Schramm-Loewner Evolution and Its Applications

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January 29, 2022

Abstract

This is a brief summary notes of SLE based on joint seminars with my friends.

1 Half-plane Capacity

We define $\mathbb{H} = \{x + iy : x \in \mathbb{R}, y > 0\}$, and give the definition of $\mathbb{H}$-hull

Definition. 1.1 (compact $\mathbb{H}$-hull)**.** *A set $A \subset \mathbb{H}$ is called a compact $\mathbb{H}$-hull if both $A = \mathbb{H} \cap \bar{A}$ and $\mathbb{H}/A$ is simply connected. We denote by $\mathcal{Q}$ the set of compact $\mathbb{H}$-hulls.*

In order to give a proper definition of Capacity, we first find the right conformal map.

Proposition. 1.1. *For any $A \in \mathcal{Q}$, there exists a unique conformal map g_A from $\mathbb{H}/A$ to $\mathbb{H}$ s.t.*

$$\lim_{z\to\infty} [g_A(z) - z] = 0 \tag{1}$$

Remark. 1.1. *For g_A, we have the following Laurent series expansion*

$$g_A(z) = z + c_{-1}z^{-1} + c_{-2}z^{-2} + \cdots \tag{2}$$

where all the coefficients are real numbers.

Once we have 2, we can define half-plane capacity,

Definition. 1.2. *For any $A \in \mathcal{Q}$, we define the half-plane capacity to be*

$$hcap(A) := \lim_{z\to\infty} z(g_A(z) - z) \tag{3}$$

which is exactly

$$g_A(z) := z + \frac{hcap(A)}{z} + O\left(\frac{1}{|z|^2}\right)$$

We then list without proof of some properties:

- (Scaling) $g_{rA}(z) = rg_A\left(\frac{z}{r}\right)$ and $\text{hcap}(rA) = r^2\text{hcap}(A)$.
- (Translation) $g_{A+x}(z) = g_A(z-x)+x$ and $\text{hcap}(A+x) = \text{hcap}(A)$
- (Monotonicity) Suppose that $A \subset \tilde{A}$, we have $g_{\tilde{A}} = g_{g_A(\tilde{A}/A)} \circ g_A$, comparing the coefficients of Laurent expansion, we have

$$\text{hcap}(\tilde{A}) = \text{hcap}(g_A(\tilde{A}/A)) + \text{hcap}(A) \tag{4}$$

Proposition. 1.2. *$\forall A \in \mathcal{Q}$, let $\tau := \inf\{t > 0 : B_t \in \mathbb{R} \cup A\}$ and $\forall z \in \mathbb{H}\backslash A$, we have*

$$Im(z) = Im(g_A(z)) + E^z(Im(B_\tau)) \tag{5}$$

Proof of 1.2 is not hard, we may use optional sampling theorem and harmonicity of imaginary parts.
We then give two direct corollaries,

Corollary. 1.1.

$$hcap(A) = \lim_{y\to\infty} yE^{iy}[Im(B_\tau)] \tag{6}$$

Corollary. 1.2. *If $rad(A) < 1$(i.e. $A \subset \mathbb{D}$)*

$$hcap(A) = \frac{2}{\pi}\int_0^\pi E^{e^{i\theta}}[Im(B_\tau)]sin\theta d\theta \tag{7}$$

Why hcap is a "capacity"? We may explain by the following proposition.

Proposition. 1.3 (Capacity property)**.**

$$hcap(A_1) + hcap(A_2) \geq hcap(A_1 \cup A_2) + hcap(A_1 \cap A_2) \tag{8}$$

and we give a useful estimate to end the first section:

Proposition. 1.4. *There exists a constant $c > 0$, s.t. $\forall A \in \mathcal{Q}$ and $|z| \geq 2rad(A)$, we have*

$$|z - g_A(z) + \frac{hcap(A)}{z}| \leq c\frac{rad(A)hcap(A)}{|z|^2} \tag{9}$$

2 Loewner Differential Equation

2.1 Chordal Loewner Equation

Let $\gamma : [0,\infty) \to \mathbb{C}$ be a simple curve, where $\gamma(0) \in \mathbb{R}$ and $\gamma(0,\infty) \subset \mathbb{H}$. Let $g_t = g_{\gamma(0,t]}, (g_0(z) = z)$, $b(t) := hcap(\gamma(0,t])$ and $U(t) := g_t(r(t)) \in \mathbb{R}$.
Moreover, $\forall s \geq 0$, $t \geq 0$, we let $\gamma^s(t) := g_s(\gamma(s+t))$, using 4, we have

$$hcap(\gamma^s(0,t]) = b(s+t) - b(s)$$

$\forall 0 \leq s \leq t$, we denote $g_{\gamma^s(0,t-s]}$ by $g_{s,t}$, therefore,

$$g_t = g_{s,t} \circ g_s$$

Proposition. 2.1 (Chordal Loewner Equation). *Suppose that γ is a simple curve s.t. $b(t) \in C^1$ and $b(t) \to \infty$. Then $\forall z \in \mathbb{H}$, $g_t(z)$ satisfies*

$$\dot{g}_t(z) = \frac{\dot{b}(t)}{g_t(z) - U_t},\ g_0(z) = z \tag{10}$$

Moreover, if $z = \gamma(t_0)$, then the equation holds for all $t < t_0$ and $U_{t_0} = \lim_{t \to t_0} g_t(z)$.
If $z \notin \gamma(0, \infty)$, then the equation holds for all $t \geq 0$

Remark. 2.1. *Without loss of generality, we can assume that $b(t) = 2t$.*

In order to give the definition of SLE_κ, we first need the following propostion.

Proposition. 2.2. *Suppose that $\{u_t\}_{t \geq 0}$ is a family of nonnegative measure on $\mathbb{R}$ s.t. $t \to u_t$ is continuous in the weak topology and $\forall t \geq 0$, there exists M_t s.t.*

$$\sup\{u_s(\mathbb{R}) : 0 \leq s \leq t\} < M_t \textit{ and } supp(u_s) \in [-M_t, M_t],\ \forall s \leq t$$

For any $z \in \mathbb{H}$ and $t \geq 0$, let $g_t(z)$ be the solution of

$$\dot{g}_t(z) = \int_{\mathbb{R}} \frac{u_t(du)}{g_t(z) - u}$$

and let $T_z := \sup\{t :$ the equation above has solution $g_t(z) \in \mathbb{H}\}$ and $H_t := \{z : T_z > t\}$.
Then $g_t = g_{H_t}$ and $g_t(z) = z + \frac{b(t)}{z} + O\left(\frac{1}{|z|^2}\right)$, where $b(t) = \int_0^t u_s(\mathbb{R})ds$

3 Schramm-Loewner Evolution

3.1 Preliminaries: Bessel Process

Definition. 3.1 (Bessel Process). *A Bessel process is a process satisfying the following two conditions,*

- $X_0^x = x$
- $dX_t^x = \frac{a}{X_t^x} dt + dB_t$

Lemma. 3.1 (Scaling of Bessel Process). *If $x > 0$ and $Y_t = x^{-1} X^x_{x^2 t}$, then $Y_t \overset{(d)}{=} X_t^1$ (abbreviated as X_t).*

Lemma. 3.2 (Lemma 1.23 in G. Lawler's Book). *If $a < 1/2$ and $x > 0$, then w.p.1,*

$$I := \int_0^{T_x} \frac{1}{X_t^2} dt = \infty$$

Once we have scaling property of Bessel process, we then proceed to state without proof of the following proposition.

Proposition. 3.1 (Proposition 1.21 in G. Lawler's textbook)**.** *For different values of a, we have*

- *If $a \geq \frac{1}{2}$, then w.p.1. $T_x = \infty$ for all $x > 0$*
- *If $a = \frac{1}{2}$, then w.p.1 $\inf_t X_t^x = 0$ for all $x > 0$*
- *If $a > \frac{1}{2}$, then w.p.1 $X_t^x \to \infty$ for all $x > 0$*
- *If $a < \frac{1}{2}$, then w.p.1 $T_x < \infty$ for all $x > 0$*
- *If $\frac{1}{4} < a < \frac{1}{2}$ and $x < y$, then $\mathbb{P}(T_x = T_y) > 0$*
- *If $a \leq \frac{1}{4}$, then w.p.1 $T_x < T_y$ for all $x < y$*

