Lubik, Thomas A.; Schorfheide, Frank

Working Paper

Testing for indeterminacy: An application to US monetary policy

Working Paper, No. 480

Provided in Cooperation with:
Department of Economics, The Johns Hopkins University

Testing for Indeterminacy:

An Application to U.S. Monetary Policy*

Thomas A. Lubik
Department of Economics
Johns Hopkins University†

Frank Schorfheide
Department of Economics
University of Pennsylvania‡

June, 2003

JEL CLASSIFICATION: C11, C52, C62, E52

KEY WORDS: Econometric Evaluation and Testing, Rational Expectations Models, Indeterminacy, Monetary DSGE Models

---

*We would like to thank Luca Benati, Ben Bernanke, Marco Del Negro, Roger Farmer, Peter Ireland, Jinill Kim, Eric Leeper, Athanasios Orphanides, four anonymous referees, and seminar participants at the Canadian Econometrics Study Group Meeting in Quebec City, the Federal Reserve Bank of St. Louis, Johns Hopkins University, the Meetings of the Society for Computational Economics in Aix-en-Provence, the 2002 NBER Summer Institute, the North American Econometric Society Summer Meeting in Los Angeles, UCLA, University of Maryland, University of Pennsylvania, and Tilburg University for insightful comments and suggestions. The second author gratefully acknowledges financial support from the University Research Foundation of the University of Pennsylvania.

†Mergenthaler Hall, 3400 N. Charles Street, Baltimore, MD 21218. Tel.: (410) 516-5564. Fax: (410) 516-7600. Email: thomas.lubik@jhu.edu

‡McNeil Building, 3718 Locust Walk, Philadelphia, PA 19104. Tel.: (215) 898-8486. Fax: (215) 573-2057. Email: schorf@ssc.upenn.edu

Testing for Indeterminacy:

An Application to U.S. Monetary Policy

Abstract

This paper considers a prototypical monetary business cycle model for the U.S. economy, in which the equilibrium is undetermined if monetary policy is 'passive'. In previous multivariate studies it has been common practice to restrict parameter estimates to values for which the equilibrium is unique. We show how the likelihood-based estimation of dynamic stochastic general equilibrium models can be extended to allow for indeterminacies and sunspot fluctuations. We construct posterior weights for the determinacy and indeterminacy region of the parameter space and posterior estimates for the propagation of fundamental and sunspot shocks. According to the estimated New Keynesian model the monetary policy post 1982 is consistent with determinacy, whereas the pre-Volcker policy is not. We find that before 1979 indeterminacy substantially altered the propagation of monetary policy, demand, and supply shocks.

1 Introduction

Economists are increasingly making use of dynamic stochastic general equilibrium (DSGE) models for macroeconomic analysis. In order to solve these models and keep them tractable, linear rational expectations (LRE) models are typically used as local approximations. However, it is well known that LRE models can have multiple equilibria, which is often referred to as indeterminacy. Broadly speaking, indeterminacy has two consequences. First, the propagation of fundamental shocks, such as technology or monetary policy shocks, through the system is not uniquely determined. Second, sunspot shocks can influence equilibrium allocations and induce business cycle fluctuations that would not be present under determinacy.

In the popular prototypical New Keynesian monetary DSGE model (see, for instance, King (2000), and Woodford (2003)) indeterminacy can arise if the monetary policy authority follows an interest rate rule and does not raise the nominal interest rate aggressively enough in response to an increase in inflation. While in the presence of market imperfections some sunspot fluctuations could be welfare improving (Christiano and Harrison, 1999) others will lead to a substantial deterioration of welfare. Hence, a central bank that is concerned about the least favorable outcome has a strong incentive to choose a policy that leads to determinacy.

It has been suggested by Clarida, Galí, and Gertler (2000), henceforth CGG, that the U.S. monetary policy before 1979 viewed through the lens of a standard New Keynesian DSGE model was inconsistent with equilibrium determinacy. Self-fulfilling expectations are potentially one of the explanations for the high inflation episode in the 1970s. Beginning with Paul Volcker's tenure as Board Chairman and, later on, Alan Greenspan's the Federal Reserve implemented a much more aggressive rule that suppresses self-fulfilling beliefs. Studying indeterminacy can therefore contribute to our understanding of the macroeconomic instability during the 1970s and further the design of beneficial policy rules.

The first contribution of this paper is to provide econometric tools that allow for a systematic assessment of the quantitative importance of equilibrium indeterminacy and the propagation of fundamental and sunspot shocks in the context of a DSGE model. While it is well known how to construct a likelihood function of a DSGE model for the determinacy region of the parameter space, we show how the likelihood function can be extended to the indeterminacy region. Based on results obtained in Lubik and Schorfheide (2003) we index the multiple solutions that arise under indeterminacy through additional parameters that characterize the transmission of fundamental shocks and the distribution of sunspot shocks. We discuss the extent to which these additional and the original DSGE model parameters are identifiable and illustrate that even if the structural parameters are only partially identified the likelihood function provides interesting and useful information about indeterminacy and the propagation of shocks.

To summarize the information contained in the likelihood function we use a Bayesian approach. The likelihood function is combined with a prior density for the model parameters and the resulting function is interpreted as posterior density of the parameters given the data. In the directions of the parameter space in which there is no identification, that is, where the likelihood function is flat, the prior density is not updated. Nevertheless, the posterior density provides a coherent summary of prior and sample information. The Bayesian approach lets us calculate probability weights for the determinacy and indeterminacy region of the parameter space and posterior distributions for the propagation of shocks in the model.

The second contribution of this paper is an application of our econometric tools to a New Keynesian business cycle model. We revisit the question whether U.S. monetary policy was stabilizing pre- and post-Volcker. Our estimates confirm the finding of CGG that U.S. monetary policy before 1979 has contributed to aggregate instability and that policy has become markedly more stabilizing during the Volcker-Greenspan period.

Our multivariate analysis has several advantages over the univariate approach pursued by CGG. First and foremost, we are able to estimate the additional parameters that characterize the model solution under indeterminacy and can assess the importance of sunspots and the propagation of structural shocks. We offer two interpretations of the pre-Volcker dynamics of output, inflation, and interest rates in the U.S.. Under the first interpretation the presence of indeterminacy changed the propagation of the monetary policy, demand, and supply shocks, but sunspot shocks played no role. Under the second interpretation, the responses to fundamental shocks resemble the determinacy responses. Additional sunspot shocks caused substantial inflation and interest rate fluctuations but had only a small effect on aggregate output. The data slightly favor the first interpretation.

More generally, indeterminacy is a property of a dynamic system and should therefore studied through multivariate analysis. In most models the indeterminacy region is a complicated function of several parameters, not just the parameters of a monetary policy reaction function. In the context of DSGE models full-information estimators are typically more efficient than instrumental variable estimators based on single equations. Ruge-Murcia (2002) provides some simulation evidence. However, the potential presence of sunspot fluctuations may cause identification problems that are not readily transparent in a univariate analysis. We provide an example in which the parameter that marks the determinacy region of the parameter space is only identifiable under indeterminacy. Nevertheless, it is possible to infer whether data have been generated from the determinacy region.

A weakness of our approach is that it is potentially sensitive to model misspecification. The endogenous dynamics in the indeterminacy region of the parameter space are richer than in the determinacy region. Thus, propagation mechanisms omitted from the specification of the DSGE model tend to bias our posteriors toward indeterminacy. However, this weakness is shared by all system-based approaches to evaluation of indeterminacies. To check the robustness of our empirical findings we compare the pre-Volcker fit of the simple New Keynesian model to a richer model specification with habit formation and backward-looking price setters which we restrict to the determinacy region. We find that the data favor the indeterminacy interpretation provided by the simple model.

It has also been noted in the literature that the association of passive monetary policy with indeterminacy is very model specific. For instance, Dupor (2001) shows that in a continuous time model with endogenous investment passive monetary policy can be consistent with determinacy. Hence our empirical finding that prior to 1979 aggregate fluctuations of output, inflation, and interest rates are best described by indeterminacy is conditional on the model choice. However, since the New Keynesian monetary model considered in this paper has become a standard benchmark in the literature, we regard it as a good starting point for the application of our techniques.

Other empirical studies of indeterminacy fall broadly into two categories: First, calibration exercises, such as Farmer and Guo (1994), Perli (1998) or Schmitt-Grohé (1997, 2000), that attempt to quantify the extent to which sunspot shocks are helpful in matching model properties to business cycle facts. They face the common difficulty of specifying the stochastic properties of the sunspot shock. The typical practice is to choose its variance to match the observed variance of output. While these authors demonstrate the qualitative importance of indeterminacy, their quantitative conclusions are more tenuous. As we show in this paper, equilibrium indeterminacy does not imply that aggregate fluctuations are in fact driven by sunspots.

An alternative empirical approach is taken by Farmer and Guo (1995) and Salyer and Sheffrin (1998). They try to identify sunspot shocks from rational expectations residuals that are left unexplained by exogenous fundamentals. Although this approach imposes more structure than simple calibration, it cannot distinguish between omitted fundamentals and actual sunspots. All of the cited papers ask the question 'can the observed business cycle fluctuations be explained by sunspot shocks?' Our approach treats the determinacy and indeterminacy hypotheses symmetrically and quantifies the empirical evidence in favor of one against the other.

The closest theoretical and empirical precursors[1] to this paper are Leeper and Sims (1994), Kim (2000), Ireland (2001), and Rabanal and Rubio-Ramirez (2002)

[1]In a much earlier contribution Jovanovic (1989) provides a general characterization of the identification problems inherent in econometric analyses of models with multiple equilibria. Cooper (2001) surveys different empirical approaches to equilibrium indeterminacy from several fields.

who estimate monetary models similar to ours with likelihood-based techniques. Ireland (2001) finds significant evidence of a change in monetary policy behavior after 1979. However, all of the above authors explicitly rule out indeterminate equilibria in their estimation strategy. If the data are in fact best described by parameters from the indeterminacy region, a restriction of the estimates to the determinacy region will result in biased parameter estimates. This problem vanishes if the model is estimated over the entire parameter space.

The paper is structured as follows. In Section 2 we present the log-linearized New Keynesian monetary business cycle model to which our econometric tools are applied. Section 3 illustrates the econometric tools in the context of a very simple one-equation model. The representation of the indeterminacy solutions for a canonical LRE model is presented in Section 4. Section 5 discusses some of the advantages of our inferential approach. Empirical results for the New Keynesian model are presented in Section 6. Section 7 concludes and the Appendix provides computational details.

2 A Model for the Analysis of Monetary Policy

For the empirical analysis presented in this paper we consider a prototypical New Keynesian monetary DSGE model. This model can be summarized by the following three equations:

$$\widetilde{x}_t = I\!E_t[\widetilde{x}_{t+1}] - \tau(\widetilde{R}_t - I\!E_t[\widetilde{\pi}_{t+1}]) + g_t, \tag{1}$$

$$\widetilde{\pi}_t = \beta I\!E_t[\widetilde{\pi}_{t+1}] + \kappa(\widetilde{x}_t - z_t), \tag{2}$$

$$\widetilde{R}_t = \rho\widetilde{R}_{t-1} + (1-\rho)\left(\psi_1\widetilde{\pi}_t + \psi_2[\widetilde{x}_t - z_t]\right) + \epsilon_{R,t}, \tag{3}$$

where x is output, π is inflation, and R the nominal interest rate. The tilde denotes percentage deviations from a steady state or, in the case of output, from a trend path. Using log-linear approximations these equations can be derived from a micro-founded dynamic general equilibrium model. Details can be found, for instance, in King (2000) and Woodford (2003).

Equation (1) is an intertemporal Euler equation obtained from the households' optimal choice of consumption and bond holdings. Since the underlying model has no investment, output is proportional to consumption up to an exogenous process that can be interpreted as time-varying government spending or, more broadly, as preference change. The net effects of these exogenous shifts on the Euler equation are captured in the process g_t. The parameter $0 < \beta < 1$ is the households' discount factor and $\tau > 0$ can be interpreted as intertemporal substitution elasticity.

The production sector in the underlying model economy is characterized by a continuum of monopolistically competitive firms, each of which faces a downward sloping demand curve for its differentiated product. Prices are sticky due to quadratic adjustment costs in nominal prices or a Calvo-style rigidity that allows only a fraction of firms to adjust their prices. The resulting inflation dynamics are described by the expectational Phillips curve (2) with slope κ. The process z_t captures exogenous shifts of the marginal costs of production.

The third equation describes the behavior of the monetary authority. The central bank follows a nominal interest rate rule by adjusting its instrument to deviations of inflation and output from their respective target levels. The shock $\epsilon_{R,t}$ can be interpreted as unanticipated deviation from the policy rule or as policy implementation error. Its standard deviation is denoted by σ_R.

