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学 生 姓 名	**巩舒阳**
学　　　号	**201700111073**
学　　　院	**数学学院**
专　　　业	**统计学**
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指 导 教 师	**王汉超**

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Stein 方法及其在统计物理中的应用

山东大学数学学院 2017 级统计学专业　巩舒阳
指导老师　王汉超

摘要：Stein 方法作为一种正态近似方法，自从 1972 年 Charles Stein 提出后，它的理论和研究范围一直在不断扩大。为了说明一列随机变量依分布收敛于标准正态随机变量，我们可以用特征函数法，利用特征函数近似程度刻画随机变量近似程度，但是当我们在处理变换后的特征函数时，难免少了一些概率上的直觉和信息，而 Stein 方法可以弥补这一缺陷。本文首先介绍 Stein 方法的基本概念，以及 Exchangeable Pair、Zero bias 分布、Size bias 分布等一些常用技巧；然后我们介绍关于 Berry-Esseen 界的一些结果以及常用证明方法，比如集中不等式法、数学归纳法；再者我们给出参考文献[1]中的可应用到统计物理模型的L^∞界，并证明对应的非一致界，得到 Berry-Esseen 界的逼近速度；最后我们会将上述推导出的 Berry-Esseen 界应用到组合中心极限定理（Combinatorial Central Limit Theorem）、Anti-voter 模型、Quadratic forms、Lightbulb process 一些模型中。
关键词：Stein 方法、正态近似、Berry-Esseen 界、统计物理模型、非一致界

Abstract: As a method for normal approximation, since proposed by Charles Stein in 1972, Stein's method has developed considerably and penetrated into many areas such as statistical physics and random graph. Now its theory and applications are continuing to expand. In order to show a series of random variables converge in distribution to a standard normal variable, characteristic functions may be a good choice, the closeness of characteristic functions implies the closeness of distributions, however, in doing so we inevitably lose some probabilistic intuition, and Stein's method can make up for this defect. In our paper, firstly we introduce some basic definitions and concepts and some techniques such as Exchangeable Pair, Zero Bias distribution and Size Bias distribution. Then we show some results about Berry-Esseen bound while giving some frequently-used techniques. After that we proceed to list some applicable L^∞ bounds in [1], and give corresponding non-uniform bounds. In this way, we can obtain the approximation rate of Berry-Esseen bound. Finally, we apply bound we derived to some models such as Combinatorial Central Limit Theorem, Anti-voter model, Lightbulb process, quadratic forms and Lightbulb process.
Keywords: Stein's method, Normal approximations, Berry-Esseen bound, Statistical physics models, Non-uniform bound

目录

一、绪论以及基本知识

1.1 Stein 方法的历史

1972 年 Stein(参看[8])提出了 Stein 方法，其中主要想法为对于函数h，令f为以下微分方程的解

$$f'(x) - xf(x) = h(x) - Eh(Z) \qquad (1)$$

其中Z为标准正态随机变量，我们一般可以用Nh来代表$Eh(Z)$。通过比较$Eh(W)$和Nh的差距来衡量W，Z之间的差距，为估计$Eh(W)-Eh(Z)$，根据(1)，可将其转化为$E\big(f'(W)-Wf(W)\big)$来进行估计。关于这种转化思想，[4]中提出了如下定理：

命题 1.1 （证明参看[4]）令$\mathcal{M}=\big\{f:R\to R$，二次连续可微,且对任意$x\in R$满足$|f(x)|\le 1,|f'(x)|\le 1,|f''(x)|\le 1\big\}$，令$Z$为标准正态随机变量且$W$为任意随机变量，那么

$$\sup_{t\in R}|P(W\le t)-P(Z\le t)|\le 2\left(\sup_{f\in\mathcal{M}}\left|E\big(f'(W)-Wf(W)\big)\right|\right)^{\frac{1}{2}}$$

至此，我们不难发现当$E\big(f'(W)-Wf(W)\big)$越小，随机变量W和Z之间的差距越小，那么是否当$E\big(f'(W)-Wf(W)\big)=0$时，W服从标准正态分布呢？答案是肯定的，我们有如下定理：

引理 1.1 如果 W服从标准正态分布，那么对于满足$E|f'(Z)|<\infty$的绝对连续函数$f:R\to R$，有

$$Ef'(W)=E\big(Wf(W)\big) \qquad (2)$$

相反，如果对于任意有界、连续且分段连续可导函数$f:R\to R$，$E|f'(Z)|<\infty$，(2)均成立，那么W服从标准正态分布。（证明见[1]）

若要实现上面的想法，重要方法之一是可交换对（Exchangeable Pairs）。可交换对定义如下：一对随机变量(W',W)与(W,W')具有相同分布，那么(W,W')称为可交换对。我们可以举出许多可交换对的例子(参看[1])，令$\{\xi_i,1\le i\le n\}$为具有零均值且满足$\sum_{i=1}^{n}E\xi_i^2=1$的一列独立随机变量，令$W=\sum_{i=1}^{n}\xi_i$，令$\{\xi_i',1\le i\le n\}$为与$\{\xi_i,1\le i\le n\}$独立且同分布的一列随机变量，I为在$\{1,2,\cdots,n\}$具有均匀分布的随机指标，且与ξ_i、ξ_i'独立，定义$W'=W-\xi_I+\xi_I'$，那么(W',W)为可交换对且满足

$$E(W'-W|W)=\left(1-\frac{1}{n}\right)W$$

另外，为了方便，我们给出λ-Stein 对（λ-Stein pair）定义：如果可交换对(W',W)满足如下“线性回归条件”

$$E(W'|W)=(1-\lambda)W \qquad (3)$$

那么(W',W)称为λ-Stein 对。因此在以上例子中的可交换对可称为$\frac{1}{n}$-Stein 对。

如果我们使可交换对(W',W)满足以下三个条件（参看[4]）：

$$E(W'-W|W)=-\lambda W+o(\lambda)$$
$$E((W'-W)^2|W)=2\lambda+o(\lambda)$$
$$E|W'-W|^3=o(\lambda) \tag{4}$$

其中$o(\lambda)$代表比λ小很多的数量。

对于任意$f\in\mathcal{M}$，根据可交换性，我们得到

$$E\left((W'-W)\big(f(W')+f(W)\big)\right)=0,$$

根据已有条件，在满足(4)的情况下，可计算得到

$$\frac{1}{2\lambda}E\left((W'-W)\big(f(W')+f(W)\big)\right)=E\big(f'(W)-Wf(W)\big)+o(1)$$

因此根据命题 1.1，W是近似正态的。

1984 年，Bolthausen 使用了可交换对的方法获得了组合中心极限定理(Combinatorial Central Limit Theorem)中误差估计的一个重要结果（参看[11]），主要证明了$W=\sum_{i=1}^{n}a_{i\pi(i)}$所满足的中心极限定理并给出了一个误差界（其中$\pi$为$\{1,2,\cdots,n\}$上的一个随机置换）。关于组合中心极限定理这一模型，当π在共轭类（Conjugacy Classess）上为均匀分布时，我们会给出它相关的一致界和非一致界。

Barbour 在 1990 年引入了扩散（diffusion）方法（参看[19]）；1993 年 Avram 和 Bertsimas（参看[20]）运用 Stein 方法解决了几何概率中许多重要问题；1996 年 Goldstein 和 Rinott（参看[21]）引入了 Size Bias 方法；1997 年 Goldstein 和 Reinert（参看[22]）引入了 Zero Bias 方法。

进入 2000 年后，Stein 方法得到长足发展。2001 年 Chen 和 Shao（参看[6]）证明了非一致的 Berry-Esseen 界；Chatterjee 在 2008 年提出了一种新的正态近似方法（参看[2]），引入了变量T使得$E\big(Wf(W)\big)\approx E\big(f'(W)T\big)$（当$ET\approx 1$，$Var(T)$很小的时候，可认为$E\big(Wf(W)\big)\approx Ef'(W)$），在 Kantorovich-Wasserstein 距离下得到误差估计；2009 年 Chatterjee 应用 Stein 方法分析了高温下的 spin glasses（参看[7]）；后来 Chatterjee 和 Shao（参看[5]）得出了在临界温度$T=1$时 Curie-Weiss 模型中的误差估计；2017 年 Chatterjee 和 Sen（参看[3]）解决了欧式最小生成树（Minimal spanning tree）中心极限定理的误差估计问题，并且类似的给出了格子Z^d中最小生成树收敛速度问题；2019 年 Chatterjee（参看[23]）证明了随机场伊辛模型自由能的中心极限定理；2021 年，Chen, Martin Raič 和 Lê Văn Thành（参看[24]）使用 Zero Bias 方法获得了与 Jack measure 有关的一致与非一致 Kolmogorov 距离。

1.2 Stein 方法中的基本概念

引理 1.2（Stein 方程）在方程(1)中令$h(x) = 1\{x \le z\}$，即

$$f'(x) - xf(x) = 1\{x \le z\} - P(Z \le z) \qquad (5)$$

其中Z为标准正态随机变量，则(5)的解为

$$f_z(x) = \begin{cases} \sqrt{2\pi}e^{\frac{x^2}{2}}\Phi(x)\big(1-\Phi(z)\big) & 若x < z \\ \sqrt{2\pi}e^{\frac{x^2}{2}}\Phi(x)\big(1-\Phi(z)\big) & 若x \ge z \end{cases}$$

注：Stein 方程的解随着z的增大迅速衰减，这一点在我们后面寻找非一致界时起到至关重要的作用；同时，它对于证明一些一致界也起到了非常重要的作用，因为它可以将$P(W \le Z) - P(Z \le z)$转化为$f'(W) - Wf(W)$。对于一般的(1)，我们也可以表示出解的形式

$$f_h(x) = e^{\frac{x^2}{2}}\int_{-\infty}^{x}(h(t) - Nh)e^{-\frac{t^2}{2}}dt$$

Stein 方程是 Stein 方法的应用和理论研究的基石。

引理 1.3（Stein 方程解的性质）令f_z表示(5)的解，它有如下性质：

1. $xf_z(x)$关于x是递增的 (6)
2. $0 < f_z(x) < \min\left(\frac{\sqrt{2\pi}}{4}, \frac{1}{|x|}\right)$ (7)
3. $|xf_z(x)| \le 1 \qquad |rf_z(r) - |rf_z(r)| \le 1$
   $|f_z'(x)| \le 1 \qquad |f_z'(r) - f_z'(s)| \le 1$ (8)
   $|(x+r)f_z(x+r) - (x+s)f_z(x+s)| \le \left(|x| + \frac{\sqrt{2\pi}}{4}\right)(|s| + |t|)$

引理 1.4 令f_h表示(1)的解，若h为有界可测函数，那么

$$\| f_h \|_\infty \le \sqrt{\frac{\pi}{2}} \| h(\cdot) - Nh \|_\infty \qquad \| f_h' \|_\infty \le 2 \| h(\cdot) - Nh \|_\infty$$

若h绝对连续

$$\| f_h \|_\infty \le 2 \| h' \|_\infty \quad \| f_h' \|_\infty \le \sqrt{\frac{2}{\pi}} \| h' \|_\infty \quad \| f_h'' \|_\infty \le 2 \| h' \|_\infty$$

引理 1.5 对于$z \in R$，$\alpha > 0$，令f为(1)的解，其中h定义如下：

$$h(x) = \begin{cases} 1 & x \le z \\ 1 + \dfrac{z-w}{\alpha} & z < x \le z+\alpha \\ 0 & x > z+\alpha \end{cases} \qquad (9)$$

我们一般通俗地称它为“光滑”示性函数(Smoothed indicator function)。

那么对于任意的$s, t \in R$，我们有

$$0 \le f(s) \le 1 \quad |f'(s)| \le 1 \quad |f'(s) - f'(t)| \le 1$$

并且有

$$|f'(s+t) - f'(s)| \le |t|\left(1 + |s| + \frac{1}{\alpha}\int_0^1 \mathbf{1}_{[z,z+\alpha]}(s+rt)dr\right)$$

注：以上三个引理刻画了 Stein 方程的解在h取不同函数时的基本性质，不难看出，在求解 Kolmogorov 距离时我们一般会用到引理 1.3 和引理 1.5。引理 1.4 告诉我们f_h以及它的两阶导数均是有界的，不过很多时候仅仅有界可能还无法

满足我们的要求，特别是在求解非一致界时，我们还需要知道更多信息，所以很多情况下，我们需要对于给定的h求解解的性质，比如引理 1.3 的(7)告诉我们，$0 < f_z(x) < \min\left(\frac{\sqrt{2\pi}}{4}, \frac{1}{|x|}\right) < \frac{1}{|x|}$，这一点，在求解非一致界时是非常重要的，当然，若要求解逼近速度更快的非一致界，需要对于(5)的解进行更细致的分析处理；引理 1.5 在求解一些一致界时可能会用到。以上三个引理的证明我们不再赘述，读者如果感兴趣可以参考[1]。

1.3 常用技巧

1.3.1 “leave one out”方法

Leave one out，我们可以理解为在独立随机变量的和中去掉一个随机变量。我们令$\xi_1, \xi_2, \cdots, \xi_n$是均值为 0 且方差和为 1 的独立随机变量，令$W = \sum_{i=1}^{n} \xi_i$，$W^{(i)} = W - \xi_i$，我们容易得出

$$E\big(Wf(W)\big) = E\sum_{i=1}^{n} \xi_i f\big(W^{(i)} + \xi_i\big)$$

如果f可导，

$$E\big(Wf(W)\big) = E\sum_{i=1}^{n} \xi_i^2 \int_0^1 f'\big(W^{(i)} + r\xi_i\big)dr$$

同时，我们容易说明

$$Ef'(W) = E\sum_{i=1}^{n} \sigma_i^2 f'(W) = E\sum_{i=1}^{n} \xi_i^2 f'\big(W^{(i)}\big) + E\sum_{i=1}^{n} \sigma_i^2 \left(f'(W) - f'\big(W^{(i)}\big)\right)$$

作差有

$$\begin{aligned} &E\big(f'(W) - Wf(W)\big) \\ &= E\sum_{i=1}^{n} \sigma_i^2 \left(f'(W) - f'\big(W^{(i)}\big)\right) \\ &+ E\sum_{i=1}^{n} \xi_i^2 \int_0^1 \left(f'\big(W^{(i)}\big) - f'\big(W^{(i)} + r\xi_i\big)\right) dr \end{aligned}$$

使用 Hölder 不等式通过上式我们不难说明

$$E\big(f'(W) - Wf(W)\big) \le 2\, \| f'' \| \sum_{i=1}^{n} E|\xi_i|^3$$

注：以上结果比较简单基础，并不是很常用，展示以上过程是为了简要说明“leave one out”方法的思路。

1.3.2 K 函数方法（K function approach）

首先，我们定义

$$K_i(x) = E\left(\xi_i\left(1_{\{0\le x\le \xi_i\}} - 1_{\{\xi_i\le x<0\}}\right)\right) \tag{10}$$

容易发现$K_i(x)$具有如下性质：

1. $K_i(x) \ge 0$
2. $\int_{-\infty}^{\infty} K_i(x)dx = E\xi_i^2$
3. $\int_{-\infty}^{\infty} |x|K_i(x)dx = \frac{1}{2}E|\xi_i|^3$ (11)

保持W和ξ_i定义与 1.3.1 一致，因此我们有

$$\begin{aligned}
E\big(Wf(W)\big) &= \sum_{i=1}^{n} E\big(\xi_i f(W)\big) \\
&= \sum_{i=1}^{n} E\left(\xi_i\left(f(W) - f\big(W^{(i)}\big)\right)\right) \\
&= \sum_{i=1}^{n} E(\xi_i \int_0^{\xi_i} f'(W^{(i)} + r)\, dr) = \sum_{i=1}^{n}\int_{-\infty}^{\infty} E\left(f'\big(W^{(i)} + r\big)K_i(r)\right)dr
\end{aligned}$$

根据性质(11)第二条，

$$Ef'(W) = \sum_{i=1}^{n}\int_{-\infty}^{\infty} E\big(f'(W)K_i(r)\big)dr$$

作差得出

$$E\big(Wf(W) - f'(W)\big) = \sum_{i=1}^{n}\int_{-\infty}^{\infty} E\left(f'(W) - f'\big(W^{(i)} + r\big)\right)K_i(r)dr \tag{12}$$

注：在这种方法中，(12) 对任意有界的绝对连续函数成立，它在正态近似中起到了非常重要的作用。可以利用方程(1)解的性质（引理 1.4），然后再利用 Stein 方程(1)将它转化为相应的距离。

1.3.3 可交换对（Exchangeable Pairs）

关于可交换对的基本概念，我们在前面已经给出，以下直接给出一个重要引理，它的证明比较容易，可参考[1]。

引理 1.6（参考[1]） 若(W, W')是一个λ −Stein 对，我们令$\Delta = W - W'$，$EW^2 < \infty$我们有

$$EW = 0 \qquad E\Delta^2 = 2\lambda EW^2$$

对于任意满足$|f(x)| \le C(1 + |x|)$的绝对连续函数f

$$2\lambda E\big(Wf(W)\big) = E\left((W - W')\big(f(W) - f(W')\big)\right)$$

$$E\big(Wf(W)\big) = E\left(\int_{-\infty}^{\infty} f'(W + r)\widehat{K}(r)dr\right)$$

$E\big(f'(W)-Wf(W)\big)$

$$=E\left(\int_{-\infty}^{\infty}\big(f'(W)-f'(W+r)\big)\hat{K}(r)dr\right)+Ef'(W)\left(1-\frac{\Delta^2}{2\lambda}\right)$$

其中 $\hat{K}(t)=\frac{\Delta}{2\lambda}\big(1_{\{-\Delta\le t\le 0\}}-1_{\{0<t\le-\Delta\}}\big)$，不难发现，$\int_{-\infty}^{\infty}\hat{K}(r)dr=\frac{\Delta^2}{2\lambda}$

很多时候，λ-Stein 对条件(3)可能无法成立，这时，常常有如下等式成立

$$E(W-W'|W)=\lambda(W-R) \qquad (13)$$

其中R是一个具有较小阶的随机变量。用同样的方法，我们不难得出

$$E\big(Wf(W)\big)=E\int_{-\infty}^{\infty}f'(W+r)\hat{K}(r)dr+E\big(Rf(W)\big) \qquad (14)$$

1.3.4 Zero Bias

定义 随机变量X具有 0 均值且方差为σ^2，若随机变量X'满足

$$E\big(Xf(X)\big)=\sigma^2E\big(f'(X')\big) \qquad (15)$$

f为使得上述期望存在的绝对连续函数。不难证明，这样$X\to X'$就构成了一个映射（事实上可以求出X'的密度函数），根据引理 1.1，$X=X'$成立当且仅当X为 0 均值的正态变量，这也就是说，我们可以理解为零均值正态变量是这个映射唯一的“不动点”。因此，我们可以认为如果随机变量X和它的 zero bias 分布X'差距不大时，X可认为是近似正态的，这一点是比较显然的

$$P(W\le z)-P(Z\le z)=E\big(f'(W)-Wf(W)\big)=E\big(f'(W)-f'(W')\big)$$

性质：

1. $\sigma^2E|X'|=\frac{1}{2}E|X|^3$
2. 如果$a=inf support(X)$，$b=sup support(X)$，且a、b均为有限的，那么X'的支撑$support(X')=[a,b]$
3. 若$|X|\le M$，那么$|X'|\le M$
4. $(aX)'=_d aX'$

以下给出一个重要引理

定理 1.1（证明参考[1]） 设$\xi_1,\xi_2,\cdots,\xi_n$为一列 0 均值且相互独立的随机变量，且方差和（$\sum_{i=1}^{n}\sigma_i^2=1$）为 1，设$\xi_i'$具有$\xi_i$ −zero bias 分布，令I为一个随机指标，满足$P(I=i)=\sigma_i^2$，那么令

$$W'=_d W-\xi_I+\xi_I'$$

W'具有W-zero bias 分布。

注：我们可以构造W'使得$|W'-W|=|\xi_I-\xi_i'|\le\delta$，使得二者距离尽量小，从而可以做到近似正态性。

1.3.5 Size Bias

定义 设X为非负且均值不为 0 的随机变量，对于任意使得$E\big(Xf(X)\big) < \infty$的函数$f$，令$\mu = EX$，有

$$E\big(Xf(X)\big) = \mu E(X^s) \qquad (16)$$

一般地，若$W = (Y-\mu)/\sigma$，我们令$W^s = (Y^s-\mu)/\sigma$，在许多问题中，我们常用的一个等式为$E\big(Wf(W)\big) = \frac{\mu}{\sigma}E\big(f(W^s) - f(W)\big)$。对于(16)，对应的分布函数有如下关系：

$$\frac{dF^s(t)}{dF(t)} = \frac{t}{\mu}$$

性质

1. 若$0 \le X \le C$，那么$0 \le X^s \le C$
2. X^s具有密度$p^s(t) = tp(t)/\mu$
3. $(aX)^s =_d aX^s$

一般地我们可以定义$\mathbf{X} = \{X_\alpha, \alpha \in \mathcal{A}\}$，我们称$\mathbf{X}^\alpha$具有$\alpha$方向上的 bias 分布，如果

$$EX_\alpha f(\mathbf{X}) = \mu_\alpha Ef(\mathbf{X}^\alpha) \qquad (17)$$

对于任意使得上述期望存在的f成立，不难发现

$$\frac{dF^\alpha(\mathbf{x})}{dF(\mathbf{x})} = \frac{x_\alpha}{\mu_\alpha} \qquad (18)$$

引理 1.7（参考[1]） 令$\mathbf{X} = \{X_\alpha, \alpha \in \mathcal{A}\}$，其中$X_\alpha$均为非负具有有限期望的随机变量，对任意$\mathcal{B} \subset \mathcal{A}$，令

$$X_\mathcal{B} = \sum_{\beta \in \mathcal{B}} X_\beta \qquad \mu_\mathcal{B} = EX_\mathcal{B}$$

令I为一随机指标，满足

$$P(I = i) = \frac{\mu_\beta}{\mu_\mathcal{B}}$$

然后我们令

$$\mathbf{X}^\mathcal{B} = \mathbf{X}^I$$

它满足

$$E\big(X_\mathcal{B} f(\mathbf{X})\big) = \mu_\mathcal{B} Ef(\mathbf{X}^\mathcal{B})$$

特殊地，对于$X_\mathcal{A} = \sum_{\alpha \in \mathcal{A}} X_\alpha$，我们有

$$E\big(X_\mathcal{B} f(X_\mathcal{A})\big) = \mu_\mathcal{B} f(X_\mathcal{A}^\mathcal{B})$$

其中$X_\mathcal{A}^\mathcal{B} = \sum_{\alpha \in \mathcal{A}} X_\alpha^\mathcal{B}$。显然，$X_\mathcal{A}^\mathcal{A}$有$X_\mathcal{A}$-size biased 分布。

注：在构造某方向上的 bias 分布时，我们常常使用一种分解分布函数的方法，可以让我们清晰直观的了解到某随机向量的方向 bias 分布的内部结构，本文不再对此进行叙述，可参考[1]。

定理 1.2 $X_1, X_2, \cdots, X_n$为相互独立的非负随机变量，$EX_i = \mu_i$，令$W = \sum_{i=1}^n X_i$，令I为一随机指标，满足$P(I = i) = \mu_i / \sum_{j=1}^n \mu_j$，令

$$W^I =_d W - X_I + X_I^s$$

则W^I具有W-size biased 分布。

注：其实 zero bias 和 size bias 分布均为一些 bias 分布的特殊情况，比如 square bias 分布也是较为常用的一种分布，它在组合中心极限定理中有应用。

小结：除了上述几个常用技巧之外，还有其他非常好的技巧，比如 Chatterjee 在[4]中提出的 generalized perturbative approach。

1.4 Berry-Esseen 界初步

定义 Kolmogorov 距离（或L^∞界）：$\parallel F - \mathrm{G} \parallel_\infty = \sup_{x\in R} |F(x) - G(x)|$，它刻画了两个分布的差距，本节我们主要介绍$\parallel F - \Phi \parallel_\infty = \sup_{x\in R} |F(x) - \Phi(x)|$，其中$\Phi(x)$为标准正态分布的分布函数。Berry-Esseen 界由 Berry(1941)和 Esseen(1942)分别独立提出。我们假设$\xi_1, \xi_2, \cdots, \xi_n$为一列 0 均值相互独立的随机变量，满足$\sum_{i=1}^n \sigma_i^2 = \sum_{i=1}^n Var\xi_i = 1$。最简单的 Berry-Esseen 具有如下形式：

$$\parallel F - \Phi \parallel_\infty \le C \sum_{i=1}^n E|\xi_i|^3 \qquad (19)$$

目前最精确的常数C是由 Tyurin 在 2010 年提出的（参看[16]）$C = 0.4785$。在上式(19)中，我们一般用$\gamma = \sum_{i=1}^n E|\xi_i|^3$。

1.4.1 有界时的 Berry-Esseen 界

引理 1.8（参考[1]） $\xi_1, \xi_2, \cdots, \xi_n$为一列 0 均值相互独立的随机变量，满足$\sum_{i=1}^n \sigma_i^2 = \sum_{i=1}^n Var\xi_i = 1$。那么有

$$\left|\sum_{i=1}^n \int_{-\infty}^{\infty} P\left(W^{(i)} + t \le x\right) K_i(x) dx - P(Z \le x)\right| \le 2.44\gamma$$

若W'为W-zero bias 分布，那么上式可写作

$$\left|\sum_{i=1}^n P(W' \le x) - P(Z \le x)\right| \le 2.44\gamma \qquad (20)$$

如果$|\xi_i| \le \delta_0$对于$1 \le i \le n$成立，我们有

$$\sup_{x\in R} |P(W \le x) - P(Z \le x)| \le 3.3\delta_0$$

以下定理在求L^∞界（Kolmogorov 距离）中有非常重要的作用

定理 1.3（参考[1]）如果对于任意$z \in R$，存在随机变量R_1、随机函数$\widehat{K}(x) \geq 0$、不依赖于z的常数δ_0和δ_1，使得$|ER_1| \leq \delta_1$，

$$EWf_z(W) = E\int_{|t|\leq\delta_0} f_z'(W+r)\widehat{K}(r)dr + ER_1$$

那么我们有

$$\sup_{x\in R}|P(W \leq x) - P(Z \leq x)| \leq \delta_0\left(1.1 + E\left(|W|\widehat{K}_1\right)\right) + 2.7E\left|1 - \widehat{K}_1\right| + \delta_1 \qquad (21)$$

其中$\widehat{K}_1 = E(\int_{|t|\leq\delta_0}\widehat{K}(t)dt \mid W)$

注：后面我们需要用本定理推导关于可交换对的一个L^∞界。

1.4.2 无界时的 Berry-Esseen 界

这一部分主要介绍两种证明方法：集中不等式法（Concentration inequality approach）和归纳法（Inductive Approach）。

首先是集中不等式法，首先给出如下引理

引理 1.9（参考[1]） 对于任意$1 \leq i \leq n$，我们有

$$P\left(a \leq W^{(i)} \leq b\right) \leq 2\left(\sqrt{2}+1\right)\gamma + \sqrt{2}(b-a)$$

利用引理 1.8 和引理 1.9 我们给出以下定理

定理 1.4（参考[1]）$\xi_1, \xi_2, \cdots, \xi_n$为一列 0 均值相互独立的随机变量，满足$\sum_{i=1}^n Var\xi_i = 1$，对于$W$我们有

$$\sup_{x\in R}\left|P(W \leq x) - P(Z \leq x)\right)| \leq 9.4\gamma$$

注：集中不等式法是一种非常重要的方法，虽然定理 1.4 中常数不是最优的，但是这种使用集中不等式的思路是非常重要的。关于一些集中不等式的结果，可以参考 Chatterjee[10]，该文章求出了关于临界温度时 Curie-Weiss 模型磁性的集中不等式。

然后是归纳法，用数学归纳法可以得到如下定理

定理 1.5（参考[1]） $\xi_1, \xi_2, \cdots, \xi_n$为一列 0 均值相互独立的随机变量，满足$\sum_{i=1}^n Var\xi_i = 1$，我们有

$$\sup_{x\in R}\left|P(W \leq x) - P(Z \leq x)\right)| \leq 10\gamma$$

注：虽然定理 1.5 的常数依然不是最优的，甚至不如定理 1.4，但是归纳法是非常有用的，特别是当我们在删除某些变量之后仍然能够留下与原来类似的结构时。具体应用可以参考 Chen(2020)[17]。

1.5 本文行文思路以及创新点

本文主要是运用 Stein 方法求解不同类型的 Berry-Esseen 界——一致界和非一致界，并且我们会将结果应用到模型中去。我们主要运用了一些技术比如之

前我们提到的 zero bias，size bias，exchangeable pairs 等，这一般要求一些很好的情况发生，例如对于 zero bias 我们可能会要求$|W'-W|\le\delta$，不过在一些模型中，这些条件的构造并不困难，因此，我们可以将这些 Berry-Esseen 界顺利地应用到模型中。对于第二部分中一致界，我们主要是总结了前人的一些有价值的结果；主要的创新在于第三部分，在这一部分中我们给出了部分非一致界，有些结果逼近速度可以到$O(e^{-\frac{x^2}{2}})$；第四部分我们首先介绍了一些模型背景，然后给出了二、三部分结果在模型中的具体应用。

二、L^∞界（一致界）

在这一部分中，我们主要给出[1]中提到的几个关于 Berry-Esseen 不等式的结果，定理 2.1，我们需要满足$|W'-W|\le\delta$，其中W'为 zero bias 分布；定理 2.2 中我们需要$|W'-W|\le\delta$，这里指的是可交换对，定理 2.3 告诉我们当这一条件不满足时，如果容易处理$E(W'-W)^2 1_{\{|W'-W|>a\}}$也是可以的；定理 2.4 在$\lambda$-Stein 对差有界时给出了 Berry-Esseen 界；类似于定理 2.2，2.3 的思想，我们给出 size bias 中的两个结果定理 2.5 和定理 2.6。除此之外，我们还列举了[4]中关于 Kantorovich-Wasserstein 距离的结果，以及[18]中的结果。

定理 2.1（参看[1]）设W是 0 均值方差为 1 的随机变量，假设存在一个W-zero bias 分布W'使得$|W'-W|\le\delta$。那么我们有

$$\sup_{x\in R}\left|P(W\le x)-P(Z\le x)\right|\le C\delta$$

其中$C=1+\frac{1}{\sqrt{2\pi}}+\frac{\sqrt{2\pi}}{4}$。

注：该定理告诉我们如果可以在同一概率空间构造一个随机变量W'使得上述条件成立，那么W就是近似正态的。

定理 2.2（参看[1]）设W，W'为 0 均值方差为 1 的可交换对，且满足

$$E(W-W'|W)=\lambda(W-R)$$

若$|W'-W|\le\delta$，$\lambda\in(0,1)$，那么有

$$\sup_{x\in R}\left|P(W\le x)-P(Z\le x)\right|\le\delta\left(1.1+E\left(|W|\hat{K}_1\right)\right)+2.7B+\frac{\sqrt{2\pi}}{4}E|R|$$

其中$B=E\left|1-E\left(\frac{\Delta^2}{2\lambda}\middle|W\right)\right|$，$\hat{K}_1=E\left(\int_{|r|\le\delta}\hat{K}(r)dr\middle|W\right)$，$\Delta=W'-W$

注：本定理证明需要用到定理 1.3，很多时候B并不便于计算，在观察定理 1.3 以及 2.2 的证明后我们不难发现很多时候B可以更换为

$$B = E\left|1 - E\left(\frac{\Delta^2}{2\lambda}\middle|X\right)\right| \qquad (22)$$

这方便了我们的计算，因为在构造可交换对之后我们常常会处理一些细节，我们需要这些细节关于$\sigma(X)$可测，对于参数B，我们可以用不等式

$$B \le \frac{\sqrt{Var(E(\Delta^2|W))}}{2\lambda} \qquad (23)$$

来代替。我们计算 Anti-voter 模型以及 Quadratic forms 会用到我们这一点，在计算非一致界时，我们引入参数B'，有趣的是，$B' = \frac{\sqrt{Var\left(E(\Delta^2|W)\right)}}{2\lambda}$。

作为上述定理 2.2 定理的推论，我们给出如果$R = 0$，我们有：

$$\sup_{x\in R}\left|P(W \le x) - P(Z \le x)\right| \le 1.1\delta + \frac{\delta^3}{2\lambda} + 2.7B$$

令$\delta = \delta_0$，此时$\widehat{K}_1 = E\left(\frac{\Delta^2}{2\lambda}\middle|W\right)$，由上述定理显然。

接下来，我们给出当$|W' - W|$无界时对应的 Berry-Esseen 界，首先给出引理:

引理 2.1（参看[1]）令W，W'为零均值方差为 1 的λ-Stein 对，那么

$$E\Delta^2 1_{\{-a\le\Delta\le 0\}} 1_{\{z-a\le W\le z\}} \le 3\lambda a$$

定理 2.3（参看[1]） 设W，W'为 0 均值方差为 1 的可交换对，且满足

$$E(W - W'|W) = \lambda(W - R)$$

这里的$\lambda \in (0,1)$，我们有

$$\sup_{x\in R}\left|P(W \le x) - P(Z \le x)\right| \le B + 1.5a + \frac{0.41a^3}{\lambda} + \frac{E\left(\Delta^2 1_{\{|\Delta|>a\}}\right)}{2\lambda} + \frac{\sqrt{2\pi}}{4}E|R|$$

如果(W, W')是λ-Stein 对，且有$|\Delta| \le \delta$，那么

$$\sup_{x\in R}\left|P(W \le x) - P(Z \le x)\right| \le B + \frac{0.41\delta^3}{\lambda} + 1.5\delta$$

其中$\Delta = W' - W$。

注：当不存在δ使得满足$|W' - W| \le \delta$时，我们给出了$E\left(\Delta^2 1_{\{|\Delta|>a\}}\right)$项，当这一项容易处理时我们可以使用定理 2.3。

以下定理亦是关于满足$|W' - W| \le \delta$的λ-Stein 对：

定理 2.4（参看[1]）设W'，W是方差为 1 且满足$|W' - W| \le \delta$的λ-Stein 对，那么

$$\sup_{x\in R}\left|P(W \le x) - P(Z \le x)\right| \le \frac{3\delta^3}{\lambda} + 2B \qquad (24)$$

注：若要使用本定理，显然我们要构造满足上述条件的可交换对，这类构造还是比较常见的，比如$X_1, X_2, \cdots, X_n$是一列独立同分布的随机变量，$X_1', X_2', \cdots, X_n'$与$X_1, X_2, \cdots, X_n$独立同分布，满足$|X_1| \le M$，$Var(X_1) = \sigma^2$，$EX_1 = 0$。我们令

$W=\frac{\sum_{i=1}^{n}X_i}{\sqrt{n}\sigma}$,$W'=W-\frac{X_I}{\sqrt{n}\sigma}+\frac{X_I'}{\sqrt{n}\sigma}$，因此根据我们一开始提到的例子很显然能看出$(W,W')$是可交换对，满足$|W-W'|\le\frac{2M}{\sqrt{n}\sigma}=\delta$。若要应用本定理，我们需要首先找到很小的$\delta$，然后需要对$B$进行放缩。在文献 [1]中，作者还列举了关于$E|\Delta|^3$的结果，此处我们不再赘述。

当我们处理非负且具有有限期望的随机变量时，我们常常使用 size bias 方法。首先，我们给出一些定理中的常用符号

$$W=(Y-\mu)/\sigma\text{，记}W^s=(Y^s-\mu)/\sigma$$

$$D=E\left|E\left(1-\frac{\mu}{\sigma}(W^s-W)\middle|W\right)\right|\text{，}\Psi=\sqrt{Var\big(E(Y^s-Y|Y)\big)}$$

容易证明，二者有如下关系，类似上面B与$\sqrt{Var\big(E(\Delta^2|W)\big)}$的关系：

$$D\le\frac{\mu}{\sigma^2}\Psi \tag{25}$$

类似于定理 2.2，2.3 我们直接给出如下定理：

定理 2.5（参考[1]） 设Y是一个非负随机变量，满足$EY=\mu$，$VarY=\sigma^2$。如果我们可以构造Y-size biased 分布Y^s，满足$|Y^s-Y|\le M$，那么我们有

$$\sup_{x\in R}\big|P(W\le x)-P(Z\le x)\big)|\le\frac{6\mu M^2}{\sigma^3}+2D$$

定理 2.6（参考[1]） 设Y是一个非负随机变量，满足$EY=\mu$，$VarY=\sigma^2$。如果我们可以构造Y-size biased 分布Y^s，满足$Y^s\ge Y$，那么对任意的$a\ge 0$

$$\sup_{x\in R}\big|P(W\le x)-P(Z\le x)\big)|\le\frac{0.82a^2\mu}{\sigma}+a+\frac{\mu}{\sigma}E(W^s-W)1_{\{W^s-W>a\}}+D$$

进一步，如果存在a满足$W^s-W\le M$

$$\sup_{x\in R}\big|P(W\le x)-P(Z\le x)\big)|\le\frac{0.82M^2\mu}{\sigma}+D+M$$

注：在定理 2.5 中，我们需要构造$|Y^s-Y|\le M$，为证明W的近似正态性，我们需要M和D较小；如果无法满足$|Y^s-Y|\le M$，但可以构造使得Y^s满足$Y^s\ge Y$，并且此时$E(W^s-W)1_{\{W^s-W>a\}}$比较容易处理，那么我们可以使用定理 2.6。

以下定理 2.7 我们给出 Chatterjee[2]关于 Kantorovich-Wasserstein 距离的结果，首先定义 Kantorovich-Wasserstein 距离，对于概率测度P和Q，定义

$$\mathcal{W}(P,Q)=\sup\left\{\left|\int hdP-\int hdQ\right|:h\text{ 为 Lipschitz 函数，满足 }\|h\|_{\text{Lip}}\le 1\right\} \tag{26}$$

我们不难看出，Kantorovich-Wasserstein 距离强于弱收敛。

定理 2.7 （参考[2]）令$W=f(X)$，满足$E(W)=0$，且$\sigma^2\coloneqq E(W^2)\le\infty$，我们有

$$\mathcal{W}(P,Q)\le\frac{\Big(Var\big(E(T|W)\big)\Big)^{\frac{1}{2}}}{\sigma^2}+\frac{1}{2\sigma^3}\sum_{i=1}^{n}E|\Delta_i f(X)|^3$$

其中P指的是由$(W-EW)/\sqrt{Var(W)}$引入的测度，Q指的是由标准正态分布引入的测度。

注：上述符号具体表示含义参看[2]，[2]主要使用了一个变形$E\big(Wf(W)\big)\approx E(Tf'(W))$，Chatterjee 将它运用到了 Quadratic forms 模型中。

以下定理 2.8、2.9 给出了一个可交换对差无界时正态和非正态近似的结果（参看 Shao 和 Zhang[18]）：

定理 2.8（正态近似）令$\Delta^* \coloneqq \Delta^*(W,W')$为任意满足：1. $\Delta^*(W,W')=\Delta^*(W',W)$，2.$\Delta^* \geq |\Delta|$的随机变量，那么我们有

$$\sup_{x\in R}|P(W\leq x)-P(Z\leq x)|\leq E\left|1-\frac{1}{2\lambda}E(\Delta^2|W)\right|+\frac{1}{\lambda}E|E(\Delta\Delta^*|W)|+E|R|$$

定理 2.9（非正态近似） 具体符号含义见[18]，我们有

$$\sup_{x\in R}|P(W\leq x)-P(Z\leq x)|\leq E\left|1-\frac{1}{2\lambda}E(\Delta^2|W)\right|+\frac{1}{\lambda}E|E(\Delta\Delta^*|W)|+\frac{1}{c_1}E|R|$$

注：以上两个定理给出了可交换对的$|W-W'|$无界时的结果，与定理 2.3 不同，它并不关心$E\big(\Delta^2 1_{\{|\Delta|>a\}}\big)$是否便于处理。在[18]中，Shao 和 Zhang 不仅仅给出了上述两个主要定理，并且给出了它们在 Quadratic forms、General Curie-Weiss model、Counting monochromatic edges in uniformly colored graphs、Mean field Heisenberg model 等模型中的应用。

三、非一致界以及它们的证明

与一致界相比，非一致界不再描述 Kolmogorov 距离的上界，它描述的上界是与x有关的。比如 Chen 和 Shao(2001)年在[6]中提出的：

$$|F(x)-\Phi(x)|\leq C\sum_{i=1}^{n}\Big(\frac{EX_i^2 1_{\{|X_i|>1+|x|\}}}{(1+|x|)^2}+\frac{E|X_i|^3 1_{\{|X_i|\leq 1+|x|\}}}{(1+|x|)^3}\Big)$$

非一致界的得出，往往与 Stein 方程解(5)的性质有关，在这一部分，我们将会利用(5)的衰减性得出一些非一致界，其中部分非一致界可以达到“指数下降”的收敛速度。本部分中，非一致界的证明结合了部分一致界的证明思路。

以下定理 3.1 是关于 zero bias 分布的非一致界：

定理 3.1 令W为 0 均值方差为 1 随机变量，$|W|$具有有限的矩母函数（$Ee^{t|W|}<\infty$对任意t成立）且$Ee^{W^2}<\infty$，假设W'具有W-zero bias 分布，满足$|W'-W|\leq\delta$，那么对于任意的常数$M\geq 0$，存在常数$C_i(i=1,2,3)$，使得

$$|P(W \le z) - P(Z \le z)| = \begin{cases} C_1 \delta e^{-\frac{(z+2\delta)^2}{2}} & z \le -M - 2\delta \\ C_2 \delta e^{-\frac{z^2}{2}} & |z| \le M + 2\delta \\ C_3 \delta e^{-\frac{(z-2\delta)^2}{2}} & z \ge M + 2\delta \end{cases} \quad (27)$$

注：在以上定理 3.1 中，我们计算出一个“指数上平方”下降的非一致界，因此在$|z|$足够大时，$|P(W \le z) - P(Z \le z)|$会以非常快的速度下降，这一点非常有利于提升中心极限定理中误差估计的准确性。但是，想要做到指数下降并不是一件很容易的事，因为我们需要$|W|$具有有限的矩母函数并且$Ee^{W^2} < \infty$，而且Ee^{W^2}通常和n有关。在下面的证明中，M可以取任意正常数（比如$M = 1$），同时我们不难发现，在$\delta \le A$，A为一正常数时，下面证明中常数$C_i(i = 1,2,3)$可以与δ无关，仅仅与W有关，具体可以参看我们的证明过程。上面的结果可以看做定理 2.1 的变形，在后面我们介绍应用时，我们会将定理 2.1 和定理 3.1 应用到组合中心极限定理（Combinatorial central limit theorem），并对两者进行对比。

