this compare to the number of days T?
   (f) For each day $t = 1, 2, \ldots, T$, calculate the probability that the orchid will have bloomed by day t or earlier (i.e., $P(O_t = 1)$) when following the optimal policy from $(t = 1, O_1 = 0, S_1 = 0)$. Create a plot showing this cumulative bloom probability as a function of time.
   (g) Create a plot showing how the expected total utility $V_1(0, 0)$ changes as a function of the orchid price p (holding all other parameters fixed). Use a range of prices from $p = 50$ to $p = 200$. At what price does Basil's expected utility become negative?

Problem 3: Lifecycle consumption with uncertain lifetime

Consider a lifecycle consumption-savings model where an agent lives up to a maximum of J periods. Each period, the agent faces an age-specific probability of death π_j for $j = 1, \ldots, J$. The agent derives utility from consumption while alive and from leaving a bequest upon death.

The agent's problem in period j with assets a is:

$$V_j(a) = \max_{c, a'} \left\{ u(c) + \beta \left[(1 - \pi_j) V_{j+1}(a') + \pi_j \phi(a') \right] \right\}$$

subject to:

$$c + a' = (1 + r)a + y_j, \quad a' \geq 0, \quad c \geq 0$$

where $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ is period utility, $\phi(a) = \theta \frac{(a+\underline{a})^{1-\gamma_b}}{1-\gamma_b}$ is bequest utility, and $(1 - \pi_j) V_{j+1}(a')$ and $\pi_j \phi(a')$ represent continuation value if alive and bequest utility if dead, respectively.

At the terminal age J, the agent dies with certainty ($\pi_J = 1$): $V_J(a) = \max_{c,a'} \{u(c) + \beta\phi(a')\}$.

Steps to follow:

1. Assume the following parameter values:
   - $J = 60$, $\gamma = 2$, $\gamma_b = 1$, $\beta = 0.96$, $r = 0.04167$ (so $\beta(1+r) = 1$)
   - $\theta = 0.5$, $\underline{a} = 2$, $\bar{y} = 1$
   - Death probabilities: $\pi_j = \min\left\{0.0005 \times 1.14^j, 1\right\}$
   - Labor income:

   $$y_j = \bar{y} \times \begin{cases} 0.8 + 0.02j & \text{if } j \leq 40 \\ 0.3 & \text{if } j > 40 \end{cases}$$

   Discretize the asset space with 500 points from $a = 0$ to a_{max}, where a_{max} is the maximum wealth achievable if the agent saves all income (experiment to find a suitable value). Use a non-uniform grid with higher density near zero: $a_i = a_{max}\left(\frac{i-1}{n_a-1}\right)^2$ for $i = 1, \ldots, n_a$.

2. Solve the model using backward induction. For all ages $j = J, J-1, \ldots, 1$, solve for the optimal policy functions $c_j(a)$ and $a'_j(a)$.

3. Create plots of (i) consumption policy $c_j(a)$, (ii) savings policy $a'_j(a)$, and (iii) value function $V_j(a)$ for ages $j \in \{20, 30, 40, 50, 60\}$.

4. Plot lifecycle profiles (consumption, income, assets, savings) for an agent starting with $a_1 = 0$ who survives to age $J = 60$.

5. Re-solve for three income levels $\bar{y} \in \{0.5, 1.0, 2.0\}$ and plot: (i) consumption, (ii) assets, and (iii) savings rate over the lifecycle. Discuss how the bequest motive manifests differently across income levels.

6. Re-solve the model for all three income levels $\bar{y} \in \{0.5, 1.0, 2.0\}$ both with ($\theta = 0.5$) and without ($\theta = 0$) bequest motive. For each specification:

   (a) Create plots showing lifecycle asset profiles for all three income levels (on the same graph), with separate plots for with/without bequest motive

   (b) Calculate peak wealth $\max_j a_j$ for each income level under both specifications

   (c) Compute a measure of wealth inequality: the ratio of peak wealth between rich and poor agents ($\max_j a_j(\bar{y} = 2.0) / \max_j a_j(\bar{y} = 0.5)$) with and without bequest motive

The quadratic spacing concentrates grid points where they're most needed: near zero where value functions are most curved and borrowing constraints bind. For example, with $n_a = 500$ and $a_{max} = 20$, the first 100 points cover only $[0, 1.6]$ while the last 100 cover $[12.8, 20]$. When simulating lifecycle patterns, assume agents start with zero initial assets ($a_1 = 0$).