3.2 Chordal SLE

Definition. 3.2 (Chordal SLE)**.** *The chordal SLE with parameter $\kappa \geq 0$, or chordal SLE_κ, is the random collection of conformal maps g_t obtained from solving the initial value problem*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \; g_0(z) = z (z \in \mathbb{H}) \tag{11}$$

The driving function $U_t = \sqrt{\kappa}B_t$, let H_t be the domain of g_t and $K_t = \mathbb{H} \backslash H_t$ be the randomly growing hulls.

Remark. 3.1. *The definition above is reasonable if we take $u_t = 2\delta_{\sqrt{\kappa}B_t}$ in 2.2*

Remark. 3.2. *When we vary κ, we get significantly different behaviors of g_t. Besides, it's not easy to see whether the chain is generated by a path.*

Theorem. 3.1. *W.p.1 SLE_κ is generated by a path.*

Definition. 3.3 (SLE_κ path)**.** *A (chordal SLE_κ path in $\mathbb{H}$ is the random curve $\gamma(t)$ that generates chordal SLE_κ).*

Proposition. 3.2 (SLE scaling)**.** *Suppose g_t is chordal SLE_κ and $r > 0$. Then*

$$\hat{g}_t(z) := r^{-1} g_{r^2 t}(rz)$$

has the distribution of SLE_κ. Equivalently, if γ is an SLE_κ path, and $\hat{\gamma}(t) := r^{-1}\gamma(r^2 t)$, then $\hat{\gamma}$ has the distribution of SLE_κ path.

Remark. 3.3. *The Chordal SLE equation is also valid for $x \in \mathbb{R} \backslash \{0\}$ and is valid up to time $T_x = \inf\{t : x \in \bar{K}_t\}$. If $T_x < \infty$, we have $\lim_{t \to T_x -} g_t(x) - \sqrt{\kappa}B_t = 0$*

Theorem. 3.2 (SLE_κ and Bessel process)**.** *Let* $\hat{g}_t(z) = \frac{g_t(z) - \sqrt{\kappa}B_t}{\sqrt{\kappa}}$*, then it satisfies the SDE*

$$d\hat{g}_t(z) = \frac{2/\kappa}{\hat{g}_t(z)} + dW_t,\ \hat{g}_0(z) = z \tag{12}$$

3.3 Phases

Apply 3.1 and after noting 12,

Proposition. 3.3. *we have*

- *If* $\kappa \leq 4$*, then w.p.1* $T_x = \infty$ *for all* $x > 0$
- *If* $\kappa > 4$*, then w.p.1* $T_x < \infty$ *for all* $x > 0$
- *If* $\kappa \geq 8$*, then w.p.1* $T_x < T_y$ *for all* $0 < x < y$
- *If* $4 < \kappa < 8$ *and* $x < y$*, then* $\mathbb{P}(T_x = T_y) > 0$

With 3.3, we then state the properties of γ with different κ.

Proposition. 3.4. *If* $\kappa \leq 4$*, then* γ *is a simple curve with* $\gamma(0, \infty) \subset \mathbb{H}$.

Proposition. 3.5. *If* $4 < \kappa < 8$*, then w.p.1,*

$$\cup_{t>0} \bar{K}_t = \bar{\mathbb{H}}$$

but $\gamma(0, \infty) \cap \mathbb{H} \neq \mathbb{H}$*. Also,* $dist(0, \mathbb{H} \backslash \bar{K}_t) \to \infty$*. In particular,* $|\gamma(t)| \to \infty$ *as* $t \to \infty$.

Proposition. 3.6. *If* $\kappa \geq 8$*, then* γ *is space filling curve, i.e.* $\gamma[0, \infty) = \bar{\mathbb{H}}$.

Proposition. 3.7. *W.p.1, for* $0 < \kappa \leq 4$*,*

$$\lim_{t \to \infty} |\gamma(t)| = \infty$$

4 The Restriction Property of $SLE_{8/3}$

4.1 The self-avoiding walk

Suppose that $G = (V, E)$ is a graph with bounded maximal degree. The SAW in G starting from x with length n is the uniform measure on simple paths in G which starts from x and have length n. We note here that the SAW has the restriction property which says conditioned on a subgraph G', the SAW has the law of SAW in G'.