为证明定理 3.1，我们首先给出以下引理：

引理 3.1 若f_x为 Stein 方程(5)的解，我们有

$$\left(xf_z(x)\right)' = \begin{cases} \sqrt{2\pi}\left(1 - \Phi(z)\right)\left((1 + x^2)e^{\frac{x^2}{2}}\Phi(x) + \dfrac{x}{\sqrt{2\pi}}\right) & x < z \\ \sqrt{2\pi}\Phi(z)\left((1 + x^2)e^{\frac{x^2}{2}}\left(1 - \Phi(x)\right) - \dfrac{x}{\sqrt{2\pi}}\right) & x > z \end{cases}$$

以上引理 3.1 证明参看[1]。

注：有了引理 3.1，我们就可以利用$\left(1 - \Phi(z)\right)$的衰减性给出指数界。

定理 3.1 的证明：

（1）对于$z > 2\delta + M$（其中$M \ge 0$，因此$\frac{1}{z-\delta} \le \frac{1}{M+\delta} \le \frac{1}{M}$），令$f$为 Stein 方程(5)的解，其中我们将$z$更换为$z - \delta$，因此我们有：

$$\begin{aligned} f'(W') &= 1_{\{W' \le z - \delta\}} - \Phi(z - \delta) + W'f(W') \\ &\le 1_{\{W \le z\}} - \Phi(z - \delta) + W'f(W') \qquad (28) \end{aligned}$$

我们对上式(28)取期望，利用 zero bias 分布的定义，我们有：

$$\begin{aligned} P(W \le z) - \Phi(z) &= P(W \le z) - \Phi(z - \delta) + \Phi(z - \delta) - \Phi(z) \\ &\ge E\left(f'(W') - W'f(W')\right) - \int_{z-\delta}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \\ &\ge E\left(Wf(W) - W'f(W')\right) - e^{-\frac{(z-\delta)^2}{2}} \frac{\delta}{\sqrt{2\pi}} \qquad (29) \end{aligned}$$

对于(29)第一项作如下划分，我们有

$$\left|E\big(Wf(W)-W'f(W')\big)\right|$$
$$\le E|Wf(W)-W'f(W')|1_{\{z\ge W+2\delta\}}$$
$$+E|Wf(W)-W'f(W')|1_{\{z-2\delta<W\}}=I_1+I_2 \tag{30}$$

对于I_1，在事件$\{z\ge W+2\delta\}$上，满足$z-\delta\ge\max\{W,W'\}$，所以我们可以用 Lagrange 中值定理，设这个满足条件的“中值”为随机变量V，显然$W+\delta\ge V$，根据(30)以及引理 3.1，再由 Hölder 不等式，对于I_1

$$I_1\le\delta E\left|\big(xf(x)\big)'_{x=V}\right|1_{\{z-2\delta\ge W\}}$$
$$=\delta E\left|\sqrt{2\pi}\big(1-\Phi(z-\delta)\big)\left((1+V^2)e^{\frac{V^2}{2}}\Phi(V)+\frac{V}{\sqrt{2\pi}}\right)\right|1_{\{z-2\delta\ge W\}}$$
$$\le\delta\big(1-\Phi(z-\delta)\big)E\sqrt{2\pi}\left|\left((1+V^2)e^{\frac{V^2}{2}}\Phi(V)+\frac{V}{\sqrt{2\pi}}\right)\right| \tag{31}$$

因为W具有有限的矩母函数且$Ee^{W^2}<\infty$，因此对于(31)，

$$I_1\le C\delta\big(1-\Phi(z-\delta)\big)\le C\delta\int_{z-\delta}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt\le C\delta\int_{z-\delta}^{\infty}\frac{1}{\sqrt{2\pi}}\cdot\frac{t}{z-\delta}\cdot e^{-\frac{t^2}{2}}dt$$
$$=\frac{C\delta}{\sqrt{2\pi}(z-\delta)}\int_{z-\delta}^{\infty}te^{-\frac{t^2}{2}}dt=\frac{C\delta}{\sqrt{2\pi}(z-\delta)}e^{-\frac{(z-\delta)^2}{2}} \tag{32}$$

对于I_2，根据 Chebyshev 不等式，我们有

$$I_2=E|Wf(W)-W'f(W')|1_{\{z-2\delta<W\}}$$
$$\le\frac{E|Wf(W)-W'f(W')|e^{\frac{W^2}{2}}1_{\{z-2\delta<W\}}}{e^{\frac{(z-2\delta)^2}{2}}} \tag{33}$$

因为W具有有限的矩母函数且$Ee^{W^2}<\infty$，根据引理 1.3 的性质(8)，对于(33)，我们有

$$I_2\le\frac{E\left(\left(|W|+\frac{\sqrt{2\pi}}{4}\right)|W'-W|e^{\frac{W^2}{2}}1_{\{z-2\delta<W\}}\right)}{e^{\frac{(z-2\delta)^2}{2}}}$$
$$\le\frac{E\left(\left(|W|+\frac{\sqrt{2\pi}}{4}\right)e^{\frac{W^2}{2}}1_{\{z-2\delta<W\}}\right)}{e^{\frac{(z-2\delta)^2}{2}}}\cdot\delta\le C\delta e^{-\frac{(z-2\delta)^2}{2}}$$

因此对于(30)，我们有

$$\left|E\big(Wf(W)-W'f(W')\big)\right|\le\frac{C\delta}{\sqrt{2\pi}(z-\delta)}e^{-\frac{(z-\delta)^2}{2}}+C\delta e^{-\frac{(z-2\delta)^2}{2}} \tag{34}$$

对于(29)，我们有

$$P(W \le z) - \Phi(z) \ge -e^{-\frac{(z-\delta)^2}{2}} \frac{\delta}{\sqrt{2\pi}} - \frac{C\delta}{\sqrt{2\pi}(z-\delta)} e^{-\frac{(z-\delta)^2}{2}} - C\delta e^{-\frac{(z-2\delta)^2}{2}}$$

$$\ge -e^{-\frac{(z-2\delta)^2}{2}} \frac{\delta}{\sqrt{2\pi}} - \frac{C\delta}{\sqrt{2\pi}M} e^{-\frac{(z-2\delta)^2}{2}} - C\delta e^{-\frac{(z-2\delta)^2}{2}}$$

$$\ge -C\delta e^{-\frac{(z-2\delta)^2}{2}} \tag{35}$$

这里常数C的值可能会有变化。

对于另一个方向的不等式，我们令f为 Stein 方程(5)的解，我们将其中的z更换为$z+\delta$，因此我们有

$$f'(W') = 1_{\{W' \le z+\delta\}} - \Phi(z+\delta) + W'f(W')$$

$$\ge 1_{\{W \le z\}} - \Phi(z+\delta) + W'f(W') \tag{36}$$

对(36)取期望，根据 zero bias 分布定义

$$P(W \le z) - \Phi(z) = P(W \le z) - \Phi(z+\delta) + \Phi(z+\delta) - \Phi(z)$$

$$\le E\big(f'(W') - W'f(W')\big) + \int_z^{z+\delta} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$\le E\big(Wf(W) - W'f(W')\big) + e^{-\frac{z^2}{2}} \frac{\delta}{\sqrt{2\pi}} \tag{37}$$

对于上式第一项，与之前的证明方法类似

$$\left|E\big(Wf(W) - W'f(W')\big)\right|$$

$$\le E|Wf(W) - W'f(W')|1_{\{z \ge W\}} + E|Wf(W) - W'f(W')|1_{\{z < W\}}$$

$$= I_1 + I_2 \tag{38}$$

对I_1，在事件$\{z \ge W\}$上，我们可以推出$z+\delta \ge \max\{W, W'\}$，因此可以使用 Lagrange 中值定理，同样设“中值”为随机变量V，不难发现$W+\delta \ge V$，根据引理 3.1，由 Hölder 不等式，对于I_1

$$I_1 \le \delta E\left|\big(xf(x)\big)'_{x=V}\right| 1_{\{z \ge W\}}$$

$$= \delta E\left|\sqrt{2\pi}\big(1-\Phi(z)\big)\left((1+V^2)e^{\frac{V^2}{2}}\Phi(V) + \frac{V}{\sqrt{2\pi}}\right)\right| 1_{\{z \ge W\}}$$

$$\le \delta\big(1-\Phi(z)\big)E\left|\sqrt{2\pi}\left((1+V^2)e^{\frac{V^2}{2}}\Phi(V) + \frac{V}{\sqrt{2\pi}}\right)\right| \tag{39}$$

因为W具有有限的矩母函数且$Ee^{W^2} < \infty$，因此对于(39)，

$$I_1 \le C\delta\big(1-\Phi(z)\big) \le C\delta \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \le C\delta \int_z^\infty \frac{1}{\sqrt{2\pi}} \cdot \frac{t}{z} \cdot e^{-\frac{t^2}{2}} dt$$

$$= \frac{C\delta}{\sqrt{2\pi}z} \int_z^\infty te^{-\frac{t^2}{2}} dt = \frac{C\delta}{\sqrt{2\pi}z} e^{-\frac{z^2}{2}} \tag{40}$$

根据 Chebyshev 不等式，对于I_2

$$I_2 = E|Wf(W) - W'f(W')|1_{\{z<W\}}$$
$$\leq \frac{E|Wf(W) - W'f(W')|e^{\frac{W^2}{2}}1_{\{z<W\}}}{e^{\frac{z^2}{2}}} \tag{41}$$

由于W具有有限的矩母函数且$Ee^{W^2} < \infty$，根据引理 1.3 的性质(8)，对于(41)，我们有

$$I_2 \leq \frac{E\left(\left(|W| + \frac{\sqrt{2\pi}}{4}\right)|W' - W|e^{\frac{W^2}{2}}1_{\{z<W\}}\right)}{e^{\frac{z^2}{2}}} \leq \frac{E\left(\left(|W| + \frac{\sqrt{2\pi}}{4}\right)e^{\frac{W^2}{2}}1_{\{z<W\}}\right)}{e^{\frac{z^2}{2}}} \cdot \delta$$
$$\leq C\delta e^{-\frac{z^2}{2}}$$

因此，对于(38)，

$$\left|E\left(Wf(W) - W'f(W')\right)\right| = I_1 + I_2 \leq \frac{C\delta}{\sqrt{2\pi}z}e^{-\frac{z^2}{2}} + C\delta e^{-\frac{z^2}{2}}$$
$$\leq \frac{C\delta}{\sqrt{2\pi}M}e^{-\frac{z^2}{2}} + C\delta e^{-\frac{z^2}{2}} \tag{42}$$

对于(37)

$$P(W \leq z) - \Phi(z) \leq E\left(Wf(W) - W'f(W')\right) + e^{-\frac{z^2}{2}}\frac{\delta}{\sqrt{2\pi}}$$
$$\leq \frac{C\delta}{\sqrt{2\pi}M}e^{-\frac{z^2}{2}} + C\delta e^{-\frac{z^2}{2}} + e^{-\frac{z^2}{2}}\frac{\delta}{\sqrt{2\pi}} = C\delta e^{-\frac{z^2}{2}} \tag{43}$$

所以结合(35)和(43)，我们有

$$-C\delta e^{-\frac{(z-2\delta)^2}{2}} \leq P(W \leq z) - \Phi(z) \leq C\delta e^{-\frac{z^2}{2}} \leq C\delta e^{-\frac{(z-2\delta)^2}{2}} \tag{44}$$

即

$$|P(W \leq z) - \Phi(z)| \leq C\delta e^{-\frac{(z-2\delta)^2}{2}} \tag{45}$$

（2）对于$z \leq -M - 2\delta$（其中$M \geq 0$），令f为 Stein 方程(5)的解，同样地，此处我们将z更换为$z - \delta$，因此我们有：

$$f'(W') = 1_{\{W' \leq z-\delta\}} - \Phi(z - \delta) + W'f(W')$$
$$\leq 1_{\{W \leq z\}} - \Phi(z - \delta) + W'f(W') \tag{46}$$

对(46)取期望，利用 zero bias 分布的定义，我们有：

$$P(W \leq z) - \Phi(z) = P(W \leq z) - \Phi(z - \delta) + \Phi(z - \delta) - \Phi(z)$$
$$\geq E\left(f'(W') - W'f(W')\right) - \int_{z-\delta}^{z}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt$$
$$\geq E\left(Wf(W) - W'f(W')\right) - e^{-\frac{z^2}{2}}\frac{\delta}{\sqrt{2\pi}} \tag{47}$$

对于(29)第一项作如下划分，我们有

$$\left|E\big(Wf(W)-W'f(W')\big)\right|$$
$$\le E|Wf(W)-W'f(W')|1_{\{z\ge W\}}+E|Wf(W)-W'f(W')|1_{\{z<W\}}$$
$$=I_1+I_2 \tag{48}$$
对于I_2，在事件$\{z<W\}$上，满足$z-\delta<\min\{W,W'\}$，所以我们可以用 Lagrange 中值定理，设这个满足条件的“中值”为随机变量V，根据引理 3.1，再由 Hölder 不等式，对于I_2

$$I_2\le \delta E\left|\big(xf(x)\big)'_{x=V}\right|1_{\{z<W\}}$$
$$=\delta E\left|\sqrt{2\pi}\Phi(z-\delta)\left((1+V^2)e^{\frac{V^2}{2}}\big(1-\Phi(V)\big)-\frac{V}{\sqrt{2\pi}}\right)\right|1_{\{z<W\}}$$
$$\le \delta\Phi(z-\delta)\sqrt{2\pi}E\left|(1+V^2)e^{\frac{V^2}{2}}\big(1-\Phi(V)\big)-\frac{V}{\sqrt{2\pi}}\right| \tag{49}$$

由于W具有有限的矩母函数且$Ee^{W^2}<\infty$，那么对于(49)，

$$I_2\le C\delta\Phi(z-\delta)\le C\delta\int_{-\infty}^{z-\delta}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt\le C\delta\int_{-\infty}^{z-\delta}\frac{1}{\sqrt{2\pi}}\cdot\frac{t}{z-\delta}\cdot e^{-\frac{t^2}{2}}dt$$
$$=\frac{C\delta}{\sqrt{2\pi}(z-\delta)}\int_{-\infty}^{z-\delta}te^{-\frac{t^2}{2}}dt=-\frac{C\delta}{\sqrt{2\pi}(z-\delta)}e^{-\frac{(z-\delta)^2}{2}} \tag{50}$$

对于I_1，根据 Chebyshev 不等式，我们有

$$I_1=E|Wf(W)-W'f(W')|1_{\{z\ge W\}}$$
$$\le\frac{E|Wf(W)-W'f(W')|e^{\frac{W^2}{2}}1_{\{z\ge W\}}}{e^{\frac{z^2}{2}}} \tag{51}$$

因为W具有有限的矩母函数且$Ee^{W^2}<\infty$，根据引理 1.3 的性质(8)，对于(51)，有

$$I_2\le\frac{E\left(\left(|W|+\frac{\sqrt{2\pi}}{4}\right)|W'-W|e^{\frac{W^2}{2}}1_{\{z\ge W\}}\right)}{e^{\frac{z^2}{2}}}\le\frac{E\left(\left(|W|+\frac{\sqrt{2\pi}}{4}\right)e^{\frac{W^2}{2}}1_{\{z\ge W\}}\right)}{e^{\frac{z^2}{2}}}\cdot\delta$$
$$\le C\delta e^{-\frac{z^2}{2}}$$

对于(48)，我们有

$$\left|E\big(Wf(W)-W'f(W')\big)\right|\le-\frac{C\delta}{\sqrt{2\pi}(z-\delta)}e^{-\frac{(z-\delta)^2}{2}}+C\delta e^{-\frac{z^2}{2}} \tag{52}$$

对于(29)，我们有

$$
\begin{aligned}
P(W\le z)-\Phi(z) &\ge -e^{-\frac{z^2}{2}}\frac{\delta}{\sqrt{2\pi}}+\frac{C\delta}{\sqrt{2\pi}(z-\delta)}e^{-\frac{(z-\delta)^2}{2}}-C\delta e^{-\frac{z^2}{2}}\\
&\ge -e^{-\frac{z^2}{2}}\frac{\delta}{\sqrt{2\pi}}-\frac{C\delta}{\sqrt{2\pi}M}e^{-\frac{(z-\delta)^2}{2}}-C\delta e^{-\frac{z^2}{2}}\\
&\ge -C\delta e^{-\frac{z^2}{2}} \qquad (53)
\end{aligned}
$$

关于另一个方向的不等式，我们令f为 Stein 方程(5)的解，将其中的z更换为$z+\delta$，我们有

$$
\begin{aligned}
f'(W') &= 1_{\{W'\le z+\delta\}}-\Phi(z+\delta)+W'f(W')\\
&\ge 1_{\{W\le z\}}-\Phi(z+\delta)+W'f(W') \qquad (54)
\end{aligned}
$$

对(54)取期望，根据 zero bias 分布定义

$$
\begin{aligned}
P(W\le z)-\Phi(z) &= P(W\le z)-\Phi(z+\delta)+\Phi(z+\delta)-\Phi(z)\\
&\le E\big(f'(W')-W'f(W')\big)+\int_z^{z+\delta}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt\\
&\le E\big(Wf(W)-W'f(W')\big)+e^{-\frac{(z+\delta)^2}{2}}\frac{\delta}{\sqrt{2\pi}} \qquad (55)
\end{aligned}
$$

对于(55)第一项，

$$
\begin{aligned}
&\left|E\big(Wf(W)-W'f(W')\big)\right|\\
&\quad\le E|Wf(W)-W'f(W')|1_{\{z+2\delta\ge W\}}\\
&\quad+E|Wf(W)-W'f(W')|1_{\{z+2\delta<W\}}\\
&\quad=I_1+I_2 \qquad (56)
\end{aligned}
$$

对I_2，在事件$\{z+2\delta<W\}$上，我们可以推出$z+\delta<\min\{W,W'\}$，因此可以使用 Lagrange 中值定理，我们设“中值”为随机变量V，根据引理 3.1，由 Hölder 不等式，对于I_2

$$
\begin{aligned}
I_2 &\le \delta E\left|\big(xf(x)\big)'_{x=V}\right|1_{\{z+2\delta<W\}}\\
&=\delta E\left|\sqrt{2\pi}\Phi(z+\delta)\left((1+V^2)e^{\frac{V^2}{2}}\big(1-\Phi(V)\big)-\frac{V}{\sqrt{2\pi}}\right)\right|1_{\{z+2\delta<W\}}\\
&\le \delta\sqrt{2\pi}\Phi(z+\delta)E\left|(1+V^2)e^{\frac{V^2}{2}}\big(1-\Phi(V)\big)-\frac{V}{\sqrt{2\pi}}\right| \qquad (57)
\end{aligned}
$$

由于W具有有限的矩母函数且$Ee^{W^2}\le\infty$，对于(57)，

$$
\begin{aligned}
I_2 &\le C\delta\Phi(z+\delta)\le C\delta\int_{-\infty}^{z+\delta}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt\le C\delta\int_{-\infty}^{z+\delta}\frac{1}{\sqrt{2\pi}}\cdot\frac{t}{z+\delta}\cdot e^{-\frac{t^2}{2}}dt\\
&=\frac{C\delta}{\sqrt{2\pi}(z+\delta)}\int_{-\infty}^{z+\delta}te^{-\frac{t^2}{2}}dt=-\frac{C\delta}{\sqrt{2\pi}(z+\delta)}e^{-\frac{(z+\delta)^2}{2}} \qquad (58)
\end{aligned}
$$

根据 Chebyshev 不等式，对于I_1

$$I_1 = E|Wf(W) - W'f(W')|1_{\{z+2\delta\geq W\}}$$

$$\leq \frac{E|Wf(W) - W'f(W')|e^{\frac{W^2}{2}}1_{\{z+2\delta\geq W\}}}{e^{\frac{(z+2\delta)^2}{2}}} \tag{59}$$

因为W具有有限矩母函数且$Ee^{W^2} \leq \infty$，根据引理 1.3 的性质(8)，由 Hölder 不等式，对于(59)，我们有

$$I_1 \leq \frac{E\left(\left(|W| + \frac{\sqrt{2\pi}}{4}\right)|W' - W|e^{\frac{W^2}{2}}1_{\{z+2\delta\geq W\}}\right)}{e^{\frac{(z+2\delta)^2}{2}}}$$

$$\leq \frac{E\left(\left(|W| + \frac{\sqrt{2\pi}}{4}\right)e^{\frac{W^2}{2}}1_{\{z+2\delta\geq W\}}\right)}{e^{\frac{(z+2\delta)^2}{2}}} \cdot \delta \leq C\delta e^{-\frac{(z+2\delta)^2}{2}}$$

因此，对于(56)，

$$\left|E\left(Wf(W) - W'f(W')\right)\right| = I_1 + I_2 \leq C\delta e^{-\frac{(z+2\delta)^2}{2}} - \frac{C\delta}{\sqrt{2\pi}(z+\delta)}e^{-\frac{(z+\delta)^2}{2}}$$

$$\leq C\delta e^{-\frac{(z+2\delta)^2}{2}} + \frac{C\delta}{\sqrt{2\pi}M}e^{-\frac{(z+\delta)^2}{2}} \tag{60}$$

对于(55)

$$P(W \leq z) - \Phi(z) \leq E\left(Wf(W) - W'f(W')\right) + e^{-\frac{(z+\delta)^2}{2}}\frac{\delta}{\sqrt{2\pi}}$$

$$\leq C\delta e^{-\frac{(z+2\delta)^2}{2}} + \frac{C\delta}{\sqrt{2\pi}M}e^{-\frac{(z+\delta)^2}{2}} + e^{-\frac{(z+\delta)^2}{2}}\frac{\delta}{\sqrt{2\pi}}$$

$$= C\delta e^{-\frac{(z+2\delta)^2}{2}} \tag{61}$$

结合(53)和(61)，我们有

$$-C\delta e^{-\frac{(z+2\delta)^2}{2}} \leq -C\delta e^{-\frac{z^2}{2}} \leq P(W \leq z) - \Phi(z) \leq C\delta e^{-\frac{(z+2\delta)^2}{2}} \tag{62}$$

即

$$|P(W \leq z) - \Phi(z)| \leq C\delta e^{-\frac{(z+2\delta)^2}{2}} \tag{63}$$

（3）当$|z| \leq M + 2\delta$时，由定理 2.1 知

$$|P(W \leq x) - P(Z \leq x)| \leq C\delta$$

其中$C = 1 + \frac{1}{\sqrt{2\pi}} + \frac{\sqrt{2\pi}}{4}$。我们有

$$|P(W \leq z) - P(Z \leq z)| \leq C\delta e^{-\frac{z^2}{2}} \cdot e^{\frac{z^2}{2}} \leq C\delta e^{-\frac{z^2}{2}} \cdot e^{\frac{(M+2\delta)^2}{2}}$$

$$\leq C\delta e^{-\frac{z^2}{2}} \tag{64}$$

综上所述，

$$|P(W\le z)-P(Z\le z)|=\begin{cases}C_1\delta e^{-\frac{(z+2\delta)^2}{2}} & z\le -M-2\delta\\ C_2\delta e^{-\frac{z^2}{2}} & |z|\le M+2\delta\\ C_3\delta e^{-\frac{(z-2\delta)^2}{2}} & z\ge M+2\delta\end{cases}$$

至此，我们证明了定理 3.1。

以下定理 3.2 非一致界是关于可交换对的结果：

定理 3.2 若(W,W')是 0 均值方差为 1 的可交换对并且满足

$$E(W-W'|W)=\lambda(W-R)，其中\lambda\in(0,1)$$

$|W|$具有有限的矩母函数（$Ee^{t|W|}<\infty$对任意t成立）且$Ee^{W^2}<\infty$，二者之差 $|W-W'|\le\delta$。那么对于任意常数$M>0$，存在仅依赖于δ,W,M但不依赖于z的常数C，使得下式成立

$$|P(W\le z)-P(Z\le z)|\le\begin{cases}C\left(B'+\frac{\delta^3}{2\lambda}+\frac{\delta^2}{2\lambda}+(ER^2)^{\frac{1}{2}}\right)e^{-\frac{z^2}{2}}, & |z|\ge M\\ C\left(B'+1.5\delta+\frac{0.41\delta^3}{\lambda}+\frac{\sqrt{2\pi}}{4}(ER^2)^{\frac{1}{2}}\right)e^{-\frac{z^2}{2}}, & |z|<M\end{cases}$$

其中，$B'=\left(E\left(E\left(\left(1-\frac{(W-W')^2}{2\lambda}\right)\middle|W\right)\right)^2\right)^{\frac{1}{2}}$。

注：上面的定理给出了可交换对的“指数上平方”速度下降的非一致界，有了它，我们可以给出更好的误差估计。在实际应用中，我们可以取M为一个常数（比如$M=1$），并且根据下面的证明我们不难说明当存在常数A使得$\delta\le A$（这里的A不依赖于n）时，常数C可以不依赖于δ，至此我们使得常数C仅依赖于W。如果我们可以找到常数K（K亦不依赖于n）使得与W相关的期望均小于K，那么我们的常数C就是一致的（关于n和z）。不过做到这一点并不容易，由于是非一致界，很多时候我们更关心它关于z的收敛速度。在后面的应用中，我们回尝试将它使用到 Anti-voter model 和 Quadratic forms 模型中。

为证定理 3.2，我们先给出引理 3.2

引理 3.2 令f_z为 Stein 方程(5)的解，那么对于它的导数f_z'，f_z''，我们有

$$f_z'(x)=\begin{cases}\left(\sqrt{2\pi}xe^{\frac{x^2}{2}}\Phi(x)+1\right)\left(1-\Phi(z)\right) & x<z\\ \left(\sqrt{2\pi}xe^{\frac{x^2}{2}}\left(1-\Phi(x)\right)-1\right)\Phi(z) & x>z\end{cases}$$

$$f_z''(x)=\begin{cases}\left(\left(\sqrt{2\pi}(1+x^2)e^{\frac{x^2}{2}}\Phi(x)+x\right)(1-\Phi(z))\right. & x<z\\ \left(\left(\sqrt{2\pi}(1+x^2)e^{\frac{x^2}{2}}(1-\Phi(x))-x\right)\Phi(z)\right. & x>z\end{cases}$$

引理 3.2 证明，根据 Stein 方程(5)以及引理 1.2，容易得出上述结果。

定理 3.2 证明：

在本证明中C为仅依赖于δ,W,M且不依赖于z的常数，它的值可能会发生变化。对任意的$z\in R$，我们令f为 Stein 方程(5)的解，我们有

$$
\begin{aligned}
0&=E\left((W-W')\big(f(W')+f(W)\big)\right)\\
&=E\left((W-W')\big(f(W')-f(W)\big)\right)+2E\big(f(W)(W-W')\big)\\
&=E\left((W-W')\big(f(W')-f(W)\big)\right)+2E\big(f(W)E(W-W'|W)\big)\\
&=E\left((W-W')\big(f(W')-f(W)\big)\right)+2\lambda E\big((W-R)f(W)\big)\\
&=E\left((W-W')\big(f(W')-f(W)\big)\right)+2\lambda E\big(Wf(W)\big)-2\lambda E\big(Rf(W)\big)
\end{aligned}
$$

因此，我们得到

$$E\big(Wf(W)\big)=\frac{1}{2\lambda}E\left((W-W')\big(f(W)-f(W')\big)\right)+E\big(Rf(W)\big)\quad(65)$$

根据(5)和(65)，我们有

$$
\begin{aligned}
|P(W\le z)-P(Z\le z)|&=\left|E\big(f'(W)-Wf(W)\big)\right|\\
&=\left|E\left(f'(W)-\frac{1}{2\lambda}E\left((W-W')\big(f(W)-f(W')\big)\right)-E\big(Rf(W)\big)\right)\right|\\
&=E\left|\left(f'(W)\left(1-\frac{(W-W')^2}{2\lambda}\right)\right.\right.\\
&\quad\left.\left.+\frac{f'(W)(W'-W)^2-\big(f(W')-f(W)\big)(W'-W)}{2\lambda}-Rf(W)\right)\right|\\
&\coloneqq|E(J_1+J_2+J_3)|\le|EJ_1|+|EJ_2|+|EJ_3|\qquad(66)
\end{aligned}
$$

(1)当$z\ge M$时

对于EJ_1

$$|EJ_1| = \left|E\left(f'(W)\left(1-\frac{(W-W')^2}{2\lambda}\right)\right)\right| = \left|E\left(f'(W)E\left(\left(1-\frac{(W-W')^2}{2\lambda}\right)\middle|W\right)\right)\right|$$

$$\leq E\left|f'(W)E\left(\left(1-\frac{(W-W')^2}{2\lambda}\right)\middle|W\right)\right|$$

$$= E\left|f'(W)E\left(\left(1-\frac{(W-W')^2}{2\lambda}\right)\middle|W\right)\right|1_{\{z>W\}}$$

$$+ E\left|f'(W)E\left(\left(1-\frac{(W-W')^2}{2\lambda}\right)\middle|W\right)\right|1_{\{z\leq W\}} = EJ_{11} + EJ_{12}$$

对于EJ_{11}，我们有

$$EJ_{11} = E\left|f'(W)E\left(\left(1-\frac{(W-W')^2}{2\lambda}\right)\middle|W\right)\right|1_{\{z>W\}}$$

$$= E\left|\left(\sqrt{2\pi}We^{\frac{W^2}{2}}\Phi(W)+1\right)\big(1\right.$$

$$\left.-\Phi(z)\big)E\left(\left(1-\frac{(W-W')^2}{2\lambda}\right)\middle|W\right)\right|1_{\{z>W\}}$$

$$= \big(1-\Phi(z)\big)E\left|\left(\sqrt{2\pi}We^{\frac{W^2}{2}}\Phi(W)+1\right)E\left(\left(1-\frac{(W-W')^2}{2\lambda}\right)\middle|W\right)\right|$$

根据 Cauchy-Schwarz 不等式，我们有

$$EJ_{11} \leq \big(1-\Phi(z)\big)\left(E\left(\sqrt{2\pi}We^{\frac{W^2}{2}}\Phi(W)\right.\right.$$

$$\left.\left.+1\right)^2\right)^{\frac{1}{2}}\left(E\left(E\left(\left(1-\frac{(W-W')^2}{2\lambda}\right)\middle|W\right)\right)^2\right)^{\frac{1}{2}} \leq C\big(1-\Phi(z)\big)B'$$

$$= C\int_z^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt\cdot B' \leq C\int_z^\infty \frac{t}{z}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt\cdot B'$$

$$= \frac{CB'}{\sqrt{2\pi}}\cdot\frac{1}{z}\cdot e^{-\frac{z^2}{2}} \tag{67}$$

其中$B' = \left(E\left(E\left(\left(1-\frac{(W-W')^2}{2\lambda}\right)\middle|W\right)\right)^2\right)^{\frac{1}{2}}$。

对于EJ_{12}，由 Chebyshev 不等式以及引理 1.3 的(8)我们有

$$EJ_{12}=E\left|f'(W)E\left(\left(1-\frac{(W-W')^2}{2\lambda}\right)\middle|W\right)\right|1_{\{z\leq W\}}$$

$$\leq\frac{E\left|f'(W)E\left(\left(1-\frac{(W-W')^2}{2\lambda}\right)\middle|W\right)e^{\frac{W^2}{2}}\right|}{e^{\frac{z^2}{2}}}$$

$$\leq\frac{E\left|E\left(\left(1-\frac{(W-W')^2}{2\lambda}\right)\middle|W\right)e^{\frac{W^2}{2}}\right|}{e^{\frac{z^2}{2}}}$$

由 Cauchy-Schwarz 不等式

$$EJ_{12}\leq\frac{\left(E\left(E\left(\left(1-\frac{(W-W')^2}{2\lambda}\right)\middle|W\right)\right)^2\right)^{\frac{1}{2}}\left(Ee^{W^2}\right)^{\frac{1}{2}}}{e^{\frac{z^2}{2}}}\leq CB'\cdot e^{-\frac{z^2}{2}}\quad(68)$$

由(67)和(68)，我们有

$$|EJ_1|\leq CB'\cdot e^{-\frac{z^2}{2}}+\frac{CB'}{\sqrt{2\pi}}\cdot\frac{1}{z}\cdot e^{-\frac{z^2}{2}}\qquad(69)$$

对于EJ_2

$$|EJ_2|\leq E\left|\frac{f'(W)(W'-W)^2-\big(f(W')-f(W)\big)(W'-W)}{2\lambda}\right|$$

$$=\frac{1}{2\lambda}E\left|f'(W)(W'-W)^2-\big(f(W')-f(W)\big)(W'-W)\right|$$
$$=\frac{1}{2\lambda}E\left|f'(W)(W'-W)^2-\big(f(W')-f(W)\big)(W'-W)\right|1_{\{z>W+\delta\}}$$
$$+\frac{1}{2\lambda}E\left|f'(W)(W'-W)^2-\big(f(W')-f(W)\big)(W'-W)\right|1_{\{z\leq W+\delta\}}$$
$$=EJ_{21}+EJ_{22}\qquad(70)$$

对于EJ_{21}，在事件$\{z>W+\delta\}$，显然有$z>\max\{W,W'\}$，由 Taylor 展开知，存在位于W，W'随机变量V（$z>V$），使得

$$f(W')=f(W)+f'(W)(W'-W)+\frac{f''(V)}{2}(W'-W)^2\qquad(71)$$

因此

$$EJ_{21}=\frac{1}{2\lambda}E\left|\frac{f''(V)}{2}\cdot|W'-W|^3\right|\le\frac{\delta^3}{4\lambda}E|f''(V)|$$

$$=\frac{\delta^3}{4\lambda}E\left|\left(\sqrt{2\pi}(1+V^2)e^{\frac{V^2}{2}}\Phi(V)+V\right)\big(1-\Phi(z)\big)\right|$$

$$\le\frac{C\delta^3}{4\lambda}\big(1-\Phi(z)\big)\le\frac{C\delta^3}{4\lambda\sqrt{2\pi}}\cdot\frac{1}{z}\cdot e^{-\frac{z^2}{2}} \tag{72}$$

对于EJ_{22}，根据引理 1.3

$$\begin{aligned}EJ_{22}&=\frac{1}{2\lambda}E\left|f'(W)(W'-W)^2-\big(f(W')-f(W)\big)(W'-W)\right|1_{\{z\le W+\delta\}}\\&\le\frac{1}{2\lambda}E|f'(W)(W'-W)^2|1_{\{z\le W+\delta\}}\\&\quad+\frac{1}{2\lambda}E\left|\big(f(W')-f(W)\big)(W'-W)\right|1_{\{z\le W+\delta\}}\\&\le\frac{\delta^2}{2\lambda}Ee^{\frac{(W+\delta)^2}{2}}\cdot e^{-\frac{z^2}{2}}+\frac{\delta^2}{2\lambda}\cdot Ee^{\frac{(W+\delta)^2}{2}}\cdot e^{-\frac{z^2}{2}}\\&\le C\frac{\delta^2}{2\lambda}e^{-\frac{z^2}{2}}+C\frac{\delta^2}{2\lambda}e^{-\frac{z^2}{2}}=C\frac{\delta^2}{2\lambda}e^{-\frac{z^2}{2}}\end{aligned} \tag{73}$$

结合(72)和(73)

$$|EJ_2|\le\frac{C\delta^3}{2\lambda}\cdot\frac{1}{z}\cdot e^{-\frac{z^2}{2}}+C\frac{\delta^2}{2\lambda}e^{-\frac{z^2}{2}} \tag{74}$$

对于EJ_3

$$\begin{aligned}|EJ_3|\le E|Rf(W)|&=E|Rf(W)|1_{\{z>W\}}+E|Rf(W)|1_{\{z\le W\}}\\&=EJ_{31}+EJ_{32}\end{aligned}$$

对于EJ_{31}，EJ_{32}，由引理 3.1、Chebyshev 不等式以及 Cauchy-Schwarz 不等式

$$EJ_{31}=E\left(R\sqrt{2\pi}e^{\frac{W^2}{2}}\Phi(W)\big(1-\Phi(z)\big)\right)=\sqrt{2\pi}\big(1-\Phi(z)\big)E\left(Re^{\frac{W^2}{2}}\Phi(W)\right)$$

$$\le\sqrt{2\pi}\big(1-\Phi(z)\big)(ER^2)^{\frac{1}{2}}\left(E\left(e^{W^2}\Phi^2(W)\right)\right)^{\frac{1}{2}}\le C(ER^2)^{\frac{1}{2}}\cdot\frac{1}{z}\cdot e^{-\frac{z^2}{2}}$$

$$EJ_{32}=E|Rf(W)|1_{\{z\le W\}}\le\frac{\sqrt{2\pi}}{4}E|R|1_{\{z\le W\}}=\frac{\sqrt{2\pi}}{4}E|R|1_{\{z\le W\}}\frac{e^{\frac{W^2}{2}}}{e^{\frac{z^2}{2}}}$$

$$\le\frac{\sqrt{2\pi}}{4}\frac{E|R|e^{\frac{W^2}{2}}}{e^{\frac{z^2}{2}}}\le\frac{\sqrt{2\pi}}{4}\frac{(ER^2)^{\frac{1}{2}}\big(Ee^{W^2}\big)^{\frac{1}{2}}}{e^{\frac{z^2}{2}}}\le C(ER^2)^{\frac{1}{2}}\cdot e^{-\frac{z^2}{2}}$$

所以对于EJ_3

$$|EJ_3|\le C(ER^2)^{\frac{1}{2}}\cdot\frac{1}{z}\cdot e^{-\frac{z^2}{2}}+C(ER^2)^{\frac{1}{2}}\cdot e^{-\frac{z^2}{2}} \tag{75}$$

结合不等式(69)(74)(75)，我们有

$$
\begin{aligned}
&|P(W \le z) - P(Z \le z)| \\
&\le CB' \cdot e^{-\frac{z^2}{2}} + \frac{CB'}{\sqrt{2\pi}} \cdot \frac{1}{z} \cdot e^{-\frac{z^2}{2}} + \frac{C\delta^3}{2\lambda} \cdot \frac{1}{z} \cdot e^{-\frac{z^2}{2}} + C\frac{\delta^2}{2\lambda} e^{-\frac{z^2}{2}} \\
&+ C(ER^2)^{\frac{1}{2}} \cdot \frac{1}{z} \cdot e^{-\frac{z^2}{2}} + C(ER^2)^{\frac{1}{2}} \cdot e^{-\frac{z^2}{2}} \\
&\le CB' \cdot e^{-\frac{z^2}{2}}\left(1 + \frac{1}{M}\right) + \frac{C\delta^3}{2\lambda} \cdot \frac{1}{M} \cdot e^{-\frac{z^2}{2}} + C\frac{\delta^2}{2\lambda} e^{-\frac{z^2}{2}} \\
&+ C(ER^2)^{\frac{1}{2}} \cdot e^{-\frac{z^2}{2}}\left(1 + \frac{1}{M}\right) \\
&= CB' \cdot e^{-\frac{z^2}{2}} + \frac{C\delta^3}{2\lambda} e^{-\frac{z^2}{2}} + C\frac{\delta^2}{2\lambda} e^{-\frac{z^2}{2}} \\
&+ C(ER^2)^{\frac{1}{2}} \cdot e^{-\frac{z^2}{2}} \qquad (76)
\end{aligned}
$$

(2)当$z \le -M$时，证明方法与$z \ge M$一致，我们不难得出

$$
\begin{aligned}
&|P(W \le z) - P(Z \le z)| \\
&\le CB' \cdot e^{-\frac{z^2}{2}} + \frac{C\delta^3}{2\lambda} e^{-\frac{z^2}{2}} + C\frac{\delta^2}{2\lambda} e^{-\frac{z^2}{2}} + C(ER^2)^{\frac{1}{2}} \cdot e^{-\frac{z^2}{2}} \qquad (77)
\end{aligned}
$$

(3)当$|z| < M$时，

首先，在定理 2.3 中，取$a = \delta$，根据Hölder不等式，我们有如下结果

$$
\begin{aligned}
|P(W \le z) - P(Z \le z)| &\le B + 1.5\delta + \frac{0.41\delta^3}{\lambda} + \frac{\sqrt{2\pi}}{4} E|R| \\
&\le B' + 1.5\delta + \frac{0.41\delta^3}{\lambda} + \frac{\sqrt{2\pi}}{4} (ER^2)^{\frac{1}{2}} \\
&\le \left(B' + 1.5\delta + \frac{0.41\delta^3}{\lambda} + \frac{\sqrt{2\pi}}{4} (ER^2)^{\frac{1}{2}}\right) e^{\frac{M^2}{2}} \cdot e^{-\frac{z^2}{2}} \\
&\le C\left(B' + 1.5\delta + \frac{0.41\delta^3}{\lambda} + \frac{\sqrt{2\pi}}{4} (ER^2)^{\frac{1}{2}}\right) e^{-\frac{z^2}{2}} \qquad (78)
\end{aligned}
$$

综上，结合(76)(77)(78)我们有

$$
|P(W \le z) - P(Z \le z)| \le \begin{cases} C\left(B' + \frac{\delta^3}{2\lambda} + \frac{\delta^2}{2\lambda} + (ER^2)^{\frac{1}{2}}\right) e^{-\frac{z^2}{2}}, & |z| \ge M \\ C\left(B' + 1.5\delta + \frac{0.41\delta^3}{\lambda} + \frac{\sqrt{2\pi}}{4} (ER^2)^{\frac{1}{2}}\right) e^{-\frac{z^2}{2}}, & |z| < M \end{cases}
$$

以下我们给出关于 size bias 分布的结果：

定理 3.3 若Y^s具有Y-size biased 分布，且满足$|Y^s - Y| \le A$，$|W|$具有有限的矩母函数（$Ee^{t|W|} < \infty$对任意t成立）且$Ee^{W^2} < \infty$。那么对于任意常数$M > 0$，存在仅依赖于σ, A, M且不依赖于z的常数C，使得下式成立，

$$|P(W\le z)-P(Z\le z)|\le\begin{cases}C\left(D'+\frac{\mu A}{\sigma^2}+\frac{\mu A^2}{\sigma^3}\right)e^{-\frac{z^2}{2}}, |z|\ge M\\ C\left(\frac{6\mu A^2}{\sigma^3}+2D\right)e^{-\frac{z^2}{2}}, \quad |z|<M\end{cases}$$

这里，$D'=\left(E\left(E\left(1-\frac{\mu}{\sigma}(W^s-W)|W\right)\right)^2\right)^{\frac{1}{2}}$。

注：我们给出了使用 size bias 分布给出的“指数上平方”下降的非一致界，这里$\frac{A}{\sigma}$很多时候是小于固定常数的。注意到，在定理 3.1，3.2，3.3 中我们均要求$Ee^{W^2}<\infty$，这一点是较难做到的，其实根据我们的证明可以发现，关于z的下降速度越快，对于$Eg(W)$的存在性要求越高，这里我们为了获得指数下降要求$Ee^{W^2}<\infty$。