We assume that both g_t and z_t evolve according to univariate AR(1) processes with coefficients ρ_g and ρ_z, respectively:

$$g_t = \rho_g g_{t-1} + \epsilon_{g,t}, \quad z_t = \rho_z z_{t-1} + \epsilon_{z,t}. \tag{4}$$

We allow for non-zero correlation ρ_{gz} between the innovations $\epsilon_{g,t}$ and $\epsilon_{z,t}$ and denote their standard deviations by σ_g and σ_z. The parameters of the log-linearized DSGE model are collected in the vector

$$\theta = [\psi_1, \psi_2, \rho_R, \kappa, \tau, \rho_g, \rho_z, \rho_{gz}, \sigma_R, \sigma_g, \sigma_z]$$

with domain Θ. The linear rational expectations model comprised of Equations (1) to (4) can be rewritten in the canonical form

$$\Gamma_0(\theta) s_t = \Gamma_1(\theta) s_{t-1} + \Psi(\theta)\epsilon_t + \Pi(\theta)\eta_t, \tag{5}$$

where

$$\begin{aligned} s_t &= [\widetilde{x}_t, \widetilde{\pi}_t, \widetilde{R}_t, I\!\!E_t[\widetilde{x}_{t+1}], I\!\!E_t[\widetilde{\pi}_{t+1}], g_t, z_t]' \\ \epsilon_t &= [\epsilon_{R,t}, \epsilon_{g,t}, \epsilon_{z,t}]' \\ \eta_t &= [(\widetilde{x}_t - I\!\!E_{t-1}[\widetilde{x}_t]), (\widetilde{\pi}_t - I\!\!E_{t-1}[\widetilde{\pi}_t])]'. \end{aligned}$$

In our model the dimension of s_t is $n = 7$. There are $l = 3$ fundamental shocks and the vector of rational expectations forecast errors, η_t, has dimension $k = 2$. We assume that in addition to the fundamental shock ϵ_t the agents observe an exogenous sunspot shock ζ_t.

It is well known in the literature that this log-linear model can give rise to self-fulfilling expectations if the central bank does not raise the nominal interest rate aggressively enough in response to inflation. In this case not just the fundamental shocks ϵ_t but also the sunspot shock ζ_t can influence the dynamics of output, inflation, and interest rates. More formally, since the model (5) is linear and the only sources of uncertainty are the shocks ϵ_t and ζ_t the forecast errors for output and inflation can be expressed as as

$$\eta_t = A_1 \epsilon_t + A_2 \zeta_t \tag{6}$$

where A_1 is $k \times l$ and A_2 is $k \times 1$. Solution algorithms for LRE systems construct a mapping from the shocks to the expectation errors either implicitly, e.g., Blanchard and Kahn (1980), or explicitly, e.g., Sims (2002).

Since the transversality conditions of the underlying optimization problems impose restrictions on the growth rates of s_t, it is common to consider only solutions to the LRE system for which s_t is non-explosive. Three cases can be distinguished: (i) non-existence of a stable solution, (ii) existence of a unique stable solution (determinacy) in which A_1 is determined by the structural parameters θ and $A_2 = 0$, and (iii) existence of multiple stable solutions (indeterminacy) in which A_1 is not uniquely determined by θ and A_2 can be non-zero. Loosely speaking, the monetary DSGE model described by Equations (1) to (3) has a unique stable solution if the central bank raises the real interest rate in response to inflation ($\psi_1 > 1$) and has multiple stable solutions otherwise. The former policy is often called 'active', whereas the latter is regarded as 'passive'.

In the existing literature system-based econometric inference has been limited to the subset of the parameter space for which the stable solution is unique. The novelty of this paper is to extend estimation and inference to the indeterminacy region of the parameter space. In particular we want to be able to assess the evidence of determinacy versus indeterminacy and estimate the propagation of fundamental and sunspot shocks under indeterminacy.

3 Econometric Inference

In order to extend the estimation of LRE models to the indeterminacy region of the parameter space we have to overcome two challenges. First, we need a convenient representation for the multiplicity of solutions. Here we build upon results that we have derived in Lubik and Schorfheide (2003). Second, we have to pay careful attention to identification issues. Some of the identification problems are trivial. For instance, under determinacy the variance of the sunspot shock ζ_t is not identifiable since $A_2 = 0$ and the sunspot shock has no influence on the endogenous variables. Other identification problems are less transparent. For example, in some instances one of the model parameters marks the determinacy region, but is only identifiable under indeterminacy. Nevertheless, it is typically possible to learn from the data whether they have been generated under determinacy or indeterminacy. We discuss these challenges and our solution in the context of the following single equation model

$$y_t = \frac{1}{\theta} I\!E_t[y_{t+1}] + \epsilon_t, \tag{7}$$

where $\epsilon_t \sim iid(0,1)$ and $\theta \in \Theta = [0,2]$. This model can be cast in the canonical form (5) by introducing the conditional expectation $\xi_t = I\!E_t[y_{t+1}]$ and the forecast error $\eta_t = y_t - \xi_{t-1}$. Thus,

$$\xi_t = \theta\xi_{t-1} - \theta\epsilon_t + \theta\eta_t. \tag{8}$$

We chose this very simple example in order to make the properties of our econometric approach transparent. Extensions to the monetary DSGE model of Section 2 will be discussed in Sections 4 and 5.

The stability properties of the difference equation (8) hinge on the value of the parameter θ. If $\theta > 1$ (determinacy) the only stable solution[2] is of the form $\xi_t = 0$, which obtains if $\eta_t = \epsilon_t$ and

$$y_t = \epsilon_t \tag{9}$$

Thus, for $\theta > 1$ the endogenous variable follows an *iid* process. Its stochastic properties do not depend on the value of θ. We denote the determinacy region of the parameter space by $\Theta^D = (1, 2]$.

If $\theta \leq 1$ (indeterminacy) the stability requirement imposes no restrictions on the rational expectations forecast error η_t:

$$\eta_t = \widetilde{M}\epsilon_t + \zeta_t \tag{10}$$

Here $\widetilde{M}$ is a parameter, unrelated to θ, that arises because the effect of the fundamental shock ϵ_t is not determined. Moreover, part of the forecast error may be due to a sunspot shock ζ_t that is unrelated to the fundamental disturbance ϵ_t. For simplicity we assume in this section that $\zeta_t = 0$ for all t. Replacing $I\!\!E_t[y_{t+1}]$ in (7) by $y_{t+1} - \widetilde{M}\epsilon_{t+1}$ leads to an ARMA(1,1) representation for y_t that depends on both θ and $\widetilde{M}$:

$$y_t - \theta y_{t-1} = \widetilde{M}\epsilon_t + \theta\epsilon_{t-1}. \tag{11}$$

While y_t is generally serially correlated under indeterminacy, it reduces to the *iid* process $y_t = \epsilon_t$ that is invariant to θ in the special case of $\widetilde{M} = 1$. If the lagged variables y_{t-1} and ϵ_{t-1} are regarded as 'states', then the $\widetilde{M} = 1$ solution can be interpreted as minimal-state-variable (MSV) solution in the spirit of McCallum (1983). For the remainder of this section we will center the indeterminacy solutions around $\widetilde{M} = 1$ and use the reparameterization $\widetilde{M} = 1 + M$. The indeterminacy region of the parameter space is labelled $\Theta^I = [0, 1]$.

[2]We regard the random walk case of $\theta = 1$ as 'stable' in this example.

The goal of the econometric analysis is to summarize the sample information about θ and M. More specifically, we are interested in assessing the hypothesis $\theta \in \Theta^D$ versus $\theta \in \Theta^I$, and estimate the propagation of the shock ϵ_t, which under indeterminacy depends on both θ and M. Our inference is, of course, constrained by the restricted identifiability of the model parameters.

Let us consider the likelihood function for a sample of observations $Y^T = [y_1, \ldots, y_T]'$. The likelihood $\mathcal{L}(\theta, M|Y^T)$ is the joint probability density function of Y^T given the parameters. We will assume that ϵ_t is normally distributed and split the likelihood function into two parts which correspond to the determinacy and the indeterminacy region of the parameter space, respectively. Let $f(x) = \{x < a\}$ be the indicator function that is one if $x < a$ and zero otherwise. Then

$$\mathcal{L}(\theta, M|Y^T) = \{\theta \in \Theta^I\}\mathcal{L}_I(\theta, M|Y^T) + \{\Theta \in \Theta^D\}\mathcal{L}_D(Y^T), \tag{12}$$

where

$$\begin{aligned} \mathcal{L}_D(Y^T) &= (2\pi)^{-T/2} \exp\left\{-\frac{1}{2}Y_T'Y_T\right\} \\ \mathcal{L}_I(\theta, M|Y^T) &= (2\pi)^{-T/2}|\Gamma_Y(\theta, M)|^{-1/2} \exp\left\{-\frac{1}{2}Y_T'\Gamma_Y^{-1}(\theta, M)Y_T\right\} \end{aligned}$$

and $\Gamma_Y(\theta, M)$ denotes the covariance matrix of the vector Y^T under the ARMA representation (11). In the determinacy region of the parameter space the likelihood function is invariant to θ and M, that is, $\mathcal{L}_D$ is constant. In other words: the parameters θ and M are not identifiable. Moreover, for $M = 0$ the indeterminacy likelihood is constant as a function of θ:

$$\mathcal{L}_I(\theta, M = 0|Y^T) = \mathcal{L}_D(Y^T). \tag{13}$$

To illustrate the shape of the likelihood function we generate two data sets from the one-equation LRE model, denoted by $Y^T(D)$ and $Y^T(I)$. The sample size is $T = 30$. $Y^T(D)$ is generated from the determinacy region of the parameter space and consists of $iid\mathcal{N}(0,1)$ draws, whereas $Y^T(I)$ is generated based on $\theta = 0.8$ and $M = 1$. Figure 1 depicts the surface of the log-likelihood functions (standardized by their respective maxima) for the two samples. Although both likelihood functions are flat for $\theta > 1$ they do provide useful information. Since the autocovariances of the *iid* sample $Y^T(D)$ are near zero the likelihood function is small in the region of the parameter space where $\theta \leq 1$ and M is very different from zero. Lack of serial correlation is interpreted as evidence for determinacy. The indeterminacy sample $Y^T(I)$, on the other hand, exhibits serial correlation. Hence, the likelihood function is high for parameter values that are consistent with the degree of serial correlation in the sample. It also appears to be informative with respect to both θ and M.

If the goal of the econometric analysis were pure testing of determinacy versus indeterminacy, this could in principle be achieved with a likelihood ratio test of the form

$$LR = \frac{\sup_{0 \leq \theta \leq 1, M} \; \mathcal{L}_I(\theta, M | Y^T)}{\mathcal{L}_D(Y^T)}. \tag{14}$$

The sampling distribution of LR under the null hypothesis and the resulting critical values are non-standard because the parameters θ and M are non-identifiable under the null hypothesis. Andrews and Ploberger (1994) show that an optimal test for this problem is of the form

$$LR_{ave} = \int \frac{\sup_{0 \leq \theta \leq 1, M} \; \mathcal{L}_I(\theta, M | Y^T)}{\mathcal{L}_D(Y^T)} w(\theta, M) \; d\theta \cdot dM \tag{15}$$

where $w(\theta, M)$ assigns large weight to alternatives against which the test is supposed to have high power. To construct such a test statistic one has to know the directions of the parameter space that lack identification. It is trivial in this one-equation example, but can become prohibitively complicated in large LRE models with many parameter restrictions.

However, rather than asking the question 'are the data consistent with equilibrium determinacy' it is more useful to construct probability weights for the determinacy and indeterminacy regions of the parameter space conditional on the observed data. These probabilities can then be used to weight the parameter estimates and the predictions that the model delivers conditional on the two regions of the parameter space. Hence, we will follow a Bayesian approach by placing a prior distribution with density $p(\theta, M)$ on the parameters θ and M and conducting inference based on the posterior distribution of the parameters given the data Y^T. This posterior distribution can be calculated with Bayes theorem:

$$p(\theta, M|Y^T) = \frac{(\{\theta \in \Theta^I\}\mathcal{L}_I(\theta, M|Y^T) + \{\theta \in \Theta^D\}\mathcal{L}_D(Y^T))p(\theta, M)}{\int \mathcal{L}(\theta, M|Y^T)p(\theta, M)d\theta \cdot dM}. \tag{16}$$

Notice that conditional on $\theta \in \Theta^D$ (determinacy) the shape of the posterior density is the same as the prior density in this simple example. More generally, the prior density is only updated in the directions of the parameter space for which the data are informative and remains unchanged in directions in which the likelihood is flat (see, for instance, Poirier (1998)). Nevertheless, the posterior always delivers a coherent summary of the information contained in both prior and sample. The posterior probability of indeterminacy is given by[3]

$$\pi_T(I) = \int \{\theta \in \Theta^I\}p(\theta, M|Y^T)d\theta \cdot dM. \tag{17}$$

If we assign a prior distribution to the parameters θ and M that is uniform on the range $[0:2] \otimes [-1:5]$ then we can interpret the likelihood plots in Figure 1 as probability densities. The implied prior probability of indeterminacy is

$$\pi_0(I) = \int \{\theta \in \Theta^I\}p(\theta, M)d\theta \cdot dM = 0.5. \tag{18}$$

In the second row of Figure 1 we plot standardized (by their respective maxima) marginal posterior density functions

$$p(\theta|Y^T) \propto \frac{1}{12}\{\theta \in \Theta^I\} \int \mathcal{L}_I(\theta, M|Y^T)dM + \frac{1}{2}\{\theta \in \Theta^D\}\mathcal{L}_D(Y^T), \tag{19}$$

where $\propto$ denotes proportionality. The ratio of the area underneath the marginal density to the left and right of $\theta = 1$ corresponds to the posterior odds ratio $\pi_T(I)/\pi_T(D)$ in favor of indeterminacy.