4.2 Statement and characterization of the restriction property

In the rest of what follows, we assume $\kappa \leq 4$, therefore by 3.4, we know that SLE_κ is simple. We also let $\mathcal{Q}_+ := \{A \in \mathcal{Q} : \bar{A} \cap (-\infty, 0] = \phi\}$ and $\mathcal{Q}_- := \{A \in \mathcal{Q} : \bar{A} \cap [0, \infty) = \phi\}$. For $A \in Q_\pm = Q_+ \cup Q_-$, we let $\psi_A = g_A - g_A(0)$. Moreover, by 3.7, we know that $\lim_{t\to\infty} |\gamma(t)| = \infty$, which implies

$$0 < \mathbb{P}(\gamma[0, \infty) \cap A = \phi) < 1$$

We let $V_A = \{\gamma[0, \infty) \cap A = \phi\}$. We say that an SLE_κ satisfies the restriction if

$$\psi_A \circ \gamma | V_A \overset{d}{=} SLE_\kappa \text{ in } \mathbb{H} \text{ from } 0 \text{ to } \infty$$

Lemma. 4.1. *Suppose there exists $\alpha > 0$ so that $\mathbb{P}(V_A) = (\psi'_A(0))^\alpha$. Then SLE_κ satisfies restriction.*

Proof. Proof of Lemma 4.1,
Assume that $\mathbb{P}(V_A) = (\psi'_A(0))^\alpha$ for all $A \in Q_\pm$. Suppose that $A, B \in Q_\pm$, we "test" the following probability,

$$\begin{aligned}
\mathbb{P}(\psi_A \circ \gamma[0, \infty) \cap B = \phi | V_A) &= \frac{\mathbb{P}(\psi_A \circ \gamma[0, \infty) \cap B = \phi, \gamma[0, \infty) \cap A = \phi)}{\mathbb{P}(\gamma[0, \infty) \cap A = \phi)} \\
&= \frac{\mathbb{P}(\gamma[0, \infty) \cap (A \cup \psi_A^{-1}(B)) = \phi)}{\mathbb{P}(\gamma[0, \infty) \cap A = \phi)} \\
&= \frac{\left(\psi'_{A\cup\psi_A^{-1}(B)}(0)\right)^\alpha}{(\psi'_A(0))^\alpha} = \frac{((\psi_B \circ \psi_A)'(0))^\alpha}{(\psi'_A(0))^\alpha} \\
&= (\psi'_B(0))^\alpha = V_B
\end{aligned}$$

□

4.3 Brownian Excursion

Roughly speaking, a Brownian excursion in D from x to y is a Brownian motion starting from x and conditioned to stay in D until exiting at y.
Firstly, we define Brownian excursion in $\mathbb{H}$ from 0 to ∞ and extend to other domains using conformal mapping and conformal invariance of B.M.
The construction proceeds as follows:

- Suppose first that $B = (B^1, B^2)$ is a B.M. starting from $i\epsilon$.
- We then condition B on the event that $Im(B_t)$ hits $R > 0$ large before hitting 0. By gambler's ruin, we know that this event has probability ϵ/R.

- Then let $R \to \infty$ and $\epsilon \to 0$
- The limit $\hat{B} = (\hat{B}^1, \hat{B}^2)$ is given by taking $\hat{B}^1$ the standard Brownian motion and $\hat{B}^2$ the BES^3 process starting from 0.