定理 3.3 证明：

关于 size bias，我们之前引入了一些记号W, W^s, D, Ψ 等，在本证明中，我们保持这些符号的意义不变。类似于定理 3.2 的证明，我们会相对应的引入记号D'。上述结果的证明与定理 3.2 较为类似。

首先，对于$|P(W\le z)-P(Z\le z)|$，设f为 Stein 方程(5)的解，我们有

$$\begin{aligned}|P(W\le z)-P(Z\le z)| &= \left|E\big(f'(W)-Wf(W)\big)\right|\\
&=\left|E\left(f'(W)-\frac{\mu}{\sigma}E\big(f(W^s)-f(W)\big)\right)\right|\\
&=\left|E\left(f'(W)\left(1-\frac{\mu}{\sigma}(W^s-W)\right)-\frac{\mu}{\sigma}\big(f(W^s)-f(W)\big)\right.\right.\\
&\quad\left.\left.+\frac{\mu}{\sigma}f'(W)(W^s-W)\right)\right|\\
&=\left|E\left(f'(W)\left(1-\frac{\mu}{\sigma}(W^s-W)\right)\right.\right.\\
&\quad\left.\left.-\frac{\mu}{\sigma}\big(f(W^s)-f(W)-f'(W)(W^s-W)\big)\right)\right|\\
&\le E\left|f'(W)\left(1-\frac{\mu}{\sigma}(W^s-W)\right)\right|\\
&\quad+\frac{\mu}{\sigma}E\left|\big(f(W^s)-f(W)-f'(W)(W^s-W)\big)\right|=EJ_1+EJ_2 \qquad (79)\end{aligned}$$

(1)若$z\ge M$

对于EJ_1，我们有，

$$EJ_1 = E\left|f'(W)E\left(1-\frac{\mu}{\sigma}(W^s-W)|W\right)\right|$$
$$= E\left|f'(W)E\left(1-\frac{\mu}{\sigma}(W^s-W)|W\right)\right|1_{\{W<z\}}$$
$$+ E\left|f'(W)E\left(1-\frac{\mu}{\sigma}(W^s-W)|W\right)\right|1_{\{W\geq z\}}$$
$$= EJ_{11}+EJ_{12} \qquad (80)$$

对于EJ_{11}，根据引理 3.2，

$$EJ_{11} = E\left|f'(W)E\left(1-\frac{\mu}{\sigma}(W^s-W)|W\right)\right|1_{\{W<z\}}$$
$$= \left(1-\Phi(z)\right)E\left|\left(\sqrt{2\pi}We^{\frac{W^2}{2}}\Phi(W)+1\right)E\left(1-\frac{\mu}{\sigma}(W^s-W)|W\right)\right|1_{\{W<z\}}$$

根据 Cauchy-Schwarz 不等式，对于上式我们有，

$$EJ_{11} \leq \left(1-\Phi(z)\right)\left(E\left(\sqrt{2\pi}We^{\frac{W^2}{2}}\Phi(W)+1\right)^2\right)^{\frac{1}{2}}\left(E\left(E\left(1-\frac{\mu}{\sigma}(W^s-W)|W\right)\right)^2\right)^{\frac{1}{2}} \leq CD'\left(1-\Phi(z)\right)$$
$$\leq \frac{CD'}{z}e^{-\frac{z^2}{2}} \qquad (81)$$

其中，$D' = \left(E\left(E\left(1-\frac{\mu}{\sigma}(W^s-W)|W\right)\right)^2\right)^{\frac{1}{2}}$。

对于EJ_{12}，根据引理 1.3 和 Cauchy-Schwarz 不等式，我们有，

$$EJ_{12} = E\left|f'(W)E\left(1-\frac{\mu}{\sigma}(W^s-W)|W\right)\right|1_{\{W\geq z\}}$$
$$\leq E\left|E\left(1-\frac{\mu}{\sigma}(W^s-W)|W\right)\right|1_{\{W\geq z\}}$$
$$\leq \frac{E\left|E\left(1-\frac{\mu}{\sigma}(W^s-W)|W\right)\right|e^{\frac{W^2}{2}}1_{\{W\geq z\}}}{e^{\frac{z^2}{2}}}$$
$$\leq \frac{\left(E\left(E\left(1-\frac{\mu}{\sigma}(W^s-W)|W\right)\right)^2\right)^{\frac{1}{2}}\left(Ee^{W^2}\right)^{\frac{1}{2}}}{e^{\frac{z^2}{2}}} \leq CD'e^{-\frac{z^2}{2}} \qquad (82)$$

对于(80)，

$$EJ_1 \leq \frac{CD'}{z}e^{-\frac{z^2}{2}} + CD'e^{-\frac{z^2}{2}} \qquad (83)$$

对于EJ_2，我们有，

$$
\begin{aligned}
EJ_2 &= \frac{\mu}{\sigma}E\left|\left(f(W^s)-f(W)-f'(W)(W^s-W)\right)\right| \\
&= \frac{\mu}{\sigma}E\left|\left(f(W^s)-f(W)-f'(W)(W^s-W)\right)1_{\left\{z>W+\frac{A}{\sigma}\right\}}\right| \\
&+ \frac{\mu}{\sigma}E\left|\left(f(W^s)-f(W)-f'(W)(W^s-W)\right)1_{\left\{z\le W+\frac{A}{\sigma}\right\}}\right| \\
&= EJ_{21}+EJ_{22} \qquad (84)
\end{aligned}
$$

对于EJ_{21}，在事件$\left\{z>W+\frac{A}{\sigma}\right\}$上，我们有$z>\max\{W,W^s\}$，所以我们可以用Taylor 展开，存在$W^s$和$W$之间的随机变量$V$，

$$
f(W^s)=f(W)+f'(W)(W^s-W)+\frac{f''(V)}{2}(W^s-W)^2 \qquad (85)
$$

所以，

$$
\begin{aligned}
EJ_{21} &\le \frac{\mu}{\sigma}E\left|\frac{f''(V)}{2}(W^s-W)^2\right| \le \frac{\mu A^2}{2\sigma^3}E|f''(V)| \\
&= \frac{\mu A^2}{2\sigma^3}\left(1-\Phi(z)\right)E\left|\left(\sqrt{2\pi}(1+V^2)e^{\frac{V^2}{2}}\Phi(V)+V\right)\right| \\
&\le C\frac{\mu A^2}{\sigma^3}\cdot\frac{1}{z}e^{-\frac{z^2}{2}} \qquad (86)
\end{aligned}
$$

对于EJ_{22}，我们有，

$$
\begin{aligned}
EJ_{22} &= \frac{\mu}{\sigma}E\left|\left(f(W^s)-f(W)-f'(W)(W^s-W)\right)1_{\left\{z\le W+\frac{A}{\sigma}\right\}}\right| \\
&\le \frac{\mu}{\sigma}E|f(W^s)-f(W)|1_{\left\{z\le W+\frac{A}{\sigma}\right\}}+\frac{\mu}{\sigma}E|f'(W)(W^s-W)|1_{\left\{z\le W+\frac{A}{\sigma}\right\}}
\end{aligned}
$$

由于$|f'(x)|\le 1$，

$$
EJ_{22} \le \frac{2\mu A}{\sigma^2}E1_{\left\{z\le W+\frac{A}{\sigma}\right\}} \le \frac{2\mu A}{\sigma^2}\cdot Ee^{\frac{\left(M+\frac{A}{\sigma}\right)^2}{2}}\cdot e^{-\frac{z^2}{2}} \le C\frac{\mu A}{\sigma^2}e^{-\frac{z^2}{2}} \quad (87)
$$

所以对于EJ_2，

$$
EJ_2 \le C\frac{\mu A}{\sigma^2}e^{-\frac{z^2}{2}}+C\frac{\mu A^2}{\sigma^3}\cdot\frac{1}{z}e^{-\frac{z^2}{2}} \qquad (88)
$$

结合(83)(88)，由于$z\ge M$，

$$
\begin{aligned}
|P(W\le z)-P(Z\le z)| &\le \frac{CD'}{z}e^{-\frac{z^2}{2}}+CD'e^{-\frac{z^2}{2}}+C\frac{\mu A}{\sigma^2}e^{-\frac{z^2}{2}}+C\frac{\mu A^2}{\sigma^3}\cdot\frac{1}{z}e^{-\frac{z^2}{2}} \\
&\le CD'e^{-\frac{z^2}{2}}+C\frac{\mu A}{\sigma^2}e^{-\frac{z^2}{2}}+C\frac{\mu A^2}{\sigma^3}e^{-\frac{z^2}{2}} \qquad (89)
\end{aligned}
$$

(2)对于$z\le -M$，同理我们有上式成立。

(3)对于$|z|<M$，由定理 2.5，

$$|P(W \le z) - P(Z \le z)| \le \frac{6\mu A^2}{\sigma^3} + 2D \le \left(\frac{6\mu A^2}{\sigma^3} + 2D\right) e^{\frac{M^2}{2}} \cdot e^{-\frac{z^2}{2}}$$

$$\le C\left(\frac{6\mu A^2}{\sigma^3} + 2D\right) e^{-\frac{z^2}{2}} \tag{90}$$

结合(89)(90)，我们有，

$$|P(W \le z) - P(Z \le z)| = \begin{cases} C\left(D' + \frac{\mu A}{\sigma^2} + \frac{\mu A^2}{\sigma^3}\right) e^{-\frac{z^2}{2}}, |z| \ge M \\ C\left(\frac{6\mu A^2}{\sigma^3} + 2D\right) e^{-\frac{z^2}{2}}, \quad |z| < M \end{cases}$$

至此，完成了定理 3.3 的证明。

注：在以上定理 3.1、3.2、3.3 证明中，我们虽然说明了$|P(W \le z) - P(Z \le z)|$可以以$O\left(e^{-\frac{z^2}{2}}\right)$速度下降，但是我们的代价是$C$过大（不难从证明中看出，$C$应该和$Ee^{W^2}$差不多），这时，$C$就很可能和$n$有关，导致我们给出的非一致 Berry-Esseen 可应用性不强。并且我们需要指出，$O\left(e^{-\frac{z^2}{2}}\right)$是本证明方法可以达到的最快非一致下降速度。

因此，如果我们要求C不依赖W，我们需要用到$EW^2 = 1$这一条件，不需要额外的条件（$|W|$具有有限的矩母函数（$Ee^{t|W|} < \infty$对任意t成立）且$Ee^{W^2} < \infty$）。利用上面的证明方法（几乎完全一致，仅仅在放缩 Chebyshev 不等式时不一致），就可以得到$O\left(\frac{1}{z}\right)$的非一致逼近速度，这一点是比较容易的。

以定理 3.2 为例，它对应的$O\left(\frac{1}{z}\right)$非一致界为：

定理 3.2 的变形 若(W, W')是 0 均值方差为 1 的可交换对并且满足

$$E(W - W'|W) = \lambda(W - R)，其中\lambda \in (0,1)$$

二者之差$|W - W'| \le \delta$。那么对于任意常数$M > 0$，存在仅依赖于δ, M但不依赖于z的常数C，使得下式成立

$$|P(W \le z) - P(Z \le z)| \le \begin{cases} \frac{C}{z}\left(B' + \frac{\delta^3}{2\lambda} + \frac{\delta^2}{2\lambda} + (ER^2)^{\frac{1}{2}}\right), & |z| \ge M \\ \frac{C}{z}\left(B' + 1.5\delta + \frac{0.41\delta^3}{\lambda} + \frac{\sqrt{2\pi}}{4}(ER^2)^{\frac{1}{2}}\right), & |z| < M \end{cases}$$

其中，$B' = \left(E\left(E\left(\left(1 - \frac{(W-W')^2}{2\lambda}\right)\middle|W\right)\right)^2\right)^{\frac{1}{2}}$。

它的证明方法与定理 3.2 基本一致，使用了$EW^2=1$这一条件，得到的常数C只与M,δ有关，而M可以任意取，$\delta\le A$，一般成立，所以可以认为C与n无关。由于前人$O\left(\frac{1}{z}\right)$的非一致界已经有不少结果，所以在本文我们主要证明了$O\left(e^{-\frac{z^2}{2}}\right)$的结果。

四、Stein 方法的应用

在第四部分，我们介绍 Stein 方法中 Berry-Esseen 界的一些应用。在此之前，许多人已经在这方面已经做出了非常不错的结果，比如 Chatterjee[3]中关于最小生成树（Minimal Spanning Tree）的 Kantorovich-Wasserstein 距离的结果。主要有：共轭类上的组合中心极限定理（Combinatorial central limit theorem on conjugacy classess）、Anti-voter model、二次型（Quadratic forms）、Lightbulb process、Pattern in graphs and permutations。我们主要采取[1]中一致界和我们证明的非一致界对比的方式进行，深入探究非一致界的特性。

4.1 共轭类上的组合中心极限定理(Combinatorial central limit theorem on conjugacy classess)

我们令$A=\{a_{ij}\}_{i=1}^{n}$为一个数矩阵（本部分所有a_{ij}均是非随机的），组合中心极限定理关心如下随机变量：

$$Y=\sum_{i=1}^{n}a_{i,\pi(i)}\qquad(91)$$

其中π属于对称群$\mathcal{S}_n$。很多时候，我们要求 π在对称群上是均匀分布的，关于均匀分布的情况，本部分不再赘述，具体可以参考[1]。本节，我们讨论当π在各共轭类（Conjugacy classess）上服从均匀分布时的情况。

首先，我们介绍循环类（Cycle type）的概念。为便于理解，我们举个例子，以$\mathcal{S}_8$为例，令$\pi=(1\ 2\ 4)(3\ 5)(6\ 7)(8)$，它具有的映射关系为：$1\to2\to4\to1$，$3\to5\to3$，$8\to8$。其中$(3\ 5)$和$(6\ 7)$是 2-cycle，$(1\ 2\ 4)$是 3-cycle。我们用$c_q(\pi)$代表$\pi$中 q-cycle 的个数，令

$$c(\pi)=\left(c_1(\pi),c_2(\pi),\cdots,c_n(\pi)\right)\qquad(92)$$

若$c(\pi)=c(\sigma)$，我们称π和σ具有相同的循环类。我们不难发现，π和σ具有相同的 cycle type 当且仅当它们是共轭的，即$\pi=\rho^{-1}\sigma\rho$。我们称概率测度P在 Cycle type 上为常数当且仅当

$$P(\pi)=P(\rho^{-1}\pi\rho)，对任意的\pi，\rho成立$$

对于$\mathcal{S}_n$，我们令

$$\mathcal{N}_n=\left\{(c_1,c_2,\cdots,c_n)\in N^n:\sum_{i=1}^{n} ic_i=n\right\}$$

令$N(c)$代表具有 cycle type c置换的个数，我们有

$$N(c)=n!\prod_{j=1}^{n}\left(\frac{1}{j}\right)^{c_j}\frac{1}{c_j!}$$

令$\mathcal{U}(c)$为在 cycle type c上的均匀分布，即

$$P(\pi)=\begin{cases}\frac{1}{N(c)}, & c(\pi)=c\\ 0, & 其他\end{cases}\tag{93}$$

引理 4.1（参考[1]）如果P在不同 cycle type 上分别保持均匀分布，那么我们有

$$P=\sum_{c\in\mathcal{N}_n}\rho_c\mathcal{U}(c)\qquad 其中\rho_c=P(c(\pi)=c)$$

证明略。我们所关心的就是这个分布P。

以下我们给出[1]中关于此问题的一致界结果。

定理 4.1（参考[1]）设$n\geq 5$且$\{a_{ij}\}_{i,j=1}^{n}$为一对称矩阵，即

$$a_{ij}=a_{ji}$$

我们令π具有引理 4.1 中的性质并且没有固定点（不存在$\pi(k)=k$）。那么对于$W=(Y-EY)/\sigma_\rho$，

$$\sup_{z\in R}\left|P(W\leq z)-P(Z_\rho\leq z)\right|\leq 40C\left(1+\frac{1}{\sqrt{2\pi}}+\frac{\sqrt{2\pi}}{4}\right)\sum_{c\in\mathcal{N}_n}\frac{\rho_c}{\sigma_c}$$

在这里，

$$\sigma_\rho^2=\sum_{c\in\mathcal{N}_n}\rho_c\sigma_c^2,\qquad \mathcal{L}(Z_\rho)=\sum_{c\in\mathcal{N}_n}\rho_c\mathcal{L}\left(\frac{Z_c}{\sigma_\rho}\right),\qquad Z_c\sim N(0,\sigma_c^2)$$

$$C=\max_{i\neq j}\left|a_{ij}-2a_{io}+a_{oo}\right|,a_{io}=\frac{1}{n-2}\sum_{j=1}^{n}a_{ij},a_{oo}=\frac{1}{(n-1)(n-2)}\sum_{ij}a_{ij}$$

证明略。

以下给出我们得出关于非一致界的结果：

为证明定理 4.2 上述定理，我们先给出引理 4.2，

引理 4.2 设$n\geq 5$且$\{a_{ij}\}_{i,j=1}^{n}$为一对称矩阵，即

$$a_{ij}=a_{ji}$$

我们令π具有(93)中的性质并且没有固定点。那么对于$W_c=(Y-EY)/\sigma_c$，

$$|P(W_c \le z) - P(Z \le z)| \le \begin{cases} C_1 \dfrac{40\max\limits_{i\ne j}\left|a_{ij} - 2a_{io} + a_{oo}\right|}{\sigma_c} e^{-\frac{(z+2\delta_c)^2}{2}} & z \le -M - 2\delta_c \\ C_2 \dfrac{40\max\limits_{i\ne j}\left|a_{ij} - 2a_{io} + a_{oo}\right|}{\sigma_c} e^{-\frac{z^2}{2}} & |z| \le M + 2\delta_c \\ C_3 \dfrac{40\max\limits_{i\ne j}\left|a_{ij} - 2a_{io} + a_{oo}\right|}{\sigma_c} e^{-\frac{(z-2\delta_c)^2}{2}} & z \ge M + 2\delta_c \end{cases}$$

其中$\delta_c = \frac{40\max_{i\ne j}\left|a_{ij}-2a_{io}+a_{oo}\right|}{\sigma_c}$，$Z \sim N(0,1)$

引理 4.2 的证明：
首先根据(Lemma 6.10, [1])，我们构造出 zero bias 分布满足

$$|W' - W| = \frac{|Y' - Y|}{\sigma_c} \le \frac{40\max\limits_{i\ne j}\left|a_{ij} - 2a_{io} + a_{oo}\right|}{\sigma_c}$$

根据定理 3.1，令$\delta = \frac{40\max_{i\ne j}\left|a_{ij}-2a_{io}+a_{oo}\right|}{\sigma_c}$，我们可以得出上述结果。

定理 4.2 设$n \ge 5$且$\left\{a_{ij}\right\}_{i,j=1}^{n}$为一对称矩阵，即

$$a_{ij} = a_{ji}$$

令π满足引理 4.1 的条件并且没有固定点（不存在$\pi(k) = k$），我们令$W = \frac{Y-EY}{\sigma_\rho}$，对任意常数$M \ge 0$，我们有，

$$\left|P(W \le z) - P\left(Z_\rho \le z\right)\right|$$

$$\le \begin{cases} C_1 40\max\limits_{i\ne j}\left|a_{ij} - 2a_{io} + a_{oo}\right| e^{-\frac{(z+2\delta)^2}{2}} \sum\limits_{c\in\mathcal{N}_n} \dfrac{\rho_c}{\sigma_c} & z \le -M - 2\delta \\ C_2 40\max\limits_{i\ne j}\left|a_{ij} - 2a_{io} + a_{oo}\right| e^{-\frac{z^2}{2}} \sum\limits_{c\in\mathcal{N}_n} \dfrac{\rho_c}{\sigma_c} & |z| \le M + 2\delta \\ C_3 40\max\limits_{i\ne j}\left|a_{ij} - 2a_{io} + a_{oo}\right| e^{-\frac{(z-2\delta)^2}{2}} \sum\limits_{c\in\mathcal{N}_n} \dfrac{\rho_c}{\sigma_c} & z \ge M + 2\delta \end{cases}$$

注：与之前一致，定理 4.2 中常数C_i是不依赖于z但是依赖于Y, M, δ的，由于常数M是任意取的，δ一般较小，常数C_i一般只依赖于Y，而且如果要获得上述$O\left(e^{-\frac{z^2}{2}}\right)$的收敛速度，往往$C_i$都会比较大。如果我们要求$C_i$不依赖于$Y$，我们只能将上述收敛速度降至$O\left(\frac{1}{z}\right)$。

定理 4.2 的证明：

同定理 4.1 的证明，我们可以有$EY=0$。
由引理 4.1，我们有，

$$P(Y\le z)=\sum_{c\in\mathcal{N}_n}\rho_c P(Y_c\le z) \qquad (94)$$

并且Z_ρ满足，

$$P\left(Z_\rho\le z\right)=\sum_{c\in\mathcal{N}_n}\rho_c P\left(\frac{Z_c}{\sigma_\rho}\le z\right) \qquad (95)$$

令$W=\frac{Y}{\sigma_\rho}$，结合(94)(95)，我们有，

$$\sup_{z\in R}\left|P(W\le z)-P\left(Z_\rho\le z\right)\right|=\sup_{z\in R}\left|P(Y\le z)-P\left(\sigma_\rho Z_\rho\le z\right)\right|$$

$$\le\sum_{c\in\mathcal{N}_n}\rho_c|P(Y_c\le z)-P(Z_c\le z)|$$

$$=\sum_{c\in\mathcal{N}_n}\rho_c|P(W_c\le z)-P(Z\le z)| \quad (96)$$

由引理 4.2，我们令$\delta=\max\limits_{c\in\mathcal{N}_n}\delta_c$，我们有，

$$|P(W\le z)-P(Z\le z)|$$

$$=\begin{cases} C_1 40\max\limits_{i\neq j}\left|a_{ij}-2a_{io}+a_{oo}\right|e^{-\frac{(z+2\delta)^2}{2}}\sum\limits_{c\in\mathcal{N}_n}\frac{\rho_c}{\sigma_c} & z\le -M-2\delta \\ C_2 40\max\limits_{i\neq j}\left|a_{ij}-2a_{io}+a_{oo}\right|e^{-\frac{z^2}{2}}\sum\limits_{c\in\mathcal{N}_n}\frac{\rho_c}{\sigma_c} & |z|\le M+2\delta \\ C_3 40\max\limits_{i\neq j}\left|a_{ij}-2a_{io}+a_{oo}\right|e^{-\frac{(z-2\delta)^2}{2}}\sum\limits_{c\in\mathcal{N}_n}\frac{\rho_c}{\sigma_c} & z\ge M+2\delta \end{cases}$$

至此，我们完成了定理 4.2 的证明。

4.2 Anti-voter 模型

在本部分中，我们考虑 finite graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$，并且是 r-regular（每个$v\in\mathcal{V}$都是 r 度）的。我们考虑如下马氏链：$\{\mathbf{X}^{(t)}: t=0,1,\cdots\}$，它的状态空间为$\{-1,1\}^{\mathcal{V}}$，首先在从$\mathcal{V}$中等可能选取一个点$v$，然后从$v$相邻的点中等可能的选取一个不同的点$w$（从$N_v=\{w:\{v,w\}\in\mathcal{E}\}$中选）。然后我们令，

$$X_u^{(t+1)}=\begin{cases} X_u^{(t)}, & v\neq v \\ -X_w^{(t)}, & u=v \end{cases} \qquad (97)$$

我们先给出[1]中的关于 Anti-voter 的结果，

定理 4.3（参考[1]） 令$\{\mathbf{X}^{(t)}: t=0,1,\cdots\}$为一个图上的具有平稳分布的马氏链。这个图满足如下性质：(1)具有n个顶点；(2)是 r-regular 的；(3)既不是 n cycle 也不是二分（bipartite）的。我们令U表示$\mathbf{X}$上的符号和，即

$$U=\sum_{v\in\mathcal{V}} X_v$$

我们令，

$$W=\frac{U}{\sigma},\quad \sigma^2=Var(U)$$

然后，我们令U'为$\mathbf{X}$向前一步的符号和。那么，(U,U')是一个$2/n$-Stein 对并且满足：

$$|U-U'|\leq 2,\ \ E((U-U')^2|\mathbf{X})=8(a+b)/rn$$

其中a代表端点均为$+1$的边的个数，b代表端点均为-1的边的个数。
我们有，

$$\sup_{z\in R}|P(W\leq z)-P(Z\leq z)|\leq\frac{12n}{\sigma^3}+\frac{\sqrt{Var(Q)}}{r\sigma^2}$$

其中

$$Q=\sum_{v\in\mathcal{V}}\sum_{w\in N_v} X_vX_w$$

注：上述结果是很有意义的，当$\sigma=O(\sqrt{n})$，$Var(Q)=O(\sqrt{n})$时，上述结果具有$O(\sqrt{n})$的一致收敛速度。当然σ和$Var(Q)$和特定的图有很大关系，上述结果可以在特定的图上成立。
以下我们证明非一致界的结果：

定理 4.4 令$\{\mathbf{X}^{(t)}: t=0,1,\cdots\}$为一个图上的具有平稳分布的马氏链。这个图满足如下性质：(1)具有n个顶点；(2)是 r-regular 的；(3)既不是 n cycle 也不是二分（bipartite）的。我们令U表示$\mathbf{X}$上的符号和，即

$$U=\sum_{v\in\mathcal{V}} X_v$$

我们令，

$$W=\frac{U}{\sigma},\quad \sigma^2=Var(U)$$

然后，我们令U'为$\mathbf{X}$向前一步的符号和。那么，(U,U')是一个$2/n$-Stein 对并且满足：

$$|U-U'|\leq 2,\ \ E((U-U')^2|\mathbf{X})=8(a+b)/rn$$

其中a代表端点均为$+1$的边的个数，b代表端点均为-1的边的个数。
对任意$M>0$，存在依赖于M,n，不依赖于z的常数C，使下式成立，

$$|P(W \le z) - P(Z \le z)| \le \begin{cases} C\left(\frac{\sqrt{Var(Q)}}{2r\sigma^2} + \frac{2n}{\sigma^3} + \frac{n}{\sigma^2}\right)e^{-\frac{z^2}{2}}, & |z| \ge M \\ C\left(\frac{\sqrt{Var(Q)}}{2r\sigma^2} + \frac{3}{\sigma} + 1.64\frac{n}{\sigma^3}\right)e^{-\frac{z^2}{2}}, & |z| < M \end{cases}$$

其中，Q同定理 4.3。

注：定理 4.4 给出了关于 Anti-voter 模型的非一致 Berry-Esseen 界。如果我们找到一个图使得，$Var(Q) = O(n)$，$\sigma > O(n)$，也就是说各X_v之间相关性较大时，我们可以使得括号内的项一致收敛。这里的C是一个较大的且一般与n有关的常数，若要获得与n无关的C，我们可以将非一致收敛速度调整至$O\left(\frac{1}{z}\right)$。与定理 4.3 相比，定理 4.4 一致收敛能力减弱，但是考虑了很快的非一致收敛。

定理 4.4 的证明：

关于(U, U')的可交换性，可直接参考 Lemma 6.11, [1]。$E((U - U')^2|\mathbf{X}) = 8(a+b)/rn$参考 Theorem 6.6, [1]证明前半部分。

关于B'，我们已知，

$$B' = \left(E\left(E\left(\left(1 - \frac{(W - W')^2}{2\lambda}\right)\middle|W\right)\right)^2\right)^{\frac{1}{2}}$$

所以，

$$\begin{aligned} B'^2 &= E\left(E\left(\left(1 - \frac{(W - W')^2}{2\lambda}\right)\middle|W\right)\right)^2 \\ &= E\left(E\left(\left(1 - \frac{(W - W')^2}{2\lambda}\right)\middle|W\right)\right)^2 - \left(E\left(1 - \frac{(W - W')^2}{2\lambda}\right)\right)^2 \\ &= \frac{Var(E((W' - W)^2|W))}{4\lambda^2} \le \frac{Var(E((W' - W)^2|\mathbf{X}))}{4\lambda^2} = \frac{Var\left(\frac{2Q}{rn\sigma^2}\right)}{4\lambda^2} \\ &= \frac{4}{r^2n^2\sigma^4} \cdot \frac{Var(Q)}{4\lambda^2} = \frac{Var(Q)}{\lambda^2 r^2 n^2 \sigma^4} \end{aligned}$$

因此，

$$B' \le \frac{\sqrt{Var(Q)}}{\lambda rn\sigma^2} = \frac{\sqrt{Var(Q)}}{2r\sigma^2} \quad (98)$$

又$\delta = \frac{2}{\sigma}$,

因此我们有，

$$|P(W \le z) - P(Z \le z)| \le \begin{cases} C\left(\dfrac{\sqrt{Var(Q)}}{2r\sigma^2} + \dfrac{2n}{\sigma^3} + \dfrac{n}{\sigma^2}\right)e^{-\frac{z^2}{2}}, & |z| \ge M \\ C\left(\dfrac{\sqrt{Var(Q)}}{2r\sigma^2} + \dfrac{3}{\sigma} + 1.64\dfrac{n}{\sigma^3}\right)e^{-\frac{z^2}{2}}, & |z| < M \end{cases}$$

至此，我们完成了定理 4.4 的证明。

4.3 Quadratic forms

考虑一列 0 均值，方差为 1 且具有有限四阶矩的独立同分布随机变量 $X_1, X_2, \cdots, X_n$。令$A = \{a_{ij}\}_{i,j=1}^{n}$为一对称矩阵，

$$W_n = \sum_{1 \le i \ne j \le n} a_{ij} X_i X_j \qquad (99)$$

许多文献都曾讨论过关于W_n的中心极限定理，比如 de Jong[9]，证明了当

$$\sigma_n^{-4} Tr(A^4) \to 0, \qquad \sigma_n^{-2} \max_{1 \le i \le n} \sum_{1 \le j \le n} a_{ij}^2 \to 0$$

时所满足的中心极限定理。Shao 和 Zhang 在[18]中也曾讨论过。

在这里，我们用定理 3.2 获得一个关于W_n的非一致 Berry-Esseen 界。

定理 4.5 令$X_1, X_2, \cdots, X_n$为一列 0 均值，方差为 1，有界（$|X_i| \le A, 1 \le i \le n$）的独立同分布随机变量，$A = \{a_{ij}\}_{i,j=1}^{n}$为对角线为 0（$a_{ii} = 0$）的对称矩阵，令 $W_n = \sum_{1 \le i \ne j \le n} a_{ij} X_i X_j$，$\sigma_n^2 = 2\sum_{i,j=1}^{n} a_{ij}^2$，对任意$M > 0$，存在$C$，$C_0$，使得：

$$|P(W \le z) - P(Z \le z)| \le \begin{cases} C\left(\dfrac{C_0 A^4}{\sigma_n^2}\sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^2\right)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n} a_{ik}a_{jk}\right)^2} + \dfrac{16nA^6}{\sigma_n^3}\left(\max_{1 \le i \le n}\left(\sum_{j=1}^{n}|a_{ij}|\right)\right)^3 + \dfrac{4nA^4}{\sigma_n^2}\left(\max_{1 \le i \le n}\left(\sum_{j=1}^{n}|a_{ij}|\right)\right)^2\right)e^{-\frac{z^2}{2}}, & |z| \ge M \\ C\left(\dfrac{C_0 A^4}{\sigma_n^2}\sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^2\right)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n} a_{ik}a_{jk}\right)^2} + \dfrac{6A^2}{\sigma_n}\max_{1 \le i \le n}\left(\sum_{j=1}^{n}|a_{ij}|\right) + \dfrac{13.12nA^6}{\sigma_n^3}\left(\max_{1 \le i \le n}\left(\sum_{j=1}^{n}|a_{ij}|\right)\right)^3\right)e^{-\frac{z^2}{2}}, & |z| < M \end{cases}$$

这里，C_0是一个与任何量均无关的常数，C的要求与定理 3.2 一致。

注：这里C是一个与n有关较大的常数，为防止混淆，可写作C_n，若要获得一致的C，我们需要调整非一致收敛速度至$O\left(\frac{1}{z}\right)$。如果我们可以取矩阵$A$使得定理 4.5 中括号内是一致收敛，那么它就具有了一致和非一致收敛性，将会是非常不错的结果。关于满足上述条件的X_i不难寻找，比如，我们可以取$P(X_i = -1) = P(X_i = 1) = \frac{1}{2}$的两点分布，通过本定理得到它对应的 Quadratic form 的渐进分布以及误差。

为证明定理 4.5，我们先证明引理 4.3，

引理 4.3 令I为一随机指标，在$\{1,2,\cdots,n\}$上服从均匀分布，$\{X_1,X_2,\cdots,X_n\}$为一列 0 均值，方差为 1，有界（$|X_i|\le A, 1\le i\le n$）的独立同分布随机变量。$\{X_1',X_2',\cdots,X_n'\}$为与$\{X_1,X_2,\cdots,X_n\}$独立且具有相同分布的随机变量。$I$与以上随机变量均独立。对于$W_n=\sum_{1\le i\ne j\le n} a_{ij}X_iX_j/\sigma_n$，令

$$W_n' = W_n - \frac{2\sum_{j=1}^n a_{Ij}X_IX_j}{\sigma_n} + \frac{2\sum_{j=1}^n a_{Ij}X_I'X_j}{\sigma_n} \quad (100)$$

那么(W_n,W_n')是$2/n$-Stein 对。除此之外，$(W_n,W_n{}')$满足：

$$|W_n - W_n'| \le \frac{4A^2}{\sigma_n}\max_{1\le i\le n}\left(\sum_{j=1}^n |a_{ij}|\right) \quad (101)$$

引理 4.3 证明：显然二者是可交换的，下证他们是$2/n$-Stein 对：
令

$$\Delta = W_n - W_n' = \frac{2}{\sigma_n}\sum_{j=1}^n a_{Ij}X_j\left(X_I - X_I'\right) \quad (102)$$

我们作如下推导，

$$\begin{aligned}
E(\Delta|W_n) &= E\left(\frac{2}{\sigma_n}\textstyle\sum_{j=1}^n a_{Ij}X_j(X_I - X_I')\left|W_n\right.\right)\\
&= \frac{2}{n\sigma_n}\sum_{i=1}^n E\left(\textstyle\sum_{j=1}^n a_{Ij}X_j(X_i - X_i')\left|W_n\right.\right)\\
&= \frac{2}{n\sigma_n}\sum_{i=1}^n E\left(E\left(\textstyle\sum_{j=1}^n a_{Ij}X_j(X_i - X_i')\left|W_n, X_1,\cdots,X_n\right.\right)\middle|W_n\right)\\
&= \frac{2}{n\sigma_n}\sum_{i=1}^n E\left(\textstyle\sum_{j=1}^n a_{ij}X_iX_j - \sum_{j=1}^n a_{ij}X_jE(X_i')\left|W_n\right.\right)\\
&= \frac{2}{n\sigma_n}\sum_{i=1}^n E\left(\textstyle\sum_{j=1}^n a_{ij}X_iX_j\left|W_n\right.\right) = \frac{2}{n}W_n
\end{aligned}$$

另外，

$$|\Delta| \le \frac{2}{\sigma_n}\sum_{j=1}^n |a_{Ij}||X_j|(|X_i| + |X_I'|) \le \frac{4A^2}{\sigma_n}\max_{1\le i\le n}\left(\sum_{j=1}^n |a_{ij}|\right)$$

至此，我们完成了引理 4.3 的证明。

定理 4.5 证明：
首先对于B'，我们有，

$$B' = \left(E\left(E\left(\left(1 - \frac{(W-W')^2}{2\lambda}\right)\middle|W\right)\right)^2\right)^{\frac{1}{2}} = \frac{\sqrt{Var(E(\Delta^2|W))}}{2\lambda}$$

对于$Var(E(\Delta^2|W))$，令$\mathcal{F} = \sigma(X_1, X_2, \cdots, X_n)$，我们有，

$$Var(E(\Delta^2|W)) \le Var(E(\Delta^2|\mathcal{F})) \qquad (103)$$

首先，我们计算$E(\Delta^2|\mathcal{F})$，

$$E(\Delta^2|\mathcal{F}) = \frac{4}{\sigma_n^2} E\left(\left(\sum_{j=1}^n a_{Ij} X_j (X_I - X_I')\right)^2 \middle| \mathcal{F}\right) = \frac{4}{n\sigma_n^2} \sum_{i=1}^n E\left(\left(\sum_{j=1}^n a_{ij} X_j (X_i - X_i')\right)^2 \middle| \mathcal{F}\right)$$

$$= \frac{4}{n\sigma_n^2} \sum_{i=1}^n \left(\left(\sum_{j=1}^n a_{ij} X_i X_j \right)^2 - 2 \sum_{j=1}^n a_{ij} X_i X_j \, E\left(\sum_{j=1}^n a_{ij} X_i' X_j \middle| \mathcal{F}\right) \right.$$

$$\left. + E\left(\left(\sum_{j=1}^n a_{ij} X_i' X_j\right)^2 \middle| \mathcal{F}\right) \right) = \frac{4}{n\sigma_n^2} \sum_{i=1}^n \left(\left(\sum_{j=1}^n a_{ij} X_i X_j \right)^2 + \left(\sum_{j=1}^n a_{ij} X_j \right)^2 E{X_i'}^2 \right)$$

$$= \frac{4}{n\sigma_n^2} \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} X_j \right)^2 (X_i^2 + 1) \qquad (104)$$

对于上式中的$\sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} X_j\right)^2 (X_i^2 + 1)$，

$$Var\left(\sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} X_j \right)^2 (X_i^2 + 1) \right)$$

$$= \sum_{i=1}^n Var\left(\left(\sum_{j=1}^n a_{ij} X_j \right)^2 (X_i^2 + 1) \right)$$

$$+ \sum_{i \ne l} cov\left(\left(\sum_{j=1}^n a_{ij} X_j \right)^2 (X_i^2 + 1), \left(\sum_{k=1}^n a_{lk} X_k \right)^2 (X_l^2 + 1) \right) \qquad (105)$$

对于上式第一项，由于对角线元素均为 0，

$$\sum_{i=1}^{n} Var\left(\left(\sum_{j=1}^{n} a_{ij}X_j\right)^2 (X_i^2+1)\right) \le \sum_{i=1}^{n} E\left(\sum_{j=1}^{n} a_{ij}X_j\right)^4 (X_i^2+1)^2$$

$$= \sum_{i=1}^{n} E\left(\sum_{j=1}^{n} a_{ij}X_j\right)^4 E(X_i^2+1)^2$$

$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^4 EX_1^4 + 6\sum_{\substack{r,s=1\\ r\neq s}}^{n} a_{ir}^2 a_{is}^2 (EX_1^2)^2\right)(EX_1^4+2EX_1^2+1)$$

$$\le \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^4 + 6\sum_{\substack{r,s=1\\ r\neq s}}^{n} a_{ir}^2 a_{is}^2\right) EX_1^4(EX_1^4+2EX_1^2+1)$$

$$\le A^4(A^4+3)\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^4 + 6\sum_{\substack{r,s=1\\ r\neq s}}^{n} a_{ir}^2 a_{is}^2\right)$$

$$= A^4(A^4+3)\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^4 + 6\left(\sum_{j=1}^{n} a_{ij}^2\right)^2\right)$$

$$\le CA^8 \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^2\right)^2 \qquad (106)$$

为了计算方便，定义，

$$M_i = (X_i^2+1)\left(\sum_{j=1}^{n} a_{ij}X_j\right)^2, M_i^{(k)} = (X_i^2+1)\left(\sum_{j\neq k} a_{ij}X_j\right)^2 \qquad (107)$$

对于(105)的第二项，根据 Shao 和 Zhang[18]，我们可以作如下分解，分别进行计算，

$$cov\left(\left(\sum_{j=1}^{n} a_{ij}X_j\right)^2 (X_i^2+1), \left(\sum_{k=1}^{n} a_{lk}X_k\right)^2 (X_l^2+1)\right) = cov(M_i, M_l)$$

$$= cov\left(M_i - M_i^{(l)}, M_l - M_l^{(i)}\right) + cov\left(M_i^{(l)}, M_l\right) + cov\left(M_i, M_l^{(i)}\right) - cov\left(M_i^{(l)}, M_l^{(i)}\right) \qquad (108)$$

(1) 首先计算$cov\left(M_i - M_i^{(l)}, M_l - M_l^{(i)}\right)$

$$M_i - M_i^{(l)} = \left(X_i^2 + 1\right)\left(\left(\sum_{j=1}^{n} a_{ij} X_j\right)^2 - \left(\sum_{j\neq l} a_{ij} X_j\right)^2\right)$$

$$= \left(X_i^2 + 1\right) a_{il} X_l \left(2\sum_{j\neq l} a_{ij} X_j + a_{il} X_l\right) \qquad (109)$$

同理可求,

$$M_l - M_l^{(i)} = (X_l^2 + 1)\left(\left(\sum_{j=1}^{n} a_{lj} X_j\right)^2 - \left(\sum_{j\neq i} a_{lj} X_j\right)^2\right)$$

$$= (X_l^2 + 1) a_{li} X_i \left(2\sum_{j\neq i} a_{lj} X_j + a_{li} X_i\right) \qquad (110)$$

结合(109)(110)

$$cov\left(M_i - M_i^{(l)}, M_l - M_l^{(i)}\right)$$

$$= cov\left((X_i^2+1)a_{il}X_l\left(2\sum_{j\neq l} a_{ij}X_j + a_{il}X_l\right), (X_l^2 + 1)a_{li}X_i\left(2\sum_{j\neq i} a_{lj}X_j + a_{li}X_i\right)\right)$$

$$= cov\left((X_i^2+1)a_{il}X_l 2\sum_{j\neq l} a_{ij}X_j + (X_i^2+1)a_{il}^2X_l^2, (X_l^2+1)a_{li}X_i 2\sum_{j\neq i} a_{lj}X_j + (X_l^2+1)a_{li}^2X_i^2\right)$$

$$= cov\left((X_i^2+1)a_{il}X_l 2\sum_{j\neq l} a_{ij}X_j , (X_l^2+1)a_{li}X_i 2\sum_{j\neq i} a_{lj}X_j\right) + cov\left((X_i^2+1)a_{il}^2X_l^2, (X_l^2+1)a_{li}^2X_i^2\right)$$

$$= E\left(4a_{il}^2(X_i^2+1)X_iX_l(X_l^2+1)\left(\sum_{j\neq l} a_{ij}X_j\right)\left(\sum_{j\neq i} a_{lj}X_j\right)\right) + a_{il}^4 cov\left((X_i^2+1)X_l^2, (X_l^2+1)X_i^2\right)$$

$$= 4a_{il}^2(EX_1^3)^2\sum_{k=1}^{n} a_{ik}a_{lk} + a_{il}^4((EX_1^4+1)^2 - 4) \qquad (111)$$