For each state (j, a), search over all feasible values of a' on your grid.

(d) Discuss: Does the bequest motive amplify or dampen wealth inequality? Why? How does the luxury good nature of bequests contribute to this effect?

Problem 4: Human capital accumulation with education choice

Consider an agent who lives for an infinite number of periods and derives utility from consumption and disutility from education. The agent's state variable is human capital $h \geq 0$.

Each period, the agent earns labor income equal to:

$$y = w \cdot f(h) \cdot (1 - e)$$

where:

- w is a stochastic wage shock that follows an AR(1) process in logs: $\log w' = \rho \log w + \varepsilon$, where $\varepsilon \sim N(0, \sigma^2)$
- $f(h)$ is a function of human capital (to be specified)
- $e \in \{0, 0.01, 0.02, \ldots, 1.0\}$ is the fraction of time devoted to education (a discrete choice)

Human capital evolves according to:

$$h' = h + e - \delta$$

where $\delta > 0$ is the depreciation of human capital (an absolute amount per period, not a rate).

The agent has standard CRRA preferences over consumption:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

The agent's value function satisfies the Bellman equation:

$$V(h, w) = \max_{e \in \{0, 0.01, 0.02, \ldots, 1.0\}} \left\{ u(w \cdot f(h) \cdot (1 - e)) - \psi e + \beta \mathbb{E}_{w'|w} V(h', w') \right\}$$

subject to $h' = h + e - \delta$, where $\psi > 0$ is the direct utility cost of devoting time to education.

1. Write down the Bellman equation clearly, specifying:
   - The state variables: (h, w)
   - The choice variable: $e \in \{0, 0.01, 0.02, \ldots, 1.0\}$
   - The law of motion for human capital: $h' = h + e - \delta$
   - How to compute $\mathbb{E}_{w'|w} V(h', w')$ using the discretized wage process

2. Assume the following parameter values:

   - $f(h) = \min\{h^\alpha + 0.1, \bar{f}\}$ with $\alpha = 0.1$ and $\bar{f} = 1.2$. This function increases with human capital at a diminishing rate for $h < (\bar{f} - 0.1)^{1/\alpha} \approx 13.8$, then becomes flat at $\bar{f}$ for higher values of h.
   - $\delta = 0.1$ (depreciation of 0.1 units per period)
   - $\gamma = 1.5$ (CRRA coefficient)
   - $\beta = 0.95$ (discount factor)
   - $\rho = 0.98$ (persistence of wage shock)
   - $\sigma = 0.15$ (standard deviation of innovation)
   - $\psi = 0.5$ (utility cost of education)

   Discretize the state space as follows:

   - Education: Use increments of 0.01 for the education choice grid, so $e \in \{0, 0.01, 0.02, \ldots, 1.0\}$.
   - Human capital grid: Use increments of 0.01, so $h \in \{0, 0.01, 0.02, \ldots, h_{\max}\}$.
   - Wage shock: Use the Rouwenhorst method to discretize the AR(1) process into a Markov chain with $n_w = 7$ states.

3. Solve for the value function and policy function using value function iteration. Experiment with the tolerance level and maximum number of iterations to ensure convergence.

4. Create plots showing:

   (a) The optimal education choice $e^*(h, w)$ as a function of human capital h for three different wage levels: low ($w = w_1$), medium ($w = w_4$), and high ($w = w_7$).

   (b) The value function $V(h, w)$ as a function of h for the same three wage levels.

   (c) Consumption $c^*(h, w) = w \cdot f(h) \cdot (1 - e^*(h, w))$ as a function of h for the same three wage levels.

5. Starting from initial conditions $(h_0, w_0) = (1, w_4)$, simulate 1100 periods total following the optimal policy. Discard the first 100 periods as a burn-in.

   (a) Plot the evolution of the first 100 periods after burn-in (i.e., periods 101-200) for: (i) human capital h_t, (ii) wage shock w_t, (iii) education time e_t, and (iv) consumption c_t. Show 5 different paths on the same plot to illustrate variability due to wage shocks.

(b) Using all 900 periods after burn-in across all 5 paths (4,500 observations total), calculate:

(a) The correlation between labor earnings $y_t = w_t \cdot f(h_t) \cdot (1 - e_t)$ and education effort e_t. Is this correlation positive or negative? Explain the economic intuition.

Think about the opportunity cost effect: when earning potential is high, what happens to education vs. work time?

(b) The correlation between the wage shock w_t and education effort e_t. Is this correlation positive or negative? Explain the economic intuition.