Consider the determinacy sample $Y^T(D)$. The likelihood function is large for parameter values that imply little serial correlation. This includes $\theta > 1$ and $\theta < 1, M \approx 0$. Since the indeterminacy region also contains parameter combinations that imply substantial serial correlation the marginal posterior density of θ

[3]The posterior probability of determinacy is $\pi_T(D) = 1 - \pi_T(I)$. Notice that in this example the ratio of $\pi_T(D)/\pi_T(I)$ is essentially equivalent to Andrews and Ploberger's likelihood ratio statistic, except that the weight function $w(\theta, M)$ is replaced by a prior density.

is substantially lower for $\theta < 1$ than it is for $\theta > 1$. Thus, the posterior odds strongly favor determinacy. Just as the likelihood ratio test of Andrews and Ploberger has no power against the $\theta < 1$ and $M = 0$ indeterminacy alternative, the Bayesian procedure will interpret a long *iid* sample as evidence in favor of determinacy as long as the prior distribution for M is continuous and assigns zero probability to $M = 0$. If the data exhibit serial correlation then there are values of $\theta < 1$ and $M \neq 0$ for which the likelihood function is substantially larger than in the determinacy region which is also reflected in the marginal posterior density of θ. Even based on our short sample the posterior probability of indeterminacy is essentially one.[4]

The findings of this section can be summarized as follows. In the context of a one-equation model we have shown that an additional parameter M, which is unrelated to θ is needed to characterize the dynamics of y_t. We have discussed the resulting identification problems and illustrated how a Bayesian approach can be used to make inference with respect to determinacy versus indeterminacy and the parameters that determine the law of motion for y_t. In the following section we will provide a characterization for the solutions of the canonical LRE model (5) under indeterminacy.

4 Solution of the Canonical LRE Model

In solving the LRE system (5) we closely follow the approach of Sims (2002), extended in Lubik and Schorfheide (2003).[5] To keep the exposition simple we assume

[4]The posterior probabilities provide a consistent test for indeterminacy. If $\theta \in \Theta^D$ then $\pi_T(I)$ will converge in probability to zero as the sample size T tends to infinity. If $\theta \in \Theta^I$ then $\pi_T(I)$ will converge to one. The latter convergence will fail on a subset of the parameter space, namely $\theta < 1$ and $M = 0$, which has probability zero under our prior distribution. A formal proof can be constructed by deriving a large sample approximation of $\int \mathcal{L}_I(\theta, M|Y^T)p(\theta, M)d\theta \cdot dM$ as in Phillips (1996), Kim (1998), and Fernandez-Villaverde and Rubio-Ramirez (2001) and then applying the argument in Hannan (1980) to treat the identification problem that arises in 'determinacy' samples.

[5]Sims (2002) solution procedure generalizes the method proposed by Blanchard and Kahn (1980). In particular, it does not require the researcher to separate the list of endogenous variables into

that the matrix Γ_0 in Eq. (5) is invertible.[6] The system can be rewritten as

$$s_t = \Gamma_1^*(\theta)s_{t-1} + \Psi^*(\theta)\epsilon_t + \Pi^*(\theta)\eta_t. \tag{20}$$

Replace Γ_1^* by its Jordan decomposition $J\Lambda J^{-1}$ and define the vector of transformed model variables $w_t = J^{-1}s_t$. Let the i'th element of w_t be $w_{i,t}$ and denote the i'th row of $J^{-1}\Pi^*$ and $J^{-1}\Psi^*$ by $[J^{-1}\Pi^*]_{i.}$ and $[J^{-1}\Psi^*]_{i.}$, respectively. The model can be rewritten as a collection of AR(1) processes

$$w_{i,t} = \lambda_i w_{i,t-1} + [J^{-1}\Pi^*]_{i.}\epsilon_t + [J^{-1}\Psi^*]_{i.}\eta_t. \tag{21}$$

We will refer to the $w_{i,t}$'s as latent 'states.' Define the set of stable AR(1) processes as

$$I_s(\theta_{(1)}) = \left\{ i \in \{1, \dots n\} \middle| \left|\lambda_i(\theta_{(1)})\right| \leq 1 \right\} \tag{22}$$

and let $I_x(\theta_{(1)})$ be its complement. Let Ψ_x^J and Π_x^J be the matrices composed of the row vectors $[J^{-1}\Psi^*]_{i.}$ and $[J^{-1}\Pi^*]_{i.}$ that correspond to unstable eigenvalues, i.e., $i \in I_x(\theta_{(1)})$. To ensure stability of s_t the expectation errors η_t have to satisfy

$$\Psi_x^J\epsilon_t + \Pi_x^J\eta_t = 0 \tag{23}$$

for all t. Equation (23) has either no solution, one solution (determinacy), or multiple solutions (indeterminacy). We will assume that the parameter space Θ is restricted to the set of θ's for which at least one solution to Equation (23) exists. To solve the potentially underdetermined system of equations (23) for η_t it is convenient to proceed with a singular value decomposition of the matrix Π_x^J:

$$\Pi_x^J = \begin{bmatrix} U_{.1} & U_{.2} \end{bmatrix} \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V'_{.1} \\ V'_{.2} \end{bmatrix} = \underbrace{U}_{m\times m}\underbrace{D}_{m\times k}\underbrace{V'}_{k\times k} = \underbrace{U_{.1}}_{m\times r}\underbrace{D_{11}}_{r\times r}\underbrace{V'_{.1}}_{r\times k}, \tag{24}$$

where D_{11} is a diagonal matrix and U and V are orthonormal matrices. Here we used m to denote the number of unstable eigenvalues and r is the number of non-zero

'jump' and 'predetermined' variables. It recognizes that it is the structure of the coefficient matrices that implicitly pins down the solution. Instead of imposing ex ante which individual variables are 'predetermined', Sims' algorithm determines endogenously the linear combinations of variables that have to be 'predetermined' for a solution to exist.

[6] If Γ_0 is singular, a generalized complex Schur decomposition (QZ) can be used to manipulate the system, see Sims (2002) and the appendix of this paper.

singular values of Π_x^J. Recall that k is the dimension of the vector of forecast errors η_t and l denotes the number of exogenous shocks. Let p be the dimension of the sunspot shock ζ_t. The following proposition is proved in Lubik and Schorfheide (2003):

Proposition 1 *If there exists a solution to Eq. (23) that expresses the forecast errors as function of the fundamental shocks ϵ_t and sunspot shocks ζ_t, it is of the form*

$$\begin{aligned} \eta_t &= \eta_1 \epsilon_t + \eta_2 \zeta_t \\ &= (-V_{.1} D_{11}^{-1} U_{.1}' \Psi_x^J + V_{.2} \widetilde{M}) \epsilon_t + V_{.2} M_\zeta \zeta_t, \end{aligned} \tag{25}$$

where $\widetilde{M}$ is an $(k-r) \times l$ matrix, M_ζ is a $(k-r) \times p$ matrix, and the dimension of $V_{.2}$ is $k \times (k-r)$. The solution is unique if $k = r$ and $V_{.2}$ is zero.

The representation for the rational expectations forecast errors leads to the following law of motion for s_t:

$$\begin{aligned} s_t &= \Gamma_1^*(\theta) s_{t-1} + [\Psi^*(\theta) - \Pi^*(\theta) V_{.1}(\theta) D_{11}^{-1}(\theta) U_{.1}'(\theta) \Psi_x^J(\theta)] \epsilon_t \\ &\quad + \Pi^*(\theta) V_{.2}(\theta)(\widetilde{M} \epsilon_t + M_\zeta \zeta_t). \end{aligned} \tag{26}$$

Under determinacy $V_{.2} = 0$ and the terms in the second line of Equation (26) drop out. In this case the dynamics of x_t are purely a function of the parameter vector θ. Indeterminacy introduces additional parameters and changes the nature of the solution in two dimensions. First, the propagation of the structural shocks ϵ_t is not uniquely determined as it depends on the matrix $\widetilde{M}$. Second, the dynamics of s_t are potentially affected ($M_\zeta \neq 0$) by the sunspot shocks ζ_t. In the monetary DSGE model of Section 2 the degree of indeterminacy $k - r$ is at most 1. Hence, we set $p = 1$ and impose the normalization $M_\zeta = 1$. The standard deviation of the sunspot shock, denoted by σ_ζ, is treated as additional parameter. Since it is not possible to identify the covariances of the sunspot shock with the fundamental shocks in addition to $\widetilde{M}$ we use the normalization $I\!E[\epsilon_t \zeta_t] = 0$.

In Section 3 we reparameterized the indeterminacy solutions by $\widetilde{M} = 1 + M$, such that $M = 0$ corresponded to $y_t = \epsilon_t$. While we consider all possible values of $\widetilde{M}$

in our estimation procedure we specify a prior distribution that is centered around one particular solution. We do this by replacing $\widetilde{M}$ with $M^*(\theta)+M$ and setting the prior mean for M equal to zero. We find it desirable to choose $M^*(\theta)$ such that the impulse responses $\partial s_t/\partial\epsilon_t'$ are continuous at the boundary between the determinacy and indeterminacy region. According to our prior mean, small changes of θ do not lead to drastic changes in the propagation of fundamental shocks.

One candidate for $M^*(\theta)$ is the minimal-state-variable solution considered in Section 3. Suppose in a model with a one-dimensional indeterminacy it is possible to identify an eigenvalue function $\lambda_{i^*}(\theta)$ such that $|\lambda_{i^*}(\theta)|$ is greater than one in the determinacy region ($\theta \in \Theta^D$) and less than one in the indeterminacy region ($\theta \in \Theta^I$). In this case a baseline solution could be constructed by solving the system of equations

$$[J^{-1}\Pi^*]_i\epsilon_t + [J^{-1}\Psi^*]_i\eta_t = 0 \tag{27}$$

for $i \in I_x(\theta)$ and $i = i^*$. This solution has the feature that it eliminates as many 'states' $w_{i,t}$ as the determinacy solution.

While it is possible to define and track eigenvalue functions in the model presented in Section 2, it is difficult in larger systems. Moreover, there is no guarantee that this procedure yields economically plausible impulse response for values of θ that are not in the immediate vicinity of the determinacy region. Hence, we proceed with an alternative method. For every vector $\theta \in \Theta^I$ we construct a vector $\tilde{\theta} = g(\theta)$ that lies on the boundary of the determinacy region and choose $M^*(\theta)$ such that the response of s_t to ϵ_t conditional on θ mimics the response conditional on $\tilde{\theta}$. Thus, we compare

$$\begin{aligned} \frac{\partial s_t}{\partial \epsilon_t'}(\theta, M) &= \Psi^*(\theta) - \Pi^*(\theta)V_{.1}(\theta)D_{11}^{-1}(\theta)U_{.1}'(\theta)\Psi_x^J(\theta) + \Pi^*(\theta)V_{.2}(\theta)\widetilde{M} \\ & \quad B_1(\theta) + B_2(\theta)\widetilde{M} \end{aligned} \tag{28}$$

to

$$\frac{\partial s_t}{\partial \epsilon_t'}(g(\theta), \cdot) = B_1(g(\theta)). \tag{29}$$

In our application we minimize the discrepancy using a least squares criterion and choose

$$M^*(\theta) = [B_2(\theta)'B_2(\theta)]^{-1}B_2(\theta)' * [B_1(g(\theta)) - B_1(\theta)] \tag{30}$$

The function $g(\theta)$ is obtained by replacing ψ_1 in the vector θ with

$$\tilde{\psi}_1 = 1 - \frac{\beta\psi_2}{\kappa}\left(\frac{1}{\beta} - 1\right), \tag{31}$$

which marks the boundary between the determinacy and indeterminacy region for the model presented in Section 2.[7] We will refer to the solution $\widetilde{M} = M^*(\theta)$ as baseline indeterminacy solution. The impulse response analysis in Section 6 will reveal that, by and large, the responses under the baseline indeterminacy solution are similar to the determinacy responses, expect for the scaling of the interest rate response which is sensitive to the choice of ψ_1.

While our baseline indeterminacy solution provides a plausible benchmark, our estimation under indeterminacy is not restricted to this specific solution. We only use it to center our prior distribution for $\widetilde{M}$ in Equation (26). Based on the solution described in this section we construct a likelihood function $\mathcal{L}(\theta, M, \sigma_\zeta | Y^T)$ that is used for Bayesian inference as described in Section 3.

5 Discussion

Before proceeding with the empirical analysis we will discuss some of the virtues and limitations of our inference approach and compare it to alternatives that have been proposed in the literature. To focus the discussion we consider a special case of the model presented in Section 2. Suppose that the monetary authority does not attempt to smooth the nominal interest rate ($\rho_R = 0$) and only targets current inflation ($\psi_2 = 0$). Moreover, the exogenous processes g_t and z_t have no serial correlation, that is, $\rho_g = \rho_z = 0$. Define the conditional expectations $\xi_t^x = I\!E_t[\widetilde{x}_{t+1}]$,

[7]The derivation of this formula is relegated to a Technical Appendix that is available from the authors upon request.

$\xi_t^\pi = I\!E_t[\widetilde{\pi}_{t+1}]$ and the vector $\xi_t = [\xi_t^x, \xi_t^\pi]'$. The vector ξ_t evolves according to

$$\xi_t = \underbrace{\begin{bmatrix} 1+\frac{\kappa\tau}{\beta} & \tau(\psi_1 - \frac{1}{\beta}) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}}_{\Gamma_1^*} \xi_{t-1} + \underbrace{\begin{bmatrix} \tau & -1 & 0 \\ 0 & 0 & \kappa \end{bmatrix}}_{\Psi^*} \epsilon_t + \underbrace{\begin{bmatrix} 1+\frac{\kappa\tau}{\beta} & \tau(\psi_1 - \frac{1}{\beta}) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}}_{\Pi^*} \eta_t. \tag{32}$$

Since the simplified model has a block-triangular structure the vector of expectation errors η_t can be determined by applying the methods described in Section 4 to the two-dimensional subsystem (32).