(2) 计算$cov\left(M_i^{(l)}, M_l^{(i)}\right)$

$$cov\left(M_i^{(l)}, M_l^{(i)}\right) = cov\left(\left(X_i^2+1\right)\left(\sum_{j\neq l}^{n} a_{ij}X_j\right)^2, \left(X_l^2+1\right)\left(\sum_{j\neq i}^{n} a_{lj}X_j\right)^2\right)$$

$$= 4E\left(\left(\sum_{j\neq l}^{n} a_{ij}X_j\right)^2\left(\sum_{j\neq i}^{n} a_{lj}X_j\right)^2\right) - 4E\left(\sum_{j\neq l}^{n} a_{ij}X_j\right)^2 E\left(\sum_{j\neq i}^{n} a_{lj}X_j\right)^2$$

$$= 4EX_1^4\sum_{j=1}^{n} a_{ij}^2 a_{lj}^2 + 8\sum_{j\neq k}^{n} a_{ij}a_{ik}a_{lj}a_{lk}$$

$$-4\left(\sum_{j=1}^{n} a_{ij}^2\right)\left(\sum_{j=1}^{n} a_{lj}^2\right) \qquad (112)$$

(3) 计算$cov\left(M_i^{(l)}, M_l\right)$， $cov\left(M_i, M_l^{(i)}\right)$

$$cov\left(M_i, M_l^{(i)}\right) = cov\left(\left(X_i^2+1\right)\left(\sum_{j\neq l}^{n} a_{ij}X_j + a_{il}X_l\right)^2, \left(X_l^2+1\right)\left(\sum_{j\neq i}^{n} a_{lj}X_j\right)^2\right)$$

$$= cov\left(\left(X_i^2+1\right)\left(\sum_{j\neq l}^{n} a_{ij}X_j + a_{il}X_l\right)^2, \left(X_l^2+1\right)\left(\sum_{j\neq i}^{n} a_{lj}X_j\right)^2\right)$$

$$= cov\left(\left(X_i^2+1\right)\left(\left(\sum_{j\neq l}^{n} a_{ij}X_j\right)^2 + 2a_{il}X_l\sum_{j\neq l}^{n} a_{ij}X_j + a_{il}^2X_l^2\right), \left(X_l^2+1\right)\left(\sum_{j\neq i}^{n} a_{lj}X_j\right)^2\right)$$

$$= cov\left(M_i^{(l)}, M_l^{(i)}\right) + cov\left(2a_{il}X_l\sum_{j\neq l}^{n} a_{ij}X_j\left(X_i^2+1\right), \left(X_l^2+1\right)\left(\sum_{j\neq i}^{n} a_{lj}X_j\right)^2\right)$$

$$+ cov\left(a_{il}^2X_l^2\left(X_i^2+1\right), \left(X_l^2+1\right)\left(\sum_{j\neq i}^{n} a_{lj}X_j\right)^2\right)$$

$$= cov\left(M_i^{(l)}, M_l^{(i)}\right) + 2a_{il}E\left(X_l\left(X_l^2+1\right)\left(X_i^2+1\right)\sum_{j\neq l}^{n} a_{ij}X_j\left(\sum_{j\neq i}^{n} a_{lj}X_j\right)^2\right)$$

$$+ a_{il}^2E\left(X_l^2\left(X_l^2+1\right)\left(X_i^2+1\right)\left(\sum_{j\neq i}^{n} a_{lj}X_j\right)^2\right)$$

$$- a_{il}^2E\left(X_l^2\left(X_i^2+1\right)\right)E\left(\left(X_l^2+1\right)\left(\sum_{j\neq i}^{n} a_{lj}X_j\right)^2\right)$$

$$= cov\left(M_i^{(l)}, M_l^{(i)}\right) + 4a_{il}(EX_1^3)^2\sum_{j=1}^{n} a_{ij}a_{lj}^2 + 2a_{il}^2(EX_1^4+1)\sum_{j\neq i} a_{lj}^2 - 4a_{il}^2\sum_{j\neq i} a_{lj}^2$$

$$= cov\left(M_i^{(l)}, M_l^{(i)}\right) + 4a_{il}(EX_1^3)^2\sum_{j=1}^{n} a_{ij}a_{lj}^2$$

$$+ 2a_{il}^2(EX_1^4-1)\sum_{j\neq i} a_{lj}^2 \qquad (113)$$

交换i和l的位置，我们有，

$$cov\left(M_l, M_i^{(l)}\right) = cov\left(M_i^{(l)}, M_l^{(i)}\right) + 4a_{il}(EX_1^3)^2\sum_{j=1}^{n} a_{lj}a_{ij}^2$$

$$+ 2a_{il}^2(EX_1^4-1)\sum_{j\neq l} a_{ij}^2 \qquad (114)$$

根据(111)(112)(113)(114)，

$$cov\left(\left(\sum_{j=1}^{n} a_{ij}X_j\right)^2 (X_i^2+1), \left(\sum_{k=1}^{n} a_{lk}X_k\right)^2 (X_l^2+1)\right) = cov(M_i, M_l)$$

$$= 4a_{il}^2(EX_1^3)^2\sum_{k=1}^{n} a_{ik}a_{lk} + a_{il}^4((EX_1^4+1)^2-4) + 4a_{il}(EX_1^3)^2\sum_{j=1}^{n} a_{ij}a_{lj}^2$$

$$+ 2a_{il}^2(EX_1^4-1)\sum_{j\neq i} a_{lj}^2 + 4a_{il}(EX_1^3)^2\sum_{j=1}^{n} a_{lj}a_{ij}^2 + 2a_{il}^2(EX_1^4-1)\sum_{j\neq l} a_{ij}^2$$

$$+ 4EX_1^4\sum_{j=1}^{n} a_{ij}^2a_{lj}^2 + 8\sum_{j\neq k} a_{ij}a_{ik}a_{lj}a_{lk} - 4\left(\sum_{j=1}^{n} a_{ij}^2\right)\left(\sum_{j=1}^{n} a_{lj}^2\right)$$

所以对于(105)第二项，

$$\sum_{i\neq l} cov\left(\left(\sum_{j=1}^{n} a_{ij}X_j\right)^2 (X_i^2+1), \left(\sum_{k=1}^{n} a_{lk}X_k\right)^2 (X_l^2+1)\right)$$

$$\leq CA^8\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^2\right)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n} a_{ik}a_{jk}\right)^2\right) \quad (115)$$

所以结合(106)(115)有，

$$Var\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}X_j\right)^2 (X_i^2+1)\right) \leq CA^8\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^2\right)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n} a_{ik}a_{jk}\right)^2\right)$$

所以，

$$Var\big(E(\Delta^2|\mathcal{F})\big) = Var\left(\frac{4}{n\sigma_n^2}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}X_j\right)^2 (X_i^2+1)\right)$$

$$\leq \frac{16}{n^2\sigma_n^4} CA^8\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^2\right)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n} a_{ik}a_{jk}\right)^2\right) \quad (116)$$

这里的C是一个与任何量均无关的常数。

$$B' = \frac{\sqrt{Var(E(\Delta^2|W))}}{2\lambda} = \frac{n}{4}\cdot\frac{4}{n\sigma_n^2}\cdot CA^4\sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^2\right)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n} a_{ik}a_{jk}\right)^2}$$

$$= \frac{C_0A^4}{\sigma_n^2}\sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}^2\right)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n} a_{ik}a_{jk}\right)^2} \quad (117)$$

由前面推导，我们知道，

$$|\Delta| \le \frac{4A^2}{\sigma_n} \max_{1\le i\le n}\left(\sum_{j=1}^{n}|a_{ij}|\right) = \delta \qquad (118)$$

根据定理 3.2,我们有，

$$|P(W \le z) - P(Z \le z)|$$

$$\le \begin{cases} C\left(\frac{C_0A^4}{\sigma_n^2}\sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}^2\right)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n}a_{ik}a_{jk}\right)^2} + \frac{16nA^6}{\sigma_n^3}\left(\max_{1\le i\le n}\left(\sum_{j=1}^{n}|a_{ij}|\right)\right)^3 + \frac{4nA^4}{\sigma_n^2}\left(\max_{1\le i\le n}\left(\sum_{j=1}^{n}|a_{ij}|\right)\right)^2\right)e^{-\frac{z^2}{2}}, & |z| \ge M \\ C\left(\frac{C_0A^4}{\sigma_n^2}\sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}^2\right)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n}a_{ik}a_{jk}\right)^2} + \frac{6A^2}{\sigma_n}\max_{1\le i\le n}\left(\sum_{j=1}^{n}|a_{ij}|\right) + \frac{13.12nA^6}{\sigma_n^3}\left(\max_{1\le i\le n}\left(\sum_{j=1}^{n}|a_{ij}|\right)\right)^3\right)e^{-\frac{z^2}{2}}, & |z| < M \end{cases}$$

这里C_0是一个与任何量都无关的常数，C的要求与定理 3.2 一致。
至此，我们证明了定理 4.5。

4.4 Lightbulb Process

首先，我们介绍一下 Lightbulb process。假设有n个开关，每个开关都连接一个灯泡，按下开关意味着打开灯或者关上灯。对应地，我们假设依次有n个时刻，假设一开始（第0步）开关均为关闭的，在第i($i = 1,2,\cdots,n$)个时刻，我们从n个开关中随机挑选i个开关并按下（这意味着切换灯的开关状态）。我们关心的是在第n个时刻结束时，亮着的灯泡数目Y。
我们给出Y期望和方差的表达式：

$$EY = \mu = \frac{n}{2}\left(1 - \prod_{i=1}^{n}\left(1 - \frac{2i}{n}\right)\right) \qquad (119)$$

$$\sigma^2 = \frac{n}{4}\left(1 - \prod_{i=1}^{n}\left(1 - \frac{4i}{n} + \frac{4i(i-1)}{n(n-1)}\right)\right) + \frac{n^2}{4}\left(\prod_{i=1}^{n}\left(1 - \frac{4i}{n} + \frac{4i(i-1)}{n(n-1)}\right) - \prod_{i=1}^{n}\left(1 - \frac{2i}{n}\right)^2\right) \qquad (120)$$

显然，n为偶数时，$\mu = n/2$；n为奇数时，$\mu = (n/2)\big(1 + O(e^{-n})\big)$，这一点根据 Stirling 公式容易得出。可以证明，对于任意的n，有$\sigma^2 = (n/2)\big(1 + O(e^{-n})\big)$。

以下我们给出[1]中关于 Lightbulb 的一致界结果：
定理 4.6（参考[1]）令Y为时刻终止时刻n时亮着的灯泡数目，μ和σ^2为它的期望和方差，具体见(119)(120)。令$W = (Y - \mu)/\sigma$，对于任意偶数n，我们有：

$$\sup_{z\in R}|P(W \le z) - P(Z \le z)| \le \frac{n}{2\sigma^2}\Psi + \frac{1.64n}{\sigma^3} + \frac{2}{\sigma} \qquad (121)$$

其中对于$n \ge 6$，Ψ满足：

$$\Psi \le \frac{1}{2\sqrt{n}} + \frac{1}{2n} + e^{-\frac{n}{2}} \qquad (122)$$

具体证明见[1]。

注：上式具有$O(\frac{1}{\sqrt{n}})$的一致收敛速度。

以下给出我们证明的非一致界结果：

定理 4.7 令Y为时刻终止时刻n时亮着的灯泡数目，μ和σ^2为它的期望和方差，令$W = (Y-\mu)/\sigma$，对于任意偶数n，$M > 0$，存在不依赖于z常数C，我们有：

$$|P(W \le z) - P(Z \le z)| = \begin{cases} C\left(\frac{\mu}{\sigma^2}\Psi + \frac{2\mu}{\sigma^2} + \frac{4\mu}{\sigma^3}\right)e^{-\frac{z^2}{2}}, |z| \ge M \\ C\left(\frac{24\mu}{\sigma^3} + 2\frac{\mu}{\sigma^2}\Psi\right)e^{-\frac{z^2}{2}}, \qquad |z| < M \end{cases}$$

其中$D' = \left(E\left(E\left(1 - \frac{\mu}{\sigma}(W^s - W)|W\right)\right)^2\right)^{\frac{1}{2}}$，$\Psi$同(122)。

定理 4.7 的证明：

根据[1]中Y^s的构造，我们可以取$A = 2$。同时，不难证明我们引入的D'（此关系类似于B与我们引入的B'）满足：

$$D \le \frac{\mu}{\sigma^2}\Psi = D' \qquad (123)$$

根据定理 3.3，我们可以得到上述结果。

注：在本定理中，常数C是和n有关的，若希望C与n无关，我们利用定理 3.3 证明方法获得$O\left(\frac{1}{z}\right)$非一致收敛速度的界。对于括号内的项，不难发现，只能做到有界，不具有一致收敛性，与定理 4.6 中$O(1/\sqrt{n})$的一致收敛速度相比，定理 4.7 的非一致收敛性减弱，不过我们可以获得一个非一致的收敛速度。

4.5 Patterns in graphs and permutations

在这一部分我们介绍图和置换中的正态近似效果，并给出非一致界。

定义（相依邻域 dependency neighborhood）：考虑一类随机变量$\mathbf{X} = \{X_\alpha, \alpha \in \mathcal{A}\}$。若$\mathcal{B}_\alpha \subset \mathcal{A}$满足：$X_\alpha$和$\{X_\beta: \beta \notin \mathcal{B}_\alpha\}$相互独立。那么称$\mathcal{B}_\alpha$为$\alpha$的相依邻域。

我们给出[1]中如下一致界结果：

定理 4.8（参考[1]）令$\mathbf{X} = \{X_\alpha, \alpha \in \mathcal{A}\}$为一类在$[0, M/2]$取值的随机变量，令

$$Y = \sum_{\alpha\in\mathcal{A}} X_\alpha, \qquad \mu = \sum_{\alpha\in\mathcal{A}} EX_\alpha, \qquad \sigma^2 = Var(Y)$$

$$p_\alpha = \frac{EX_\alpha}{\sum_{\beta\in\mathcal{A}} EX_\beta}, \qquad \bar{p} = \max_{\alpha\in\mathcal{A}} p_\alpha, b = \max_{\alpha\in\mathcal{A}} |\mathcal{B}_\alpha|$$

令$\mathbf{X}^\alpha$具有$\mathbf{X}$的α-size biased 分布，若存在$\mathcal{F} \supset \sigma(Y)$和$\mathcal{D} \subset \mathcal{A} \times \mathcal{A}$，使得：

若$(\alpha_1,\alpha_2)\notin\mathcal{D}$，那么对于任意的$(\beta_1,\beta_2)\in\mathcal{B}_{\alpha_1}\times\mathcal{B}_{\alpha_2}$，

$cov\left(E\left(X_{\beta_1}^{\alpha_1}-X_{\beta_1}\middle|\mathcal{F}\right),E\left(X_{\beta_2}^{\alpha_2}-X_{\beta_2}\middle|\mathcal{F}\right)\right)=0$。令$W=(Y-\mu)/\sigma$，它满足，

$$\sup_{z\in R}|P(W\le z)-P(Z\le z)|\le\frac{6\mu b^2M^2}{\sigma^3}+\frac{2\mu\bar{p}bM\sqrt{|\mathcal{D}|}}{\sigma^2}$$

然后我们给出我们的非一致界结果：

定理 4.9 各符号含义与定理 4.8 相同，我们令$W=(Y-\mu)/\sigma$，对任意$M>0$，存在不依赖于z的C，它满足，

$$|P(W\le z)-P(Z\le z)|=\begin{cases}C\left(\frac{\mu\bar{p}bM\sqrt{|\mathcal{D}|}}{\sigma^2}+\frac{\mu bM}{\sigma^2}+\frac{\mu b^2M^2}{\sigma^3}\right)e^{-\frac{z^2}{2}},|z|\ge M\\ C\left(\frac{6\mu b^2M^2}{\sigma^3}+\frac{2\mu\bar{p}bM\sqrt{|\mathcal{D}|}}{\sigma^2}\right)e^{-\frac{z^2}{2}},\qquad |z|<M\end{cases}$$

定理 4.9 的证明：

根据[1]，我们可以构造Y^s满足

$$|Y^s-Y|\le bM,\qquad \Psi\le\bar{p}bM\sqrt{|\mathcal{D}|}$$

代入定理 3.3，我们即得到上述结果。

注：上述常数C会依赖于其他的量，特别是W，若希望它不依赖于W，我们可以获得以$O\left(\frac{1}{z}\right)$下降的非一致界。括号内的项与定理 4.8 相比，一致收敛性减弱了，但是如果我们能够找到一个图使得括号内的量很小，我们也是可以得到一致收敛性的，虽然这个一致收敛的速度是次优的。回顾定理 4.7，亦是应用了定理 3.3，二者括号内的一致收敛性能力均减弱了。

五、总结：创新与不足

5.1 创新之处

本文主要贡献是求出了关于 zero bias、size bias 和有界可交换对的非一致界，并且将这些非一致界用到了组合中心极限定理、Anti-voter 模型、Quadratic forms、Lightbulb process、Patterns in graphs and permutations，并与这些模型的一致界进行深入对比和探讨。在证明非一致界时，我们创新性的引入了与模型参数B,D相对应的B',D'，并且这两个新的参数可以对应到许多模型中。与此同时，我们的证明方法也非常新颖，并且可推广性强：如果我们更强调关于 Berry-Esseen 关于z的变化情况，而不关心前面的常数C，我们可以获得$O\left(e^{-\frac{z^2}{2}}\right)$的非一致界；如果我们需要$C$与$W$无关，而不是非常注重关于$z$的变化情况，我们可以获得$O\left(\frac{1}{z}\right)$的非一致界。

5.2 不足之处以及改进思路

首先，定理 3.1、定理 3.2、定理 3.3 的证明方法大致相同，我们虽然可以从中得出一个非一致的$O\left(e^{-\frac{z^2}{2}}\right)$项，但是我们同时放大了前面的常数$C$，而且这个常数很多情况下依赖于常数$n$，为了弥补这个缺点，我们只得利用$EW^2=1$这一条件将非一致项调整至$O\left(\frac{1}{z}\right)$；除此之外，虽然在某些模型中我们可以保持非一致界中的一致收敛性，但是我们获得的非一致界的一致收敛性（即前面所讲的“括号内的内容”）减弱了，比如定理 3.3 的两个应用 Lightbulb process 和 Patterns in graphs and permutations，二者的一致收敛能力都减弱了。

关于改进思路，关于非一致项$O\left(\frac{1}{g(z)}\right)$和前面常数$C$，我们可以尝试尽量保持$C$关于$n$的一致性和$O\left(\frac{1}{g(z)}\right)$的阶尽量小的性质，兼顾二者，比如是否可以使$C$与$n$无关的同时获得$O\left(\frac{1}{z^k}\right), k\geq 2$的收敛速度。再者，我们需要兼顾非一致界中的一致收敛项，尽量保持它的一致收敛能力。

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关于渗流模型的介绍以及 THE UNIVERSAL RELATION BETWEEN SCALING EXPONENTS IN FIRST-PASSAGE PERCOLATION - SOURAV CHATTERJEE 总结（译文以及译文原文部分）

巩舒阳

2021 年 5 月 1 日

摘要

本文首先简要介绍一些渗流模型的基本概念以及一些应用，然后主要对 Sourav Chatterjee 的一篇文章进行翻译和总结，介绍了波动指数 χ 与游动指数 ξ 的关系 $\chi = 2\xi - 1$，并且在假定两个指数均存在的情况下给出此关系的严格证明。以下根据原文顺序进行总结（其中，第二部分证明思路参看原文即可，总结不再提及），主要对一些基本概念进行解释，以及对于原文证明的补充和完善。

1 关于渗流模型 (Percolation)

1.1 渗流模型的引入

问题引入：将一个较大且多孔的石头浸没在水中，那么石头中心被浸湿的概率是多少？
为了建立一个模型，Broadbent 和 Hammersley(1957) 首先提出了渗流模型，

考虑格子 $\mathbb{Z}^2$ 以及 $p(0 \le p \le 1)$，对于格子中的每条边，它是“开”（此处可认为水流可以通过）的概率 p，是“闭”（水流无法通过）的概率为 $1-p$，并且每个边的状态是独立的。渗流模型主要是关于“开”的边进行讨论。我们称一个点 x 是可被浸湿的当且仅当存在一个从石头边界到 x 一条开路径，同时，由于路径的大小相对于石头来讲可以忽略不计，所以石头中心被浸湿的概率和中心点是一个有无穷个格子点构成的开路径的端点的概率差不多，进一步地，可以认为这个概率与存在无穷连通的类的概率相关（关于类的定义，稍后提及）。

1.2 一些基本概念

临界值 p_c 临界值 p_c 满足当 $p < p_c$，所有的连通的类均是有限的；$p > p_c$，存在一个无限的连通的类。
当 p 较大时，称为“体积效应”；p 较小时，称为“表面效应”。
记所有边构成的集合为 $\mathbb{E}^d$，将格子架记为 $\mathbb{L}^d$，可表示为 $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$
连通类（open clusters） 若 C 为 $\mathbb{L}^d$ 的一个子集，且对于 C 任意两点，总存在一个“开”的路径连接二者，那么称 C 是一个连通类。

一些渗流模型 键渗流模型 (bond percolation)：边是开的或者闭的。点渗流模型（site percolation）：点是开的或者是闭的。在这里指出，相对于键渗流模型，点渗流模型是一个更广泛的概念。混合模型（mixed models）：点和边都有可能是闭的。非齐性模型（inhomogeneous models）：不同边有可能是开的概率不同。大范围模型（long-range models）：有可能发生远距离渗流。依赖模型（dependent percolation）：不同边的状态有可能不独立。

为什么要研究渗流（Why Percolation）？ 主要有以下四个原因：(i) 容易用模型表示并且在随机媒介中的预测中具有现实意义。(ii) 在渗流模型领域可以发展许多数学方法和技巧。(iii) 有许多难以解决的有趣猜想（比如 Sourav Chatterjee 这篇文章，$\chi \le 2\xi - 1$ 不等式的证明困扰了人们很长时间）。(iv) 有许多以 $\mathbb{Z}^d$ 为下标的相关随机变量可以研究。同时，渗流模型理论是随机无序媒介（disordered media）研究的基石，有以下四个例子：1. 无序电子网络（disordered electrical network）：计算一个混合物的电阻，主要由材料 A 和材料 B 构成，该混合物可以看做一个格子架，每条边以概率 p 为材料 A，否则为材料 B，且每条边的状态相互独立，若材料 B 是一个完

美的绝缘体，这就与讨论的渗流模型相同了。2. 铁磁性（ferromagnetism）：与 Ising model 相关。3. 果园失火（Epidemics and fires in orchards）4. 圆片规模集成（Wafer-scale integration）

2 文章中概念解释以及一些结果

考虑 $\mathbb{R}^d$ 空间以及欧式距离 $|\cdot|$，考虑一个子集“整数格子” $\mathbb{Z}^d$。如果两个 $\mathbb{Z}^d$ 中的点 x, y 满足 $|x-y|=1$，称二者为最近邻。$E(\mathbb{Z}^d)$ 为所有最近邻的连接键组成的集合。$t=(t_e)_{e\in E(\mathbb{Z}^d)}$ 为一些独立同分布非负随机变量构成的集合。随机变量 t_e 称为 e 的“通过时间”，亦可称为“边权重”，称 t 为一个“环境”。路径 P 的总共通过时间 $t(P)$，为 P 中边通过时间的和。“第一通过时间” $T(x,y)$ 为从 x 到 y 所有格子路径中的最小通过时间。本文中格子路径为自回避路径。

若通过时间是连续型随机变量，那么以概率 1 的，存在一个最优路径 $G(x,y)$，$D(x,y)$ 为路径上的点与连接 x 与 y 的线段最大距离。

波动指数 χ 是描述 $T(x,y)$ 波动性质的一个常数。粗略的讲，$T(x,y)-\mathbb{E}T(x,y)$ 的阶为 $|x-y|^\chi$，游动指数 ξ 描述的是 $D(x,y)$ 的大小，$D(x,y)$ 的阶为 $|x-y|^\xi$。

关于 χ 和 ξ 有许多猜测，其中主要的的猜测为：χ 和 ξ 的值可能会与维数有关，但是总是满足关系

$$\chi = 2\xi - 1$$

本文给出了这一猜想的完整证明。

当 $d=2$ 时，一个猜想认为 $\chi=1/3, \xi=2/3$；同时，还有人认为 d 充分大时，$\chi=0$ 成立。关系 $\chi=2\xi-1$ 经常被称为关于 χ 和 ξ 的“KPZ 关系”。之前有许多关于 χ 和 ξ 严谨的结果，其中一个是 Kesten [18，定理 1]，证明了在任意维数下都有 $\chi \le 1/2$，以下是 Benjamini，Kailai 和 Schramm 对于 Kesten 得出结论的进一步结果，证明了在 $d \ge 2$，且边权重随机变量仅取两个值时，关于 χ、ξ 有以下结果：

$$\sup_{v\in\mathbb{Z}^d, |v|>1} \frac{VarT(0,v)}{|v|/log|v|} < \infty \tag{1}$$

Benaïm 和 Rossignol 将这个结果扩展到近似伽马分布中（近似伽马分布包含许多常见的分布），给出“近似伽马分布”的定义：

定义 对于一个正的随机变量 X，如果它在一个区间 I 上（可以无界）有一个连续的概率密度 h 满足，对任意 $y \in I$，$\Phi^{'} \circ \Phi(H(y)) \leq A\sqrt{y}h(y)$，$A$ 是一个常数,Φ 是标准正态分布的分布函数,那么称 X 具有“近似伽马分布”。

通过时间波动的下界是由 Newman 和 Piza[26] 以及 Pemantle 和 Peres[27] 提出的，说明了在 $d = 2$ 时，$VarT(0, v)$ 至少以 $\log|v|$ 的速度增长。同时如果可以证出最优路径以较高的概率位于狭窄的“圆柱”内，就可以得出一个更好的下界。

对于游动指数 ξ，主要结果来自于 Licea，Newman 和 Piza，证明了 $\xi^{(2)} \geq 1/2$ 并且 $d = 2$ 时有 $\xi^{(3)} \geq 3/5$，这里的 $\xi^{(2)}$ 和 $\xi^{(3)}$ 可能与本文 ξ 相等。

除了 ξ 和 χ 的结果，还有一些不等式结果。Wehr 和 Aizenman[29] 在一个相关的模型中证明了不等式 $\chi \geq (1 - (d - 1)\xi)/2$，这一不等式的第一通过渗流模型版本由 Licea，Newman 和 Piza[23] 证明。与证明 $\chi = 2\xi - 1$ 最接近的是 Newman 和 Piza[26] 的结果：$\chi' \geq 2\xi - 1$。这里的 χ' 可能与 χ 相等，这一点也被 Howard 在不同的假设下观测出。

在本文之后，Antonio Auffinger 和 Michael Damron 在没有使用近似伽马分布假设条件下，当 $\chi > 0$，给出了不等式 $\chi \leq 2\xi - 1$ 的证明，进而说明了 $\chi = 2\xi - 1$ 成立，将边权重分布由“近似伽马”扩充到了全部连续型分布；同时，在 Antonio Auffinger 和 Michael Damron 的文章里还提出了一种按照方向定义这两个指数的方法，并提出了关于上述等式的一般形式 $\chi^u = \kappa^u\xi^u - (\kappa^u - 1)$（这里 κ^u 称为弯曲指数）在某些问题被证明的条件下可以成立。Antonio Auffinger 和 Michael Damron 亦使用了本文 (Sourav Chatterjee) 所使用的“圆柱”方法，通过使用了两个“并联圆柱”的方法，避免了使用“近似伽马”条件。可以说，本文中关于不等式 $\chi \leq 2\xi - 1$ 的严格证明意义是十分重大的。

下面的定理在假定 χ,ξ 存在且边权重的分布是近似伽马分布的前提下，说明了 χ 与 ξ 的意义，证明关系 $\chi = 2\xi - 1$ (我认为本文最主要的贡献是合理地刻画了 χ 和 ξ 在什么意义下存在，并在这种意义下证明了 $\chi = 2\xi - 1$)。

定理 1.1 假设边权重分布是近似伽马的, 并且在原点的附近有一个有限的矩母函数。令 χ_a 和 ξ_a 为满足以下条件最小的实数：对于任意 $\chi^{'} > \chi_a, \xi^{'} > \xi_a$，存在 $\alpha > 0$，使得

$$\sup_{v \in \mathbb{Z}^d \backslash \{0\}} \mathbb{E}exp(\alpha \frac{|T(0, v) - \mathbb{E}T(0, v)|}{|v|^{\chi^{'}}}) < \infty \tag{A1}$$

$$\sup_{v \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E} exp(\alpha \frac{D(0,v)}{|v|^{\xi'}}) < \infty \tag{A2}$$

令 χ_b, ξ_b 为满足以下条件最小的数：对任意 $\chi' < \chi_b, \xi' < \xi_b$，存在 $C > 0$，使得

$$\inf_{v \in \mathbb{Z}^d, |v| > C} \frac{Var(T(0,v))}{|v|^{2\chi'}} > 0 \tag{A3}$$

$$\inf_{v \in \mathbb{Z}^d, |v| > C} \frac{\mathbb{E}D(0,v)}{|v|^{\xi'}} > 0 \tag{A4}$$

由以上条件可推出，$0 \le \chi_b \le \chi_a \le 1/2, 0 \le \xi_b \le \xi_a \le 1$，且 $\chi_a \ge 2\xi_b - 1$。同时如果有 $\chi_a = \chi_b, \xi_a = \xi_b$，这两个数分别用 χ 和 ξ 表示，那么一定满足 $\chi = 2\xi - 1$。

3 证明 $0 \le \chi_b \le \chi_a \le 1/2, 0 \le \xi_b \le \xi_a \le 1$

3.1 $\chi_b \le \chi_a, \xi_b \le \xi_a$

以 $\chi_b \le \chi_a$ 为例，用反证法，若 $\chi_b > \chi_a$，那么必存在 $\chi_a < \chi' < \chi'' < \chi_b$，由 (A1) 可知，

$$\sup_{v \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E} exp(\alpha \frac{|T(0,v) - \mathbb{E}T(0,v)|}{|v|^{\chi'}}) < \infty$$

对指数项泰勒展开，取平方项，即有：

$$\frac{Var(T(0,v))}{|v|^{2\chi'}} \le M$$

所以，对于 $\mathbb{Z}^d$ 中任一趋于无穷的序列 v_n

$$\frac{Var(T(0,v_n))}{|v_n|^{2\chi''}} = \frac{Var(T(0,v))}{|v|^{2\chi'}} |v_n|^{2(\chi' - \chi'')}$$

因此有 $\lim_{n \to \infty} \frac{Var(T(0,v_n))}{|v_n|^{2\chi''}} = 0$，这与 (A3) 矛盾。
$\xi_b \le \xi_a$ 证明同理。

3.2 $\xi_b \ge 0$

令 E_0 代表与原点相接的边组成的集合，$\mathcal{F}_0$ 代表由 $(t_e)_{e \notin E_0}$ 生成的 σ-代数。同时，由于边权重分布是非退化的，所以存在 $c_1 < c_2$，使得 $\mathbb{P}(t_e < c_1) > 0$

且 $\mathbb{P}(t_e > c_2) > 0$，由于 E_0 中边的数目是有限的，所以直接得出

$$\mathbb{P}(\max_{e \in E_0} t_e < c_1) = (\mathbb{P}(t_e < c_1))^{2d} > 0, \mathbb{P}(\min_{e \in E_0} t_e > c_2) = (\mathbb{P}(t_e > c_2))^{2d} > 0 \tag{2}$$

令 $(t_e^{'})_{e \in E_0}$ 为一列独立同分布新的边权重分布。若 $e \notin E_0$，那么 $t_e^{'} = t_e$，$T'(0,v)$ 为新环境 t' 中的第一通过时间。若 $t_e < c_1, t_e' > c_2$ for all $e \in E_0$，就有 $T'(0,v) > T(0,v) - c_1 + c_2$，因此有：

$$\begin{aligned}
\mathbb{E}(T(0,v) - T'(0,v))^2 &= \mathbb{E}(\mathbb{E}((T(0,v) - T'(0,v))^2 | \mathcal{F}_0)) \\
&= \mathbb{E}(\mathbb{E}((T(0,v) - \mathbb{E}(T(0,v)|\mathcal{F}_0))^2 | \mathcal{F}_0)) \\
&+ \mathbb{E}(\mathbb{E}((T'(0,v) - \mathbb{E}(T'(0,v)|\mathcal{F}_0))^2 | \mathcal{F}_0)) \\
&+ 2\mathbb{E}(\mathbb{E}((T(0,v) - \mathbb{E}(T(0,v)|\mathcal{F}_0))(T'(0,v) - \mathbb{E}(T'(0,v)|\mathcal{F}_0)) | \mathcal{F}_0)) \\
&= 2\mathbb{E}Var(T(0,v)|\mathcal{F}_0) = 2Var(T(0,v))
\end{aligned}$$

进一步地，有：

$$\begin{aligned}
\mathbb{E}(T(0,v) - T'(0,v))^2 &= \int_\Omega (T(0,v) - T'(0,v))^2 dP \ge \int_{\{T(0,v) - T'(0,v) > c_2 - c_1\}} (T(0,v) - T'(0,v))^2 dP \\
&\ge \int_{\{T(0,v) - T'(0,v) > c_2 - c_1\}} (c_2 - c_1)^2 dP = (c_2 - c_1)^2 \mathbb{P}(T(0,v) - T'(0,v) > c_2 - c_1) \\
&\ge (c_2 - c_1)^2 \mathbb{P}(\max_{e \in E_0} t_e < c_1) \mathbb{P}(\min_{e \in E_0} t_e' > c_2) > 2\delta > 0
\end{aligned}$$

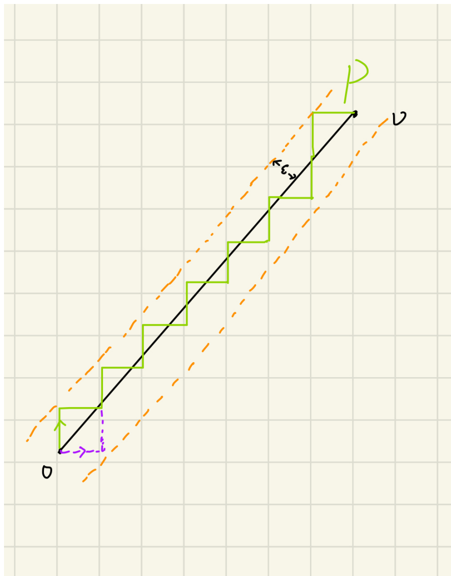
所以由此推出 $Var(T(0,v)) > \delta$，又由定理 1.1(A3):

$$\inf_{v \in \mathbb{Z}^d, |v| > C} \frac{Var(T(0,v))}{|v|^{2\chi'}} > 0$$

直接得出 $\chi_b \ge 0$

3.3 $\xi_b \ge 0$

对于足够小的 $\epsilon > 0$，在距离连接 $0, v$ 的直线 ϵ 范围内，至多有一条格子路径，如下图所示：



上图中的绿色路径 P 为最优路径，如果按照紫色路径走，其他一样，在边权重分布连续独立同分布情况下，则这两种路径为最优路径的概率相同。有如下关系：

$$\mathbb{P}(P = G(0,v)) < \delta < 1$$

这里的 δ 与 v 无关。
由此，进行如下计算：

$$\begin{aligned} \mathbb{E}D(0,v) &= \int_{\Omega} D(0,v)dP \ge \int_{P \neq G(0,v)} D(0,v)dP \\ &= \int_{P \neq G(0,v)} \epsilon dP \ge \epsilon \mathbb{P}(P \neq G(0,v)) > \epsilon(1-\delta) \end{aligned} \tag{3}$$

所以可推出 $\xi_b \ge 0$

3.4 $\chi_a \le 1/2$

首先注意到对于一个具有 $|v|_1$ 个边的路径 P，有：

$$\mathbb{E}T(0,v) \le \mathbb{E}t(P) = \mathbb{E}(edge_1 + edge_2 + \cdots + edge_{|v|_1}) = C_0|v|_1 \tag{4}$$

对于 $v \in \mathbb{Z}^d, |v|_1 \ge 2$，任意 $0 \le t \le |v|_1$，有:

$$\mathbb{P}(|T(0,v) - \mathbb{E}T(0,v)| \ge t\sqrt{\frac{|v|_1}{log|v|_1}}) \le C_1 e^{-C_2 t} \tag{5}$$

然后固定一个具有 $|v|_1$ 个边的路径 P，存在 $|v|_1 > C_3$，对于任意 $t > |v|_1$，

$$\begin{aligned}
&\mathbb{P}(|T(0,v) - \mathbb{E}T(0,v)| \ge t\sqrt{\frac{|v|_1}{log|v|_1}}) \\
&\le \mathbb{P}(T(0,v) \ge \mathbb{E}T(0,v) + t\sqrt{\frac{|v|_1}{log|v|_1}}) \\
&\le \mathbb{P}(T(0,v) \ge C_0'|v|_1 + t\sqrt{\frac{|v|_1}{log|v|_1}}) \\
&\le \mathbb{P}(t(P) \ge C_0'|v|_1 + t\sqrt{\frac{|v|_1}{log|v|_1}}) \\
&= \int_{C_0'|v|_1 + t\sqrt{\frac{|v|_1}{log|v|_1}}} dF_{t(P)}(s) \le \int_{C_0'|v|_1 + t\sqrt{\frac{|v|_1}{log|v|_1}}} \frac{e^s}{e^{C_0'|v|_1 + t\sqrt{\frac{|v|_1}{log|v|_1}}}} dF_{t(P)}(s) \\
&\le \mathbb{E}(e^{t(P)}) e^{-C_0'|v|_1 - t\sqrt{\frac{|v|_1}{log|v|_1}}} = (\mathbb{E}e^{edge})^{|v|_1} e^{-C_0'|v|_1 - t\sqrt{\frac{|v|_1}{log|v|_1}}} \\
&\le e^{C_4|v|_1 - C_4' t\sqrt{\frac{|v|_1}{log|v|_1}}}
\end{aligned} \tag{6}$$

由此可以得出，存在 C_5, C_6, C_7，使得对于任意 $|v|_1 > C_5$，有:

$$\begin{aligned}
&\mathbb{E}exp(C_6 \frac{|T(0,v) - \mathbb{E}T(0,v)|}{\sqrt{|v|_1/log|v|_1}}) = \int_0^\infty \mathbb{P}(exp(C_6 \frac{|T(0,v) - \mathbb{E}T(0,v)|}{\sqrt{|v|_1/log|v|_1}}) > t)dt \\
&= 1 + \int_1^\infty \mathbb{P}(exp(C_6 \frac{|T(0,v) - \mathbb{E}T(0,v)|}{\sqrt{|v|_1/log|v|_1}}) > t)dt \\
&\le 1 + \int_1^{e^{|v|_1}} C_1 e^{-C_2 logt} dt + e^{C_4|v|_1} \int_{e^{|v|_1}}^\infty t^{-C_4'\sqrt{|v|_1/log|v|_1}} dt \\
&\le C_7
\end{aligned} \tag{7}$$

所以有:

$$\sup_{v \in \mathbb{Z}^d, v \ne 0} \mathbb{E}exp(C_6 \frac{|T(0,v) - \mathbb{E}T(0,v)|}{\sqrt{|v|_1/log|v|_1}}) < \infty$$

由此即得 $\chi_a \le 1/2$

3.5 $\xi_a \le 1$

由控制收敛定理，$e^{-\theta t_e} \le 1$，对于任意边 e，

$$\lim_{\theta\to\infty} \mathbb{E}(e^{-\theta t_e}) = 0$$

所以有：

$$\begin{aligned}
\mathbb{P}(t(P) \le cm) = \mathbb{P}(e^{-t(P)/c} \ge e^{-m})& \\
= \int_{e^{-m}}^{\infty} dF_{e^{-t(P)/c}}(s) \le \int_{e^{-m}}^{\infty} e^{-t(P)/c} e^m dF_{e^{-t(P)/c}}(s)& \\
\le (e\mathbb{E}(e^{-t_e/c}))^m&
\end{aligned}$$

所以对于任意 $\delta > 0$，存在足够小的常数 c，使得

$$\mathbb{P}(t(P) \le cm) \le \delta^m \tag{8}$$

由于至多有 $(2d)^m$ 条有 m 条边的路径（从 0 开始，每走一步都有 $2d$ 种选择），取 c 足够小，于是有：

$$\mathbb{P}(t(P) \le cm\text{，对于某个有 m 个边的路径 P 成立}) \le (2d)^m\delta^m \le 2^{-m-1}$$

因此可以推出

$$\begin{aligned}
&\mathbb{P}(t(P) \le cm\text{对于某个有} \ge m\text{边的路径 P 成立}) \\
&\le \mathbb{P}(t(P) \le c \times \text{P 的边数，对于有} \ge m\text{的路径 P 成立}) \\
&\le 2^{-m-1} + 2^{-m-2} + \cdots + 2^{-m-k} + \cdots \le 2^{-m}
\end{aligned} \tag{9}$$

存在常数 B，若 $D(0,v) \ge t|v|$，则 $G(0,v)$ 至少有 $Bt|v|$ 条边，因此有：

$$\mathbb{P}(D(0,v) \ge t|v|) \le \mathbb{P}(T(0,v) \ge Bt|v|/c) + 2^{-Bt|v|} \tag{10}$$

然后放缩 $\mathbb{P}(T(0,v) \ge Bt|v|/c)$，与之前一样，固定一个具有 $|v|_1$ 个边的路径 P，

$$\begin{aligned}
&\mathbb{P}(T(0,v) \ge Bt|v|/c) \le \mathbb{P}(t(P) \ge Bt|v|/c) \\
&= \int_{Bt|v|/c}^{\infty} dF_{t(P)}(s) \le \int_{Bt|v|/c}^{\infty} \frac{e^s}{e^{Bt|v|/c}} dF_{t(P)}(s) \\
&\le \mathbb{E}(e^{t(P)}) e^{-Bt|v|/c} = (\mathbb{E}e^{t_e})^{|v|_1} e^{-Bt|v|/c} \\
&\le e^{C|v| - Bt|v|/c}
\end{aligned}$$

因此有 $\mathbb{P}(D(0,v)\geq t|v|)\leq e^{C|v|-Bt|v|/c}+2^{-Bt|v|}=e^{C|v|-Bt|v|/c}+e^{-B't|v|}$

$$\begin{aligned}
&\mathbb{E}(exp(\alpha\frac{D(0,v)}{|v|}))=\int_0^\infty \mathbb{P}(exp(\alpha\frac{D(0,v)}{|v|})>t)dt\\
&=1+\int_1^{e^\alpha}\mathbb{P}(exp(\alpha\frac{D(0,v)}{|v|})>t)dt+\int_{e^\alpha}^\infty \mathbb{P}(exp(\alpha\frac{D(0,v)}{|v|})>t)dt\\
&\leq e^\alpha+\int_{e^\alpha}^\infty e^{C|v|-Blog(t)|v|/\alpha c}+e^{-B'log(t)|v|/\alpha}dt\\
&=e^\alpha+\int_{e^\alpha}^\infty e^{C|v|}t^{-B|v|/\alpha c}+t^{-B'|v|/\alpha}dt<C'
\end{aligned}$$

从上述结果可以直接看出 $\xi_a\leq 1$

4 亚历山大次可加逼近理论

对于 $x\in\mathbb{Z}^d$，定义 $h(x):=\mathbb{E}T(0,x)$，显然 $h(x)$ 具有次可加性，再定义

$$g(x):=\lim_{n\to\infty}\frac{h(nx)}{n}$$

（以上极限存在，是因为 $h(\cdot)$ 是次可加函数，根据 fekete 引理得出）
对于 $x\in\mathbb{Q}^d$，可对于 $nx\in\mathbb{Z}^d$ 进行逼近，然后进行连续延拓到 $\mathbb{R}^d$ 上，由次可加性：

$$g(x)\leq h(x)\text{，对于任意}x\in\mathbb{Z}^d\text{成立}\tag{11}$$

设 $x=a_1e_1+\cdots+a_de_d$，设 $|x|_\infty=\max_i|a_i|=a_j$，所以有：

$$g(x)\geq g(a_de_d)=|a_d|g(e_1)=|x|_\infty g(e_1)$$

同时

$$g(x)\leq|a_1|g(e_1)+\cdots+|a_d|g(e_d)=|x|_1g(e_1)$$

所以有

$$|x|_\infty\leq g(x)/g(e_1)\leq|x|_1\tag{12}$$

定理 4.1 与定理 1.1 设定一致，对于任意 $\chi'>\chi_a$，存在 $C>0$ 使得对于任意 $x\in\mathbb{Z}^d,|x|>1$ ，有

$$g(x)\leq h(x)\leq g(x)+C|x|^{\chi'}log|x|$$

固定 $\chi'>\chi_a,0<\chi'<1$，令 $B_0:=\{x:g(x)\leq 1\}$，给定 $x\in\mathbb{R}^d$，令 H_x 代表垂直于 $g(x)B_0$ 边界的超平面，令 H_x^0 代表平行于 H_x 且过原点的平面。

在 $\mathbb{R}^d$ 存在唯一线性泛函 g_x，满足：

$$g_x(y) = 0 \text{ 对于任意} y \in H_x^0, g_x(x) = g(x)$$

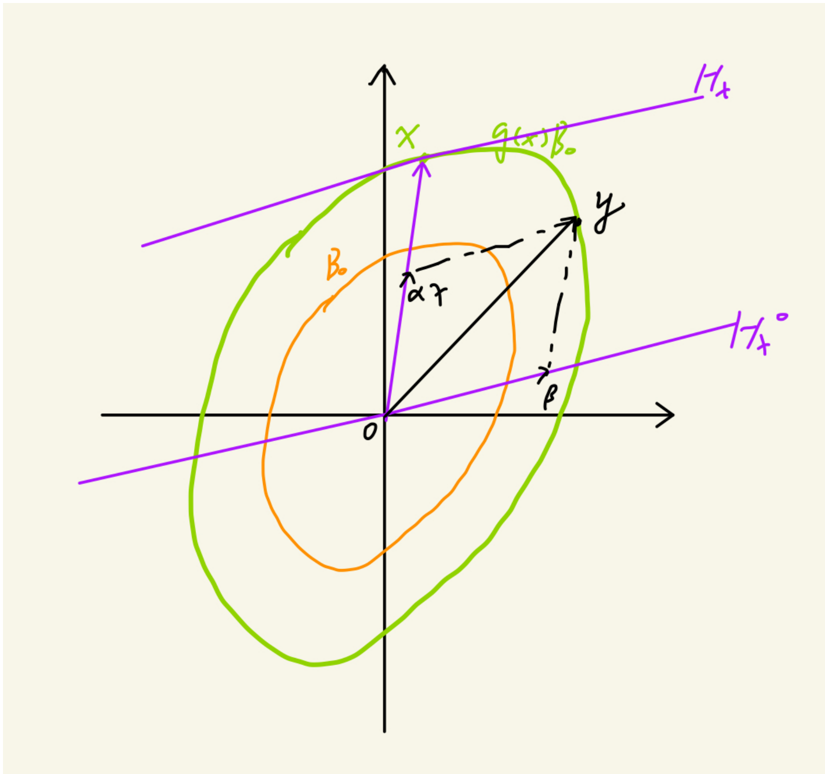
令

$$\begin{aligned} &Q_x(C, K) \\ &\quad := \{y \in \mathbb{Z}^d : |y| \le K|x|, g_x(y) \le g(x), h(y) \le g_x(y) + C|x|^{\chi'} log|x|\} \end{aligned} \tag{13}$$

引理 4.2(Alexander [2], Theorem 1.8) 考虑定理 4.1 的设定。假定对于 $M > 1$，$C > 0$，$K > 0$，$a > 1$，以下成立：对于每个 $x \in \mathbb{Q}^d, |x| \ge M$，存在一个整数 $n \ge 1$，存在一个从 0 到 nx 的格子路径 γ，并且在 γ 有一列序列 $0 = v_0, v_1, \cdots, v_m = nx$ 使得 $m \le an$ 且对任意的 $1 \le i \le m$，有 $v_i - v_{i-1} \in Q_x(C, K)$。则定理 4.1 结论成立。在证明引理 4.2 条件之前，需要给出一些定义和初步结果。
定义

$$s_x(y) := h(y) - g_x(y), y \in \mathbb{Z}^d$$

以下说明 $s_x(y) \ge 0$
由于 $g(x)$ 是一个范数，$g((x+y)/2) \le (g(x)+g(y))/2$，所以 B_0 是一个“凸球”。如图所示



只需证对于 $y \in \partial(g(x)B_0)$，结论成立

$$|g_x(y)| = |g_x(\alpha x + \beta)| = |g_x(\alpha x)| = g(\alpha x) \le g(y)$$

由次可加性，$g(y) \le h(y)$，证毕。
再由 g 的次可加性以及 g_x 的线性性质。

$$s_x(y+z) \le s_x(y) + s_x(z) \tag{14}$$

令 $C_1 := 320d^2/\alpha$，其中 α 在定理 1.1 中，定义：

$$\begin{aligned} Q_x &:= Q_x(C_1, 2d+1), \\ G_x &:= \{y \in \mathbb{Z}^d : g_x(y) > g(x)\} \\ \Delta_x &:= \{y \in Q_x : y\text{邻近}\mathbb{Z}^d \backslash Q_x, y\text{不邻近于}G_x\} \\ D_x &:= \{y \in Q_x : y\text{邻近}G_x\} \end{aligned} \tag{15}$$

引理 4.3 假设定理 1.1 条件成立。则存在常数 C_2 使得若 $|x| \ge C_2$，以下成立

$$(i)\text{若}y \in Q_x\text{，则}g(y) \le 2g(x) \text{ 且}|y| \le 2d|x|$$
$$(ii)\text{若}y \in \Delta_x\text{，则}s_x(y) \ge C_1|x|^{\chi'}(log|x|)/2$$
$$(iii)\text{若}y \in D_x\text{，则}g_x(y) \ge 5g(x)/6$$

证明：
(i) 用反证法，假设 $g(y) > 2g(x)$ 且 $g_x(y) \le g(x)$，有：

$$2g(x) < g(y) \le h(y) = g_x(y) + s_x(y) \le g(x) + s_x(y)$$

又有 $|x|_\infty \le g(x)/g(e_1) \le |x|_1$，所以：

$$s_x(y) > g(x) > C_1|x|^{\chi'}log|x| \text{ 假设}|x| \ge C_2$$

这说明 $y \notin Q_x$，矛盾，所以 (i) 的第一个结论成立。对于第二个结论：

$|y|_\infty g(e_1) \le g(y) \le 2g(x) \le 2|x|_1 g(e_1)$
$|y|_\infty \ge \sqrt{\frac{d|y|_\infty^2}{d}} \ge \frac{|y|}{\sqrt{d}}$
$|x|_1 = \sqrt{(|a_1| + \cdots + |a_d|)^2} = \sqrt{\sum_{i=1}^{d}|a_i|^2 + 2\sum_{i<j}|a_i||a_j|} \le \sqrt{\sum_{i=1}^{d}|a_i|^2 + \sum_{i<j}(|a_i|^2 + |a_j|^2)} =$
$\sqrt{d\sum_{i=1}^{d}|a_i|^2} = \sqrt{d}|x|$

所以第二个结论亦成立。
(ii) 由于 $y \in \Delta_x$，存在 $1 \le i \le d$，使得 $z = y \pm e_i$，$z \in \mathbb{Z}^d \cap Q_x^c \cap G_x^c$，由 (i) 知，$|y| \le 2d|x|$，所以有 $|z| \le (2d+1)|x|$，若有 $z \notin Q_x$，那么必有 $s_x(z) > C_1|x|^{\chi'}log|x|$。
所以，若 $|x| \ge C_2$

$$\begin{aligned} s_x(y) &\ge s_x(z) - s_x(\pm e_i) \\ &\ge C_1|x|^{\chi'}log|x| - (h(\pm e_i) + g_x(\pm e_i)) \\ &\ge C_1|x|^{\chi'}log|x| - h(\pm e_i) - g(\pm e_i) \end{aligned}$$

当 $|x| \ge C_2$ 时，$s_x(y) \ge C_1|x|^{\chi'}log|x|/2$ 成立。
(iii) 存在 $1 \le i \le d$，使得 $z = y \pm e_i$，$z \in \mathbb{Z}^d \cap G_x$

$$\begin{aligned}g_x(y) &= g_x(z) - g_x(\pm e_i) \geq g_x(z) - g(\pm e_i)\\ &\geq g(x) - g(\pm e_i) = 5g(x)/6 + (g(x)/6 - g(\pm e_i))\\ &\geq 5g(x)/6 + (g(e_1)|x|_\infty/6 - g(\pm e_i)) \geq 5g(x)/6\end{aligned}$$

证毕。

将引理 4.2. 中的 $m+1$ 个点称为“标记点”。如果对 m 不加限制，很容易在路径 γ 递推出一列标记点。令 $v_0 = 0$，若已推出 v_i，令 v'_{i+1} 是 v_i 后第一个满足 $v'_{i+1} - v_i \notin Q_x$ 的点。若 v'_{i+1} 存在，则令 v_{i+1} 是 v'_{i+1} 前面的最后一个点；否则，令 $v_{i+1} = nx$，若 $|x|$ 足够大，很容易验证所有原点邻近点都属于 Q_x，因此推出 $v_{i+1} \neq v_i$（若 $v_{i+1} = v_i$，则 $v'_{i+1} - v_i \notin Q_x$，而它又为原点最近邻）。因此，该递推必然在有限步之内结束。将这些标记点的集合成为路径 γ 的 Q_x- 结构。

将 Q_x- 结构简记作 (v_i)，将 Q_x 中邻近 Q_x^c 的点分为以下两类：

$$\begin{aligned}S((v_i)) &:= \{i : 0 \leq i \ < m-1, v_{i+1} - v_i \in \Delta_x\}\\ L((v_i)) &:= \{i : 0 \leq i \ < m-1, v_{i+1} - v_i \in D_x\}\end{aligned}$$

推论 4.4. 假设定理 1.1. 条件成立。存在一个常数 C_3，使得若 $|x| \geq C_3$，则对于足够大的 n 存在从 0 到 nx 的格子路径，这条路径有含有 $\leq 2n+1$ 个顶点的 Q_x- 结构。

证明:

令 $Y_i := \mathbb{E}T(v_i, v_{i+1}) - T(v_i, v_{i+1})$，由定理 1.1. 以及引理 4.2.(i) 存在 $C_4 := \alpha/(2d)^{\chi'} \geq \alpha/2d$ 和 C_5 使得:

$$\mathbb{E}exp(C_4|Y_i|/|x|^{\chi'}) \leq \mathbb{E}exp(\alpha|Y|_i/|v_i - v_{i+1}|^{\chi'}) \leq C_5 \tag{16}$$

令 $Y'_0, Y'_1, \cdots, Y'_{m-1}$ 为独立的随机变量且 Y'_i 与 Y_i 有相同的分布。令 $T(0, w; (v_j))$ 为从 0 至 w 的含有 Q_x- 结构路径的最小通过时间。由 [17,equation(4.13)] 或者 [1, Theorem 2.3]，对于任意 $t \geq 0$，

$$\mathbb{P}(\sum_{i=0}^{m-1} Y'_i \geq t) \geq \mathbb{P}(\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, v_m; (v_j)) \geq t)$$

由 (16)

$$\mathbb{P}(\sum_{i=0}^{m-1} Y_i' \geq t) = \int_t^{\infty} dF_{\sum_{i=0}^{m-1} Y_i'}(s) \leq \int_t^{\infty} exp(C_4 \sum_{i=0}^{m-1} |Y_i|/|x|^{\chi'})/exp(C_4 t/|x|^{\chi'}) dF_{\sum_{i=0}^{m-1} Y_i'}(s)$$
$$\leq \mathbb{E}(exp(C_4 \sum_{i=0}^{m-1} |Y_i|/|x|^{\chi'}))exp(-C_4 t/|x|^{\chi'}) \leq exp(-C_4 t/|x|^{\chi'}) C_5^m$$

令 $C_6 := 20d^2/\alpha, t = C_6 m|x|^{\chi'}\log|x|$, 因此存在常数 C_7, 对于任意 $|x| \geq C_7$:

$$\mathbb{P}(\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, v_m; (v_j)) \geq C_6 m|x|^{\chi'}\log|x|) \leq C_5^m e^{-C_6 C_4 m log|x|} \leq (C_5 e^{-10d log|x|})^m$$

至多有 $(C_8|x|^d)^m$ 个有 $m+1$ 个顶点的 Q_x- 结构，存在 C_9, C_{10} 有

$$\mathbb{P}(\text{对于有 m+1 个顶点的}Q_x-\text{结构成立}$$
$$\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, v_m; (v_j)) \geq C_6 m|x|^{\chi'}\log|x|) \leq e^{-C_{10} m log|x|}$$

因此存在 C_{11}，对于 $|x| \geq C_{11}$

$$\mathbb{P}(\text{对于有 m+1 个顶点的且}m \geq 1\text{的}Q_x-\text{结构成立}$$
$$\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, v_m; (v_j)) \geq C_6 m|x|^{\chi'}\log|x|) \leq 2e^{-C_{10} log|x|} \tag{17}$$

现在令 $\omega := \{t_e : e \text{ 是}\mathbb{Z}^d \text{ 中的一个边}\}$ 为一个固定的通过时间集合，由于 $v_{i+1} - v_i \in Q_x$，

$$mg(x) \geq \sum_{i=0}^{m-1} g_x(v_{i+1} - v_i) = g_x(nx) = ng(x)$$

因此得出

$$n \leq m \tag{18}$$

然后证随着 $n \to \infty$，有 $\mathbb{P}(T(0, nx) \leq ng(x) + n) \to 1$ 令 $X_n = \frac{T(0,nx)}{n}$，则 $g(x) = \lim_{n\to\infty} \mathbb{E}X_n$ 由 (1)

$$VarX_n = \frac{VarT(0, nx)}{n^2} \leq \frac{C}{nlogn}$$

因此

$$\mathbb{P}(|X_n - \lim_{n\to\infty} \mathbb{E}X_n| > \epsilon) \leq \mathbb{P}(|X_n - \mathbb{E}X_n| > \epsilon/2) + \mathbb{P}(|\mathbb{E}X_n - \lim_{n\to\infty} \mathbb{E}X_n| > \epsilon/2)$$

当 n 充分大时 $\mathbb{P}(|\mathbb{E}X_n - \lim_{n\to\infty}\mathbb{E}X_n| > \epsilon/2) = 0$，并且由切比雪夫不等式，有

$$\mathbb{P}(|X_n - \lim_{n\to\infty}\mathbb{E}X_n| > \epsilon) \le \frac{4}{\epsilon^2}VarX_n \le \frac{4C}{\epsilon^2 nlogn}$$

取 $\epsilon = 1$，得出 $\lim_{n\to\infty}\mathbb{P}(|T(0,nx) - ng(x)| > n) = 0$，因此 $\mathbb{P}(T(0,nx) \le ng(x)+n) \to 1$。

所以由 (17) 以及上式知：存在一个 ω 和从 0 至 nx 路径的 Q_x- 结构 (v_j) 使得

$$T(0,nx;(v_j)) = T(0,nx) \le n + ng(x) \tag{19}$$

并且

$$\sum_{i=0}^{m-1}\mathbb{E}T(v_i,v_{i+1}) - T(0,nx;(v_j)) < C_6 m|x|^{\chi'}\log|x| \tag{20}$$

因此存在 C_{12}，若 $|x| \ge C_{12}$

$$\sum_{i=0}^{m-1}\mathbb{E}T(v_i,v_{i+1}) < ng(x) + n + C_6 m|x|^{\chi'}log|x| \le ng(x) + 2C_6 m|x|^{\chi'}log|x| \tag{21}$$

又有

$$\begin{aligned}\sum_{i=0}^{m-1}\mathbb{E}T(v_i,v_{i+1}) &= \sum_{i=0}^{m-1}(g_x(v_{i+1}-v_i) + s_x(v_{i+1}-v_i)) \\ &\ge g_x(nx) + C_1|S((v_i))||x|^{\chi'}(log|x|/2)\end{aligned}$$

由 (21)(22)，得出

$$|S((v_i))| \le 4C_6 m/C_1 \le m/4 \tag{22}$$

同时求 $|L((v_i))|$

$$\begin{aligned}\sum_{i=0}^{m-1}\mathbb{E}T(v_i,v_{i+1}) &= \sum_{i=0}^{m-1}(g_x(v_{i+1}-v_i) + s_x(v_{i+1}-v_i)) \\ &\ge 5|L((v_i))|g(x)/6\end{aligned}$$

存在 C_{13}，对于 $|x| \ge C_{13}$

$$|L((v_i))| \le 6n/5 + \frac{12C_6 m|x|^{\chi'}log|x|}{5g(x)} \le 6n/5 + \frac{12C_6 m|x|^{\chi'}log|x|}{5g(e_1)|x|/\sqrt{d}} \le m/8 + 6n/5$$

所以

$$m = |L((v_i))| + |S((v_i))| + 1 \leq 6n/5 + 3m/8 + 1$$

对于足够大的 n

$$m \leq 48n/25 + 1 \leq 2n$$

推论 4.4. 证毕

定理 4.1. 证明总结 引理 4.2. 和推论 4.4 证明了 x 足够大的情况。对于所有 $|x| > 1$，只需要提高 C 的值。

5 曲率边界

命题 5.1. $g(\cdot)$ 与之前定义一致并且假定边权重分布是连续的。那么存在 $x_0 \in \mathbb{R}^d, |x_0| = 1$，常数 $C \geq 0$ 和一个过原点且垂直于 x_0 的超平面 H_0，使得对任意 $z \in H_0$，有

$$|g(x_0 + z) - g(x_0)| \leq C|z|^2$$

同时，还存在 $x_1 \in \mathbb{R}^d, |x_1| = 1$ 和一个过原点垂直于 x_1 的超平面 H_1，使得对于任意 $z \in H_1$，有

$$g(x_1 + z) \geq \sqrt{1 + |z|^2} g(x_1)$$

证明:

令 $B(0,r)$ 表示半径为 r 的欧式球，令 $B_g(0,r) := \{x : g(x) \leq r\}$。令 r 为满足 $B_g(0,r) \supseteq B(0,1)$ 最小数，x_0 为二者的交点，H_0 为过原点且与在 x_0 处 $\partial B_g(0,r)$ 相切的平面平行的超平面，$x_0 + H_0$ 也与 $B(0,1)$ 在 x_0 处相切，所以 x_0 与 H_0 垂直。令 $y := (x_0 + z)/|x_0 + z|$，则 y 位于 $\partial B(0,1)$ 上因此也在 $B_g(0,r)$ 内，因此:

$$g(x_0) = r \geq g(y) = \frac{1}{|x_0 + z|} g(x_0 + z) = \frac{1}{\sqrt{1 + |z|^2}} g(x_0 + z)$$

所以

$$g(x_0 + z) - g(x_0) = (\sqrt{1 + |z|^2} - 1) g(x_0) = \frac{|z|^2}{1 + \sqrt{1 + |z|^2}} g(x_0) \leq C|z|^2$$

又 $x_0 + z \notin B_g(0,r)$，所以 $g(x_0 + z) \geq g(x_0)$，证明上述第一个论断。图示见图 5.1.1 对于第二个论断，证明类似。令 r 为满足 $B_g(0,r) \subseteq B(0,1)$ 的

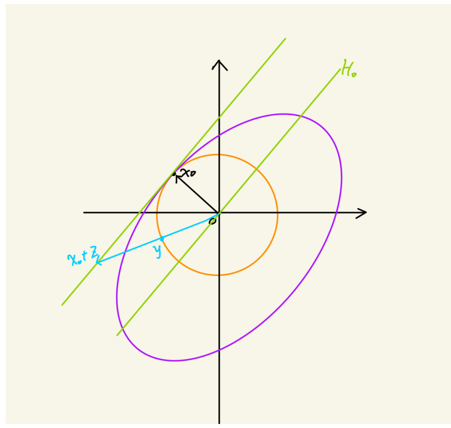


图 5.1.1

最大数，x_1 为二者边界的交。令 H_1 为过原点且平行于 $\partial B(0,1)$ 在 x_1 处切平面的超平面。令 $y := (x_0+z)/|x_0+z|$，y 位于 $\partial B(0,1)$ 上。所以有：

$$g(x_1) = r \leq g(y) = \frac{1}{|x_1+z|} g(x_1+z) = \frac{1}{\sqrt{1+|z|^2}} g(x_1+z)$$

第二部分论断证明图示见 5.1.2 证毕。

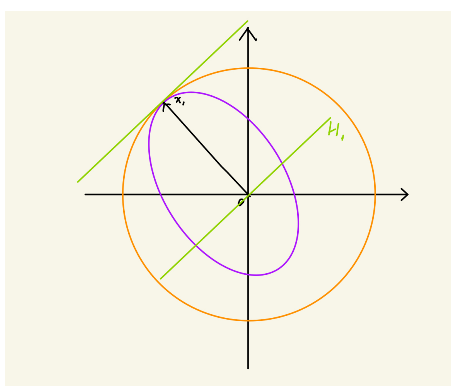


图 5.1.2

6 证明 $\chi_a \geq 2\xi_b - 1$

用反证法，若 $2\xi_b - 1 > \chi_a$，令 ξ' 满足：

$$\frac{1+\chi_a}{2} < \xi' < \xi_b$$

所以 $\xi' < 1$，x_1, H_1 与命题 5.1 中相同。n 是一个正整数，C 代表不依赖于 n 的常数并且可能不同。同时，"n 足够大" 这一条件在以下证明中将被省略。

令 y 为 $\mathbb{Z}^d$ 中离 nx_1 最近的点，所以

$$|y - nx_1| \leq \sqrt{d} \tag{23}$$

令 L 为过 0 与 nx_1 的直线，L' 为连接这两个点的线段。令 $V := \{v \in \mathbb{Z}^d :$ v 与 L' 距离位于 $[n^{\xi'}, 2n^{\xi'}]$ 范围内$\}$

首先证明结论：存在常数 C，是的对任意 $v \in V$，

$$g(v) + g(nx_1 - v) \geq g(nx_1) + Cn^{2\xi'-1} \tag{24}$$

令 w 为 v 在 L 上的垂直投影，要证明上述结论，考虑以下三种情况： 首先考虑 $w \in L'$，注意到 $w/|w| = x_1$，令 $v' := v/|w|$ 且 $z := v' - x_1 = (v - w)/|w|$，如下图所示 注意 $z \in H_1$，由命题 5.1，

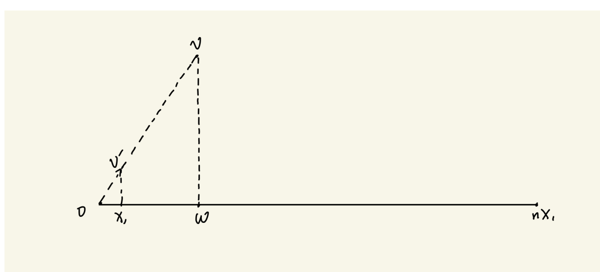


图 6.1

$$g(v') = g(x_1 + z) \geq \sqrt{1 + |z|^2} g(x_1)$$

所以有

$$g(v) \geq |w| \sqrt{1 + |z|^2} g(x_1) \tag{25}$$

然后 $w' := nx_1 - w$，注意到 $w'/|w'| = x_1$，令 $v'' := (nx_1 - v)/|w'|$，并且 $z' := v'' - x_1 = (w - v)/|w'|$，显然 $z' \in H_1$，所以有：

$$g(v'') = g(x_1 + z') \geq \sqrt{1 + |z'|^2}g(x_1)$$

因此有

$$g(nx_1 - v) \geq |w'|\sqrt{1 + |z'|^2}g(x_1) \tag{26}$$

由于 $v \in V$，因此 $|v - w| \geq n^{\xi'}$，同时 $|w| + |w'| = n$，由于

$$|z| = \frac{|v - w|}{|w|}; |z'| = \frac{|w - v|}{|w'|}$$

因此显然

$$\min\{|z|, |z'|\} \geq n^{\xi' - 1}$$

结合上面的结果，注意到 $\xi' < 1$，有

$$\begin{aligned} g(v) + g(nx_1 - v) &\geq (|w| + |w'|)\sqrt{1 + n^{2\xi' - 2}}g(x_1) \\ &= \sqrt{1 + n^{2\xi' - 2}}g(nx_1) \\ &= g(nx_1) + g(nx_1)(\sqrt{1 + n^{2\xi' - 2}} - 1) \\ &= g(nx_1) + g(nx_1)\frac{n^{2\xi' - 2}}{\sqrt{1 + n^{2\xi' - 2}} + 1} \\ &\geq g(nx_1) + \frac{g(e_1)|nx_1|}{\sqrt{d}}\frac{n^{2\xi' - 2}}{\sqrt{1 + n^{2\xi' - 2}} + 1} \\ &\geq g(nx_1) + Cn^{2\xi' - 1} \end{aligned}$$

再考虑 $w \in L \backslash L'$，位于靠近 nx_1 的一侧，如下图 6.2 所示 与之前一致

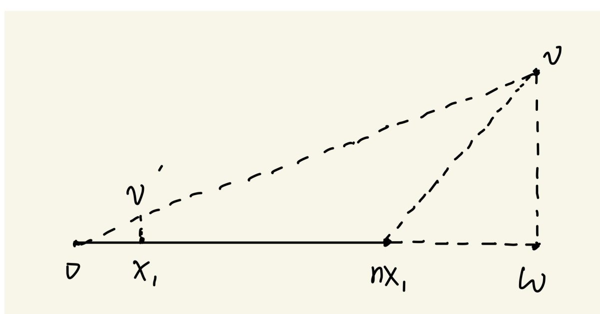


图 6.2

$$g(v') = g(x_1 + z) \geq \sqrt{1+|z|^2}g(x_1)$$

因此

$$g(v) \geq |w|\sqrt{1+|z|^2}g(x_1)$$

在这种情况下

$$|v - nx_1|^2 = (|w| - n)^2 + |v - 2|^2 = (|w| - n)^2 + |w|^2|z|^2$$

同时显然有 $n \leq |w| \leq 3n$，以下两条件必有一成立：$1.|w|^2|z|^2 > n^{2\xi'}/2$（这意味着 $|z|^2 \geq Cn^{2\xi'-2}$），$2.|w| \geq n + n^{\xi'}/\sqrt{2}$。在 1、2 下均有

$$g(v) \geq g(nx_1) + Cn^{2\xi'-1}$$

若 $w \in L\backslash L'$ 并靠近 0，同上

$$g(nx_1 - v) \geq g(nx_1) + Cn^{2\xi'-1}$$

至此，我们证明了 (24)。
结合定理 4.1，$2\xi' - 1 > \chi_a$ 以及 (24)，如果 n 足够大，则对于任意 $v \in V$，

$$h(v) + h(y - v) \geq h(y) + Cn^{2\xi'-1} \tag{27}$$

证明：

$$\begin{aligned}
h(v) + h(y - v) &\geq g(v) + g(y - v) \geq g(v) + g(nx_1 - v) - g(y - nx_1) \\
&\geq g(nx_1) + Cn^{2\xi'-1} - g(y - nx_1) \\
&\geq g(y) + Cn^{2\xi'-1} - 2g(y - nx_1) \\
&\geq g(y) + Cn^{2\xi'-1} - C_1 \\
&\geq h(y) - C_2|y|^{2\xi'-\beta_1}log|y| + Cn^{2\xi'-1} - C_1(\beta_1\text{可以取} >1 \text{ 的数}) \\
&\geq h(y) - C_2|y|^{2\xi'-\beta_2} + Cn^{2\xi'-1} - C_1(1 < \beta_2 < \beta_1) \\
&\geq h(y) + Cn^{2\xi'-1}(n \text{ 足够大})
\end{aligned}$$

证毕。
现在选择 χ_1, χ_2，使得 $\chi_a < \chi_1 < \chi_2 < 2\xi' - 1$，由定理 1.1 的 (A1)，存在常数 C，使得对于足够大的 n，

$$\mathbb{P}(T(0, y) > h(y) + n^{\chi_2}) \leq e^{-Cn^{\chi_2 - \chi_1}}$$

证明:

$$\begin{aligned}
\mathbb{P}(T(0,y) > h(y) + n^{\chi_2}) &\leq \mathbb{P}(|T(0,y) - h(y)| > n^{\chi_2}) \\
&= \mathbb{P}(exp(\alpha\frac{|T(0,y)-h(y)|}{|y|^{\chi_1}}) > exp(\alpha n^{\chi_2}/|y|^{\chi_1})) \\
&= \mathbb{P}(exp(\alpha\frac{|T(0,y)-h(y)|}{|y|^{\chi_1}}) > e^{Cn^{\chi_2-\chi_1}}) \\
&\leq \int_{\mathbf{1}\{exp(\alpha\frac{|T(0,y)-h(y)|}{|y|^{\chi_1}}) > e^{Cn^{\chi_2-\chi_1}}\}} exp(\alpha\frac{|T(0,y)-h(y)|}{|y|^{\chi_1}})/e^{Cn^{\chi_2-\chi_1}} dP \\
&\leq e^{-C_1n^{\chi_2-\chi_1}}\mathbb{E}exp(\alpha\frac{|T(0,y)-h(y)|}{|y|^{\chi_1}}) \leq e^{-C_1n^{\chi_2-\chi_1}}
\end{aligned}$$

证毕。

对于任意 $v \in V$，$|v|$ 和 $|y-v|$ 都以 Cn 为上界。因此同上有

$$\mathbb{P}(T(0,v) < h(v) - n^{\chi_2}) \leq e^{-Cn^{\chi_2-\chi_1}}$$
$$\mathbb{P}(T(v,y) < h(y-v) - n^{\chi_2}) \leq e^{-Cn^{\chi_2-\chi_1}}$$

结合 (27)，对于任意 $v \in V$，有

$$\begin{aligned}
\mathbb{P}(T(0,y) = T(0,v) + T(v,y)) &\leq \mathbb{P}(\{T(0,y) > h(y) + n^{\chi_2}\} \cup \{T(0,v) + T(v,y) < h(y) + C_1n^{2\xi'-1}\}) \\
&\leq \mathbb{P}(T(0,y) > h(y) + n^{\chi_2}) \\
&+ \mathbb{P}(T(0,v) + T(v,y) < h(y) + C_1n^{2\xi'-1})(n \text{ 足够大}) \\
&\leq \mathbb{P}(T(0,y) > h(y) + n^{\chi_2}) \\
&+ \mathbb{P}(T(0,v) + T(v,y) < h(y) + Cn^{2\xi'-1} - 2n^{\chi_2}) \\
&\leq \mathbb{P}(T(0,y) > h(y) + n^{\chi_2}) \\
&+ \mathbb{P}(T(0,v) + T(v,y) < h(v) + h(y-v) - 2n^{\chi_2}) \\
&\leq \mathbb{P}(T(0,y) > h(y) + n^{\chi_2}) \\
&+ \mathbb{P}(T(0,v) < h(v) - n^{\chi_2}) + \mathbb{P}(T(v,y) < h(y-v) - n^{\chi_2}) \\
&\leq e^{-Cn^{\chi_2-\chi_1}} + e^{-Cn^{\chi_2-\chi_1}} + e^{-Cn^{\chi_2-\chi_1}} = e^{-Cn^{\chi_2-\chi_1}}
\end{aligned}$$

由于易看出 V 中元素数目为 Cn^{κ}，那么

$$\mathbb{P}(\text{对于某个}v \in V\text{，}T(0,y) = T(0,v) + T(v,y)\text{成立}) \leq e^{-Cn^{\chi_2-\chi_1}}$$

注意到若最优路径通过 V，那么必存在 v，使得 $T(0,y) = T(0,v) + T(v,y)$ 成立。且若 $D(0,y) > n^{\xi'}$，那么最优路径一定通过 V，所以有

$$\mathbb{P}(D(0,y) > n^{\xi'}) \leq e^{-Cn^{\chi_2-\chi_1}}$$

由上述结论、定理 1.1 的 (A2) 以及 $H\ddot{o}lder$ 不等式，有

$$\begin{aligned}\mathbb{E}D(0,y) &= \int_{\{D(0,y)>n^{\xi'}\}} D(0,y)dP + \int_{\{D(0,y)\le n^{\xi'}\}} D(0,y)dP \\ &\le n^{\xi'} + \mathbb{E}(D(0,y)\mathbf{1}_{\{D(0,y)>n^{\xi'}\}}) \\ &\le n^{\xi'} + \sqrt{\mathbb{E}(D^2(0,y)\mathbb{P}(D(0,y)>n^{\xi'}))} \\ &\le n^{\xi'} + C_1 n^{C_1} e^{-C_2 n^{\chi_2-\chi_1}}\end{aligned}$$

只需要取 $\xi'<\xi''<\xi_b$，即可推出与定理 1.1 的 (A4) 矛盾，所以 $\chi_a>2\xi_b-1$。

7 当 $0<\chi<1/2$ 时，证明 $\chi\le 2\xi-1$

本文剩余部分，假定 $\chi_a=\chi_b,\xi_a=\xi_b$，并且用 χ,ξ 来表示。
依然使用反证法。假设 $0<\chi<1/2,\chi>2\xi-1$。固定 $\chi_1<\chi<\chi_2$。选择 ξ' 使得

$$\xi<\xi'<\frac{1+\chi}{2}$$

定义

$$\begin{aligned}\beta' &:= \frac{1}{2}+\frac{\xi'}{1+\chi} \\ \beta &:= 1-\frac{\chi}{2}+\frac{\chi}{2}\beta' \\ \epsilon &:= (1-\beta)(1-\frac{\chi}{2})\end{aligned}$$

以下为关于 β',β,ϵ 之间的一些不等关系：
由于 $0<\frac{\xi'}{1+\chi}<\frac{1}{2}$，所以

$$\frac{1}{2}<\beta'<1 \tag{28}$$

由于 $\chi<1,\xi'<(1+\chi)/2<1$，

$$\beta'>\frac{1}{2}+\frac{\xi'}{2}>\xi' \tag{29}$$

由于 β 是 1 和 β' 的一个线性组合并且 $0<\chi<1/2$，因此

$$\beta'<\beta<1 \tag{30}$$

由于 $0<\chi<1/2$ 且 $0<\beta<1$，

$$0<\epsilon<1-\beta \tag{31}$$

由于 β' 是 1 与 $2\xi'/(1+\chi)\in(0,1)$ 的算术平均，所以 $\beta'>2\xi'/(1+\chi)$，因此

$$\begin{gathered}2\xi'-\beta'<2\xi'-\frac{2\xi'}{1+\chi}\\ \frac{2\xi'}{1+\chi}\chi<\beta'\chi\end{gathered}\tag{32}$$

由 (30)，这意味着

$$2\xi'-\beta<2\xi'-\beta'<\beta'\chi<\beta\chi\tag{33}$$

然后，由 (28)，

$$1-\beta+\beta'\chi=\frac{\chi}{2}(1+\beta')<\chi\tag{34}$$

最后，由 (28)，

$$\beta\chi+1-\beta-\epsilon=\beta\chi+(1-\beta)\frac{\chi}{2}<\chi\tag{35}$$

令 q 为一足够大的正整数。在本章节证明中，省略“q 足够大”。常数 C 为任意不依赖于 q 的常数，但有可能依赖于其他常数。

令 r 为一个介于 $\frac{1}{2}q^{(1-\beta-\epsilon)/\epsilon}$ 与 $2q^{(1-\beta-\epsilon)/\epsilon}$ 之间的整数。由 (31) $1-\beta-\epsilon>0$。令 $k=rq$。令 a 为一个介于 $q^{\beta/\epsilon}$ 与 $2q^{\beta/\epsilon}$ 之间的实数。令 $n=ak$，注意到 $n=arq$。所以有 $\frac{1}{2}q^{1/\epsilon}\le n\le 4q^{1/\epsilon}$。由此我们可以看出有只依赖于 β,ϵ 的常数 C_1,C_2，使得

$$C_1n^{\epsilon}\le q\le C_2n^{\epsilon}\tag{36}$$

$$C_1n^{1-\beta}\le k\le C_2n^{1-\beta}\tag{37}$$

$$C_1n^{\beta}\le a\le C_2n^{\beta}\tag{38}$$

$$C_1n^{1-\beta-\epsilon}\le r\le C_2n^{1-\beta-\epsilon}\tag{39}$$

令 $b:=n^{\beta'}$，由 (30)，在 q 很大时，b 相对 a 可以忽略。同时注意 r,k,q 是正整数，但 a,n,b 不一定是。

令 x_0 和 H_0 与命题 5.1 一致，对于 $0\le i\le k$，定义

$$\begin{aligned}U_i'&:=H_0+iax_0\\ V_i'&:=H_0+(ia+a-b)x_0\end{aligned}$$

令 U_i 为 $\mathbb{Z}^d$ 中距离 U_i' 距离小于等于 $\sqrt{d}$ 的点集。令 V_i 为 $\mathbb{Z}^d$ 中与 V_i' 距离小于等于 $\sqrt{d}$ 的点集。

对于 $0\le i\le k$，令 y_i 为 $\mathbb{Z}^d$ 中距离 iax_0 最小的点，令 z_i 为 $\mathbb{Z}^d$ 中距离 $(ia+a-b)x_0$ 最小的点。显然 $y_i\in U_i, z_i\in V_i$。以下图 7.1 为示意图 (此处假设 $U_i=U_i', V_i=V_i'$)

令 U_i^o 为 U_i 中距离 y_i 小于等于 $n^{\xi'}$ 的点构成的点集，同样定义 V_i^o。

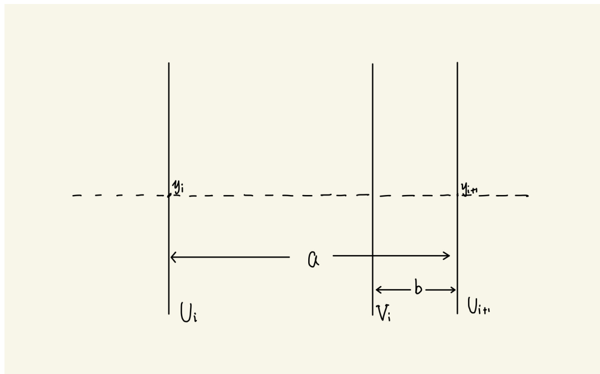


图 7.1 U_i, V_i, y_i, z_i 位置示意图

对于任意 $A,B\subseteq\mathbb{Z}^d$，令 $T(A,B)$ 为由 A 至 B 的最小通过时间，令 $G(A,B)$ 为由 A 至 B 的最优路径，所以 $T(A,B)$ 是 $G(A,B)$ 中的边权重之和。

固定两个整数 $0\le l<m\le k$ 使得 $m-l\ge 3$。考虑 $G:=G(y_l,y_m)$。由于 $x_0\notin H_0$，所以显然 G 一定会通过每个 $U_i,V_i,l\le i\le m-1$。对于每个 $l\le i<m$, 令 u_i' 为 U_i 中第一个被 G 通过的点, 令 v_i' 为 V_i 中第一个被 G 通过的点, 令 $u_m':=y_m$, 由此, 按顺序得到一列 $u_l',v_l',u_{l+1}',v_{l+1}',\cdots,v_{m-1}',u_m'$。图 7.2 为示意图。令 T_i' 为 G 中 u_i',v_i' 之间的通过时间。

令 $E:=\{$对于任意 i，$u_i'\in U_i^o, v_i'\in V_i^o$成立$\}$，如果 E 成立，显然有

$$T_i'\ge T(U_i^o,V_i^o)$$

同时，G 中 v_i',u_{i+1}' 之间通过时间必须大于或等于 $T(v_i',u_{i+1}')$，所以

$$\begin{aligned} T(y_l,y_m) &\ge \sum_{i=l}^{m-1} T_i' + \sum_{i=l}^{m-1} T(v_i',u_{i+1}') \\ &\ge \sum_{i=l}^{m-1} T(U_i^o,V_i^o) + \sum_{i=l}^{m-1} T(v_i',u_{i+1}') \end{aligned} \tag{40}$$

然后，对于每个 $0\le i<k$，令 $G_i:=G(U_i^o,V_i^o)$，且 u_i,v_i 为 G_i 的端点。令 $G_i':=G(v_i,u_{i+1})$, 以下图 7.3 为示意图 将 $G(y_l,v_l),G_l',G_{l+1}G_{l+1}',\cdots,G_{m-1},G(v_{m-1},y_m)$

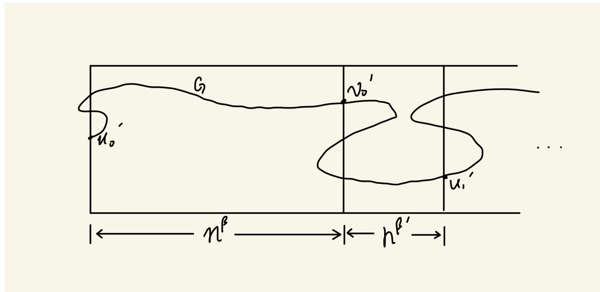


图 7.1 $u_0', v_0', \cdots$ 位置示意图

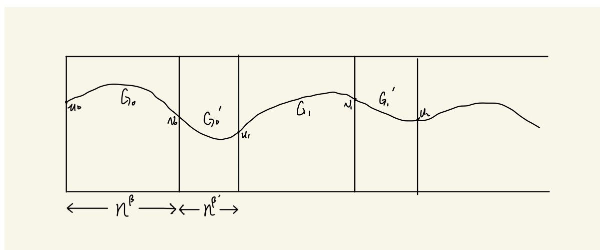


图 7.3 路径 G_i, G_i'