The dynamics of the system depend on the eigenvalues of Γ_1^*. It can be shown that both eigenvalues are unstable if $\psi_1 > 1$ (see, for instance, Bullard and Mitra (2002) or Lubik and Marzo (2003)). In this case the only stable solution is $\xi_t = 0$ which uniquely determines the forecast errors $\eta_t = -\Pi^*\Psi^*\epsilon_t$. Tedious but straightforward algebraic manipulations lead to the following law of motion for output, inflation, and interest rates:

$$\begin{bmatrix} \widetilde{x}_t \\ \widetilde{\pi}_t \\ \widetilde{R}_t \end{bmatrix} = \frac{1}{1+\kappa\tau\psi_1} \begin{bmatrix} -\tau & 1 & \tau\kappa\psi_1 \\ -\kappa\tau & \kappa & -\kappa \\ 1 & \kappa\psi_1 & -\kappa\psi_1 \end{bmatrix} \begin{bmatrix} \epsilon_{R,t} \\ \epsilon_{g,t} \\ \epsilon_{z,t} \end{bmatrix}. \tag{33}$$

The model exhibits no dynamics because the solution suppresses the two roots of the autoregressive matrix Γ_1^*. An unanticipated monetary contraction leads to a one-period fall in output and inflation. An Euler-equation shock $\epsilon_{g,t}$ increases output, inflation, and interest rate. A Phillips-curve shock $\epsilon_{z,t}$ raises output, but lowers inflation and interest rates for one period.

If $\psi_1 < 1$ only one of the eigenvalues is unstable and the evolution of the endogenous variables can be described as follows:

$$\begin{bmatrix} \widetilde{x}_t \\ \widetilde{\pi}_t \\ \widetilde{R}_t \end{bmatrix} = \frac{1}{1+\kappa\tau\psi_1} \begin{bmatrix} -\tau & 1 & \tau\kappa\psi_1 & (\lambda_2 - 1 - \kappa\tau\psi_1) \\ -\kappa\tau & \kappa & -\kappa & \kappa\lambda_2 \\ 1 & \kappa\psi_1 & -\kappa\psi_1 & \psi_1\kappa\lambda_2 \end{bmatrix} \begin{bmatrix} \epsilon_t \\ \zeta_t \end{bmatrix} \tag{34}$$
$$+ \begin{bmatrix} (\beta(\lambda_2 - 1) - \tau\kappa)/\kappa \\ 1 \\ \psi_1 \end{bmatrix} w_{1,t-1},$$

where $w_{1,t}$ follows the AR(1) process

$$w_{1,t} = \lambda_1(\theta) w_{1,t-1} + \mu_1(\theta)(M\epsilon_t + \zeta_t)$$

Here $\lambda_1(\theta)$ is the stable eigenvalue of $\Gamma_1^*(\theta)$ and μ_1 is a function of the parameter vector θ. We implicitly chose $M^*(\theta)$ in the notation of Section 4 such that for $M = 0$ the structural shocks have no persistent effect on the endogenous variables. If $M \neq 0$ or $\sigma_\zeta > 0$ output, inflation, and interest rates will be serially correlated.

The stated goal of our empirical analysis is two-fold. We want to assess the evidence in favor of determinacy versus indeterminacy and we want to estimate the model parameters to understand how shocks are being propagated in the system. There are two pieces of information that can help us distinguish between determinacy and indeterminacy, namely direct information about θ and the autocovariance pattern of the observations. The simplified New Keynesian model (33, 34) and the example in Section 3 illustrate that even in the absence of direct information about θ it is possible to detect indeterminacy through the serial correlation of the data. This insight translates to larger models. Under indeterminacy the number of stable eigenvalues is generally larger than under determinacy. In the terminology of Section 4, fewer 'states' $w_{i,t}$ are suppressed. Thus, one can expect a richer autocovariance pattern that cannot be reproduced with parameters from the determinacy region. The cross coefficient restrictions generated by the DSGE model might generate further evidence on determinacy versus indeterminacy. Equation (33) suggests that the covariance matrix of output, inflation, and interest rates can provide direct information about ψ_1.

Our likelihood-based approach will exploit both sources of information simultaneously when we construct the posterior weights for the two regions of the parameter space. Using the information from the autocovariances has one disadvantage. It makes the inference sensitive to model misspecification. In the simplified version (33, 34) of the monetary model serial correlation in the data is interpreted as evidence in favor of indeterminacy. However, an alternative explanation for serial correlation could be that the supply and demand shocks are serially correlated rather than independent over time. Under this type of misspecification the posterior weights that we are constructing are potentially biased toward indeterminacy. In our empirical analysis we will verify the robustness of our conclusions by considering a model with richer dynamics, restricted to determinacy, as an alternative to the model outlined in Section 2.

CGG assess the indeterminacy hypothesis by estimating a univariate monetary policy reaction function and examining the magnitude of the estimated coefficient ψ_1. While this procedure is more robust against model misspecification, it has several drawbacks. First, single equation instrumental variable estimates are typically much less efficient than full information system estimates, in particular in models with many cross-parameter restrictions.[8] Second, CGG's analysis evolves around point estimates, downplaying the role of associated standard errors. It does not provide a statistical measure for the likelihood that the pre-1979 observations were generated from a sunspot equilibrium.

Third, the quality of the available instruments can in principle be closely linked to the determinacy hypothesis. In the simplified version of the DSGE model considered in this section the single-equation estimation of ψ_1 fails under determinacy. Since $\widetilde{\pi}_t$ is correlated with the monetary policy shock $\epsilon_{R,t}$ an instrument is needed. However, current output is also correlated with the monetary policy shock. The only remaining possibility is to use lagged values of inflation, output, or interest rates as instruments. This instrumental variable approach can only be successful if $\psi_1 < 1$ and $M \neq 0$ or $\sigma_\zeta > 0$ such that there is serial correlation in the endogenous variables. To be fair, the absence of serial correlation in demand and supply shocks is an unrealistic assumption. Nevertheless, the argument suggests that the equilibrium properties of a model should be examined carefully to find an informative set of instruments. The most efficient, albeit sensitive to model misspecification, set of instruments is embodied in the likelihood function.

To study the propagation of shocks under indeterminacy we need parameter estimates of the entire vector θ and the matrix M. While estimates of θ could in

[8] See Ruge-Murcia (2002) for some simulation evidence in the context of DSGE models.

principle be obtained based on generalized method of moments estimates of Equations (1) to (3) subject to the availability of appropriate instruments, the estimation of M requires a full-information approach such as the one proposed in this paper.

6 Empirical Results

The log-linearized monetary DSGE model described in Section 2 is fitted to quarterly post-war U.S. data on output, inflation, and nominal interest rates.[9] Both inflation and interest rates are annualized. To make our empirical analysis comparable to other studies, such as CGG, we use the HP filter to remove a smooth trend from the output series. Results based on linear trend extraction are briefly discussed at the end of this section. In line with the monetary policy literature we consider the following sample periods: a pre-Volcker sample from 1960:I to 1979:II, a Volcker-Greenspan sample from 1979:III to 1997:IV, and a post-1982 sample from 1982:IV to 1997:IV that excludes the Volcker-disinflation period.

Inflation in the pre-Volcker years is marked by a substantial upward shift and increase in volatility for which a variety of explanations have been offered. Orphanides (2002) suggests that the Federal Reserve misjudged trend productivity growth in the 1970s and overestimated potential output. Since actual output appeared relatively low the Fed loosened its monetary policy which led to high inflation. Sargent (1999) argues that a perceived inflation-output trade-off could have led central bankers to raise inflation targets. In the absence of an actual trade-off this policy caused mainly high inflation rates. Other authors blame oil price shocks for the rise in inflation. A fourth explanation for the experience in the pre-Volcker years, set forth for instance by CGG, is that passive monetary policy failed to suppress self-fulfilling inflation expectations. In the early 1980s the Fed switched to an active

[9] The time series are extracted from the DRI·WEFA database. Output is log real per capita GDP (GDPQ), HP detrended over the period 1955:I to 1998:IV. We multiply deviations from trend by 100 to convert them into percentages. Inflation is annualized percentage change of CPI-U (PUNEW). Nominal interest rate is average Federal Funds Rate (FYFF) in percent.

monetary policy, raising the real interest rate in response to inflation deviations from target, thus eliminating fluctuations due to self-fulfilling expectations.

Our simple monetary model is not detailed enough to completely disentangle the four competing hypotheses. For instance, the central bank does not have to solve a signal extraction problem that could capture the notion of mis-measured potential output. Nevertheless, we can gain some interesting insights. Our analysis is based on the assumption that the target inflation rate, which equals the steady state inflation rate, stayed constant in the pre-Volcker years but possibly shifted in the early 1980s as steady state inflation is estimated for each subsample separately. Although the New Keynesian model does not distinguish energy and goods prices, it has the ability to capture oil price shocks as drops in z_t which, broadly speaking, shift marginal costs of production. By estimating the DSGE model over both the determinacy and indeterminacy region of the parameter space we can assess the hypothesis of passive-monetary policy and self-fulfilling expectations and study the propagation of fundamental and sunspot shocks under indeterminacy for the pre-Volcker as well as the Volcker-Greenspan years.

Observed output deviations from trend, inflation, and interest rates are stacked in the vector y_t. The measurement equation that relates y_t to the vector of model variables s_t is of the form:

$$y_t = \begin{bmatrix} 0 \\ \pi^* \\ r^* + \pi^* \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \end{bmatrix} s_t, \tag{35}$$

where π^* and r^* are annualized steady state inflation and real interest rates (percentages). Abstracting from the small effect of long-run output growth on the relationship between the discount factor β that appears in Equation (1) and r^* we replace the discount factor by $\beta = (1+r^*/100)^{-1/4}$. The measurement equation (35) together with the law of motion (26) for s_t provide a state-space model for the observables y_t. The Kalman filter (see, for instance, Hamilton (1994)) can be used to evaluate the likelihood function $\mathcal{L}(\theta, M, \sigma|Y^T)$. This likelihood function is combined with a prior distribution for the parameters θ, M, and σ_ζ and computational methods described in the Appendix are used to conduct posterior inference.

The empirical analysis is structured as follows. We first review the prior distribution, then present posterior probabilities for determinacy versus indeterminacy together with estimates of all the model parameters. To gain insights into the propagation of shocks we study impulse response functions and variance decompositions under indeterminacy. Finally, we assess the robustness of our analysis to model specification and detrending method.

6.1 Prior Distribution

The specification of the prior distribution is summarized in Table 1, which reports prior densities, means, standard deviations, and 90% probability intervals for the elements of θ, M, and σ_ζ.[10] It is assumed that the parameters are *a priori* independent. In choosing priors for the policy parameters we adopted an agnostic approach. The prior for the inflation coefficient ψ_1 is centered at 1.1 and implies a probability interval from 0.35 to 1.85. The interval for the output gap coefficient ψ_2 ranges from 0.5 to 0.45. The degree of interest rate smoothing lies between 17% and 83%.

Our prior for the annual real interest rate is centered at 2% with a standard deviation of 1. Steady state inflation ranges from 1% to 7%. The slope coefficient in the Phillips-curve is chosen to be consistent with the range of values typically found in the New-Keynesian Phillips-curve literature (see, for instance, Rotemberg and Woodford (1997), Galí and Gertler (1999), and Sbordone (2002)). Its mean is set at 0.5, but we allow the slope to vary widely in the unit interval. The prior for $1/\tau$ is centered at 2, which makes the representative agents in the underlying model more risk averse than agents with log-utility. The intervals for the autocorrelation parameters ρ_g and ρ_z imply a fairly high degree of persistence of the exogenous processes.

[10] Subsequent interval statements about parameters have the following interpretation: we report the shortest (connected) intervals that have – according to our prior/posterior – a 90% coverage probability.

The coefficients of the matrix M that appear in the indeterminacy solution have standard normal distributions. Thus, our prior is centered at the baseline solution described in Section 4. The prior for M, and the parameters that characterize the distribution of the exogenous shocks are best assessed indirectly through their implications for the volatility of output, inflation, and interest rates. Our prior implies that the contribution of the monetary policy shock to output fluctuations lies between 0 and 18%. Supply and demand shocks $\epsilon_{a,t}$ and $\epsilon_{z,t}$ may explain as little as 1% or as much as 80% of the output variation. The sunspot shock $\epsilon_{\zeta,t}$ plays hardly any role for output fluctuations but may cause up to 10% of the variation in inflation and nominal interest rates. According to our prior the standard deviation of inflation lies between 1% and 16%, which indicates that the prior for M assigns some mass in regions of the parameter space that imply a large effect of fundamental shocks on price movements.

We refer to the prior distribution described in Table 1 as *Prior 1*. *Prior 2* is obtained by imposing $M = 0$ and restricting the likelihood function in the indeterminacy region to the baseline solution described in Section 4. A third prior imposes $\sigma_\zeta = 0$, which means that there are no sunspot shocks under indeterminacy so that only the propagation of structural shocks is affected.

6.2 Estimation Results

The DSGE models is estimated under Priors 1 to 3 for the three samples. We will first examine the probability mass assigned to the determinacy and indeterminacy region. According to the priors the probability of determinacy is 0.527. One advantage of our framework is that it lets us take into account the possible dependence of the determinacy region on all elements of the parameter vector θ. To obtain the posterior probabilities for the two regions of the parameter space it is convenient to define the following (marginal) data densities

$$p^s(Y^T) = \int \{\theta \in \Theta^s\} \mathcal{L}(\theta, M, \sigma_\zeta | Y^T) p(\theta, M, \sigma_\zeta) d\theta \cdot dM \cdot d\sigma_\zeta \quad s \in \{D, I\} \tag{36}$$

by integrating the likelihood function over region s with respect to the parameters θ, M, and σ_ζ.[11] It can be seen from Equations (16) and (17) that the posterior probability of indeterminacy is given by

$$\pi_T(I) = \frac{p^I(Y^T)}{p^I(Y^T) + p^D(Y^T)}. \tag{37}$$

Table 2 reports $\ln p^s(Y^T)$ and the resulting posterior probabilities by prior and subsample.

The posterior probabilities reveal striking differences between the three subsamples. The pre-Volcker posterior concentrates (almost) all of its probability mass in the indeterminacy region. The evidence from the Volcker-Greenspan sample is mixed. Depending on the choice of prior the probability of determinacy ranges from 0.38 to 0.7. These estimates could be strongly influenced by the Volcker disinflation period which is better characterized by nonborrowed-reserve targeting than by an interest rate rule. The inflation rate drops from 15% in 1980:I to about 6% in 1982. Hence, the a sample which excludes the disinflation period is considered as an alternative. Under all three priors the posterior probability of determinacy is around 0.98 for the post-1982 sample.

The log-data densities of Table 2 can also be used to compare the different specifications of the DSGE model under indeterminacy. Under Prior 2 the model is restricted to the baseline indeterminacy solution. The odds of the unrestricted version (Prior 1) versus the $M = 0$ version (Prior 2) are 2 to 1 for the pre-Volcker sample and 1.5 to 1 for the post-1982 sample. This finding suggests that the fit of the model can be improved by deviating from the baseline solution and altering the propagation of the structural shocks. Under Prior 3 the variance of the sunspot shock is restricted to zero. Thus, agents do not react to an additional source of uncertainty. The only effect of indeterminacy is to change the transmission of structural shocks. For all three samples the 'indeterminacy without sunspots' version of the model (Prior 3) is weakly preferred to the unrestricted specification.

[11] The data density intrinsically penalizes the likelihood function under indeterminacy for the presence of the additional parameters M and σ_ζ. The Schwarz (1978) approximation of a Bayesian data density makes the penalty explicit.

Table 3 contains posterior estimates of the structural parameters for the pre-Volcker sample (Prior 1, Prior 2) conditional on indeterminacy and for the post-1982 sample conditional on determinacy. The post-1982 posterior indicates that the monetary policy followed the Taylor principle. According to the posterior mean the central bank raises the nominal rate by 2.2% in response to a 1% discrepancy between actual and desired inflation. Active inflation targeting is supported by a substantial degree of output gap targeting ($\hat{\psi}_2 = 0.3$) and interest rate smoothing ($\hat{\rho}_R = 0.84$).

Depending on the choice of prior the pre-Volcker sample leads to estimates of ψ_1 around 0.8 to 0.9. The Bayesian confidence sets for ψ_2 range from 0.05 to 0.3. The estimated steady state inflation rate appears slightly larger for the pre-Volcker than for the post-1982 samples. However, there is substantial uncertainty about π^*. The real rate was markedly lower before 1980, between 0.6% and 1.6%, than post 1982 when it was between 2.2% and 3.8% according to our posterior. The posterior mean estimate of the slope κ of the Phillips curve is 0.77 with a confidence interval ranging from 0.4 to 1.1 which is on the high side, but not unreasonably so.

Under Prior 2 (pre-Volcker) the estimated correlation between the shocks $\epsilon_{g,t}$ and $\epsilon_{z,t}$ appears unreasonably large as it implies almost perfect correlation. Other estimates of the covariance parameters as well as the vector M that determines the relationship between fundamental shocks and forecast errors are best discussed in the context of impulse response functions (IRF) and variance decompositions. Recall that conditional on determinacy the likelihood function is invariant to M and σ_ζ. Hence the posterior distribution for these parameters is identical to the prior distribution and the corresponding entries for Table 3 are left blank.

6.3 Propagation of Shocks

The system-based estimation approach pursued in this paper allows us to study the propagation of fundamental and sunspot shocks in the model. Figure 2 graphs the posterior mean responses (and pointwise 90% confidence bands) of output, inflation and interest rates to a (negative) one-standard-deviation sunspot shock. The responses are based on the pre-Volcker posterior obtained under Prior 1. Under an inflationary sunspot belief the expected real rate declines and the expected output growth is negative. The fall in the real rate stimulates current consumption and therefore output. According to the Phillips curve, this is consistent with positive current inflation which validates the initial assumption of sunspot-driven positive inflation expectations. As inflation falls toward its steady state in subsequent periods, the positive interest rate policy keeps the real rate low and output returns to its steady state.. According to our estimates the sunspot shock has a much bigger effect on inflation and interest rates than it has on output. Since both output and inflation rise the sunspot shock cannot be interpreted as stagflation shock during the 1970s in itself.

Figure 3 depicts the propagation of a monetary policy shock. Impulse responses are computed based on three different parameter estimates, all based on the pre-Volcker sample: (i) conditional on indeterminacy under Prior 1, (ii) conditional on indeterminacy under Prior 2, and (iii) conditional on determinacy. The estimated indeterminacy responses under Prior 2 are hardly distinguishable from the responses under indeterminacy. This suggests that the baseline solution around which our prior is centered extends the determinacy solution to the indeterminacy region without substantially altering the propagation of the fundamental shocks. In the absence of the $M = 0$ restriction the estimated indeterminacy responses differ from the baseline responses and reflect the non-zero estimates of $M = [M_{R\zeta}, M_{g\zeta}, M_{z\zeta}]$ reported in Table 3. Since $\hat{M}_{R\zeta}$ is negative the propagation of $\epsilon_{R,t}$ is given by a linear combination of the baseline IRFs and an inflationary sunspot shock as shown in Figure 2.

Overall, in response to an unanticipated tightening of monetary policy output drops initially by 0.15%, the interest rate rises by 70 basis points and inflation falls 0.5% below steady state. Subsequently, inflation rises to about 0.2% whereas output and inflation return to their steady states. Thus, according to our pre-Volcker estimates under Prior 1, an increase in the nominal rate can have a slightly inflationary effect. Figure 3 highlights that indeterminacy can alter the propagation of fundamental shocks. Since the estimate of $M_{r\zeta}$ is fairly imprecise the confidence bands for the inflation and interest rate responses are wide. Given the the odds of unrestricted M versus $M = 0$ are roughly 2 to 1 there is a substantial degree of uncertainty about the effects of a monetary policy shock, even conditional on our tightly parameterized DSGE model.

The shock $\epsilon_{g,t}$ shifts the consumption Euler equation and can broadly be interpreted as demand shock. Under determinacy a positive shock $\epsilon_{g,t}$ raises both output and inflation. The positive estimate of $M_{g\zeta}$ indicates that in the unrestricted indeterminacy version of the model the demand shock is less inflationary than under the $M = 0$ indeterminacy solution. A positive supply shock $\epsilon_{z,t}$ reduces the marginal costs of production, increases output and lowers inflation. The presence of indeterminacy creates an inflationary effect of $\epsilon_{z,t}$ since $\hat{M}_{z\zeta}$ is negative. The impulse response functions for the Euler-equation and Phillips-curve shocks are depicted in Figures 4 and 5.

Variance decompositions for output (deviations from trend), inflation, and interest rates are summarized in Table 4. For the pre-Volcker sample we report posteriors conditional on indeterminacy under Priors 1 and 2. The posterior for the post-1982 sample is conditional on determinacy. Since we allow $\epsilon_{g,t}$ and $\epsilon_{z,t}$ to have non-zero correlation we need to orthogonalize the two shocks before computing variance decompositions. We assume that the orthogonalized supply shock only affects $\epsilon_{z,t}$, whereas the orthogonalized demand shock shifts both the Euler equation as well as the price-setting equation. The rationale for this assumption is that shocks to the marginal utility of consumption affect the labor supply decision and, in equilibrium, the wage rate which is a component of marginal costs.

According to our posterior estimates neither monetary policy shocks nor sunspot shocks have a notable impact on output fluctuations. Since the estimated correlation between $\epsilon_{g,t}$ and $\epsilon_{z,t}$ under Prior 2 conditional on indeterminacy is near one, the $M = 0$ version of the model attributes almost all of the output fluctuations to the orthogonalized demand shock. The unrestricted pre-Volcker estimates as well as the post-1982 estimates imply that most of the output fluctuations are due to shocks that only affect the marginal costs of production. If M is restricted to zero then the sunspot shock ζ_t explains between 50% and 90% of the variation in inflation and nominal interest rates in the pre-Volcker period. Without this restriction the contribution drops to 0 to 15%.

In summary, our empirical results shows that post 1982 monetary policy is sufficiently anti-inflationary to rule out any indeterminacy. The pre-Volcker years, however, were characterized by a monetary policy that violated the Taylor-principle. According to the New Keynesian DSGE model this policy led to indeterminacy of aggregate business cycle dynamics. Indeterminacy has two effects: the propagation of fundamental shocks is not uniquely determined and sunspot shocks that are unrelated to the fundamental shocks may affect business cycle fluctuations. We consider priors that assign different weights to the two effects. Someone who believes that the propagation of the fundamental shocks is similar under determinacy and indeterminacy (Prior 2) will conclude that sunspot shocks explain a sizeable fraction of inflation and interest rate variation. Someone who believes that indeterminacy may fundamentally alter the propagation of structural shocks via the expectation formation mechanism (Prior 1) will conclude that sunspots have played only a small (or no) role as a source of business cycle fluctuations.

6.4 Robustness Analysis

We argued in Sections 4 and 5 that LRE models generate richer dynamics under indeterminacy than determinacy because fewer autoregressive roots are suppressed. The posterior weight on the determinacy region is based on the one hand on the autocorrelation pattern in the observed data and on the other hand on the information contained in the cross-coefficient restrictions implied by the DSGE model. If the DSGE model lacks important propagation mechanisms that can generate sufficiently rich dynamics than the posterior distribution of its parameters might be unduly biased toward the indeterminacy region. Hence, as a robustness check we will estimate a model with less restrictive dynamics than the standard New Keynesian model presented in Section 2.

A more elaborate consumption Euler equation can be obtained by introducing habit formation. Equation (1) can be derived from a period utility function in which consumption C_t enters as

$$u_{C,t} = \frac{C_t^{1-1/\tau}}{1-1/\tau}. \tag{38}$$

Previous research has found evidence that consumption relative to a habit stock rather than the level of consumption itself should enter the period utility function. Hence, we follow Fuhrer (2000) and introduce multiplicative habit formation. We consider a period utility function of the form

$$u_{C,t} = \frac{[C_t/C_{t-1}^{\gamma}]^{1-1/\tau}}{1-1/\tau}, \tag{39}$$

which for $\gamma = 0$ reduces to (38). The term C_{t-1}^{γ} can be interpreted as habit stock. It can be shown that the introduction of habit leads to the following modified Euler equation:[12]

$$\begin{aligned}
&\left[\tau + \frac{1+\beta\gamma^2+\gamma}{1-\beta\gamma}(1-\tau)\right]\widetilde{x}_t \\
&= \frac{\gamma(1-\tau)}{1-\beta\gamma}\widetilde{x}_{t-1} + \left[\tau + \frac{1+\beta\gamma(1+\gamma)}{1-\beta\gamma}(1-\tau)\right] I\!E_t[\widetilde{x}_{t+1}] \\
&\quad -\frac{\beta\gamma}{1-\beta\gamma}(\tau-1) I\!E_{t+2}[\widetilde{x}_{t+2}] - \tau(\widetilde{R}_t - I\!E_t[\widetilde{\pi}_{t+1}]) + g_t,
\end{aligned} \tag{40}$$

which for $\gamma = 0$ reduces to Equation (1).

The price setting equation (2) can be derived from the assumption that only a fraction of monopolistically competitive firms is able to reoptimize their price in response to shocks in any given period while the remaining firms adjust their prices according to the steady state inflation rate. It is well known that (2) is unable to endogenously generate the observed persistence in inflation. Gali and Gertler (1999) propose a hybrid Phillips curve that involves lagged inflation on the right-hand side:

$$\widetilde{\pi}_t = \frac{\beta}{1+\beta\omega} I\!E_t[\widetilde{\pi}_{t+1}] + \frac{\omega}{1+\beta\omega}\widetilde{\pi}_{t-1} + \frac{\kappa}{1+\beta\omega}(\widetilde{x}_t - z_t). \tag{41}$$

[12]The derivation is relegated to a Technical Appendix that is available upon request.

Such an equation can be justified by assuming that a fraction ω of the firms that are unable to reoptimize their price in a given period adjust their price charged in the previous period by the lagged inflation rate rather than the steady state inflation rate.

We estimated a LRE model based on Equations (40), (41), (3), and (4). Our prior for ω is uniform on the interval $[0, 1]$ whereas γ is distributed according to a Gamma-distribution with mean 1 and standard deviation 0.4. Preliminary estimates indicated that the simple price setting equation ($\omega = 0$) is preferred to the hybrid Phillips curve. In the estimates for the pre-Volcker sample reported in Table 5 we impose the restriction $\omega = 0$. According to the posterior distribution the habit persistence parameter γ lies between 0.35 and 0.8. The estimates of the remaining parameters are by and large in line with the estimates reported for the standard New Keynesian model in Table 3.

Most interesting is whether the enriched model overturns our findings with respect to the indeterminacy hypothesis. Log-data densities for the habit specification conditional on determinacy are reported in Table 6. We revisit the pre-Volcker and the Volcker-Greenspan sample. For both samples the habit model fits better than the no-habit specification restricted to determinacy. The last column of Table 6 contains posterior probabilities for a comparison of the habit model restricted to the determinacy region and the no-habit version estimated over both the determinacy and the indeterminacy region. For the pre-Volcker sample the standard model under indeterminacy provides a better description of the data than the richer model restricted to the determinacy region. Thus, the conclusion of Section 2 with respect to indeterminacy is not overturned.