连接起来构成一条从 y_l 至 y_m 的路径，所以有

$$T(y_l, y_m) \le T(y_l, v_l) + \sum_{i=l+1}^{m-1} T(U_i^o, V_i^o) + \sum_{i=l}^{m-2} T(v_i, u_{i+1}) + T(v_{m-1}, y_m) \quad (41)$$

定义

$$\Delta_{l,m} := T(y_l, y_m) - \sum_{i=l}^{m-1} (T(U_i^o, V_i^o) + T(V_i^o, U_{i+1}^o))$$

结合 (40)，(41)，有

$$\begin{aligned}|\Delta_{l,m}| \le &\sum_{i=l}^{m-1} |T(V_i^o, U_{i+1}^o) - T(v_i', u_{i+1}')| + \sum_{i=l}^{m-2} |T(V_i^o, U_{i+1}^o) - T(v_i, u_{i+1})| \\ &+ |T(U_l^o, V_l^o) - T(y_l, v_l)| + |T(V_{m-1}^o, U_m^o) - T(v_{m-1}, y_m)|\end{aligned}$$

定义

$$M_i := \max_{v, v' \in V_i^o,\ u, u' \in U_{i+1}^o} |T(v, u) - T(v', u')|$$

$$N_i := \max_{u, u' \in U_i^o, v, v' \in V_i^o,} |T(v, u) - T(v', u')|$$

如果事件 E 发生，则有

$$|\Delta_{l,m}| \le 2\sum_{i=l}^{m-1} M_i + N_l \tag{42}$$

令 $||\cdot||_p$ 代表 L^p 范数，容易看出 $||\Delta_{l,m}||_4 \le n^C$，取任意 $\xi_1 \in (\xi, \xi')$，由定理 1.1 的 (A2)，有

$$\begin{aligned}
\mathbb{P}(E^c) \le \mathbb{P}(D(0,nx_0) \ge n^{\xi'}) &= \int_{\{D(0,nx_0)\ge n^{\xi'}\}} dP \\
&\le \int_{\{D(0,nx_0)\ge n^{\xi'}\}} exp(\alpha\frac{D(0,nx_0)}{n^{\xi_1}})/exp(\alpha n^{\xi'-\xi_1})dP \\
&\le e^{-\alpha n^{\xi'-\xi_1}}\mathbb{E}(exp(\alpha\frac{D(0,nx_0)}{n^{\xi_1}})) \le e^{-Cn^{\xi'-\xi_1}}
\end{aligned}$$

再由 (42)，存在 C_3, C_4，

$$\begin{aligned}
||\Delta_{l,m}||_2 &\le ||\Delta_{l,m}1_{E^c}||_2 + ||\Delta_{l,m}1_E||_2 \\
&\le ||\Delta_{l,m}||_4(\mathbb{P}(E^c))^{1/4} + ||\Delta_{l,m}1_E||_2 \\
&\le n^{C_3}e^{-C_4 n^{\xi'-\xi_1}} + 2\sum_{i=l}^{m-1}||M_i||_2 + ||N_l||_2
\end{aligned} \tag{43}$$

固定 $0 \le i \le k-1$ 和 $v \in V_i^o, u \in U_{i+1}^o$，令 x 为 V_i' 中离 v 最近的点，y 为 U_{i+1}' 中离 u 最近的点，所以存在 $z, z' \in H_0$，使得 $|z|, |z'| \le Cn^{\xi'}$ 并且 $x = (ia+a-b)x_0 + z$，$y = (ia+a)x_0 + z'$，所以有

$$\begin{aligned}
|g(y-x) - g(bx_0)| &\le |g(bx_0 + z' - z) - g(bx_0)| \\
&= b|g(x_0 + (z'-z)/b) - g(x_0)| \\
&\le \frac{C|z'-z|^2}{b} \le Cn^{2\xi'-\beta'}
\end{aligned}$$

所以

$$\begin{aligned}
|g(u-v) - g(u'-v')| &\le |g(u-v) - g(x-y)| + |g(u'-v') - g(x'-y')| \\
&+ |g(y-x) - g(bx_0)| + |g(y'-x') - g(bx_0)| \\
&\le Cn^{2\xi'-\beta'}
\end{aligned}$$

又注意到 $|y-x| \le C(n^{\beta'}+n^{\xi'}) \le Cn^{\beta'}$，再由定理 4.1

$$\begin{aligned}|\mathbb{E}T(v,u)-\mathbb{E}T(v',u')| &= |h(v-u)-g(v-u)-(h(v'-u')-g(v'-u'))+g(v-u)-g(v'-u')| \\ &\le |h(v-u)-g(v-u)|+|h(v'-u')-g(v'-u')|+|g(v-u)-g(v'-u')| \\ &\le Cn^{2\xi'-\beta'}+Cn^{\beta'\chi_2}logn\end{aligned}$$

由 (32)，有

$$|\mathbb{E}T(v,u)-\mathbb{E}T(v',u')| \le Cn^{\beta'\chi_2}logn \tag{44}$$

令

$$M := \max_{v\in V_i^o, u\in U_{i+1}^o} \frac{|T(v,u)-\mathbb{E}T(v,u)|}{|u-v|^{\chi_2}}$$

由定理 1.1 的 (A1)

$$\begin{aligned}\mathbb{E}(e^{\alpha M}) &\le \sum_{v\in V_i^o, u\in U_{i+1}^o} \mathbb{E}(exp(\alpha\frac{|T(v,u)-\mathbb{E}T(v,u)|}{|u-v|^{\chi_2}})) \\ &\le C|V_i^o||U_{i+1}^o| \le Cn^C\end{aligned}$$

所以容易得出 $\mathbb{P}(M>t) \le Cn^Ce^{-\alpha t}$，由此

$$\begin{aligned}\mathbb{E}(M^2) = \int_0^\infty 2t\mathbb{P}(M>t)dt &= \int_0^{\frac{C}{\alpha}logn+\frac{logC}{\alpha}} 2tdt \\ &+ \int_{\frac{C}{\alpha}logn+\frac{logC}{\alpha}}^\infty 2tCn^Ce^{-\alpha t}dt \le (Clogn)^2\end{aligned}$$

所以 $||M||_2 \le Clogn$ 令

$$M' := \max_{v\in V_i^o, u\in U_{i+1}^o} |T(v,u)-\mathbb{E}T(v,u)|$$

显然 $|u-v| \le C(n^{\beta'}+n^{\xi'}) \le Cn^{\beta'}$，所以

$$M' = \max_{v\in V_i^o, u\in U_{i+1}^o} \frac{|T(v,u)-\mathbb{E}T(v,u)|}{|u-v|^{\chi_2}}|u-v|^{\chi_2} \le MCn^{\beta'\chi_2} \tag{36}$$

所以有

$$||M'||_2 \le Cn^{\beta'\chi_2}logn$$

再结合 (44)，有

$$||M_i||_2 \le Cn^{\beta'\chi_2}logn$$

类似地，有

$$||N_i||_2 \le Cn^{\beta\chi_2}logn$$

结合 (43)，有

$$||\Delta_{l,m}||_2 \le Cn^{\beta\chi_2}logn + C(m-l)n^{\beta'\chi_2}logn \tag{45}$$

根据 $\Delta_{l,m}$ 的定义以及三角不等式

$$|T(y_0,y_k) - \sum_{j=0}^{r-1} T(y_{jq}, y_{(j+1)q})| \le |\Delta_{0,k}| + \sum_{j=0}^{r-1} |\Delta_{jq,(j+1)q}|$$

由 (45),(39),(37)，有

$$\begin{aligned} ||T(y_0,y_k) - \sum_{j=0}^{r-1} T(y_{jq}, y_{(j+1)q})||_2 &\le ||\Delta_{0,k}||_2 + \sum_{j=0}^{r-1} ||\Delta_{jq,(j+1)q}||_2 \\ &\le C(r+1)n^{\beta\chi_2}logn + Ckn^{\beta'\chi_2}logn \\ &\le Cn^{1-\beta-\epsilon+\beta\chi_2}logn + Cn^{1-\beta+\beta'\chi_2}logn \end{aligned} \tag{46}$$

对于两个随机变量 X,Y，由三角不等式及 $H\ddot{o}lder$ 不等式有，

$$\begin{aligned} |\sqrt{Var(X)} - \sqrt{Var(Y)}| &= |||X-\mathbb{E}X||_2 - ||Y-\mathbb{E}Y||_2| \\ &\le ||(X-\mathbb{E}X)-(Y-\mathbb{E}Y)||_2 \\ &\le ||X-Y||_2 + |\mathbb{E}X-\mathbb{E}Y| \le 2||X-Y||_2 \end{aligned} \tag{47}$$

由 (46) 有

$$\begin{aligned} &|(VarT(y_0,y_k))^{1/2} - (Var\sum_{j=0}^{r-1} T(y_{jq}, y_{(j+1)q}))^{1/2}| \\ &\qquad \le Cn^{1-\beta-\epsilon+\beta\chi_2}logn + Cn^{1-\beta+\beta'\chi_2}logn \end{aligned} \tag{48}$$

对于任意 $x,y\in\mathbb{Z}^d$，$T(x,y)$ 是关于边权重的一个增函数，所以根据 Harris-FKG 不等式，$Cov(T(x,y),T(x',y'))\ge 0$ 对于任意 $x,y,x',y'\in\mathbb{Z}^d$ 成立。所以根据定理 1.1 的 (A3)，以及 (38),(39) 和 (36)

$$\begin{aligned} Var\sum_{j=0}^{r-1} T(y_{jq}, y_{(j+1)q}) &\ge \sum_{j=0}^{r-1} VarT(y_{jq}, y_{(j+1)q}) \\ &\ge C\sum_{j=0}^{r-1} |y_{jq} - y_{(j+1)q}|^{2\chi_1} \\ &\ge Cr(aq)^{2\chi_1} \ge Cn^{(1-\beta-\epsilon)+(\beta+\epsilon)2\chi_1} \end{aligned} \tag{49}$$

由 (34),(35)，若 χ_1 和 χ_2 距离 χ 足够近，那么 χ_1 严格大于 $1-\beta-\epsilon+\beta\chi_2$ 和 $1-\beta+\beta'\chi_2$，由 (48)，(49)，以及 $1-\beta-\epsilon+(\beta+\epsilon)2\chi_1 > 2\chi_1$ 有

$$\begin{aligned}
VarT(y_0,y_k) &= ((VarT(y_0,y_k))^{1/2})^2 \\
&= ((VarT(y_0,y_k))^{1/2} - (Var\sum_{j=0}^{r-1}T(y_{jq},y_{(j+1)q}))^{1/2} + (Var\sum_{j=0}^{r-1}T(y_{jq},y_{(j+1)q}))^{1/2})^2 \\
&\ge ((Var\sum_{j=0}^{r-1}T(y_{jq},y_{(j+1)q}))^{1/2} - |(VarT(y_0,y_k))^{1/2} - (Var\sum_{j=0}^{r-1}T(y_{jq},y_{(j+1)q}))^{1/2}|)^2 \\
&\ge Cn^{1-\beta-\epsilon+(\beta+\epsilon)2\chi_1}
\end{aligned}$$

由于 $\chi < 1/2$，所以当 χ_1 与 χ 足够近时，有

$$(1-\beta-\epsilon)+(\beta+\epsilon)2\chi_1 > 2\chi$$

由于 $|y_0 - y_k| \le Cak \le Cn$，令 $q \to \infty$，因此 $n \to \infty$，取 $2\chi < 2\chi' < 1-\beta-\epsilon+(\beta+\epsilon)2\chi_1$，那么有

$$\begin{aligned}
\frac{VarT(y_0,y_k)}{|y_0-y_k|^{2\chi'}} &\ge \frac{VarT(y_0,y_k)}{Cn^{2\chi'}} = \frac{VarT(y_0,y_k)}{Cn^{1-\beta-\epsilon+(\beta+\epsilon)2\chi_1}}n^{1-\beta-\epsilon+(\beta+\epsilon)2\chi_1-2\chi'} \\
&\ge Cn^{1-\beta-\epsilon+(\beta+\epsilon)2\chi_1-2\chi'} \to \infty
\end{aligned}$$

所以显然对于任意 α，定理 1.1 的 (A1) 均不成立，矛盾，证毕。

8 当 $\chi = 1/2$ 时，证明 $\chi \le 2\xi - 1$

用反证法，假设 $\chi = 1/2$ 且 $\chi > 2\xi - 1$。$\chi_1, \chi_2, x_0, H_0, \xi', \beta, \beta', \epsilon, q, a, r, k, n, y_i, z_i$ 定义与上一部分一致。注意到上一部分仅在最后用到了 $\chi < 1/2$ 这个假设。所以只需对最后部分证明作改动。本证明，C 代表任意不依赖于 q 的常数。

对实数 $m \ge 1$，令 w_m 为 $\mathbb{Z}^d$ 中距离 mx_0 最近的点，并注意到 $y_i = w_{ia}$。令

$$f(m) := VarT(0, w_m)$$

显然存在 C_0，使得 $f(m) \le C_0 m$ 对于任意 m 成立。再由 (A3)，对于任意 m，有

$$f(m) \ge C_1 m^{2\chi_1} \tag{50}$$

由 w_m 的定义以及三角不等式，有

$$\begin{aligned}
|w_{(j+1)aq} - w_{jaq} - w_{aq}| &= |w_{(j+1)aq} - (j+1)aqx_0 - (w_{jaq} - jaqx_0) - (w_{aq} - aqx_0)| \\
&\le |w_{(j+1)aq} - (j+1)aqx_0| + |w_{jaq} - jaqx_0| + |w_{aq} - aqx_0| \\
&\le 3\sqrt{d} = C
\end{aligned}$$

然后，根据 (47) 的结论

$$\begin{aligned}
|VarX - VarY| &= |\sqrt{VarX} - \sqrt{VarY}|(\sqrt{VarX} + \sqrt{VarY}) \\
&\le 2||X-Y||_2(2\sqrt{VarX} + 2||X-Y||_2)
\end{aligned} \tag{51}$$

由于第一通过时间的次可加性

$$\begin{aligned}
T(0, w_{aq}) - T(w_{jaq}, w_{(j+1)aq}) &= T(0, w_{aq}) - T(0, w_{(j+1)aq} - w_{jaq} - w_{aq} + w_{aq}) \\
&\ge -T(0, w_{(j+1)aq} - w_{jaq} - w_{aq})
\end{aligned}$$

同样地，有 $T(0, w_{aq}) - T(w_{jaq}, w_{(j+1)aq}) \le T(0, w_{(j+1)aq} - w_{jaq} - w_{aq})$
因此有

$$||T(w_{jaq}, w_{(j+1)aq})||_2 \le ||T(0, w_{(j+1)aq} - w_{jaq} - w_{aq})||_2 \le C$$

再由 (51)，

$$\begin{aligned}
&f(aq) - Var(T(w_{jaq}, w_{(j+1)aq})) \\
&\quad \le 2||T(0, w_{jaq}) - T(w_{jaq}, w_{(j+1)aq})||_2(2\sqrt{f(aq)} + 2||T(0, w_{jaq}) - T(w_{jaq}, w_{(j+1)aq})||_2) \\
&\quad \le C\sqrt{f(aq)} + C
\end{aligned}$$

所以

$$\begin{aligned}
Var(T(w_{jaq}, w_{(j+1)aq})) &\ge f(aq) - C\sqrt{f(aq)} - C \ge f(n/r) - C\sqrt{C_0 n/r} - C \\
&\ge f(n/r) - C\sqrt{n/r}
\end{aligned}$$

由于 Harris-FKG 不等式

$$Var(\sum_{j=0}^{r-1} T(w_{jaq}, w_{(j+1)aq})) \ge rf(n/r) - C\sqrt{nr} \tag{52}$$

由 (34)，(35)，若 χ_2 距离 χ 足够近，那么 $1-\beta-\epsilon+\beta\chi_2$ 和 $1-\beta+\beta'\chi_2$ 均可严格小于 $1/2$。因此由 (46)，(51) 以及 $f(n)\le Cn$，

$$
\begin{aligned}
&|f(n)-Var(\sum_{j=0}^{r-1}T(w_{jaq},w_{(j+1)aq}))|\\
&\le 2||T(0,w_n)-\sum_{j=0}^{r-1}T(w_{jaq},w_{(j+1)aq})||_2(2\sqrt{f(n)}\\
&+2||T(0,w_n)-\sum_{j=0}^{r-1}T(w_{jaq},w_{(j+1)aq})||_2)\\
&\le C(n^{1-\beta-\epsilon+\beta\chi_2}logn+n^{1-\beta+\beta'\chi_2}logn)(C\sqrt{n}+C(n^{1-\beta-\epsilon+\beta\chi_2}logn+n^{1-\beta+\beta'\chi_2}logn))\\
&\le C\sqrt{n}(n^{1-\beta-\epsilon+\beta\chi_2}logn+n^{1-\beta+\beta'\chi_2}logn)
\end{aligned}
$$

以上与 (52) 相结合，有

$$
\begin{aligned}
f(n)&\ge Var(\sum_{j=0}^{r-1}T(w_{jaq},w_{(j+1)aq}))-|f(n)-Var(\sum_{j=0}^{r-1}T(w_{jaq},w_{(j+1)aq}))|\\
&\ge rf(n/r)-C\sqrt{nr}-C\sqrt{n}(n^{1-\beta-\epsilon+\beta\chi_2}logn+n^{1-\beta+\beta'\chi_2}logn)
\end{aligned}
$$

再由 (39) 和 (50)，有

$$rf(n/r)\ge Cn^{(1-\beta-\epsilon)+(\beta+\epsilon)2\chi_1}$$

不难验证，当 χ_1 与 χ_2 足够接近 χ，且 $q\to\infty$ 时，有

$$\liminf\frac{f(n)}{rf(n/r)}\ge 1$$

特别地，对任意 $\delta>0$，存在 $q(\delta)$ 使得若 $q\ge q(\delta)$，有

$$f(n)\ge(1-\delta)rf(n/r) \tag{53}$$

固定 $\delta=(1-\beta-\epsilon)/2$ 以及选择合适的 $q(\delta)$，注意到 $q(\delta)$ 可以取得任意大。令 $m_0:=aq=n/r,m_1=n$。由 (53) 有，

$$\frac{f(m_1)}{m_1}\ge(1-\delta)\frac{f(m_0)}{m_0}$$

注意到由 (36)，如果 $q(\delta)$ 足够大，那么

$$m_1^{\epsilon/(\beta+\epsilon)}>Cq^{1/(\beta+\epsilon)}>q(\delta)$$

现在递推给出一列递增序列 $m_2, m_3, \cdots$，
假设 m_{i-1} 已经按照如下要求定义

$$m_{i-1}^{\epsilon/(\beta+\epsilon)} > q(\delta) \tag{54}$$

令

$$q_i := [m_{i-1}^{\epsilon/(\beta+\epsilon)}] + 1$$

所以由 (54)，$q_i \geq q(\delta)$，令 $a_i := m_{i-1}/q_i$，所以如果 $q(\delta)$ 足够大,

$$a_i \geq \frac{2}{3} m_{i-1}^{\beta/(\beta+\epsilon)} \geq \frac{1}{2} q_i^{\beta/\epsilon}$$

且

$$a_i \leq m_{i-1}^{\beta/(\beta+\epsilon)} \leq q_i^{\beta/\epsilon}$$

令 r_i 为 $q_i^{(1-\beta-\epsilon)/\epsilon}$ 与 $2q_i^{(1-\beta-\epsilon)/\epsilon}$ 之间的整数，令 $k_i = r_i q_i$ 且 $n_i = a_i k_i = a_i r_i q_i = r_i m_{i-1}$。若我们按照文章第七部分方法得出 q_i, r_i, k_i, a_i, n_i，那么，我们有

$$f(n_i) \geq (1-\delta) r_i f(n_i/r_i) = (1-\delta) r_i f(m_{i-1})$$

定义 $m_i = n_i$，那么有

$$\frac{f(m_i)}{m_i} \geq (1-\delta) \frac{f(m_{i-1})}{m_{i-1}} \tag{55}$$

注意到 $m_i = r_i m_{i-1}$，所以 $\{m_i\}$ 是递增的。所以 (54) 对于 m_i 依然满足，所以推理可以继续下去。现在对任意 i

$$m_i = r_i m_{i-1} \geq q_i^{(1-\beta-\epsilon)/\epsilon} m_{i-1} \geq m_{i-1}^{1/(\beta+\epsilon)}$$

所以, 对任意 $i \geq 2$

$$m_i \geq m_1^{(\beta+\epsilon)^{-(i-1)}}$$

由 (1)

$$\frac{f(m_i)}{m_i} \leq \frac{C}{log m_i} \leq C_3 (\beta+\epsilon)^{i-1}$$

但是由 (55)，存在 $C_4 > 0$，使得

$$\frac{f(m_i)}{m_i} \geq C_4 (1-\delta)^{i-1}$$

由于 $1-\delta > \beta+\epsilon$，得到矛盾。

9 当 $\chi = 0$ 时，证明 $\chi \leq 2\xi - 1$

依然用反证法，假设 $\chi = 0$ 且 $2\xi - 1 < \chi$，显然 $\xi < 1/2$，选择 ξ_1, ξ', ξ'' 使得 $\xi < \xi_1 < \xi'' < \xi' < 1/2$。为证明这一点，首先给出引理

引理 9.1 假设边权重分布是连续的，令 L 表示支撑的下确界，那么存在 $M > L$，使得对任意 $x \in \mathbb{R}^d \backslash \{0\}$，有 $g(x) \geq M|x|_1$ 证明:

由于 g 是 $\mathbb{R}^d$ 上的一个范数，定义

$$M := \inf_{x \neq 0} \frac{g(x)}{|x|_1} > 0$$

存在 x，使下确界可以取到（可以考虑有界闭集单位球 E，$\frac{g(x)}{|x|_1}$ 是其上恒大于 0 的一个连续函数，所以存在下界，并可以取到下界）。取 $x \neq 0$ 使得 $g(x) = M|x|_1$。定义一列新的边权重 s_e 为 $s_e := s_e - L$。所以 s_e 是独立同分布且非负的随机变量。对于新的环境，定义 g^s, h^s, T^s。由于对于任意从 y 至 z 的路径 P，都至少有 $|z - y|_1$ 个边，所以有 $s(P) \leq t(P) - L|z - y|_1$，所以

$$T^s(y, z) \leq T(y, z) - L|z - y|_1$$

显然对任意 y，$h^s(y) \leq h(y) - L|y|_1$，考虑一列 y_n 使得 $y_n/n \to x$，我们可以看到

$$\begin{aligned} g^s(x) = \lim_{n \to \infty} \frac{h^s(y_n)}{n} &\leq \lim_{n \to \infty} \frac{h(y_n) - L|y_n|_1}{n} \\ &= g(x) - L|x|_1 = (M - L)|x|_1 \end{aligned}$$

由于 t_e 有连续分布，所以由 [17] 的一个结果，$g^s(x) > 0$ 表明 $M > L$。

证毕。 选择 $\beta, \epsilon', \epsilon$ 使得 $0 < \epsilon' < \epsilon < \beta < (\xi'' - \xi_1)/d$，$x_0, H_0$ 与命题 5.1 一致。n 是一个正整数，可取任意大。C 代表不依赖于 n 的任意常数。

选择 $z \in H_0$ 使得 $|z| \in [n^{\xi'}, 2n^{\xi'}]$，令 $v := nx_0/2 + z$。所以由命题 5.1 和 $\xi' < 1/2$

$$\begin{aligned} |g(v) - g(nx_0/2)| &= n/2|g(x_0 + 2z/n) - g(x_0)| \\ &\leq C|z|^2/n \leq Cn^{2\xi' - 1} \leq C \end{aligned} \tag{56}$$

类似有

$$|g(nx_0 - v) - g(nx_0/2)| \leq Cn^{2\xi' - 1} \leq C \tag{57}$$

令 w 为距离 v 最近的格子点, y 为距离 nx_0 最近的格子点。所以 $|w-v|, |y-nx_0|$ 均小于等于 $\sqrt{d}$ 由 (56)，(57) 以及三角不等式显然有

$$|g(y) - (g(w) + g(y-w))| \le C \tag{58}$$

以下图片给出了 y, w 以及其他一些点的位置关系。

由定理 4.1，$\chi = 0$，有 $|h(y) - g(y)|, |h(w) - g(w)|, |h(y-w) - g(y-w)| \le Cn^{\epsilon}$，再由定理 1.1 的 (A1)，结合之前的方法（此处不再详述），有 $\mathbb{P}(|T(0,w) - h(w) > n^{\epsilon}|), \mathbb{P}(|T(w,y) - h(y-w)| > n^{\epsilon}), \mathbb{P}(|T(0,y) - h(y)| > n^{\epsilon}) \le e^{-Cn^{\epsilon-\epsilon'}}$。再结合 (58)，以及三角不等式

$$\mathbb{P}(|T(0,y) - (T(0,w) + T(w,y))| > C_1 n^{\epsilon}) \le e^{-C_2 n^{\epsilon-\epsilon'}} \tag{59}$$

令 $T_o(0,y)$ 表示所有距离 0 至 y 直线不超过 $n^{\xi''}$ 的路径中的最小通过时间。由定理 1.1 的 (A2)，结合之前的技巧，有

$$\mathbb{P}(T_o(0,y) = T(0,y)) = \mathbb{P}(D(0,y) \le n^{\xi''}) \ge 1 - e^{-Cn^{\xi''-\xi_1}}$$

结合 (59)，令 E_1 为事件

$$E_1 := \{|T_o(0,y) - (T(0,w) + T(w,y))| \le C_1 n^{\epsilon}\} \tag{60}$$

所以

$$\begin{aligned}
\mathbb{P}(E_1^c) &= \mathbb{P}(|T_o(0,y) - (T(0,w) + T(w,y))| > C_1 n^{\epsilon}) \\
&\le \mathbb{P}(|T_o(0,y) - T(0,y)| + |T(0,y) - (T(0,w) + T(w,y))| > C_1 n^{\epsilon}) \\
&\le \mathbb{P}(|T_o(0,y) - T(0,y)| > Cn^{\epsilon}) + \mathbb{P}(|T(0,y) - (T(0,w) + T(w,y))| > Cn^{\epsilon}) \\
&\le \mathbb{P}(T_o(0,y) \ne T(0,y)) + \mathbb{P}(|T(0,y) - (T(0,w) + T(w,y))| > Cn^{\epsilon}) \\
&\le e^{-C_3 n^{\xi''-\xi_1}} + e^{-C_3 n^{\epsilon-\epsilon'}}
\end{aligned}$$

所以得出

$$\mathbb{P}(E_1) \ge 1 - e^{-C_3 n^{\xi''-\xi_1}} - e^{-C_3 n^{\epsilon-\epsilon'}} \tag{61}$$

令 V 为 l_1 范数下距离 w 不超过 n^{β} 的 $\mathbb{Z}^d$ 中的点构成的集合，令 ∂V 代表代表 V 在 $\mathbb{Z}^d$ 中的边界，即由至少有一个邻居在 V 外的所有 V 中的点构成的集合。令 w_1 为 $G(0,w)$ 中第一个位于 ∂V 中的点，令 w_2 为 $G(w,y)$ 中最后一个属于 ∂V 的点。令 G_1 为 $G(0,w)$ 中连接 w_1 和 w 的部分，令

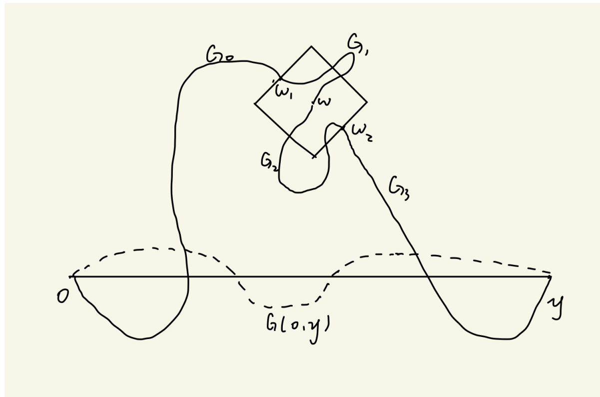


图 9.1 V, w, w_1, w_2 以及 G_0, G_1, G_2, G_3 位置示意图

G_0 为 $G(0, w)$ 中连接 0 和 w_1 的部分，令 G_3 为 $G(w, y)$ 中 w_2 到 y 的部分，G_2 为 $G(w, y)$ 中连接 w 与 w_2 的部分。并且注意到，G_0 和 G_3 完全位于 V 外。给出图示 9.1 令 L 和 M 和引理 9.1 定义一致，取 L', M' 使得 $L < L' < M' < M$。对任意 $u \in \partial V$，由引理 9.1，有 $g(u - w) \ge M|u - w|_1$，所以由定理 4.1

$$h(u - w) \ge M|u - w|_1 - C|u - w|^{\epsilon} \ge M|u - w|_1 - Cn^{\beta\epsilon}$$

显然边界上的点 u，$|u - w|_1 \ge Cn^{\beta}$，再由定理 1.1 的 (A1)，利用之前技巧（不再详述，切比雪夫不等式）有

$$\begin{aligned}
&\mathbb{P}(T(u, w) < M'|u - w|_1) \\
&\le \mathbb{P}(T(u, w) - h(u - w) < (M' - M)|u - w|_1 + Cn^{\beta\epsilon}) \\
&\le \mathbb{P}(|T(u, w) - h(u - w)| > Cn^{\beta}) \le e^{-n^{\beta - \epsilon'}/C}
\end{aligned}$$

由于 ∂V 至多有 n^C 个点，所以有

$$\mathbb{P}(\text{对于某些}u \in \partial V\text{，}T(u, w) < M'|u - w|_1) \le n^C e^{-n^{\beta - \epsilon'}/C}$$

特别地，如果 E_2, E_3 是事件

$$\begin{aligned}
E_2 &:= \{t(G_1) \ge M'|w - w_1|_1\} \\
E_3 &:= \{t(G_2) \ge M'|w - w_2|_1\}
\end{aligned}$$

所以

$$\begin{aligned}\mathbb{P}(E_2^c \cup E_3^c) &\le \mathbb{P}(E_2^c) + \mathbb{P}(E_3^c)\\ &\le \mathbb{P}(T(w,w_1) < M'|w-w_1|_1) + \mathbb{P}(T(w,w_2) < M'|w-w_2|_1)\\ &\le n^C(e^{-n^{\beta-\epsilon'}/C} + e^{-n^{\beta-\epsilon'}/C}) \le n^{C_4}e^{-n^{\beta-\epsilon'}/C_4}\end{aligned}$$

所以有

$$\mathbb{P}(E_2 \cap E_3) \ge 1 - n^{C_4}e^{-n^{\beta-\epsilon'}/C_4} \tag{62}$$

令 $E(V)$ 为 V 中点构成的边的集合，令 $(t'_e)_{e\in E(V)}$ 为 $i.i.d$ 的随机变量，与之前的边权重随机变量独立，但是有同样分布。对于 $e \notin E(V)$，令 $t'_e = t_e$。E_4 为事件

$$E_4 := \{t'_e \le L'，对于每个 e \in E(V) 成立\}$$

如果 E_4 发生，存在连接 w_1 和 w 的路径 P_1 以及连接 w 与 w_2 的路径 P_2 使得 $t'(P_1) \le L'|w-w_1|_1$ 且 $t'(P_2) \le L'|w-w_2|_1$，令 P 为连接 G_0, P_1, P_2, G_3 构成的路径，由于 $t'(G_0) = t(G_0)$ 且 $t'(G_3) = t(G_3)$。
所以在 E_4 下

$$t'(P) \le t(G_0) + t(G_3) + L'|w-w_1|_1 + L'|w-w_2|_1$$

另一方面，在 $E_2 \cap E_3$ 条件下

$$\begin{aligned}T(0,w) + T(w,y) &= t(G_0) + t(G_1) + t(G_2) + t(G_3)\\ &\ge t(G_0) + t(G_3) + M'|w-w_1|_1 + M'|w-w_2|_1\end{aligned}$$

如果 E_1, E_2, E_3, E_4 同时发生，

$$\begin{aligned}&T_o(0,y) - t'(P)\\ &\ge T_o(0,y) - (t(G_0) + t(G_3) + L'(|w-w_1|_1 + |w-w_2|_1))\\ &\ge T_o(0,y) - (T(0,w) + T(w,y)) + (M'-L')(|w-w_1|_1 + |w-w_2|_1)\\ &\ge C_5n^\beta - C_1n^\epsilon\end{aligned}$$

所以

$$T_o(0,y) \ge t'(P) + C_5n^\beta - C_1n^\epsilon$$

由于 $\beta < \xi'' < \xi'$，并且 $x_0 \notin H_0$，在距离 0 至 y 线段 $n^{\xi''}$ 范围内的边在 t' 和 t 两个不同环境中具有相同的边权重。由于 $\beta > \epsilon$，所以当 $E_1 \cap E_2 \cap E_3 \cap E_4$ 发生的条件下，由上式，可推出 $D'(0,y) \ge n^{\xi''}$。

由于 E_4 和 E_1, E_2, E_3 独立，并且 $L' > L$，$E(V)$ 中至多有 $n^{\beta d}$ 个元素，所以 $\mathbb{P}(E_4) \ge e^{-C_6 n^{\beta d}}$，与 (61),(62) 结合，有

$$\begin{aligned}\mathbb{P}(D'(0,y) \ge n^{\xi''}) &\ge \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4) \\ &= \mathbb{P}(E_1 \cap E_2 \cap E_3)\mathbb{P}(E_4) \\ &\ge (1 - e^{-C_3 n^{\xi'' - \xi_1}} - e^{-C_3 n^{\epsilon - \epsilon'}} - n^{C_4} e^{-n^{\beta - \epsilon'}/C_4}) e^{-C_6 n^{\beta d}} \\ &\ge e^{-C_7 n^{\beta d}}\end{aligned}$$

由于 $D'(0,y)$ 与 $D(0,y)$ 有相同的分布。再由定理 1.1 的 (A2)，$\mathbb{P}(D(0,y) \ge n^{\xi''}) \le e^{-C_8 n^{\xi'' - \xi_1}}$，并且 $\beta < \frac{\xi'' - \xi_1}{d}$。由此，推出矛盾，证明了 $\chi = 0$ 时，$\chi \le 2\xi - 1$。

注：本总结参考如下两个文献：

[1] Chatterjee S. The universal relation between scaling exponents in first-passage percolation. Ann. Math. (2), 177 no. 2, 663-697, 2013.

[2] Grimmett, Geoffrey. Percolation. Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 321. Springer-Verlag, Berlin, 1999.

THE UNIVERSAL RELATION BETWEEN SCALING EXPONENTS IN FIRST-PASSAGE PERCOLATION

SOURAV CHATTERJEE

Abstract. It has been conjectured in numerous physics papers that in ordinary first-passage percolation on integer lattices, the fluctuation exponent χ and the wandering exponent ξ are related through the universal relation $\chi = 2\xi - 1$, irrespective of the dimension. This is sometimes called the KPZ relation between the two exponents. This article gives a rigorous proof of this conjecture assuming that the exponents exist in a certain sense.

1. Introduction

Consider the space $\mathbb{R}^d$ with Euclidean norm $|\cdot|$, where $d \geq 2$. Consider $\mathbb{Z}^d$ as a subset of this space, and say that two points x and y in $\mathbb{Z}^d$ are nearest neighbors if $|x-y| = 1$. Let $E(\mathbb{Z}^d)$ be the set of nearest neighbor bonds in $\mathbb{Z}^d$. Let $t = (t_e)_{e\in E(\mathbb{Z}^d)}$ be a collection of i.i.d. non-negative random variables. In first-passage percolation, the variable t_e is usually called the 'passage time' through the edge e, alternately called the 'edge-weight' of e. We will sometimes refer to the collection t of edge-weights as the 'environment'. The total passage time, or total weight, of a path P in the environment t is simply the sum of the weights of the edges in P and will be denoted by $t(P)$ in this article. The first-passage time $T(x, y)$ from a point x to a point y is the minimum total passage time among all lattice paths from x to y. For all our purposes, it will suffice to consider self-avoiding paths; henceforth, 'lattice path' will refer to only self-avoiding paths.

Note that if the edge-weights are continuous random variables, then with probability one there is a unique 'geodesic' between any two points x and y. This is denoted by $G(x, y)$ in this paper. Let $D(x, y)$ be the maximum deviation (in Euclidean distance) of this path from the straight line segment joining x and y (see Figure 1).

Although invented by mathematicians [11], the first-passage percolation and related models have attracted considerable attention in the theoretical physics literature (see [21] for a survey). Among other things, the physicists are particularly interested in two 'scaling exponents', sometimes denoted by χ and ξ in the mathematical physics literature. The *fluctuation exponent*

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Key words and phrases. First-passage percolation, scaling exponents, KPZ.

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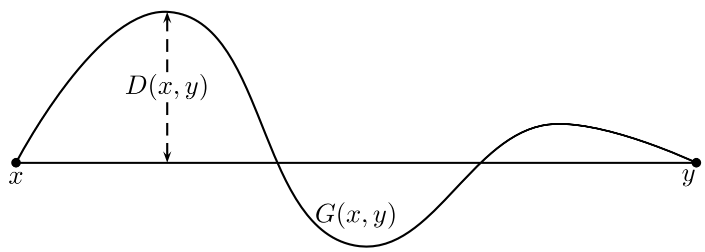

FIGURE 1. The geodesic $G(x,y)$ and the deviation $D(x,y)$.

χ is a number that quantifies the order of fluctuations of the first-passage time $T(x,y)$. Roughly speaking, for any x,y,

$$\text{the typical value of } T(x,y) - \mathbb{E}T(x,y) \text{ is of the order } |x-y|^{\chi}.$$

The *wandering exponent* ξ quantifies the magnitude of $D(x,y)$. Again, roughly speaking, for any x,y,

$$\text{the typical value of } D(x,y) \text{ is of the order } |x-y|^{\xi}.$$

There have been several attempts to give precise mathematical definitions for these exponents (see [23] for some examples) but I could not find a consensus in the literature. The main hurdle is that no one knows whether the exponents actually exist, and if they do, in what sense.

There are many conjectures related to χ and ξ. The main among these, to be found in numerous physics papers [14, 15, 16, 19, 20, 21, 24, 25, 30], including the famous paper of Kardar, Parisi and Zhang [15], is that although χ and ξ may depend on the dimension, they always satisfy the relation

$$\chi = 2\xi - 1.$$

A well-known conjecture from [15] is that when $d = 2$, $\chi = 1/3$ and $\xi = 2/3$. Yet another belief is that $\chi = 0$ if d is sufficiently large. Incidentally, due to its connection with [15], I've heard in private conversations the relation $\chi = 2\xi - 1$ being referred to as the 'KPZ relation' between χ and ξ.

There are a number of rigorous results for χ and ξ, mainly from the late eighties and early nineties. One of the first non-trivial results is due to Kesten [18, Theorem 1], who proved that $\chi \le 1/2$ in any dimension. The only improvement on Kesten's result till date is due to Benjamini, Kalai and Schramm [6], who proved that for first-passage percolation in $d \ge 2$ with binary edge-weights,

$$\sup_{v\in\mathbb{Z}^d,\ |v|>1} \frac{\mathrm{Var} T(0,v)}{|v|/\log|v|} < \infty. \tag{1}$$

Benaïm and Rossignol [5] extended this result to a large class of edge-weight distributions that they call 'nearly gamma' distributions. The definition of a nearly gamma distribution is as follows. A positive random variable X is said to have a nearly gamma distribution if it has a continuous probability density function h supported on an interval I (which may be unbounded), and its distribution function H satisfies, for all $y \in I$,

$$\Phi' \circ \Phi^{-1}(H(y)) \le A\sqrt{y}h(y),$$

for some constant A, where Φ is the distribution function of the standard normal distribution. Although the definition may seem a bit strange, Benaïm and Rossignol [5] proved that this class is actually quite large, including e.g. exponential, gamma, beta and uniform distributions on intervals.

The only non-trivial lower bound on the fluctuations of passage times is due to Newman and Piza [26] and Pemantle and Peres [27], who showed that in $d = 2$, $\mathrm{Var}T(0,v)$ must grow at least as fast as $\log|v|$. Better lower bounds can be proved if one can show that with high probability, the geodesics lie in 'thin cylinders' [7].

For the wandering exponent ξ, the main rigorous results are due to Licea, Newman and Piza [23] who showed that $\xi^{(2)} \ge 1/2$ in any dimension, and $\xi^{(3)} \ge 3/5$ when $d = 2$, where $\xi^{(2)}$ and $\xi^{(3)}$ are exponents defined in their paper which may be equal to ξ.

Besides the bounds on χ and ξ mentioned above, there are some rigorous results relating χ and ξ through inequalities. Wehr and Aizenman [29] proved the inequality $\chi \ge (1-(d-1)\xi)/2$ in a related model, and the version of this inequality for first-passage percolation was proved by Licea, Newman and Piza [23]. The closest that anyone came to proving $\chi = 2\xi - 1$ is a result of Newman and Piza [26], who proved that $\chi' \ge 2\xi - 1$, where χ' is a related exponent which may be equal to χ. This has also been observed by Howard [13] under different assumptions.

Incidentally, in the model of Brownian motion in a Poissonian potential, Wüthrich [31] proved the equivalent of the KPZ relation assuming that the exponents exist.

The following theorem establishes the relation $\chi = 2\xi - 1$ assuming that the exponents χ and ξ exist in a certain sense (to be defined in the statement of the theorem) and that the distribution of edge-weights is nearly gamma.