For the Volcker-Greenspan sample the posterior of the no-habit model assigned roughly equal weight to the determinacy and indeterminacy region of the parameter space. The log-data density for the habit specification suggests that the evidence for indeterminacy is not robust and can be overturned by the inclusion of additional dynamics into the DSGE model.

As a second robustness check we estimate the standard version of the monetary

DSGE model based on output data that are linearly detrended instead of HP detrended. While the output deviations around this less flexible trend are larger and more persistent our overall conclusion does not change. Post 1982 monetary policy is sufficiently anti-inflationary to rule out indeterminacy. The pre-Volcker years, however, monetary policy was passive which led according to the New Keynesian DSGE model to indeterminacy.

7 Conclusion

We estimate a monetary business cycle model of the U.S. economy where monetary policy is characterized by an interest rate rule that attempts to stabilize output and inflation deviations around their target levels. It is well known that the application of such a rule may lead to (local) indeterminacy, thus opening the possibility of sunspot-driven aggregate fluctuations. Although previous research has acknowledged this problem and made some attempts to deal empirically with indeterminacy, our paper is, to the best of our knowledge, the first theoretically and empirically consistent attempt to estimate a DSGE model without restricting the parameters to the determinacy region.

Using a Bayesian approach we construct posterior weights for the determinacy and indeterminacy regions of the parameter space. Our procedure takes into account the dependence of the regions on all structural parameters and not just the policy parameters. Moreover, our approach allows us to study the effect of indeterminacy on the propagation of fundamental shocks and the importance of sunspot shocks for aggregate fluctuations.

Empirical results confirm earlier studies that the behavior of the monetary authority has changed beginning with the tenure of Paul Volcker as Federal Reserve Chairman in 1979. During the Volcker-Greenspan years policy reacts very aggressively towards inflation which puts the U.S. economy into the determinacy region. On the other hand, monetary policy was much less active in the pre-Volcker period, and through the lens of a standard New Keynesian DSGE model we cannot reject the possibility of equilibrium indeterminacy. The DSGE model in this paper, albeit widely employed in the recent monetary policy literature, is highly stylized. Its overall time series fit is worse than that of a vector autoregression. Nevertheless, we believe that some interesting lessons have been learned from our empirical analysis. A fruitful avenue for future research is to apply our methods to models in which determinacy is not as closely tied to the degree of activism as in the standard New Keynesian DSGE model.

References

Andrews, Donald W.K. and Werner Ploberger (1994): "Optimal Tests when a Nuisance Parameter is Present only under the Alternative," *Econometrica*, **62**, 1383-1414.

Blanchard, Olivier J. and Charles M. Kahn (1980): "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, **48**, 1305-1311.

Bullard, James and Kaushik Mitra (2002): "Learning about Monetary Policy Rules," Forthcoming: *Journal of Monetary Economics*.

Christiano, Lawrence J. and Sharon G. Harrison (1999): "Chaos, Sunspots, and Automatic Stabilizers," *Journal of Monetary Economics*, **44**, 3-31.

Clarida, Richard, Jordi Galí, Mark Gertler (2000): "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory," *Quarterly Journal of Economics*, **115**(1), 147-180.

Cooper, Russell W. (2001): "Estimation and Indentification of Structural Parameters in the Presence of Multiple Equilibria," *Manuscript*, Department of Economics, Boston University.

Dupor, Bill (2001): "Investment and Interest Rate Policy," *Journal of Economic Theory*, **98**, 85-113.

Farmer, Roger E. A. and Jang-Ting Guo (1994): "Real Business Cycles and the Animal Spirits Hypothesis," *Journal of Economic Theory*, **63**, 42-72.

Farmer, Roger E. A. and Jang-Ting Guo (1995): "The Econometrics of Indeterminacy: An Applied Study," *Carnegie-Rochester Conference Series on Public Policy*, **43**, 225-271.

Fernandez-Villaverde, Jesus and Juan Francisco Rubio-Ramirez (2001): "Comparing Dynamic Equilibrium Models to Data," *Manuscript*, University of Pennsyvlania.

Fuhrer, Jeffrey C. (2000): "Habit Formation in Consumption and Its Implications for Monetary-Policy Models," *American Economic Review*, **90**, 367-390.

Galí, Jordi and Mark Gertler (1999): "Inflation Dynamics: A Structural Econometric Analysis," *Journal of Monetary Economics*, **44**(2), 195-222.

Geweke, John F. (1999): "Using Simulation Methods for Bayesian Econometric Models: Inference, Development, and Communication," *Econometric Reviews*, **18**, 1-126.

Hamilton, James D. (1994): *"Time Series Analysis,"* Princeton University Press, Princeton.

Hannan, E.J. (1980): "The Estimation of the Order of an ARMA Process," *Annals of Statistics*, **8**, 1071-1081.

Ireland, Peter N. (2001): "Sticky-Price Models of the Business Cycle: Specification and Stability," *Journal of Monetary Economics*, **47**, 3-18.

Jovanovic, Boyan (1989): "Observable Implications of Models with Multiple Equilibria," *Econometrica*, **57**, 1431-1438.

Kim, Jae-Young (1998): "Large Sample Properties of Posterior Densities, Bayesian Information Criterion, and the Likelihood Principle in Nonstationary Time Series Models," *Econometrica*, **66**, 359-380.

Kim, Jinill (2000): "Constructing and Estimating a Realistic Optimizing Model of Monetary Policy," *Journal of Monetary Economics*, **45**(2), 329-359.

King, Robert G. (2000): "The New IS-LM Model: Language, Logic, and Limits," Federal Reserve Bank of Richmond *Economic Quarterly*, **86**(3), 45-103.

Leeper, Eric M. and Christopher A. Sims (1994): "Toward a Modern Macroeconomic Model Useable for Policy Analysis," in: Stanley Fischer and Julio Rotemberg (eds.) *Macroeconomics Annual*, MIT Press, Cambridge.

Lubik, Thomas A. and Massimiliano Marzo (2003): "An Inventory of Monetary Policy Rules in a Simple, New-Keynesian Macroeconomic Model," *Manuscript*, Department of Economics, Johns Hopkins University.

Lubik, Thomas A. and Frank Schorfheide (2003): "Computing Sunspots in Linear Rational Expectations Models," *Journal of Economic Dynamics and Control*, forthcoming.

McCallum, Bennett T. (1983): "On Nonuniqueness in Linear Rational Expectations Models: An Attempt at Perspective," *Journal of Monetary Economics*, **11**, 134-168.

Orphanides, Athanasios (2002): "Activist Stabilization Policy and Inflation: The Taylor Rule in the 1970s," *Working Paper*, Board of Governors.

Perli, Roberto (1998): "Indeterminacy, Home Production, and the Business Cycle: A Calibrated Analysis," *Journal of Monetary Economics*, **41**, 105-125.

Phillips, Peter C.B. (1996): "Econometric Model Determination," *Econometrica*, **64**, 763-812.

Poirier, Dale (1998): "Revising Beliefs in Nonidentified Models," *Econometric Theory*, **14**, 183-209.

Rabanal, Pau and Juan Francisco Rubio-Ramirez (2002): "Nominal versus Real Wage Rigidities: A Bayesian Approach," *Manuscript*, Federal Reserve Bank of Atlanta.

Rotemberg, Julio and Michael Woodford (1997): "An Optimization-Based Econometric Framework for the Evaluation of Monetary Policy," *NBER Macroeconomics Annual*, **12**, 297-246.

Ruge-Murcia, Francisco (2002): "Methods to Estimate Dynamic Stochastic General Equilibrium Models," *Discussion Paper*, **2002-18**, Department of Economics, University of California at San Diego.

Salyer, Kevin D. and Steven M. Sheffrin (1998): "Spotting Sunspots: Some Evidence in Support of Models with Self-Fulfilling Prophecies," *Journal of Monetary Economics*, **42**, 511-523.

Sargent, Thomas J. (1999): *"The Conquest of American Inflation,"* Princeton University Press, Princeton.

Sbordone, Argia M. (2002): "Prices and Unit Labor Costs: A New Test of Price Stickiness," *Journal of Monetary Economics*, **49**, 265-292.

Schmitt-Grohé, Stephanie (1997): "Comparing Four Models of Aggregate Fluctuations due to Self-Fulfilling Fluctuations," *Journal of Economic Theory*, **72**, 96-147.

Schmitt-Grohé, Stephanie (2000): "Endogenous Business Cycles and the Dynamics of Output, Hours, and Consumption," *American Economic Review*, **90**(5), 1136-1159.

Schorfheide, Frank (2000): "Loss Function-Based Evaluation of DSGE Models," *Journal of Applied Econometrics*, **15**, 645-670.

Schwarz, Gideon (1978): "Estimating the Dimension of a Model," *Annals of Statistics*, **6**, 461-464.

Sims, Christopher A. (2002): "Solving Linear Rational Expectations Models," *Computational Economics*, **20**, 1-20.

Woodford, Michael (2003): *"Interest and Prices: Foundations of a Theory of Monetary Policy,"* Princeton University Press, forthcoming, Princeton.

A Practical Implementation

1. The matrices $\Gamma_0(\theta)$, $\Gamma_1(\theta)$, $\Psi(\theta)$, and $\Pi(\theta)$ of the canonical LRE model (5) can be derived from Equations (1) to (4). Instead of inverting $\Gamma_0(\theta)$ and using a Jordan decomposition of $\Gamma_0^{-1}\Gamma_1$ as described in Section 4, we proceed as in Sims (2002) with a generalized complex Schur decomposition (QZ) of Γ_0 and Γ_1. There exist matrices Q, Z, Λ, and Ω such that $Q'\Lambda Z' = \Gamma_0$, $Q'\Omega Z' = \Gamma_1$, $QQ' = Q'Q = ZZ' = Z'Z = I_{n\times n}$ and Λ and Ω are upper triangular. Let $w_t = Z'y_t$ and premultiply (5) by Q to obtain

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} w_{1,t-1} \\ w_{2,t-1} \end{bmatrix} + \begin{bmatrix} Q_{1.} \\ Q_{2.} \end{bmatrix} (\Psi\epsilon_t + \Pi\eta_t). \tag{42}$$

The second set of equations can be rewritten as:

$$w_{2,t} = \Lambda_{22}^{-1}\Omega_{22}w_{2,t-1} + \Lambda_{22}^{-1}Q_{2.}(\Psi\epsilon_t + \Pi\eta_t) \tag{43}$$

Without loss of generality, we assume that the system is ordered and partitioned such that the $m \times 1$ vector $w_{2,t}$ is purely explosive, where $0 \leq m \leq n$. We apply the singular value decomposition (24) to $Q_{2.}\Pi$ instead of Π_x^J and obtain η_t according to Proposition 1, where Ψ_x^J is replaced by $Q_{2.}\Psi$. To find $M^*(\theta)$ use (30), which leads to a vector autoregression law of motion for s_t.

2. Combine (26) with the measurement equation (35) to form a state space model for the observables y_t. The likelihood function $\mathcal{L}(\theta, M, \sigma_\zeta|Y^T)$ can be evaluated with the Kalman Filter.

3. Let $p(\theta, M, \sigma_\zeta)$ and $p(\theta, M, \sigma_\zeta|Y^T)$ denote prior and posterior densities of θ, respectively. Since the likelihood function $\mathcal{L}(\theta, M, \sigma_\zeta)|Y^T)$ is discontinuous at the boundary of the determinacy region we conduct the computations for the two regions of the parameter space separately.

4. A numerical-optimization procedure is used to maximize

$$p_s(\theta, M, \sigma_\zeta|Y^T) \propto \mathcal{L}_s(\theta, M, \sigma_\zeta|Y^T)p(\theta, M, \sigma_\zeta)\{\theta \in \Theta^s\}, \quad s \in \{D, I\}$$

and find the posterior mode in the two regions of the parameter space. The inverse Hessian is calculated at the posterior mode.

5. For each region, 1,000,000 draws from $p(\theta, M, \sigma_\zeta | Y^T)$ are generated with a random-walk Metropolis Algorithm. The scaled inverse Hessian serves as a covariance matrix for the Gaussian proposal distribution used in the Metropolis-Hastings algorithm. If for a particular sample a region of the parameter space does not have a (local) mode, we use the inverse Hessian obtained from one of the other samples for that region. The first 100,000 draws are discarded. The parameter draws θ are converted into impulse response functions and variance decompositions to generate the results reported in Section 6. Posterior moments are obtained by Monte-Carlo averaging. The marginal data densities for the two regions are approximated with Geweke's (1999) modified harmonic-mean estimator. Further details of these computations are discussed in Schorfheide (2000).