Theorem 1.1. *Consider the first-passage percolation model on $\mathbb{Z}^d$, $d \ge 2$, with i.i.d. edge-weights. Assume that the distribution of edge-weights is 'nearly gamma' in the sense of Benaïm and Rossignol* [5] *(which includes exponential, gamma, beta and uniform distributions, among others), and has a finite moment generating function in a neighborhood of zero. Let χ_a and ξ_a be the smallest real numbers such that for all $\chi' > \chi_a$ and $\xi' > \xi_a$, there exists $\alpha > 0$ such that*

$$\text{(A1)} \qquad \sup_{v \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}\exp\left(\alpha \frac{|T(0,v) - \mathbb{E}T(0,v)|}{|v|^{\chi'}}\right) < \infty,$$

$$\text{(A2)} \qquad \sup_{v \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}\exp\left(\alpha \frac{D(0,v)}{|v|^{\xi'}}\right) < \infty.$$

Let χ_b and ξ_b be the largest real numbers such that for all $\chi' < \chi_b$ and $\xi' < \xi_b$, there exists $C > 0$ such that

$$\inf_{v\in\mathbb{Z}^d,\ |v|>C} \frac{\mathrm{Var}(T(0,v))}{|v|^{2\chi'}} > 0, \tag{A3}$$

$$\inf_{v\in\mathbb{Z}^d,\ |v|>C} \frac{\mathbb{E}D(0,v)}{|v|^{\xi'}} > 0. \tag{A4}$$

Then $0 \le \chi_b \le \chi_a \le 1/2$, $0 \le \xi_b \le \xi_a \le 1$ and $\chi_a \ge 2\xi_b - 1$. Moreover, if it so happens that $\chi_a = \chi_b$ and $\xi_a = \xi_b$, and these two numbers are denoted by χ and ξ, then they must necessarily satisfy the relation $\chi = 2\xi - 1$.

Note that if $\chi_a = \chi_b$ and $\xi_a = \xi_b$ and these two numbers are denoted by χ and ξ, then χ and ξ are characterized by the properties that for every $\chi' > \chi$ and $\xi' > \xi$, there are some positive α and C such that for all $v \ne 0$,

$$\mathbb{E}\exp\biggl(\alpha\frac{|T(0,v)-\mathbb{E}T(0,v)|}{|v|^{\chi'}}\biggr) < C \quad\text{and}\quad \mathbb{E}\exp\biggl(\alpha\frac{D(0,v)}{|v|^{\xi'}}\biggr) < C,$$

and for every $\chi' < \chi$ and $\xi' < \xi$ there are some positive B and C such that for all v with $|v| > C$,

$$\mathrm{Var}(T(0,v)) > B|v|^{2\chi'} \quad\text{and}\quad \mathbb{E}D(0,v) > B|v|^{\xi'}.$$

It seems reasonable to expect that if the two exponents χ and ξ indeed exist, then they should satisfy the above properties.

Incidentally, a few months after the first draft of this paper was put up on arXiv, Auffinger and Damron [4] were able to replace a crucial part of the proof of Theorem 1.1 with a simpler argument that allowed them to remove the assumption that the edge-weights are nearly-gamma.

Section 2 has a sketch of the proof of Theorem 1.1. The rest of the paper is devoted to the actual proof. Proving that $0 \le \chi_b \le \chi_a \le 1/2$ and $0 \le \xi_b \le \xi_a \le 1$ is a routine exercise; this is done in Section 3. Proving that $\chi_a \ge 2\xi_b - 1$ is also relatively easy and similar to the existing proofs of analogous inequalities, e.g. in [26, 13]. This is done in Section 6. The 'hard part' is proving the opposite inequality, that is, $\chi \le 2\xi - 1$ when $\chi = \chi_a = \chi_b$ and $\xi = \xi_a = \xi_b$. This is done in Sections 7, 8 and 9.

2. Proof sketch

I will try to give a sketch of the proof in this section. I have found it very hard to aptly summarize the main ideas in the proof without going into the details. This proof-sketch represents the end-result of my best efforts in this direction. If the interested reader finds the proof sketch too obscure, I would like to request him to return to this section after going through the complete proof, whereupon this high-level sketch may shed some illuminating insights.

Throughout this proof sketch, C will denote any positive constant that depends only on the edge-weight distribution and the dimension. Let $h(x) :=$

$\mathbb{E}(T(0,x))$. The function h is subadditive. Therefore the limit

$$g(x) := \lim_{n\to\infty} \frac{h(nx)}{n}$$

exists for all $x \in \mathbb{Z}^d$. The definition can be extended to all $x \in \mathbb{Q}^d$ by taking $n \to \infty$ through a subsequence, and can be further extended to all $x \in \mathbb{R}^d$ by uniform continuity. The function g is a norm on $\mathbb{R}^d$.

The function g is a norm, and hence much more well-behaved than h. If $|x|$ is large, $g(x)$ is supposed to be a good approximation of $h(x)$. A method developed by Ken Alexander [1, 2] uses the order of fluctuations of passage times to infer bounds on $|h(x)-g(x)|$. In the setting of Theorem 1.1, Alexander's method yields that for any $\varepsilon > 0$, there exists C such that for all $x \neq 0$,

$$g(x) \le h(x) \le g(x) + C|x|^{\chi_a+\varepsilon}. \tag{2}$$

This is formally recorded in Theorem 4.1. In the proof of the main result, the above approximation will allow us to replace the expected passage time $h(x)$ by the norm $g(x)$.

In Lemma 5.1, we prove that there is a unit vector x_0 and a hyperplane H_0 perpendicular to x_0 such that for some $C > 0$, for all $z \in H_0$,

$$|g(x_0+z) - g(x_0)| \le C|z|^2.$$

Similarly, there is a unit vector x_1 and a hyperplane H_1 perpendicular to x_1 such that for some $C > 0$, for all $z \in H_1$, $|z| \le 1$,

$$g(x_1+z) \ge g(x_1) + C|z|^2.$$

The interpretations of these two inequalities is as follows. In the direction x_0, the unit sphere of the norm g is 'at most as curved as an Euclidean sphere' and in the direction x_1, it is 'at least as curved as an Euclidean sphere'.

Now take a look at Figure 2. Think of m as a fraction of n. By the definition of the direction of curvature x_1 and Alexander's approximation (2), for any $\varepsilon > 0$,

$$\begin{aligned}
&\text{Expected passage time of the path } P \\
&\ge g(mx_1+z) + g(nx_1 - (mx_1+z)) + O(n^{\chi+\varepsilon}) \\
&= mg(x_1+z/m) + (n-m)g(x_1 + z/(n-m)) + O(n^{\chi+\varepsilon}) \\
&\ge ng(x_1) + C|z|^2/n + O(n^{\chi+\varepsilon}) \\
&\ge \mathbb{E}(T(0,nx_1)) + C|z|^2/n + O(n^{\chi+\varepsilon}).
\end{aligned}$$

Suppose $|z| = n^{\xi}$. Then $|z|^2/n = n^{2\xi-1}$. Fluctuations of $T(0,nx_1)$ are of order n^{χ}. Thus, if $2\xi - 1 > \chi$, then P cannot be a geodesic from 0 to nx_1. This sketch is formalized into a rigorous argument in Section 6 to prove that $\chi_a \ge 2\xi_b - 1$.

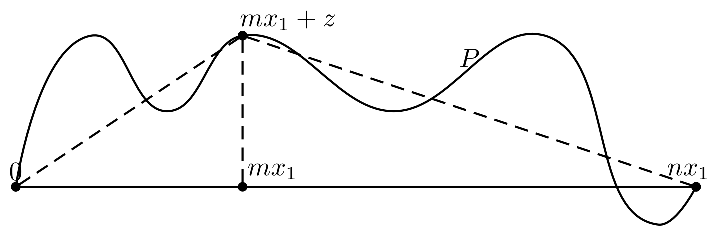

FIGURE 2. Proving $\chi \geq 2\xi - 1$

Next, let me sketch the proof of $\chi \leq 2\xi - 1$ when $\chi > 0$. The methods developed in [7] for first-passage percolation in thin cylinders have some bearing on this part of the proof. Recall the direction of curvature x_0. Let $a = n^\beta$, $\beta < 1$. Let $m = n/a = n^{1-\beta}$. Under the conditions $\chi > 2\xi - 1$ and $\chi > 0$, we will show that there is a $\beta < 1$ such that

$$(\star) \qquad T(0, nx_0) = \sum_{i=0}^{m-1} T(iax_0, (i+1)ax_0) + o(n^\chi).$$

This will lead to a contradiction, as follows. Let $f(n) := \mathrm{Var}T(0, nx_0)$. Then by Benaïm and Rossignol [5], $f(n) \leq Cn/\log n$. Under ($\star$), by the Harris-FKG inequality,

$$\begin{aligned} f(n) = \mathrm{Var}T(0, nx_0) &\geq m\mathrm{Var}T(0, ax_0) + o(n^{2\chi}) \\ &= n^{1-\beta} f(n^\beta) + o(n^{2\chi}). \end{aligned}$$

If β is chosen sufficiently small, the first term on the right will dominate the second. Consequently,

$$(\dagger) \qquad \liminf_{n\to\infty} \frac{f(n)}{n^{1-\beta} f(n^\beta)} \geq 1.$$

Choose $n_0 > 1$ and define $n_{i+1} = n_i^{1/\beta}$ for each i. Let $v(n) := f(n)/n$. Then $v(n_i) \leq C/\log n_i \leq C\beta^i$. But by ($\dagger$), $\liminf v(n_{i+1})/v(n_i) \geq 1$, and so for all i large enough, $v(n_{i+1}) \geq \beta^{1/2} v(n_i)$. In particular, there is a positive constant c such that for all i, $v(n_i) \geq c\beta^{i/2}$. Since $\beta < 1$, this gives a contradiction for i large, therefore proving that $\chi \leq 2\xi - 1$.

Let me now sketch a proof of ($\star$) under the conditions $\chi > 2\xi - 1$ and $\chi > 0$. Let $a = n^\beta$ and $b = n^{\beta'}$, where $\beta' < \beta < 1$. Consider a cylinder of width n^ξ around the line joining 0 and nx_0. Partition the cylinder into alternating big and small cylinders of widths a and b respectively. Call the boundary walls of these cylinders $U_0, V_0, U_1, V_1, \ldots, V_{m-1}, U_m$, where m is roughly $n^{1-\beta}$ (see Figure 3).

Let $G_i := G(U_i, V_i)$, that is, the path with minimum passage time between any vertex in U_i and any vertex in V_i. Let u_i and v_i be the endpoints of G_i.

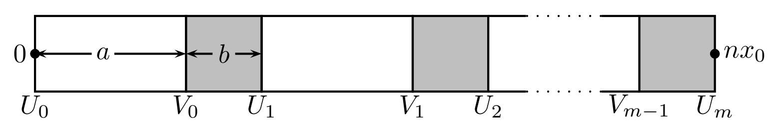

FIGURE 3. Cylinder of width n^ξ around the line joining 0 and nx_0

Let $G'_i := G(v_i, u_{i+1})$. The concatenation of the paths G'_0, G_1, G'_1, G_2, $\ldots$, G'_{m-1}, G_m is a path from U_0 to U_m. Therefore,

$$T(U_0, U_m) \le \sum_{i=1}^{m-1} T(U_i, V_i) + \sum_{i=0}^{m-1} T(v_i, u_{i+1}).$$

Next, let $G := G(U_0, U_m)$. Let u'_i be the first vertex in U_i visited by G and let v'_i be the first vertex in V_i visited by G. If G stays within the cylinder throughout, then $T(u'_i, v'_i) \ge T(U_i, V_i)$ and $T(v'_i, u'_{i+1}) \ge T(V_i, U_{i+1})$. Thus,

$$T(U_0, U_m) \ge \sum_{i=0}^{m-1} T(U_i, V_i) + \sum_{i=0}^{m-1} T(V_i, U_{i+1}).$$

Thus, if $G(U_0, U_m)$ stays in a cylinder of width n^ξ, then

$$\begin{aligned} 0 &\le T(U_0, U_m) - \sum_{i=0}^{m-1} (T(U_i, V_i) + T(V_i, U_{i+1})) \\ &\le \sum_{i=0}^{m-1} (T(v_i, u_{i+1}) - T(V_i, U_{i+1})). \end{aligned}$$

Therefore,

$$\left| T(U_0, U_m) - \sum_{i=0}^{m-1} (T(U_i, V_i) + T(V_i, U_{i+1})) \right| \le \sum_{i=0}^{m-1} M_i,$$

where $M_i := \max_{v,v' \in V_i,\ u,u' \in U_{i+1}} |T(v,u) - T(v',u')|$. Note that the errors M_i come only from the small blocks. By curvature estimate in direction x_0, for any $v, v' \in V_i$ and $u, u' \in U_{i+1}$,

$$|\mathbb{E}T(v,u) - \mathbb{E}T(v',u')| \le C(n^\xi)^2/n^{\beta'} = Cn^{2\xi-\beta'}.$$

Fluctuations of $T(v,u)$ are of order $n^{\beta'\chi}$. If $2\xi - 1 < \chi$, then we can choose β' so close to 1 that $2\xi - \beta' < \beta'\chi$. That is, fluctuations dominate while estimating M_i. Consequently, M_i is of order $n^{\beta'\chi}$. Thus, total error $=$ $n^{1-\beta+\beta'\chi}$. Since $\beta' < \beta$ and $\chi > 0$, this gives us the opportunity of choosing β', β such that the exponent is $< \chi$. This proves $(\star)$ for passage times from 'boundary to boundary'. Proving ($\star$) for 'point to point' passage times is only slightly more complicated. The program is carried out in Sections 7 and 8.

Finally, for the case $\chi = 0$, we have to prove that $\xi \geq 1/2$. This was proved by Licea, Newman and Piza [23] for a different definition of the wandering exponent. The argument does not seem to work with our definition. A proof is given in Section 9; I will omit this part from the proof sketch.

3. A priori bounds

In this section we prove the a priori bounds $0 \leq \chi_b \leq \chi_a \leq 1/2$ and $0 \leq \xi_b \leq \xi_a \leq 1$. First, note that the inequalities $\chi_b \leq \chi_a$ and $\xi_b \leq \xi_a$ are easy. For example, if $\chi_b > \chi_a$, then for any $\chi_a < \chi' < \chi'' < \chi_b$, (A1) implies that

$$\sup_{v \in \mathbb{Z}^d \setminus \{0\}} \frac{\mathrm{Var}(T(0,v))}{|v|^{2\chi'}} < \infty,$$

and hence for any sequence v_n such that $|v_n| \to \infty$,

$$\lim_{n\to\infty} \frac{\mathrm{Var}(T(0,v_n))}{|v_n|^{2\chi''}} = 0,$$

which contradicts (A3). A similar argument shows that $\xi_b \leq \xi_a$.

To show that $\chi_b \geq 0$, let E_0 denote the set of all edges incident to the origin. Let $\mathcal{F}_0$ denote the sigma-algebra generated by $(t_e)_{e \notin E_0}$. Since the edge-weight distribution is non-degenerate, there exists $c_1 < c_2$ such that for an edge e, $\mathbb{P}(t_e < c_1) > 0$ and $\mathbb{P}(t_e > c_2) > 0$. Therefore,

$$\mathbb{P}(\max_{e \in E_0} t_e < c_1) > 0, \quad \mathbb{P}(\min_{e \in E_0} t_e > c_2) > 0. \tag{3}$$

Let $(t'_e)_{e \in E_0}$ be an independent configuration of edge weights. Define $t'_e = t_e$ if $e \notin E_0$. Let $T'(0,v)$ be the first-passage time from 0 to a vertex v in the new environment t'. If $t_e < c_1$ and $t'_e > c_2$ for all $e \in E_0$, then $T'(0,v) > T(0,v) + c_2 - c_1$. Thus, by (3), there exists $\delta > 0$ such that for any v with $|v| \geq 2$,

$$\mathbb{E}\mathrm{Var}(T(0,v)|\mathcal{F}_0) = \frac{1}{2}\mathbb{E}(T(0,v) - T'(0,v))^2 > \delta.$$

Therefore $\mathrm{Var}(T(0,v)) > \delta$ and so $\chi_b \geq 0$.

To show that $\xi_b \geq 0$, note that there is an $\epsilon > 0$ small enough such that for any $v \in \mathbb{Z}^d$ with $|v| \geq 2$, there can be at most one lattice path from 0 to v that stays within distance ϵ from the straight line segment joining 0 to v. Fix such a vertex v and such a path P. If the number of edges in P is sufficiently large, one can use the non-degeneracy of the edge-weight distribution to show by an explicit assignment of edge weights that

$$\mathbb{P}(P \text{ is a geodesic}) < \delta,$$

where $\delta < 1$ is a constant that depends only on the edge-weight distribution (and not on v or P). This shows that for $|v|$ sufficiently large, $\mathbb{E}D(0,v)$ is bounded below by a positive constant that does not depend on v, thereby proving that $\xi_b \geq 0$.

Let us next show that $\chi_a \leq 1/2$. Essentially, this follows from [18, Theorem 1] or [28, Proposition 8.3], with a little bit of extra work. Below, we give a proof using [5, Theorem 5.4]. First, note that there is a constant C_0 such that for all v,

$$\mathbb{E}T(0,v) \leq C_0|v|_1, \tag{4}$$

where $|v|_1$ is the ℓ_1 norm of v. From the assumptions about the distribution of edge-weights, [5, Theorem 5.4] implies that there are positive constants C_1 and C_2 such that for any $v \in \mathbb{Z}^d$ with $|v|_1 \geq 2$, and any $0 \leq t \leq |v|_1$,

$$\mathbb{P}\biggl(|T(0,v) - \mathbb{E}T(0,v)| \geq t\sqrt{\frac{|v|_1}{\log |v|_1}}\biggr) \leq C_1 e^{-C_2 t}. \tag{5}$$

Fix a path P from 0 to v with $|v|_1$ edges. Recall that $t(P)$ denotes the sum of the weights of the edges in P. Since the edge-weight distribution has finite moment generating function in a neighborhood of zero and (4) holds, it is easy to see that there are positive constants C_3, C_4 and C_4' such that if $|v|_1 > C_3$, then for any $t > |v|_1$,

$$\begin{aligned}
&\mathbb{P}\biggl(|T(0,v) - \mathbb{E}T(0,v)| \geq t\sqrt{\frac{|v|_1}{\log |v|_1}}\biggr)\\
&\leq \mathbb{P}\biggl(T(0,v) \geq C_0|v|_1 + t\sqrt{\frac{|v|_1}{\log |v|_1}}\biggr) \qquad (6)\\
&\leq \mathbb{P}\biggl(t(P) \geq C_0|v|_1 + t\sqrt{\frac{|v|_1}{\log |v|_1}}\biggr) \leq e^{C_4|v|_1 - C_4' t\sqrt{|v|_1/\log |v|_1}}.
\end{aligned}$$

Combining (5) and (6) it follows that there are constants C_5, C_6 and C_7 such that for any v with $|v|_1 > C_5$,

$$\mathbb{E}\exp\biggl(C_6 \frac{|T(0,v) - \mathbb{E}T(0,v)|}{\sqrt{|v|_1/\log |v|_1}}\biggr) \leq C_7.$$

Appropriately increasing C_7, one sees that the above inequality holds for all v with $|v|_1 \geq 2$. In particular, $\chi_a \leq 1/2$.

Finally, let us prove that $\xi_a \leq 1$. Consider a self-avoiding path P starting at the origin, containing m edges. By the strict positivity of the edge-weight distributions, for any edge e,

$$\lim_{\theta \to \infty} \mathbb{E}(e^{-\theta t_e}) = 0.$$

Now, for any $\theta, c > 0$,

$$\mathbb{P}(t(P) \leq cm) = \mathbb{P}(e^{-t(P)/c} \geq e^{-m}) \leq (e\mathbb{E}(e^{-t_e/c}))^m.$$

Thus, given any $\delta > 0$ there exists c small enough such that for any m and any self-avoiding path P with m edges,

$$\mathbb{P}(t(P) \le cm) \le \delta^m.$$

Since there are at most $(2d)^m$ paths with m edges, therefore there exists c small enough such that

$$\mathbb{P}(t(P) \le cm \text{ for some } P \text{ with } m \text{ edges}) \le 2^{-m-1},$$

and therefore

$$\mathbb{P}(t(P) \le cm \text{ for some } P \text{ with } \ge m \text{ edges}) \le 2^{-m}. \tag{7}$$

There is a constant $B > 0$ such that for any $t \ge 1$ and any vertex $v \ne 0$, if $D(0, v) \ge t|v|$, then $G(0, v)$ has at least $Bt|v|$ edges. Therefore from (7),

$$\mathbb{P}(D(0, v) \ge t|v|) \le \mathbb{P}(T(0, v) \ge Bt|v|/c) + 2^{-Bt|v|}.$$

As in (6), there is a constant C such that if P is a path from 0 to v with $|v|_1$ edges,

$$\mathbb{P}(T(0, v) \ge Bt|v|/c) \le \mathbb{P}(t(P) \ge Bt|v|/c) \le e^{C|v|-Bt|v|/c}.$$

Combining the last two displays shows that for some α small enough,

$$\sup_{v \ne 0} \mathbb{E} \exp\left(\alpha \frac{D(0, v)}{|v|}\right) < \infty,$$

and thus, $\xi_a \le 1$.

4. Alexander's subadditive approximation theory

The first step in the proof of Theorem 1.1 is to find a suitable approximation of $\mathbb{E}T(0, x)$ by a convex function $g(x)$. For $x \in \mathbb{Z}^d$, define

$$h(x) := \mathbb{E}T(0, x). \tag{8}$$

It is easy to see that h satisfies the subadditive inequality

$$h(x + y) \le h(x) + h(y).$$

By the standard subadditive argument, it follows that

$$g(x) := \lim_{n\to\infty} \frac{h(nx)}{n} \tag{9}$$

exists for each $x \in \mathbb{Z}^d$. In fact, $g(x)$ may be defined similarly for $x \in \mathbb{Q}^d$ by taking $n \to \infty$ through a sequence of n such that $nx \in \mathbb{Z}^d$. The function g extends continuously to the whole of $\mathbb{R}^d$, and the extension is a norm on $\mathbb{R}^d$ (see e.g. [2, Lemma 1.5]). Note that by subadditivity,

$$g(x) \le h(x) \text{ for all } x \in \mathbb{Z}^d. \tag{10}$$

Since the edge-weight distribution is continuous in the setting of Theorem 1.1, it follows by a well-known result (see [17]) that $g(x) > 0$ for each

$x \neq 0$. Let e_i denote the ith coordinate vector in $\mathbb{R}^d$. Since g is symmetric with respect to interchange of coordinates and reflections across all coordinate hyperplanes, it is easy to show using subadditivity that

$$|x|_\infty \le g(x)/g(e_1) \le |x|_1 \text{ for all } x \neq 0, \tag{11}$$

where $|x|_p$ denotes the ℓ_p norm of the vector x.

How well does $g(x)$ approximate $h(x)$? Following the work of Kesten [17, 18], Alexander [1, 2] developed a general theory for tackling such questions. One of the main results of Alexander [2] is that under appropriate hypotheses on the edge-weights, there exists some $C > 0$ such that for all $x \in \mathbb{Z}^d \backslash \{0\}$,

$$g(x) \le h(x) \le g(x) + C|x|^{1/2} \log |x|.$$

Incidentally, Alexander has recently been able to obtain slightly improved results for nearly gamma edge-weights [3]. It turns out that under the hypotheses of Theorem 1.1, Alexander's argument goes through almost verbatim to yield the following result.

Theorem 4.1. *Consider the setup of Theorem* 1.1. *Let g and h be defined as in* (9) *and* (8) *above. Then for any $\chi' > \chi_a$, there exists $C > 0$ such that for all $x \in \mathbb{Z}^d$ with $|x| > 1$,*

$$g(x) \le h(x) \le g(x) + C|x|^{\chi'} \log |x|.$$

Sacrificing brevity for the sake of completeness, I will now prove Theorem 4.1 by copying Alexander's argument with only minor changes at the appropriate points.

Fix $\chi' > \chi_a$. Since $0 \le \chi_a \le 1/2$, so χ' can be chosen to satisfy $0 < \chi' < 1$.

Let $B_0 := \{x : g(x) \le 1\}$. Given $x \in \mathbb{R}^d$, let H_x denote a hyperplane tangent to the boundary of $g(x)B_0$ at x. Note that if the boundary is not smooth, the choice of H_x may not be unique. Let H_x^0 be the hyperplane through the origin that is parallel to H_x. There is a unique linear functional g_x on $\mathbb{R}^d$ satisfying

$$g_x(y) = 0 \text{ for all } y \in H_x^0, \quad g_x(x) = g(x).$$

For each $x \in \mathbb{R}^d$, $C > 0$ and $K > 0$ let

$$\begin{aligned} &Q_x(C, K) \\ &\quad := \{y \in \mathbb{Z}^d : |y| \le K|x|,\ g_x(y) \le g(x),\ h(y) \le g_x(y) + C|x|^{\chi'} \log |x|\}. \end{aligned}$$

The following key result is taken from [2].

Lemma 4.2 (Alexander [2], Theorem 1.8). *Consider the setting of Theorem* 4.1. *Suppose that for some $M > 1$, $C > 0$, $K > 0$ and $a > 1$, the following holds. For each $x \in \mathbb{Q}^d$ with $|x| \ge M$, there exists an integer $n \ge 1$, a lattice path γ from 0 to nx, and a sequence of sites $0 = v_0, v_1, \ldots, v_m = nx$ in γ such that $m \le an$ and $v_i - v_{i-1} \in Q_x(C, K)$ for all $1 \le i \le m$. Then the conclusion of Theorem* 4.1 *holds.*

Before proving that the conditions of Lemma 4.2 hold, we need some preliminary definitions and results. Define

$$s_x(y) := h(y) - g_x(y), \quad y \in \mathbb{Z}^d.$$

By the definition of g_x and the fact that g is a norm, it is easy to see that

$$|g_x(y)| \le g(y), \tag{12}$$

and by subadditivity, $g(y) \le h(y)$. Therefore $s_x(y) \ge 0$. Again from subadditivity of h and linearity of g_x,

$$s_x(y+z) \le s_x(y) + s_x(z) \ \text{ for all } y, z \in \mathbb{Z}^d. \tag{13}$$

Let $C_1 := 320d^2/\alpha$, where α is from the statement of Theorem 1.1. As in [2], define

$$\begin{aligned}
&Q_x := Q_x(C_1, 2d+1),\\
&G_x := \{y \in \mathbb{Z}^d : g_x(y) > g(x)\},\\
&\Delta_x := \{y \in Q_x : y \text{ adjacent to } \mathbb{Z}^d \backslash Q_x,\ y \text{ not adjacent to } G_x\},\\
&D_x := \{y \in Q_x : y \text{ adjacent to } G_x\}.
\end{aligned}$$

The following Lemma is simply a slightly altered copy of Lemma 3.3 in [2].

Lemma 4.3. *Assume the conditions of Theorem 1.1. Then there exists a constant C_2 such that if $|x| \ge C_2$, the following hold.*

(i) *If $y \in Q_x$ then $g(y) \le 2g(x)$ and $|y| \le 2d|x|$.*
(ii) *If $y \in \Delta_x$ then $s_x(y) \ge C_1|x|^{\chi'}(\log|x|)/2$.*
(iii) *If $y \in D_x$ then $g_x(y) \ge 5g(x)/6$.*

Proof. (i) Suppose $g(y) > 2g(x)$ and $g_x(y) \le g(x)$. Then using (10) and (12),

$$2g(x) < g(y) \le h(y) = g_x(y) + s_x(y) \le g(x) + s_x(y),$$

so from (11), $s_x(y) > g(x) > C_1|x|^{\chi'} \log|x|$ provided $|x| \ge C_2$. Thus $y \notin Q_x$ and the first conclusion in (i) follows. The second conclusion then follows from (11).

(ii) Note that $z = y \pm e_i$ for some $z \in \mathbb{Z}^d \cap Q_x^c \cap G_x^c$ and $i \le d$. From (i) we have $|y| \le 2d|x|$, so $|z| \le (2d+1)|x|$, provided $|x| > 1$. Since $z \notin Q_x$ we must then have $s_x(z) > C_1|x|^{\chi'} \log|x|$, while using (12),

$$h(\pm e_i) = s_x(\pm e_i) + g_x(\pm e_i) \ge s_x(\pm e_i) - g(\pm e_i).$$

Consequently, by (13), if $|x| \ge C_2$,

$$\begin{aligned}
s_x(y) &\ge s_x(z) - s_x(\pm e_i)\\
&\ge C_1|x|^{\chi'} \log|x| - h(\pm e_i) - g(\pm e_i)\\
&\ge C_1|x|^{\chi'}(\log|x|)/2.
\end{aligned}$$

(iii) As in (ii) we have $z = y \pm e_i$ for some $z \in \mathbb{Z}^d \cap G_x$ and $i \le d$. Therefore using (11) and (12),

$$g_x(y) = g_x(z) - g_x(\pm e_i) \ge g_x(z) - g(\pm e_i) \ge 5g(x)/6$$

for all $|x| \ge C_2$. □

Let us call the $m+1$ sites in Lemma 4.2 marked sites. If m is unrestricted, it is easy to find inductively a sequence of marked sites for any path γ from 0 to nx, as follows. One can start at $v_0 = 0$, and given v_i, let v'_{i+1} be the first site (if any) in γ, coming after v_i, such that $v'_{i+1} - v_i \notin Q_x$; then let v_{i+1} be the last site in γ before v'_{i+1} if v'_{i+1} exists; otherwise let $v_{i+1} = nx$ and end the construction. If $|x|$ is large enough, then it is easy to deduce from (11) and (12) that all neighbors of the origin must belong to Q_x and therefore $v_{i+1} \ne v_i$ for each i and hence the construction must end after a finite number of steps. We call the sequence of marked sites obtained from a self-avoiding path γ in this way, the Q_x-skeleton of γ.

Given such a skeleton $(v_0, \dots, v_m)$, abbreviated (v_i), of some lattice path, we divide the corresponding indices into two classes, corresponding to 'long' and 'short' increments:

$$S((v_i)) := \{i : 0 \le i < m-1,\ v_{i+1} - v_i \in \Delta_x\},$$
$$L((v_i)) := \{i : 0 \le i < m-1,\ v_{i+1} - v_i \in D_x\}.$$

Note that the final index m is in neither class, and by Lemma 4.3(ii),

$$j \in S((v_i)) \text{ implies } s_x(v_{j+1} - v_j) > C_1 |x|^{\chi'} (\log |x|)/2. \tag{14}$$

The next result is analogous to Proposition 3.4 in [2].

Proposition 4.4. *Assume the conditions of Theorem 1.1. There exists a constant C_3 such that if $|x| \ge C_3$ then for sufficiently large n there exists a lattice path from 0 to nx with Q_x-skeleton of $2n+1$ or fewer vertices.*

Proof. Let $(v_0, \dots, v_m)$ be a Q_x-skeleton of some lattice path and let

$$Y_i := \mathbb{E}T(v_i, v_{i+1}) - T(v_i, v_{i+1}).$$

Then by (A1) of Theorem 1.1 and Lemma 4.3(i), there are constants $C_4 := \alpha/(2d)^{\chi'} \ge \alpha/2d$ and C_5 such that for $0 \le i \le m-1$,

$$\mathbb{E}\exp(C_4 |Y_i| / |x|^{\chi'}) \le C_5. \tag{15}$$

Let $Y'_0, Y'_1, \dots, Y'_{m-1}$ be independent random variables with Y'_i having the same distribution as Y_i. Let $T(0, w; (v_j))$ be the minimum passage time among all lattice paths from 0 to a site w with Q_x-skeleton (v_j). By [17, equation (4.13)] or [1, Theorem 2.3], for all $t \ge 0$,

$$\mathbb{P}\left(\sum_{i=0}^{m-1} Y'_i \ge t\right) \ge \mathbb{P}\left(\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, v_m; (v_j)) \ge t\right).$$

Now by (15),

$$\mathbb{P}\biggl(\sum_{i=0}^{m-1} Y_i' \ge t\biggr) \le e^{-C_4 t/|x|^{\chi'}} C_5^m.$$

Let $C_6 := 20d^2/\alpha$. Taking $t = C_6 m|x|^{\chi'} \log |x|$, the above display shows that there is a constant C_7 such that for all $|x| \ge C_7$,

$$\mathbb{P}\biggl(\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, v_m; (v_j)) \ge C_6 m|x|^{\chi'} \log |x|\biggr) \le (C_5 e^{-10d \log |x|})^m.$$

From the definition of a Q_x-skeleton, it is easy to see that there is a constant C_8 such that there are at most $(C_8|x|^d)^m$ Q_x-skeletons with $m+1$ vertices. Therefore, the above display shows that there are constants C_9 and C_{10} such that when $|x| \ge C_9$,

$$\begin{aligned}\mathbb{P}\biggl(&\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, v_m; (v_j)) \ge C_6 m|x|^{\chi'} \log |x| \\ &\qquad \text{for some } Q_x\text{-skeleton with } m+1 \text{ vertices}\biggr) \le e^{-C_{10} m \log |x|}.\end{aligned}$$

This in turn yields that for some constant C_{11}, for all $|x| \ge C_{11}$,

$$\begin{aligned}\mathbb{P}\biggl(&\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, v_m; (v_j)) \ge C_6 m|x|^{\chi'} \log |x| \\ &\qquad \text{for some } m \ge 1 \text{ and some } Q_x\text{-skeleton with } m+1 \text{ vertices}\biggr) \\ &\le 2e^{-C_{10} \log |x|}.\end{aligned} \tag{16}$$

Now let $\omega := \{t_e : e \text{ is an edge in } \mathbb{Z}^d\}$ be a fixed configuration of passage times (to be further specified later) and let $(v_0, \ldots, v_m)$ be the Q_x-skeleton of a route from 0 to nx. Then since $v_{i+1} - v_i \in Q_x$,

$$mg(x) \ge \sum_{i=0}^{m-1} g_x(v_{i+1} - v_i) = g_x(nx) = ng(x).$$

Therefore

$$n \le m. \tag{17}$$

From the concentration of first-passage times,

$$\mathbb{P}(T(0, nx) \le ng(x) + n) \to 1 \quad \text{as } n \to \infty,$$

so by (16) if n is large there exists a configuration ω and a Q_x-skeleton $(v_0, \ldots, v_m)$ of a path from 0 to nx such that

$$T(0, nx; (v_j)) = T(0, nx) \le ng(x) + n \tag{18}$$

and

$$\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, nx; (v_j)) < C_6 m|x|^{\chi'} \log|x|. \tag{19}$$

Thus for some constant C_{12}, if $|x| \geq C_{12}$ then by (17), (18) and (19),

$$\begin{aligned} \sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) &< ng(x) + n + C_6 m|x|^{\chi'} \log|x| \\ &\leq ng(x) + 2C_6 m|x|^{\chi'} \log|x|. \end{aligned} \tag{20}$$

But by (14),

$$\begin{aligned} \sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) &= \sum_{i=0}^{m-1} (g_x(v_{i+1} - v_i) + s_x(v_{i+1} - v_i)) \\ &\geq g_x(nx) + C_1|S((v_i))||x|^{\chi'}(\log|x|)/2, \end{aligned}$$

which, together with (20), yields

$$|S((v_i))| \leq 4C_6 m/C_1 = m/4. \tag{21}$$

At the same time, using Lemma 4.3(iii),

$$\begin{aligned} \sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) &= \sum_{i=0}^{m-1} (g_x(v_{i+1} - v_i) + s_x(v_{i+1} - v_i)) \\ &\geq 5|L((v_i))|g(x)/6. \end{aligned}$$

With (20), (11) and the assumption that $\chi' < 1$, this implies that there is a constant C_{13} such that, provided $|x| \geq C_{13}$,

$$|L((v_i))| \leq 6n/5 + \frac{12C_6 m|x|^{\chi'} \log|x|}{6g(e_1)|x|/\sqrt{d}} \leq 6n/5 + m/8.$$

This and (21) give

$$m = |L((v_i))| + |S((v_i))| + 1 \leq 6n/5 + 3m/8 + 1,$$

which, for n large, implies $m \leq 2n$, proving the Proposition. □

Proof of Theorem 4.1. Lemma 4.2 and Proposition 4.4 prove the conclusion of Theorem 4.1 for x with sufficiently large Euclidean norm. To prove this for all x with $|x| > 1$, one simply has to increase the value of C. □

5. Curvature bounds

The unit ball of the g-norm, usually called the 'limit shape' of first-passage percolation, is an object of great interest and intrigue in this literature. Very little is known rigorously about the limit shape, except for a fundamental result about convergence to the limit shape due to Cox and Durrett [8], some qualitative results of Kesten [17] who proved, in particular, that the limit shape may not be an Euclidean ball, an important result of Durrett and

Liggett [9] who showed that the boundary of the limit shape may contain straight lines, and some bounds on the rate of convergence to the limit shape [18, 2]. In particular, it is not even known whether the limit shape may be strictly convex in every direction (except for the related continuum model of 'Riemannian first-passage percolation' [22] and first-passage percolation with stationary ergodic edge-weights [10]).

The following Proposition lists two properties of the limit shape that are crucial for our purposes.

Proposition 5.1. *Let g be defined as in (9) and assume that the distribution of edge-weights is continuous. Then there exists $x_0 \in \mathbb{R}^d$ with $|x_0| = 1$, a constant $C \geq 0$ and a hyperplane H_0 through the origin perpendicular to x_0 such that for all $z \in H_0$,*

$$|g(x_0 + z) - g(x_0)| \leq C|z|^2.$$

There also exists $x_1 \in \mathbb{R}^d$ with $|x_1| = 1$ and a hyperplane H_1 through the origin perpendicular to x_1 such that for all $z \in H_1$,

$$g(x_1 + z) \geq \sqrt{1 + |z|^2}\, g(x_1).$$

Proof. The proof is similar to that of [26, Lemma 5]. Let $B(0, r)$ denote the Euclidean ball of radius r centered at the origin and let

$$B_g(0, r) := \{x : g(x) \leq r\}$$

denote the ball of radius r centered at the origin for the norm g. Let r be the smallest number such that $B_g(0, r) \supseteq B(0, 1)$. Let x_0 be a point of intersection of $\partial B_g(0, r)$ and $\partial B(0, 1)$. Let H_0 be a hyperplane tangent to $\partial B_g(0, r)$ at x_0, translated to contain the origin. Note that $x_0 + H_0$ is also a tangent hyperplane for $B(0, 1)$ at x_0, since it touches $B(0, 1)$ only at x_0. Therefore H_0 is perpendicular to x_0. Now for any $z \in H_0$, the point $y := (x_0 + z)/|x_0 + z|$ is a point on $\partial B(0, 1)$ and hence contained in $B_g(0, r)$. Therefore

$$g(x_0) = r \geq g(y) = \frac{1}{|x_0 + z|} g(x_0 + z) = \frac{1}{\sqrt{1 + |z|^2}} g(x_0 + z).$$

Since $g(x_0 + z)$ grows like $|z|$ as $|z| \to \infty$, this shows that there is a constant C such that

$$g(x_0 + z) \leq g(x_0) + C|z|^2$$

for all $z \in H_0$. Also, since $x_0 + z \notin B_g(0, r)$ for $z \in H_0 \setminus \{0\}$, therefore $g(x_0) \leq g(x_0 + z)$ for all $z \in H_0$. This proves the first assertion of the Proposition.

For the second, we proceed similarly. Let r be the largest number such that $B_g(0, r) \subseteq B(0, 1)$. Let x_1 be a point in the intersection of $\partial B_g(0, r)$ and $\partial B(0, 1)$. Let H_1 be the hyperplane tangent to $\partial B(0, 1)$ at x_1, translated to contain the origin. Note that this is simply the hyperplane through the origin that is perpendicular to x_1. Since $B(0,1)$ contains $B_g(0,r)$, and $y := (x_1+z)/|x_1+z|$ is a point in $\partial B(0,1)$, therefore

$$g(x_1) = r \le g(y) = \frac{1}{|x_1+z|} g(x_1+z) = \frac{1}{\sqrt{1+|z|^2}} g(x_1+z).$$

This completes the argument. $\square$

6. Proof of $\chi_a \ge 2\xi_b - 1$

We will prove by contradiction. Suppose that $2\xi_b - 1 > \chi_a$. Choose ξ' such that

$$\frac{1+\chi_a}{2} < \xi' < \xi_b.$$

Note that $\xi' < 1$. Let x_1 and H_1 be as in Proposition 5.1. Let n be a positive integer, to be chosen later. Throughout this proof, C will denote any positive constant that does not depend on n. The value of C may change from line to line. Also, we will assume without mention that 'n is large enough' wherever required.

Let y be the closest point in $\mathbb{Z}^d$ to nx_1. Note that

$$|y - nx_1| \le \sqrt{d}. \tag{22}$$

Let L denote the line passing through 0 and nx_1 and let L' denote the line segment joining 0 to nx_1 (but not including the endpoints). Let V be the set of all points in $\mathbb{Z}^d$ whose distance from L' lies in the interval $[n^{\xi'}, 2n^{\xi'}]$. Take any $v \in V$. We claim that there is a constant C (not depending on n) such that for any $v \in V$,

$$g(v) + g(nx_1 - v) \ge g(nx_1) + Cn^{2\xi'-1}. \tag{23}$$

Let us now prove this claim. Let w be the projection of v onto L along H_1 (i.e. the perpendicular projection). To prove (23), there are three cases to consider. First suppose that w lies in L'. Note that $w/|w| = x_1$. Let $v' := v/|w|$ and $z := v' - x_1 = (v-w)/|w|$.

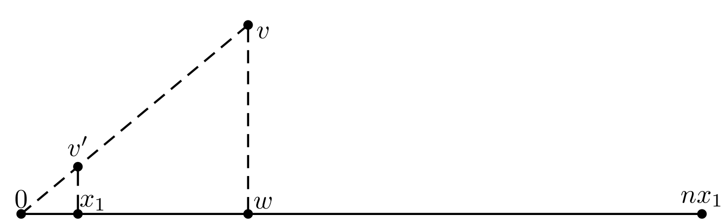

Figure 4. The relative positions of x_1, v', v, w, nx_1.