Table 1: Prior Distributions for DSGE Model Parameters

Name	Range	Density	Mean	Stdd	90% Interval
ψ_1	$I\!R^+$	Gamma	1.10	0.50	[0.33, 1.85]
ψ_2	$I\!R^+$	Gamma	0.25	0.15	[0.06, 0.43]
ρ_R	[0,1)	Beta	0.50	0.20	[0.18, 0.83]
π^*	$I\!R^+$	Gamma	4.00	2.00	[0.90, 6.91]
r^*	$I\!R^+$	Gamma	2.00	1.00	[0.49, 3.47]
κ	$I\!R^+$	Gamma	0.50	0.20	[0.18, 0.81]
τ^{-1}	$I\!R^+$	Gamma	2.00	0.50	[1.16, 2.77]
ρ_g	[0.1)	Beta	0.70	0.10	[0.54, 0.86]
ρ_z	[0.1)	Beta	0.70	0.10	[0.54, 0.86]
ρ_{gz}	[-1,1]	Normal	0.00	0.40	[-0.65, 0.65]
$M_{R\zeta}$	$I\!R$	Normal	0.00	1.00	[-1.64, 1.64]
$M_{g\zeta}$	$I\!R$	Normal	0.00	1.00	[-1.64, 1.64]
$M_{z\zeta}$	$I\!R$	Normal	0.00	1.00	[-1.64, 1.64]
σ_R	R^+	Inv. Gamma	0.31	0.16	[0.13, 0.50]
σ_g	R^+	Inv. Gamma	0.38	0.20	[0.16, 0.60]
σ_z	R^+	Inv. Gamma	1.00	0.52	[0.42, 1.57]
σ_ζ	R^+	Inv. Gamma	0.25	0.13	[0.11, 0.40]

Notes: The Inverse Gamma priors are of the form $p(\sigma|\nu, s) \propto \sigma^{-\nu-1}e^{-\nu s^2/2\sigma^2}$, where $\nu = 4$ and s equals 0.25, 0.3, 0.6, and 0.2, respectively. The prior for ρ_{gz} is truncated to ensure that the correlation lies between -1 and 1. We refer to the prior distribution in this Table as *Prior 1*. *Prior 2* is obtained by imposing $M_{R\zeta} = M_{g\zeta} = M_{z\zeta} = 0$, whereas *Prior 3* imposes $\sigma_\zeta = 0$.

Table 2: DETERMINACY VERSUS INDETERMINACY (I)

Sample	Prior	Log Data Density		Probability	
		Determ.	Indeterm.	Determ.	Indeterm.
Pre-Volcker	1	-372.4	-359.1	0.000	1.000
	2	-372.4	-359.8	0.000	1.000
	3	-372.4	-358.7	0.000	1.000
Volcker-Greenspan	1	-368.6	-368.6	0.502	0.498
	2	-368.6	-369.4	0.692	0.308
	3	-368.6	-368.1	0.379	0.621
Post 1982	1	-237.4	-241.9	0.989	0.011
	2	-237.4	-241.5	0.984	0.016
	3	-237.4	-241.3	0.980	0.020

Notes: According to the prior distribution in Table 1 the probability of determinacy is 0.527. The posterior probabilities are calculated based on the output of the Metropolis algorithm. Log Marginal data densities are approximated by Geweke's (1999) harmonic mean estimator.

Table 3: Parameter Estimation Results (I)

	Pre-Volcker (Prior 1)		Pre-Volcker (Prior 2)		Post 1982	
	Mean	Conf. Interval	Mean	Conf Interval	Mean	Conf Interval
ψ_1	0.77	[0.64, 0.91]	0.89	[0.81, 0.99]	2.19	[1.38, 2.99]
ψ_2	0.17	[0.04, 0.30]	0.15	[0.03, 0.27]	0.30	[0.07, 0.51]
ρ_R	0.60	[0.42, 0.78]	0.53	[0.43, 0.65]	0.84	[0.79, 0.89]
π^*	4.28	[2.21, 6.21]	3.98	[2.12, 5.84]	3.43	[2.84, 3.99]
r^*	1.13	[0.63, 1.62]	1.11	[0.73, 1.49]	3.01	[2.21, 3.80]
κ	0.77	[0.39, 1.12]	0.75	[0.38, 1.07]	0.58	[0.27, 0.89]
τ^{-1}	1.45	[0.85, 2.05]	2.08	[1.27, 2.84]	1.86	[1.04, 2.64]
ρ_g	0.68	[0.54, 0.81]	0.80	[0.75, 0.85]	0.83	[0.77, 0.89]
ρ_z	0.82	[0.72, 0.92]	0.69	[0.62, 0.76]	0.85	[0.77, 0.93]
ρ_{gz}	0.14	[-0.40, 0.71]	0.98	[0.96, 1.00]	0.36	[0.06, 0.67]
$M_{R\zeta}$	-0.68	[-1.58, 0.23]				
$M_{g\zeta}$	1.74	[0.90, 2.56]				
$M_{z\zeta}$	-0.69	[-0.99, -0.39]				
σ_R	0.23	[0.19, 0.27]	0.24	[0.20, 0.28]	0.18	[0.14, 0.21]
σ_g	0.27	[0.17, 0.36]	0.21	[0.16, 0.26]	0.18	[0.14, 0.23]
σ_z	1.13	[0.95, 1.30]	1.16	[0.97, 1.34]	0.64	[0.52, 0.76]
σ_ζ	0.20	[0.12, 0.27]	0.23	[0.15, 0.31]		

Notes: The table reports posterior means and 90 percent confidence intervals (in brackets). Pre-Volcker posteriors are conditional on indeterminacy, Post-1982 posteriors are conditional on determinacy. Under *Prior 2* $M_{R\zeta} = M_{g\zeta} = M_{z\zeta} = 0$. Conditional on determinacy posterior and prior means and confidence intervals for $M_{R\zeta}$, $M_{g\zeta}$, $M_{z\zeta}$, and σ_ζ are identical. The posterior summary statistics are calculated from the output of the Metropolis algorithm.

Table 4: Variance Decompositions

	Pre-Volcker (Prior 1)		Pre-Volcker (Prior 2)		Post 1982	
	Mean	Conf. Interval	Mean	Conf Interval	Mean	Conf Interval
			Output Deviations from Trend			
Policy	0.01	[0.00, 0.02]	0.01	[0.00, 0.02]	0.04	[0.01, 0.06]
Demand(*)	0.19	[0.00, 0.51]	0.97	[0.95, 0.99]	0.36	[0.09, 0.63]
Supply(*)	0.80	[0.47, 0.99]	0.01	[0.00, 0.03]	0.60	[0.33, 0.88]
Sunspot	0.00	[0.00, 0.01]	0.00	[0.00, 0.01]	0.00	[0.00, 0.00]
			Inflation			
Policy	0.08	[0.00, 0.18]	0.07	[0.02, 0.11]	0.29	[0.14, 0.43]
Demand(*)	0.23	[0.01, 0.46]	0.17	[0.03, 0.30]	0.47	[0.32, 0.63]
Supply(*)	0.59	[0.32, 0.87]	0.07	[0.00, 0.14]	0.24	[0.07, 0.40]
Sunspot	0.10	[0.01, 0.18]	0.70	[0.50, 0.92]	0.00	[0.00, 0.00]
			Interest Rates			
Policy	0.17	[0.00, 0.41]	0.08	[0.01, 0.15]	0.08	[0.03, 0.12]
Demand(*)	0.20	[0.00, 0.42]	0.16	[0.01, 0.29]	0.61	[0.38, 0.84]
Supply(*)	0.54	[0.25, 0.87]	0.05	[0.00, 0.10]	0.32	[0.12, 0.51]
Sunspot	0.08	[0.01, 0.15]	0.71	[0.51, 0.92]	0.00	[0.00, 0.00]

Notes: The table reports posterior means and 90 percent confidence intervals (in brackets). The posterior summary statistics are calculated from the output of the Metropolis algorithm. Since the estimated correlation between $\epsilon_{g,t}$ and $\epsilon_{z,t}$ is non-zero we orthogonalize the demand and supply shock for this variance decomposition using a triangular decomposition. We assume that the supply(*) shock only affects $\epsilon_{z,t}$.

Table 5: PARAMETER ESTIMATION RESULTS (II)

	Pre-Volcker (Habit)	
	Mean	Conf. Interval
ψ_1	1.07	[1.00, 1.15]
ψ_2	0.10	[0.02, 0.19]
ρ_R	0.59	[0.48, 0.72]
π^*	4.53	[3.48, 5.57]
r^*	1.14	[0.71, 1.57]
κ	0.66	[0.30, 1.01]
τ^{-1}	1.58	[1.05, 2.15]
ω	0.00	[0.00, 0.00]
γ	0.57	[0.35, 0.79]
ρ_g	0.87	[0.83, 0.91]
ρ_z	0.79	[0.74, 0.85]
ρ_{gz}	0.93	[0.88, 0.97]
σ_R	0.26	[0.21, 0.30]
σ_g	0.18	[0.13, 0.23]
σ_z	1.29	[1.06, 1.53]

Notes: The table reports posterior means and 90 percent confidence intervals (in brackets). Estimates of habit-specification are based on Pre-Volcker sample, conditional on determinacy.

Table 6: Determinacy versus Indeterminacy (II)

Sample	Spec	Log Data Density		Probability
		Determ.	Indeterm.	
Pre-Volcker	Prior 3	-372.4	-358.7	1.000
	Habit	-370.0		0.000
Volcker-Greenspan	Prior 3	-368.6	-368.1	0.038
	Habit	-364.4		0.962

Notes: The posterior probabilities are calculated based on the output of the Metropolis algorithm. Log Marginal data densities are approximated by Geweke's (1999) harmonic mean estimator.

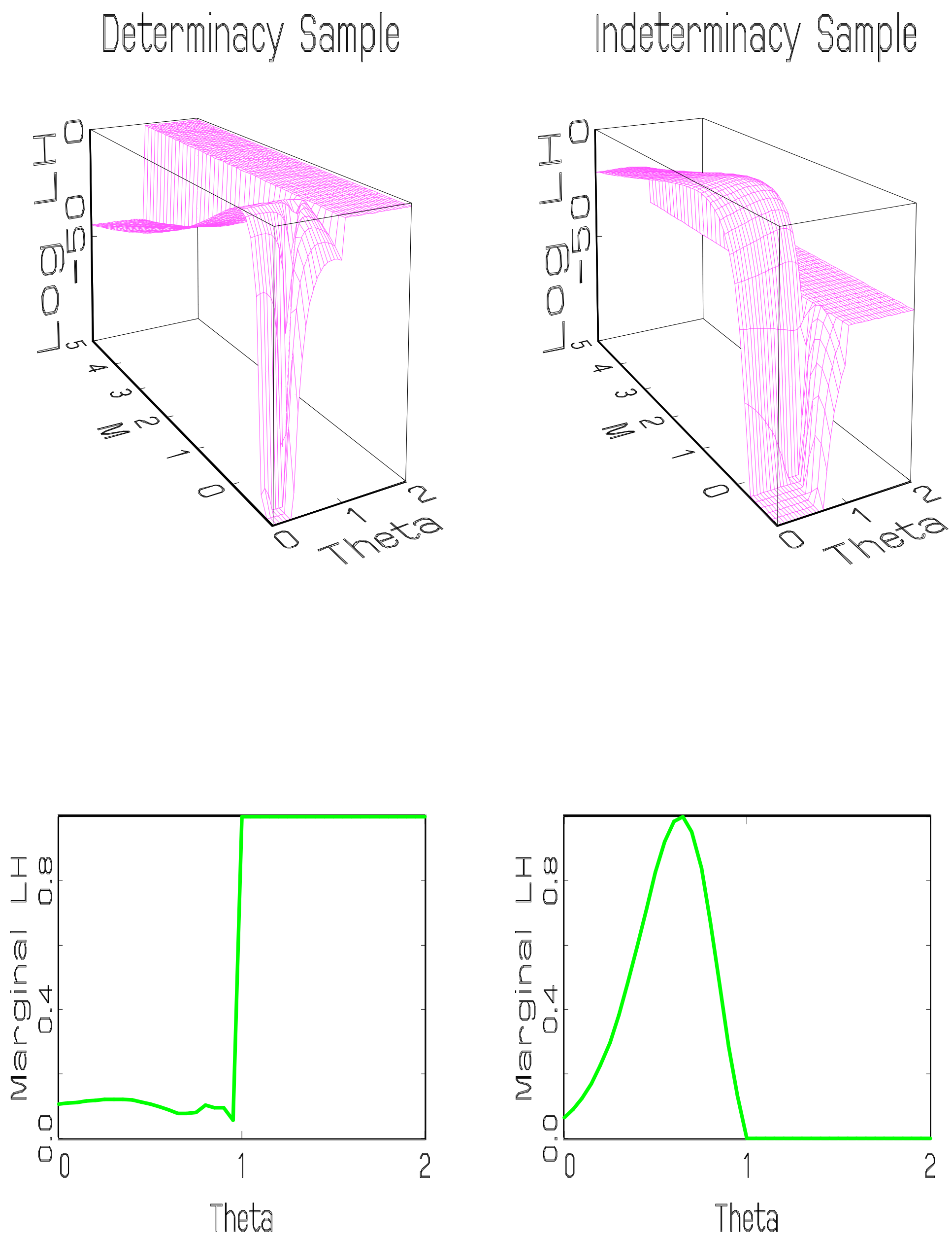

Figure 1: Likelihood / Posterior Density for Single Equation Model

Notes: First row can be interpreted as joint *log* posterior density of θ and M (under a uniform prior distribution) standardized by the posterior mode. Second row depicts the marginal posterior of density of θ, standardized by its mode.