Note that $z \in H_1$. Thus by Proposition 5.1,

$$g(v') = g(x_1+z) \ge \sqrt{1+|z|^2}\, g(x_1).$$

Consequently,

$$g(v) \geq |w|\sqrt{1+|z|^2}g(x_1). \tag{24}$$

Next, let $w' := nx_1 - w$. Note that $w'/|w'| = x_1$. let $v'' := (nx_1 - v)/|w'|$, and

$$z' := v'' - x_1 = (w - v)/|w'|.$$

Then $z' \in H_1$, and hence by Proposition 5.1,

$$g(v'') = g(x_1 + z') \geq \sqrt{1+|z'|^2}g(x_1).$$

Consequently,

$$g(nx_1 - v) \geq |w'|\sqrt{1+|z'|^2}g(x_1). \tag{25}$$

Since $v \in V$, therefore $|v - w| \geq n^{\xi'}$. Again, $|w| + |w'| = n$. Thus,

$$\min\{|z|, |z'|\} \geq n^{\xi'-1}.$$

Combining this with (24), (25), (11) and the fact that $\xi' < 1$, we have

$$\begin{aligned} g(v) + g(nx_1 - v) &\geq (|w| + |w'|)\sqrt{1+n^{2\xi'-2}}g(x_1) \\ &= \sqrt{1+n^{2\xi'-2}}g(nx_1) \\ &\geq g(nx_1) + Cn^{2\xi'-1}. \end{aligned}$$

Next, suppose that w lies in $L\backslash L'$, on the side closer to nx_1. As above, let $z := (v - w)/|w|$. As in (24), we conclude that

$$g(v) \geq |w|\sqrt{1+|z|^2}g(x_1). \tag{26}$$

By the definition of V, the distance between v and nx_1 must be greater than $n^{\xi'}$. But in this case

$$|v - nx_1|^2 = (|w| - n)^2 + |v - w|^2 = (|w| - n)^2 + |w|^2|z|^2,$$

and we also have $n \leq |w| \leq 3n$. Thus, either $|w|^2|z|^2 > n^{2\xi'}/2$ (which implies $|z|^2 \geq Cn^{2\xi'-2}$), or $|w| \geq n + n^{\xi'}/\sqrt{2}$. Since $\xi' > 2\xi' - 1$, therefore by (26), in either situation we have

$$g(v) \geq g(nx_1) + Cn^{2\xi'-1}.$$

Similarly, if w lies in $L\backslash L'$, on the side closer to 0, then

$$g(nx_1 - v) \geq g(nx_1) + Cn^{2\xi'-1}.$$

This completes the proof of (23). Now (23) combined with Theorem 4.1, (22) and the fact that $2\xi' - 1 > \chi_a$ implies that if n is large enough, then for any $v \in V$,

$$h(v) + h(y - v) \geq h(y) + Cn^{2\xi'-1}. \tag{27}$$

Choose χ_1, χ_2 such that $\chi_a < \chi_1 < \chi_2 < 2\xi' - 1$. Then by (A1) of Theorem 1.1, there is a constant C such that for n large enough,

$$\mathbb{P}(T(0, y) > h(y) + n^{\chi_2}) \leq e^{-Cn^{\chi_2 - \chi_1}}.$$

Now, for any $v \in V$, both $|v|$ and $|y-v|$ are bounded above by Cn. Therefore again by (A1),

$$\mathbb{P}(T(0,v) < h(v) - n^{\chi_2}) \le e^{-Cn^{\chi_2-\chi_1}},$$
$$\mathbb{P}(T(v,y) < h(y-v) - n^{\chi_2}) \le e^{-Cn^{\chi_2-\chi_1}}.$$

This, together with (27), shows that if n is large enough, then for any $v \in V$,

$$\mathbb{P}(T(0,y) = T(0,v) + T(v,y)) \le e^{-Cn^{\chi_2-\chi_1}}.$$

Since the size of V grows polynomially with n, this shows that

$$\mathbb{P}(T(0,y) = T(0,v) + T(v,y) \text{ for some } v \in V) \le e^{-Cn^{\chi_2-\chi_1}}.$$

Note that if the geodesic from 0 to y passes through V, then $T(0,y) = T(0,v) + T(v,y)$ for some $v \in V$. If $D(0,y) > n^{\xi'}$ then the geodesic must pass through V. Thus, the above inequality implies that

$$\mathbb{P}(D(0,y) > n^{\xi'}) \le e^{-Cn^{\chi_2-\chi_1}}.$$

By (A2) of Theorem 1.1, this gives

$$\begin{aligned}\mathbb{E}D(0,y) &\le n^{\xi'} + \mathbb{E}(D(0,y)1_{\{D(0,y)>n^{\xi'}\}}) \\ &\le n^{\xi'} + \sqrt{\mathbb{E}(D(0,y)^2)\mathbb{P}(D(0,y) > n^{\xi'})} \\ &\le n^{\xi'} + C_1 n^{C_1} e^{-C_2 n^{\chi_2-\chi_1}}.\end{aligned}$$

Taking $n \to \infty$, this shows that (A4) of Theorem 1.1 is violated (since $\xi' < \xi_b$), leading to a contradiction to our original assumption that $\chi_a < 2\xi_b - 1$. Thus, $\chi_a \ge 2\xi_b - 1$.

7. Proof of $\chi \le 2\xi - 1$ when $0 < \chi < 1/2$

In this section and the rest of the manuscript, we assume that $\chi_a = \chi_b$ and $\xi_a = \xi_b$, and denote these two numbers by χ and ξ.

Again we prove by contradiction. Suppose that $0 < \chi < 1/2$ and $\chi > 2\xi - 1$. Fix $\chi_1 < \chi < \chi_2$, to be chosen later. Choose ξ' such that

$$\xi < \xi' < \frac{1+\chi}{2}.$$

Define:

$$\begin{aligned}\beta' &:= \frac{1}{2} + \frac{\xi'}{1+\chi}. \\ \beta &:= 1 - \frac{\chi}{2} + \frac{\chi}{2}\beta'. \\ \varepsilon &:= (1-\beta)\left(1 - \frac{\chi}{2}\right).\end{aligned}$$

We need several inequalities involving the numbers β', β and ε. Since

$$0 < \frac{\xi'}{1+\chi} < \frac{1}{2},$$

therefore

$$\frac{1}{2} < \beta' < 1. \tag{28}$$

Since $\chi < 1$ and $\xi' < (1+\chi)/2 < 1$,

$$\beta' > \frac{1}{2} + \frac{\xi'}{2} > \xi'. \tag{29}$$

Since β is a convex combination of 1 and β' and $\chi > 0$,

$$\beta' < \beta < 1. \tag{30}$$

Since $0 < \chi < 1$ and $0 < \beta < 1$,

$$0 < \varepsilon < 1 - \beta. \tag{31}$$

Since β' is the average of 1 and $2\xi'/(1+\chi) \in (0,1)$, therefore β' is strictly bigger than $2\xi'/(1+\chi)$ and hence

$$\begin{aligned} 2\xi' - \beta' &< 2\xi' - \frac{2\xi'}{1+\chi} \\ &= \frac{2\xi'}{1+\chi}\chi < \beta'\chi. \end{aligned} \tag{32}$$

By (30), this implies that

$$2\xi' - \beta < 2\xi' - \beta' < \beta'\chi < \beta\chi. \tag{33}$$

Next, by (28),

$$1 - \beta + \beta'\chi = \frac{\chi}{2}(1+\beta') < \chi. \tag{34}$$

And finally by (28),

$$\beta\chi + 1 - \beta - \varepsilon = \beta\chi + (1-\beta)\frac{\chi}{2} < \chi. \tag{35}$$

Let q be a large positive integer, to be chosen later. Throughout this proof, we will assume without mention that q is 'large enough' wherever required. Also, C will denote any constant that does not depend on our choice of q, but may depend on all other parameters.

Let r be an integer between $\frac{1}{2}q^{(1-\beta-\varepsilon)/\varepsilon}$ and $2q^{(1-\beta-\varepsilon)/\varepsilon}$, recalling that by (31), $1-\beta-\varepsilon > 0$. Let $k = rq$. Let a be a real number between $q^{\beta/\varepsilon}$ and $2q^{\beta/\varepsilon}$. Let $n = ak$. Note that $n = arq$, which gives $\frac{1}{2}q^{1/\varepsilon} \le n \le 4q^{1/\varepsilon}$. From this it is easy to see that there are positive constants C_1 and C_2, depending only on β and ε, such that

$$C_1 n^{\varepsilon} \le q \le C_2 n^{\varepsilon}, \tag{36}$$

$$C_1 n^{1-\beta} \le k \le C_2 n^{1-\beta}, \tag{37}$$

$$C_1 n^{\beta} \le a \le C_2 n^{\beta}, \tag{38}$$

$$C_1 n^{1-\beta-\varepsilon} < r < C_2 n^{1-\beta-\varepsilon}. \tag{39}$$

Let $b := n^{\beta'}$. Note that by (30), b is negligible compared to a if q is large. Note also that, although r, k and q are integers, a, n and b need not be.

Let x_0 and H_0 be as in Proposition 5.1. For $0 \le i \le k$, define

$$U_i' := H_0 + iax_0 \ ,$$
$$V_i' := H_0 + (ia + a - b)x_0 \ .$$

Let U_i be the set of points in $\mathbb{Z}^d$ that are within distance $\sqrt{d}$ from U_i'. Let V_i be the set of points in $\mathbb{Z}^d$ that are within distance $\sqrt{d}$ from V_i'.

For $0 \le i \le k$ let y_i be the closest point in $\mathbb{Z}^d$ to iax_0, and let z_i be the closest point in $\mathbb{Z}^d$ to $(ia + a - b)x_0$, applying some arbitrary rule to break ties. Note that if $x \in \mathbb{R}^d$, and $y \in \mathbb{Z}^d$ is closest to x, then $|x - y| \le \sqrt{d}$. Therefore $y_i \in U_i$ and $z_i \in V_i$. Figure 5 gives a pictorial representation of the above definitions, assuming for simplicity that $U_i = U_i'$ and $V_i = V_i'$.

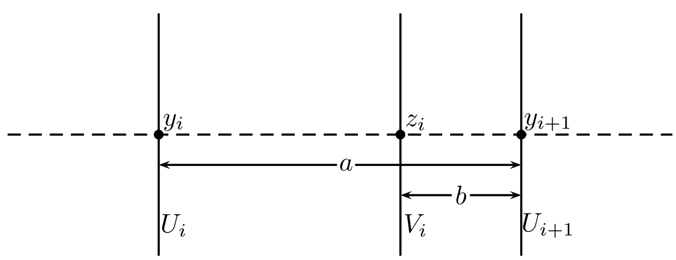

FIGURE 5. Diagrammatic representation of y_i, z_i, U_i and V_i.

Let U_i^o be the subset of U_i that is within distance $n^{\xi'}$ from y_i. Similarly let V_i^o be the subset of V_i that is within distance $n^{\xi'}$ from z_i.

For any $A, B \subseteq \mathbb{Z}^d$, let $T(A, B)$ denote the minimum passage time from A to B. Let $G(A, B)$ denote the (unique) geodesic from A to B, so that $T(A, B)$ is the sum of edge-weights of $G(A, B)$.

Fix any two integers $0 \le l < m \le k$ such that $m - l > 3$. Consider the geodesic $G := G(y_l, y_m)$. Since $x_0 \notin H_0$, it is easy to see that G must 'hit' each U_i and V_i, $l \le i \le m - 1$. Arranging the vertices of G in a sequence starting at y_l and ending at y_m, for each $l \le i < m$ let u_i' be the first vertex in U_i visited by G and let v_i' be the first vertex in V_i visited by G. Let $u_m' := y_m$. Note that G visits these vertices in the order $u_l', v_l', u_{l+1}', v_{l+1}', \ldots, v_{m-1}', u_m'$. Figure 6 gives a pictorial representation of the points u_i' and v_i' on the geodesic G. Let T_i' be the sum of edge-weights of the portion of G from u_i' to v_i'. Let E be the event that $u_i' \in U_i^o$ and $v_i' \in V_i^o$ for each i. If E happens, then clearly

$$T_i' \ge T(U_i^o, V_i^o).$$

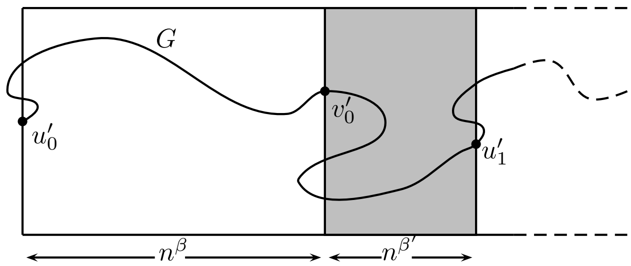

FIGURE 6. Location of $u'_0, v'_0, u'_1, v'_1, \ldots$ on the geodesic G.

Similarly, note that weight of the part of G from v'_i to u'_{i+1} must exceed or equal $T(v'_i, u'_{i+1})$. Therefore, if E happens, then

$$
\begin{aligned}
T(y_l, y_m) &\ge \sum_{i=l}^{m-1} T'_i + \sum_{i=l}^{m-1} T(v'_i, u'_{i+1}) \\
&\ge \sum_{i=l}^{m-1} T(U_i^o, V_i^o) + \sum_{i=l}^{m-1} T(v'_i, u'_{i+1}).
\end{aligned} \tag{40}
$$

Next, for each $0 \le i < k$ let $G_i := G(U_i^o, V_i^o)$. Let u_i and v_i be the endpoints of G_i. Let $G'_i := G(v_i, u_{i+1})$. Figure 7 gives a picture illustrating the paths G_i and G'_i. The concatenation of the paths $G(y_l, v_l)$, G'_l, G_{l+1}, G'_{l+1}, G_{l+2},

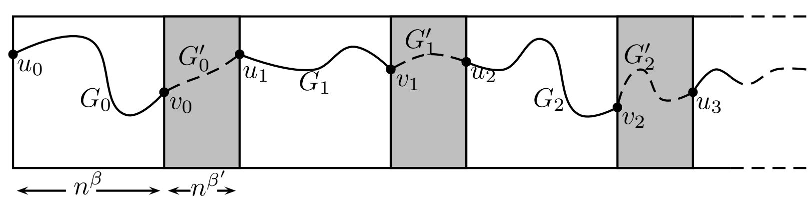

FIGURE 7. The paths $G_0, G'_0, G_1, G'_1, \ldots$.

$\ldots$, G'_{m-2}, G_{m-1}, $G(v_{m-1}, y_m)$ is a path from y_l to y_m (need not be self-avoiding). Therefore,

$$
\begin{aligned}
T(y_l, y_m) \le T(y_l, v_l) &+ \sum_{i=l+1}^{m-1} T(U_i^o, V_i^o) + \sum_{i=l}^{m-2} T(v_i, u_{i+1}) \\
&+ T(v_{m-1}, y_m).
\end{aligned} \tag{41}
$$

Define

$$\Delta_{l,m} := T(y_l, y_m) - \sum_{i=l}^{m-1} (T(U_i^o, V_i^o) + T(V_i^o, U_{i+1}^o)).$$

Combining (40) and (41) implies that if E happens, then

$$\begin{aligned}|\Delta_{l,m}| \le \sum_{i=l}^{m-1} |T(V_i^o, U_{i+1}^o) - T(v_i', u_{i+1}')| + \sum_{i=l}^{m-2} |T(V_i^o, U_{i+1}^o) - T(v_i, u_{i+1})| \\ + |T(U_l^o, V_l^o) - T(y_l, v_l)| + |T(V_{m-1}^o, U_m^o) - T(v_{m-1}, y_m)|.\end{aligned}$$

Thus, if

$$\begin{aligned}M_i &:= \max_{v,v' \in V_i^o,\ u,u' \in U_{i+1}^o} |T(v,u) - T(v',u')|, \\ N_i &:= \max_{u,u' \in U_i^o,\ v,v' \in V_i^o} |T(u,v) - T(u',v')|,\end{aligned}$$

and the event E happens, then

$$|\Delta_{l,m}| \le 2 \sum_{i=l}^{m-1} M_i + N_l. \tag{42}$$

For a random variable X, let $\|X\|_p := (\mathbb{E}|X|^p)^{1/p}$ denote its L^p norm. It is easy to see that $\|\Delta_{l,m}\|_4 \le n^C$, where recall that C stands for any constant that does not depend on our choice of the integer q, but may depend on χ, ξ, ξ' and the distribution of edge weights. Take any $\xi_1 \in (\xi, \xi')$. By (A2) of Theorem 1.1, $\mathbb{P}(E^c) \le e^{-Cn^{\xi'-\xi_1}}$. Together with (42), this shows that for some constants C_3 and C_4,

$$\begin{aligned}\|\Delta_{l,m}\|_2 &\le \|\Delta_{l,m} 1_{E^c}\|_2 + \|\Delta_{l,m} 1_E\|_2 \\ &\le \|\Delta_{l,m}\|_4 (\mathbb{P}(E^c))^{1/4} + \|\Delta_{l,m} 1_E\|_2 \\ &\le n^{C_3} e^{-C_4 n^{\xi'-\xi_1}} + 2\sum_{i=l}^{m-1} \|M_i\|_2 + \|N_l\|_2.\end{aligned} \tag{43}$$

Fix $0 \le i \le k-1$ and $v \in V_i^o$, $u \in U_{i+1}^o$. Let x be the nearest point to v in V_i' and y be the nearest point to u in U_{i+1}'. Then by definition of V_i' and U_{i+1}', there are vectors $z, z' \in H_0$ such that $|z|$ and $|z'|$ are bounded by $Cn^{\xi'}$, and $x = (ia + a - b)x_0 + z$ and $y = (ia + a)x_0 + z'$. Thus by Proposition 5.1,

$$\begin{aligned}|g(y-x) - g(bx_0)| &= |g(bx_0 + z' - z) - g(bx_0)| \\ &= b|g(x_0 + (z'-z)/b) - g(x_0)| \\ &\le \frac{C|z'-z|^2}{b} \le Cn^{2\xi'-\beta'}.\end{aligned}$$

Thus, for any $v, v' \in V_i^o$ and $u, u' \in U_{i+1}^o$,

$$|g(u-v) - g(u'-v')| \le Cn^{2\xi'-\beta'}.$$

Note also that $|y-x| \le C(n^{\beta'} + n^{\xi'}) \le Cn^{\beta'}$ by (29). This, together with Theorem 4.1, shows that for any $v, v' \in V_i^o$ and $u, u' \in U_{i+1}^o$,

$$|\mathbb{E}T(v,u) - \mathbb{E}T(v',u')| \le Cn^{2\xi'-\beta'} + Cn^{\beta'\chi_2}\log n.$$

By (32), this implies

$$|\mathbb{E}T(v,u) - \mathbb{E}T(v',u')| \le Cn^{\beta'\chi_2}\log n. \tag{44}$$

Let

$$M := \max_{v\in V_i^o,\ u\in U_{i+1}^o} \frac{|T(v,u) - \mathbb{E}T(v,u)|}{|u-v|^{\chi_2}}.$$

By (A1) of Theorem 1.1,

$$\begin{aligned}\mathbb{E}(e^{\alpha M}) &\le \sum_{v\in V_i^o,\ u\in U_{i+1}^o} \mathbb{E}\exp\biggl(\alpha\frac{|T(v,u) - \mathbb{E}T(v,u)|}{|u-v|^{\chi_2}}\biggr)\\ &\le C|V_i^o||U_{i+1}^o| \le Cn^C.\end{aligned}$$

This implies that $\mathbb{P}(M > t) \le Cn^C e^{-\alpha t}$, which in turn gives $\|M\|_2 \le C\log n$. Let

$$M' := \max_{v\in V_i^o,\ u\in U_{i+1}^o} |T(v,u) - \mathbb{E}T(v,u)|.$$

Since by (29), $|u-v| \le C(n^{\beta'} + n^{\xi'}) \le Cn^{\beta'}$ for all $v \in V_i^o$, $u \in U_{i+1}^o$, therefore $M' \le Cn^{\beta'\chi_2}M$. Thus,

$$\|M'\|_2 \le Cn^{\beta'\chi_2}\log n.$$

From this and (44) it follows that

$$\|M_i\|_2 \le Cn^{\beta'\chi_2}\log n.$$

By an exactly similar sequence of steps, replacing β' by β everywhere and using (33) instead of (32), one can deduce that

$$\|N_i\|_2 \le Cn^{\beta\chi_2}\log n.$$

Combining with (43) this gives

$$\|\Delta_{l,m}\|_2 \le Cn^{\beta\chi_2}\log n + C(m-l)n^{\beta'\chi_2}\log n, \tag{45}$$

since the exponential term in (43) is negligible compared to the rest.

Now, from the definition of $\Delta_{l,m}$, the fact that $k = rq$, and the triangle inequality, it is easy to see that

$$\biggl|T(y_0, y_k) - \sum_{j=0}^{r-1} T(y_{jq}, y_{(j+1)q})\biggr| \le |\Delta_{0,k}| + \sum_{j=0}^{r-1}|\Delta_{jq,(j+1)q}|.$$

Thus by (45), (39) and (37),

$$\begin{aligned}\left\|T(y_0,y_k)-\sum_{j=0}^{r-1}T(y_{jq},y_{(j+1)q})\right\|_2 &\le \|\Delta_{0,k}\|_2+\sum_{j=0}^{r-1}\|\Delta_{jq,(j+1)q}\|_2\\ &\le C(r+1)n^{\beta\chi_2}\log n+Ckn^{\beta'\chi_2}\log n\\ &\le Cn^{1-\beta-\varepsilon+\beta\chi_2}\log n+Cn^{1-\beta+\beta'\chi_2}\log n.\end{aligned}\tag{46}$$

For any two random variables X and Y,

$$\begin{aligned}\left|\sqrt{\mathrm{Var}(X)}-\sqrt{\mathrm{Var}(Y)}\right| &= |\|X-\mathbb{E}X\|_2-\|Y-\mathbb{E}Y\|_2|\\ &\le \|(X-\mathbb{E}X)-(Y-\mathbb{E}Y)\|_2\\ &\le \|X-Y\|_2+|\mathbb{E}X-\mathbb{E}Y|\le 2\|X-Y\|_2.\end{aligned}\tag{47}$$

Therefore it follows from (46) that

$$\begin{aligned}&\left|(\mathrm{Var}T(y_0,y_k))^{1/2}-\left(\mathrm{Var}\sum_{j=0}^{r-1}T(y_{jq},y_{(j+1)q})\right)^{1/2}\right|\\ &\qquad\qquad\le Cn^{1-\beta-\varepsilon+\beta\chi_2}\log n+Cn^{1-\beta+\beta'\chi_2}\log n.\end{aligned}\tag{48}$$

For any $x,y\in\mathbb{Z}^d$, $T(x,y)$ is an increasing function of the edge weights. So by the Harris-FKG inequality [12], $\mathrm{Cov}(T(x,y),T(x',y'))\ge 0$ for any $x,y,x',y'\in\mathbb{Z}^d$. Therefore by (A3) of Theorem 1.1 and (38), (39) and (36),

$$\begin{aligned}\mathrm{Var}\sum_{j=0}^{r-1}T(y_{jq},y_{(j+1)q}) &\ge \sum_{j=0}^{r-1}\mathrm{Var}T(y_{jq},y_{(j+1)q})\\ &\ge C\sum_{j=0}^{r-1}|y_{jq}-y_{(j+1)q}|^{2\chi_1}\\ &\ge Cr(aq)^{2\chi_1}\ge Cn^{(1-\beta-\varepsilon)+(\beta+\varepsilon)2\chi_1}.\end{aligned}\tag{49}$$

By the inequalities (34) and (35), we see that if χ_1 and χ_2 are chosen sufficiently close to χ, then χ_1 is strictly bigger than both $1-\beta-\varepsilon+\beta\chi_2$ and $1-\beta+\beta'\chi_2$. Therefore by (48) and (49), and since $1-\beta-\varepsilon+(\beta+\varepsilon)2\chi_1>2\chi_1$,

$$\mathrm{Var}T(y_0,y_k)\ge Cn^{(1-\beta-\varepsilon)+(\beta+\varepsilon)2\chi_1}.$$

By (31) and the assumption that $\chi<1/2$, we again have that if χ_1 is chosen sufficiently close to χ,

$$(1-\beta-\varepsilon)+(\beta+\varepsilon)2\chi_1>2\chi.$$

Since $|y_0-y_k|\le Cak\le Cn$ by (38) and (37), therefore taking $q\to\infty$ (and hence $n\to\infty$) gives a contradiction to (A1) of Theorem 1.1, thereby proving that $\chi\le 2\xi-1$ when $0<\chi<1/2$.

8. Proof of $\chi \le 2\xi - 1$ when $\chi = 1/2$

Suppose that $\chi = 1/2$ and $\chi > 2\xi - 1$. Define χ_1, χ_2, x_0, H_0, ξ', β, β', ε, q, a, r, k, n, y_i and z_i exactly as in Section 7, considering a, r, k and n as functions of q. Then all steps go through, except the very last, where we used $\chi < 1/2$ to get a contradiction. Therefore all we need to do is the modify this last step to get a contradiction in a different way. This is where we need the sublinear variance inequality (1). As before, throughout the proof C denotes any constant that does not depend on q.

For each real number $m \ge 1$, let w_m be the nearest lattice point to mx_0. Note that $y_i = w_{ia}$. Let

$$f(m) := \mathrm{Var} T(0, w_m).$$

Note that there is a constant C_0 such that $f(m) \le C_0 m$ for all m. Again by (A3), there is a $C_1 > 0$ such that for all m,

$$f(m) \ge C_1 m^{2\chi_1}. \tag{50}$$

Now, $|(w_{(j+1)aq} - w_{jaq}) - w_{aq}| \le C$. Again, as a consequence of (47) we have that for any two random variables X and Y,

$$\begin{aligned} \big|\mathrm{Var}(X) - \mathrm{Var}(Y)\big| &= \big|\sqrt{\mathrm{Var}(X)} - \sqrt{\mathrm{Var}(Y)}\big|\big(\sqrt{\mathrm{Var}(X)} + \sqrt{\mathrm{Var}(Y)}\big) \\ &\le 2\|X - Y\|_2\big(2\sqrt{\mathrm{Var}(X)} + 2\|X - Y\|_2\big). \end{aligned} \tag{51}$$

By (51) and the subadditivity of first-passage times,

$$\begin{aligned} \mathrm{Var}(T(w_{jaq}, w_{(j+1)aq})) &\ge f(aq) - C\sqrt{f(aq)} - C \\ &\ge f(n/r) - C\sqrt{n/r}. \end{aligned}$$

Therefore by the Harris-FKG inequality,

$$\mathrm{Var}\biggl(\sum_{j=0}^{r-1} T(w_{jaq}, w_{(j+1)aq})\biggr) \ge r f(n/r) - C\sqrt{nr}. \tag{52}$$

Now, by (34) and (35), if χ_2 is sufficiently close to χ, then both $1-\beta-\varepsilon+\beta\chi_2$ and $1 - \beta + \beta'\chi_2$ are strictly smaller than $1/2$. Therefore by (46), (51) and the fact that $f(n) \le Cn$,

$$\begin{aligned} &\biggl|f(n) - \mathrm{Var}\biggl(\sum_{j=0}^{r-1} T(w_{jaq}, w_{(j+1)aq})\biggr)\biggr| \\ &\le C\sqrt{n}(n^{1-\beta-\varepsilon+\beta\chi_2}\log n + n^{1-\beta+\beta'\chi_2}\log n). \end{aligned}$$

Combining this with (52) gives

$$f(n) \ge r f(n/r) - C\sqrt{nr} - C\sqrt{n}(n^{1-\beta-\varepsilon+\beta\chi_2}\log n + n^{1-\beta+\beta'\chi_2}\log n).$$

Again by (39) and (50),

$$r f(n/r) \ge C n^{(1-\beta-\varepsilon)+(\beta+\varepsilon)2\chi_1}.$$

Combining (39) with the last two displays, it follows that we can choose χ_1 and χ_2 so close to 1/2 that as $q \to \infty$,

$$\liminf \frac{f(n)}{rf(n/r)} \geq 1.$$

In particular, for any $\delta > 0$, there exists an integer $q(\delta)$ such that if $q \geq q(\delta)$, then

$$f(n) \geq (1-\delta)rf(n/r). \tag{53}$$

Fix $\delta = (1-\beta-\varepsilon)/2$ and choose $q(\delta)$ satisfying the above criterion. Note that $q(\delta)$ can be chosen as large as we like. Let $m_0 := aq = n/r$ and $m_1 = n$. The above inequality implies that

$$\frac{f(m_1)}{m_1} \geq (1-\delta)\frac{f(m_0)}{m_0}.$$

Note that by (36), if $q(\delta)$ is chosen sufficiently large to begin with, then

$$m_1^{\varepsilon/(\beta+\varepsilon)} > Cq^{1/(\beta+\varepsilon)} > q(\delta).$$

We now inductively define an increasing sequence $m_2, m_3, \ldots$ as follows. Suppose that m_{i-1} has been defined such that

$$m_{i-1}^{\varepsilon/(\beta+\varepsilon)} > q(\delta). \tag{54}$$

Let

$$q_i := [m_{i-1}^{\varepsilon/(\beta+\varepsilon)}] + 1,$$

where $[x]$ denotes the integer part of a real number x. By (54), $q_i \geq q(\delta)$. Let $a_i := m_{i-1}/q_i$. Then if $q(\delta)$ is chosen large enough,

$$a_i \geq \frac{2}{3}m_{i-1}^{\beta/(\beta+\varepsilon)} \geq \frac{1}{2}q_i^{\beta/\varepsilon},$$

and

$$a_i \leq m_{i-1}^{\beta/(\beta+\varepsilon)} \leq q_i^{\beta/\varepsilon}.$$

Let r_i be an integer between $q_i^{(1-\beta-\varepsilon)/\varepsilon}$ and $2q_i^{(1-\beta-\varepsilon)/\varepsilon}$. Let $k_i = r_iq_i$ and $n_i = a_ik_i = a_ir_iq_i = r_im_{i-1}$. If we carry out the argument of Section 7 with q_i, r_i, k_i, a_i, n_i in place of q, r, k, a, n, then, since $q_i \geq q(\delta)$, as before we arrive at the inequality

$$f(n_i) \geq (1-\delta)r_if(n_i/r_i) = (1-\delta)r_if(m_{i-1}).$$

Define $m_i := n_i$. Then the above inequality shows that

$$\frac{f(m_i)}{m_i} \geq (1-\delta)\frac{f(m_{i-1})}{m_{i-1}}. \tag{55}$$

Note that since r_i is a positive integer and $m_i = r_im_{i-1}$, therefore $m_i \geq m_{i-1}$. In particular, (54) is satisfied with m_i in place of m_{i-1}. This allows us to carry on the inductive construction such that (55) is satisfied for each i.

Now, the above construction shows that if the initial q was chosen large enough, then for each i,

$$m_i = r_i m_{i-1} \geq q_i^{(1-\beta-\varepsilon)/\varepsilon} m_{i-1} \geq m_{i-1}^{1/(\beta+\varepsilon)}.$$

Therefore, for all $i \geq 2$,

$$m_i \geq m_1^{(\beta+\varepsilon)^{-(i-1)}}.$$

So, by (1), there exists a constant C_3 such that

$$\frac{f(m_i)}{m_i} \leq \frac{C}{\log m_i} \leq C_3(\beta+\varepsilon)^{i-1}.$$

However, (55) shows that there is $C_4 > 0$ such that

$$\frac{f(m_i)}{m_i} \geq C_4(1-\delta)^{i-1}.$$

Since $1-\delta > \beta+\varepsilon$, we get a contradiction for sufficiently large i.

9. Proof of $\chi \leq 2\xi - 1$ when $\chi = 0$

As usual, we prove by contradiction. Assume that $\chi = 0$ and $2\xi - 1 < \chi$. Then $\xi < 1/2$. Choose ξ_1, ξ' and ξ'' such that $\xi < \xi_1 < \xi'' < \xi' < 1/2$. From this point on, however, the proof is quite different than the case $\chi > 0$. Recall that $t(P)$ is the sum of edge-weights of a path P in the environment $t = (t_e)_{e \in E(\mathbb{Z}^d)}$. This notation is used several times in this section. First, we need a simple lemma about the norm g.

Lemma 9.1. *Assume that the edge-weight distribution is continuous, and let L denote the infimum of its support. Then there exists $M > L$ such that for all $x \in \mathbb{R}^d \backslash \{0\}$, $g(x) \geq M|x|_1$, where $|x|_1$ is the ℓ_1 norm of x.*

Proof. Since g is a norm on $\mathbb{R}^d$,

$$M := \inf_{x \neq 0} \frac{g(x)}{|x|_1} > 0,$$

and the infimum is attained. Choose $x \neq 0$ such that $g(x) = M|x|_1$. Define a new set of edge-weights s_e as $s_e := t_e - L$. Then s_e are non-negative and i.i.d. Let the function g^s be defined for these new edge-weights the same way g was defined for the old weights. Similarly, define h^s and T^s. Since any path P from a point y to a point z must have at least $|z-y|_1$ many edges, therefore $s(P) \leq t(P) - L|z-y|_1$. Thus,

$$T^s(y,z) \leq T(y,z) - L|z-y|_1.$$

In particular, $h^s(y) \leq h(y) - L|y|_1$ for any y. Considering a sequence y_n in $\mathbb{Z}^d$ such that $y_n/n \to x$, we see that

$$\begin{aligned} g^s(x) &= \lim_{n\to\infty} \frac{h^s(y_n)}{n} \leq \lim_{n\to\infty} \frac{h(y_n) - L|y_n|_1}{n} \\ &= g(x) - L|x|_1 = (M-L)|x|_1. \end{aligned}$$

Since t_e has a continuous distribution, s_e has no mass at 0. Therefore, by a well-known result (see [17]), $g^s(x) > 0$. This shows that $M > L$. □

Choose β, ε' and ε so small that $0 < \varepsilon' < \varepsilon < \beta < (\xi'' - \xi_1)/d$. Choose x_0 and H_0 as in Proposition 5.1. Let n be a positive integer, to be chosen arbitrarily large at the end of the proof. Again, as usual, C denotes any positive constant that does not depend on our choice of n.

Choose a point $z \in H_0$ such that $|z| \in [n^{\xi'}, 2n^{\xi'}]$. Let $v := nx_0/2 + z$. Then by Proposition 5.1 and the fact that $\xi' < 1/2$,

$$\begin{aligned} |g(v) - g(nx_0/2)| &= (n/2)|g(x_0 + 2z/n) - g(x_0)| \\ &\leq C|z|^2/n \leq Cn^{2\xi'-1} \leq C. \end{aligned} \tag{56}$$

Similarly,

$$|g(nx_0 - v) - g(nx_0/2)| \leq Cn^{2\xi'-1} \leq C. \tag{57}$$

Let w be the closest lattice point to v and let y be the closest lattice point to nx_0. Then $|w - v|$ and $|y - nx_0|$ are bounded by $\sqrt{d}$. Therefore inequalities (56) and (57) imply that

$$|g(y) - (g(w) + g(y - w))| \leq C. \tag{58}$$

Figure 8 has an illustration of the relative locations of y and w, together with some other objects that will be defined below.

By Theorem 4.1 and the assumption that $\chi = 0$, $|h(y)-g(y)|$, $|h(w)-g(w)|$ and $|h(y-w)-g(y-w)|$ are all bounded by Cn^{ε}. Again by (A1) of Theorem 1.1 and the assumption that $\chi = 0$, the probabilities $\mathbb{P}(|T(0,w) - h(w)| > n^{\varepsilon})$, $\mathbb{P}(|T(w,y)-h(y-w)| > n^{\varepsilon})$ and $\mathbb{P}(|T(0,y)-h(y)| > n^{\varepsilon})$ are all bounded by $e^{-Cn^{\varepsilon-\varepsilon'}}$. These observations, together with (58), imply that there are constants C_1 and C_2, independent of our choice of n, such that

$$\mathbb{P}(|T(0,y) - (T(0,w) + T(w,y))| > C_1 n^{\varepsilon}) \leq e^{-C_2 n^{\varepsilon-\varepsilon'}}. \tag{59}$$

Let $T_o(0,y)$ be the minimum passage time from 0 to y among all paths that do not deviate by more than $n^{\xi''}$ from the straight line segment joining 0 and y. By assumption (A2) of Theorem 1.1,

$$\mathbb{P}(T_o(0,y) = T(0,y)) \geq 1 - e^{-Cn^{\xi''-\xi_1}}.$$

Combining this with (59), we see that if E_1 is the event

$$E_1 := \{|T_o(0,y) - (T(0,w) + T(w,y))| \leq C_1 n^{\varepsilon}\}, \tag{60}$$

where C_1 is the constant from (59), then there is a constant C_3 such that

$$\mathbb{P}(E_1) \geq 1 - e^{-C_3 n^{\xi''-\xi_1}} - e^{-C_3 n^{\varepsilon-\varepsilon'}}. \tag{61}$$

Let V be the set of all lattice points within ℓ_1 distance n^{β} from w. Let ∂V denote the boundary of V in $\mathbb{Z}^d$, that is, all points in V that have at least one neighbor outside of V. Let w_1 be the first point in $G(0,w)$ that belongs to ∂V, when the points are arranged in a sequence from 0 to w. Let w_2 be the last point in $G(w,y)$ that belongs to ∂V, when the points are arranged in a sequence from w to y. Let G_1 denote the portion of $G(0,w)$ connecting w_1 and w, and let G_2 denote the portion of $G(w,y)$ connecting w and w_2. Let G_0 be the portion of $G(0,w)$ from 0 to w_1 and let G_3 be the portion of $G(w,y)$ from w_2 to y. Note that G_0 and G_3 lie entirely outside of V. Figure 8 provides a schematic diagram to illustrate the above definitions.

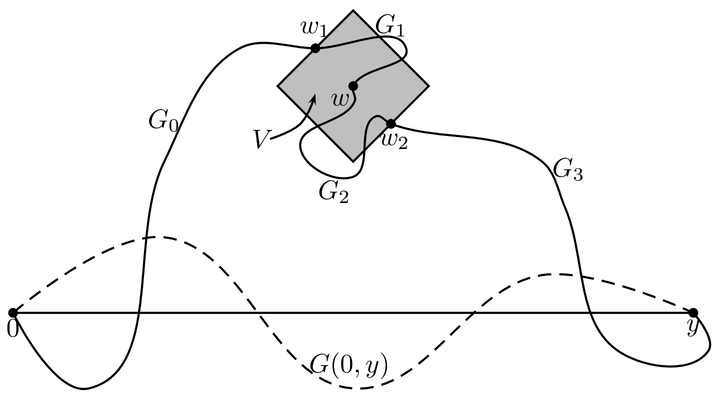

FIGURE 8. Schematic diagram for V, w, w_1, w_2 and G_0, G_1, G_2, G_3.

Let L and M be as in Lemma 9.1. Choose L', M' such that $L < L' < M' < M$. Take any $u \in \partial V$. By Lemma 9.1, $g(u-w) \geq M|u-w|_1$. Therefore by Theorem 4.1,

$$h(u-w) \geq M|u-w|_1 - C|u-w|^{\varepsilon} \geq M|u-w|_1 - Cn^{\beta\varepsilon}.$$

Now, $|u-w|_1 \geq Cn^{\beta}$. Therefore by assumption (A1) of Theorem 1.1 and the above inequality,

$$\begin{aligned}&\mathbb{P}(T(u,w) < M'|u-w|_1)\\&\leq \mathbb{P}(|T(u,w) - h(u-w)| > (M-M')|u-w|_1 - Cn^{\beta\varepsilon})\\&\leq \mathbb{P}(|T(u,w) - h(u-w)| > Cn^{\beta}) \leq e^{-n^{\beta-\varepsilon'}/C}.\end{aligned}$$

Since there are at most n^C points in ∂V, the above bound shows that

$$\mathbb{P}(T(u,w) < M'|u-w|_1 \text{ for some } u \in \partial V) \leq n^C e^{-n^{\beta-\varepsilon'}/C}.$$

In particular, if E_2 and E_3 are the events

$$E_2 := \{t(G_1) \geq M'|w-w_1|_1\},$$
$$E_3 := \{t(G_2) \geq M'|w-w_2|_1\},$$

then there is a constant C_4 such that

$$\mathbb{P}(E_2 \cap E_3) \geq 1 - n^{C_4} e^{-n^{\beta-\varepsilon'}/C_4}. \tag{62}$$

Let $E(V)$ denote the set of edges between members of V. Let $(t'_e)_{e\in E(V)}$ be a collection of i.i.d. random variables, independent of the original edge-weights, but having the same distribution. For $e \notin E(V)$, let $t'_e := t_e$. Let E_4 be the event

$$E_4 := \{t'_e \le L' \text{ for each } e \in E(V)\}.$$

If E_4 happens, then there is a path P_1 from w_1 to w and a path P_2 from w to w_2 such that $t'(P_1) \le L'|w - w_1|_1$ and $t'(P_2) \le L'|w - w_2|_1$. Let P be the concatenation of the paths G_0, P_1, P_2 and G_3. Since $t'(G_0) = t(G_0)$ and $t'(G_3) = t(G_3)$, therefore under E_4,

$$t'(P) \le t(G_0) + t(G_3) + L'|w - w_1|_1 + L'|w - w_2|_1.$$

On the other hand, under $E_2 \cap E_3$,

$$\begin{aligned} T(0,w) + T(w,y) &= t(G_0) + t(G_1) + t(G_2) + t(G_3) \\ &\ge t(G_0) + t(G_3) + M'|w - w_1|_1 + M'|w - w_2|_1. \end{aligned}$$

Consequently, if E_1, E_2, E_3, E_4 all happen simultaneously, then there is a (deterministic) positive constant C_5 such that

$$T_o(0,y) \ge t'(P) + C_5 n^{\beta} - C_1 n^{\varepsilon},$$

where C_1 is the constant in the definition (60) of E_1. Since $\beta < \xi'' < \xi'$ and $x_0 \notin H_0$, the edges within distance $n^{\xi''}$ of the line segment joining 0 and y have the same weights in the environment t' as in t. Since $\beta > \varepsilon$, this observation and the above display proves that $E_1 \cap E_2 \cap E_3 \cap E_4$ implies $D'(0,y) \ge n^{\xi''}$, where $D'(0,y)$ is the value of $D(0,y)$ in the new environment t'. (To put it differently, if $E_1 \cap E_2 \cap E_3 \cap E_4$ happens then there is a path P that has less t'-weight than the least t'-weight path within distance $n^{\xi''}$ of the straight line connecting 0 to y, and therefore $D'(0,y)$ must be greater than or equal to $n^{\xi''}$.)

Now note that the event E_4 is independent of E_1, E_2 and E_3. Moreover, since $L' > L$, there is a constant C_6 such that $\mathbb{P}(E_4) \ge e^{-C_6 n^{\beta d}}$. Combining this with (61), (62) and the last observation from the previous paragraph, we get

$$\begin{aligned} \mathbb{P}(D'(0,y) \ge n^{\xi''}) &\ge \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4) \\ &= \mathbb{P}(E_1 \cap E_2 \cap E_3)\mathbb{P}(E_4) \\ &\ge (1 - e^{-C_3 n^{\xi'' - \xi_1}} - e^{-C_3 n^{\varepsilon - \varepsilon'}} - n^{C_4} e^{-n^{\beta - \varepsilon'}/C_4}) e^{-C_6 n^{\beta d}} \\ &\ge e^{-C_7 n^{\beta d}}. \end{aligned}$$

Now $D'(0,y)$ has the same distribution as $D(0,y)$. But by (A2) of Theorem 1.1, $\mathbb{P}(D(0,y) \ge n^{\xi''}) \le e^{-C_8 n^{\xi'' - \xi_1}}$, and $\beta d < \xi'' - \xi_1$ by our choice of β. Together with the above display, this gives a contradiction, thereby proving that $\chi \le 2\xi - 1$ when $\chi = 0$.

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Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012