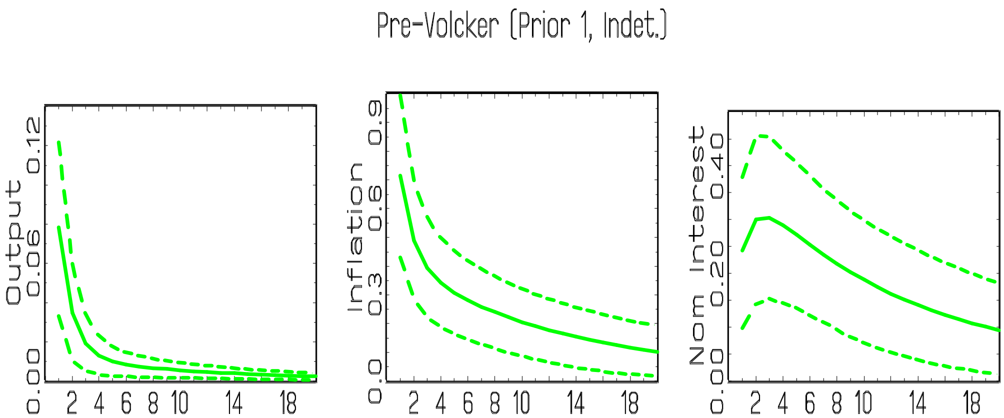

Figure 2: Impulse Responses to Sunspot Shock

Notes: Pre-Volcker sample, indeterminacy region (Prior 1). Figure depicts posterior means (solid lines) and pointwise 90% posterior confidence intervals (dashed lines) for impulse responses of output, inflation, and the nominal interest rate to a one-standard deviation shock $\epsilon_{\zeta,t}$.

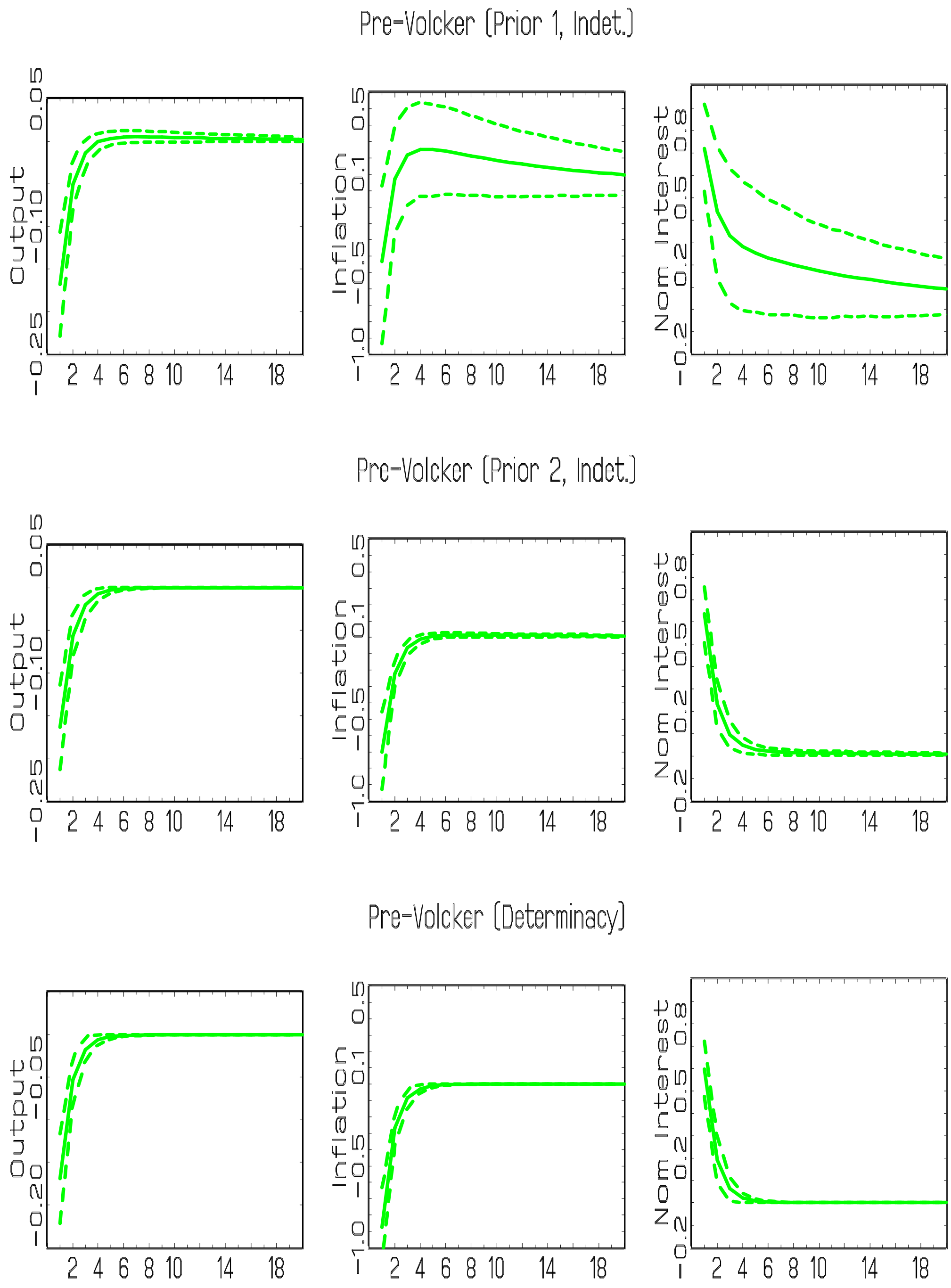

Figure 3: Impulse Responses to Monetary Policy Shock

Notes: Pre-Volcker sample. Figure depicts posterior means (solid lines) and pointwise 90% posterior confidence intervals (dashed lines) for impulse responses of output, inflation, and the nominal interest rate to a one-standard deviation shock $\epsilon_{R,t}$.

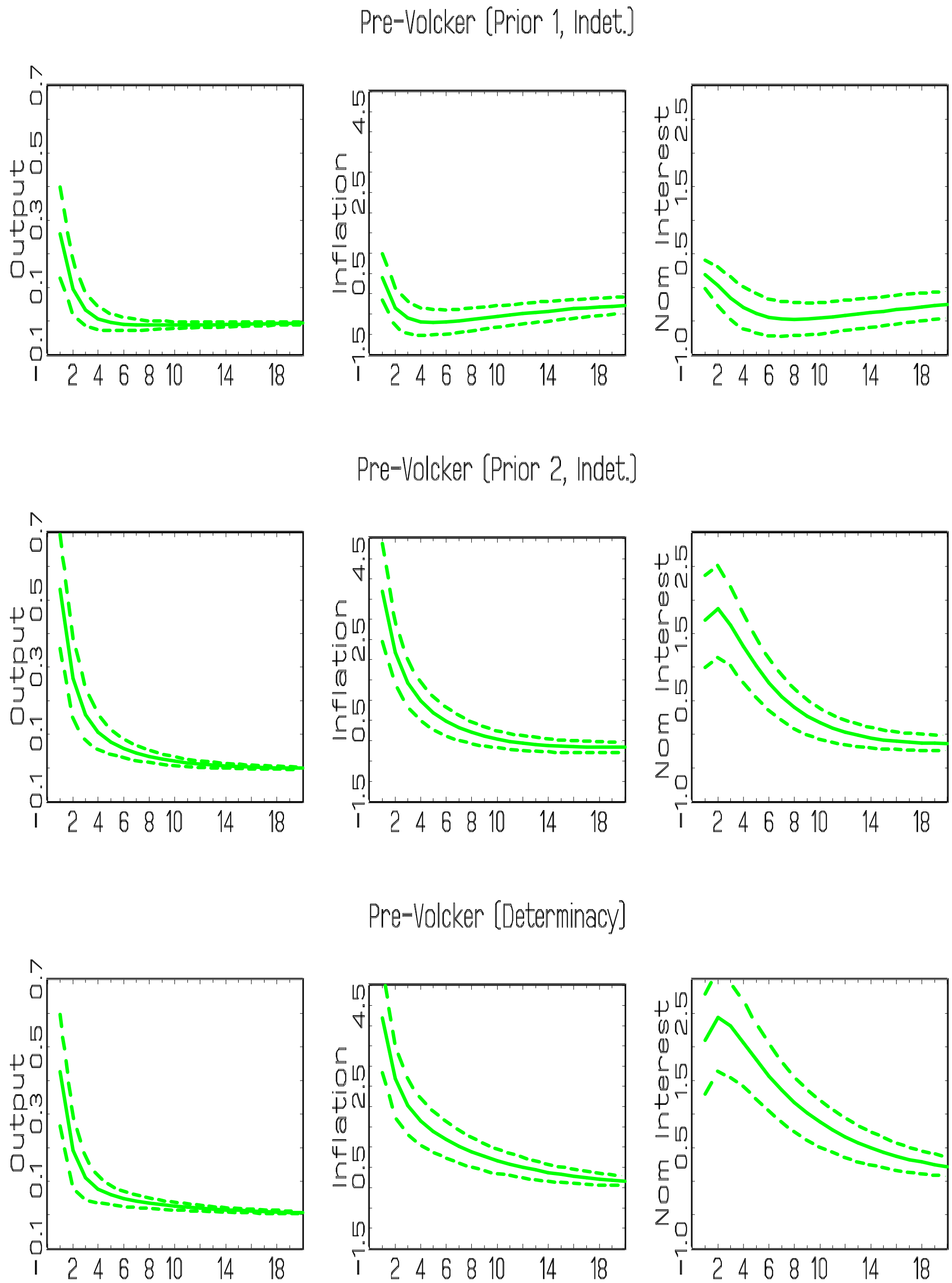

Figure 4: Impulse Responses to Demand Shock

Notes: Pre-Volcker sample. Figure depicts posterior means (solid lines) and pointwise 90% posterior confidence intervals (dashed lines) for impulse responses of output, inflation, and the nominal interest rate to a one-standard deviation shock $\epsilon_{g,t}$.

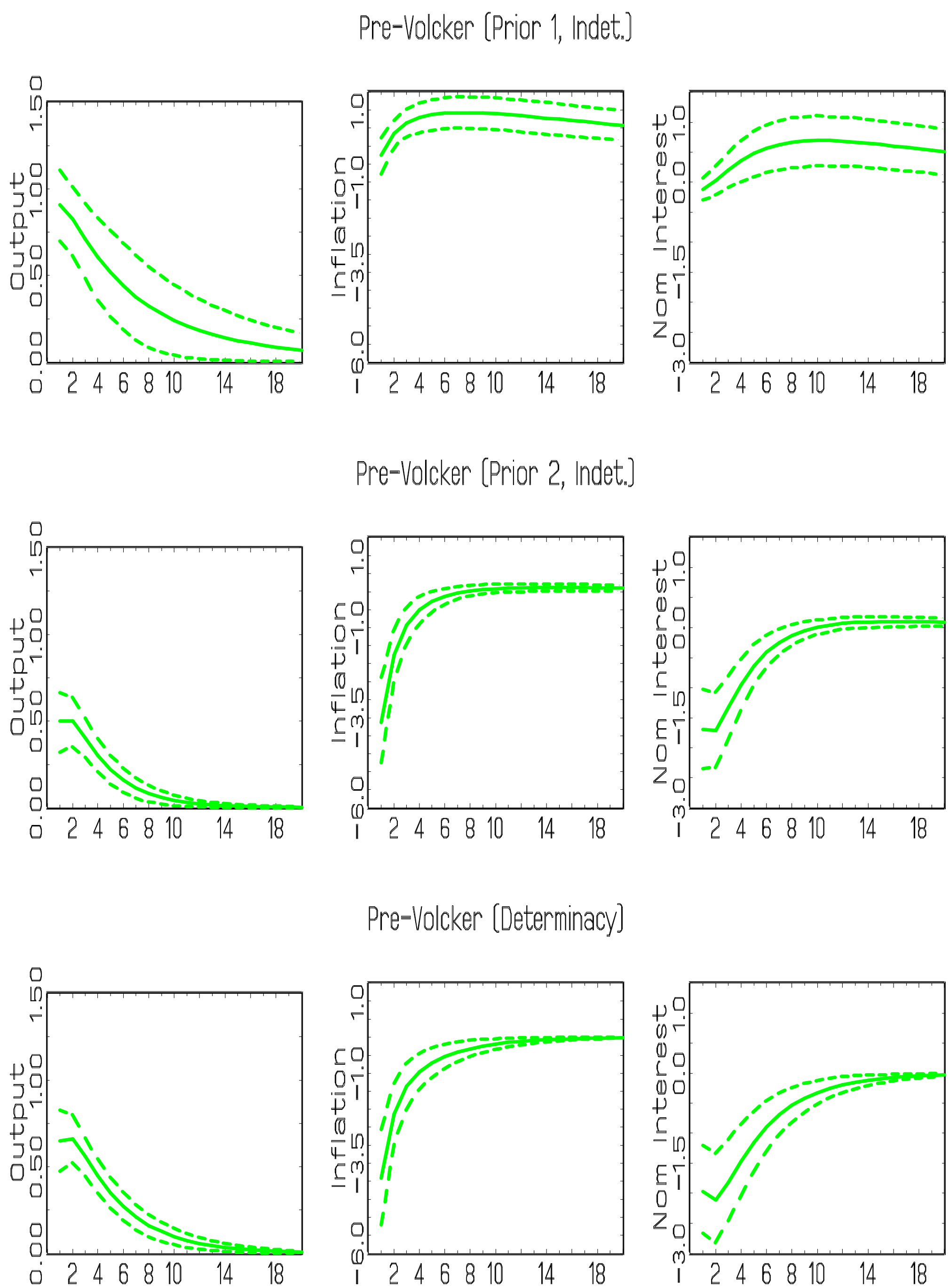

Figure 5: Impulse Responses to Supply Shock

Notes: Pre-Volcker sample. Figure depicts posterior means (solid lines) and pointwise 90% posterior confidence intervals (dashed lines) for impulse responses of output, inflation, and the nominal interest rate to a one-standard deviation shock $\epsilon_{z,t}$.