Proposition. 4.1. *Suppose that $A \in \mathcal{Q}$ and g_A is as usual. If $z \in \mathbb{R} \backslash \bar{A}$, then we have that*

$$\mathbb{P}_z[\hat{B}[0, \infty) \cap A = \phi] = g_A'(z)$$

Proof. For each $R > 0$, let $\mathcal{I}_R = \{z \in \mathbb{H} : Im(z) = R\}$. Recall that for g_A, we have that $|g_A(z) - z| \leq 3rad(A)$ for all $z \in \mathbb{H} \backslash A$. It follows that

$$g_A(\mathcal{I}_R) \subset \{z \in \mathbb{H} : R - 3rad(A) \leq Im(z) \leq R + 3rad(A)\}$$

Let B be a complex B.M. and $\hat{B}$ be a Brownian excursion. Let

$$\sigma_R = \inf\{t \geq 0 : Im(B_t) = R\} \text{ and } \hat{\sigma}_R = \inf\{t \geq 0 : Im(\hat{B}_t) = R\}$$

For $z \in \mathbb{H} \backslash A$, we note that

$$\begin{aligned} \mathbb{P}_z[\hat{B}[0, \infty) \cap A = \phi] &= \lim_{R \to \infty} \mathbb{P}_z[\hat{B}[0, \hat{\sigma}_R) \cap A = \phi] \\ &= \lim_{R \to \infty} \frac{\mathbb{P}_z[B[0, \sigma_R) \cap (A \cup \mathbb{R}) = \phi]}{\mathbb{P}_z[B[0, \sigma_R) \cap \mathbb{R} = \phi]} \end{aligned}$$

For the denominator, using gambler's ruin formula, we have

$$\mathbb{P}_z[B[0, \sigma_R) \cap \mathbb{R} = \phi] = \frac{Im(z)}{R}$$

. Moreover, by the conformal invariance of Brownian motion, for the numerator,

$$\mathbb{P}_{g_A(z)}[B[0, \sigma_{R+3rad(A)}) \cap \mathbb{R} = \phi] \leq \mathbb{P}_z[B[0, \sigma_R) \cap (A \cup \mathbb{R}) = \phi] \leq \mathbb{P}_{g_A(z)}[B[0, \sigma_{R-3rad(A)}) \cap \mathbb{R} = \phi]$$

Applying the Gambler's ruin formula to the LHS and RHS of the inequality above, we thus see that,

$$\frac{Im(g_A(z))}{R + 3rad(A)} \leq \mathbb{P}_z[B[0, \sigma_R) \cap (A \cup \mathbb{R}) = \phi] \leq \frac{Im(g_A(z))}{R - 3rad(A)}$$

Therefore,

$$\mathbb{P}_z[\hat{B}[0, \infty) \cap A = \phi] = \frac{Im(g_A(z))}{Im(z)}$$

The proposition follows by taking $Im(z) \to 0$. $\square$

4.4 Restriction theorem for $SLE_{8/3}$

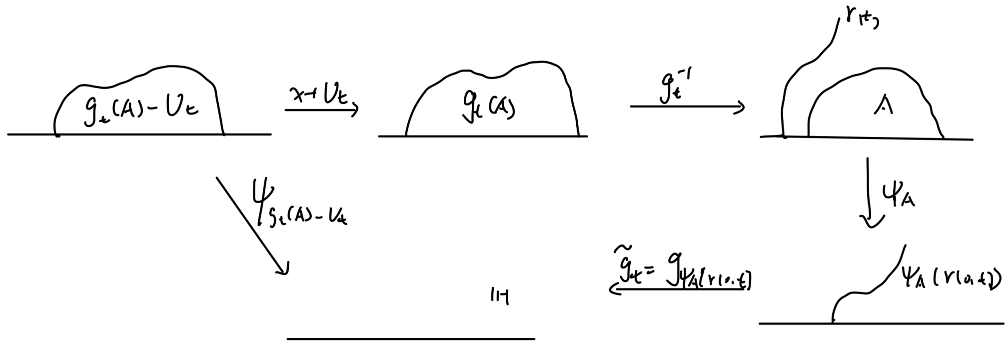
For this section, we let driving function to be $U(t) = \sqrt{\kappa}B_t$ and $\mathcal{F}_t = \sigma(U_s : s \le t)$. Consider the process $\widetilde{M}_t := \mathbb{P}(V_A|\mathcal{F}_t)$. Then it follows $\widetilde{M}$ is a bounded martingale and $\widetilde{M}_0 = \mathbb{P}(V_A)$ and $\widetilde{M}_t \to \mathbf{1}_{V_A}$ by martingale convergence theorem. Let $\tau := \inf\{t \ge 0 : \gamma(t) \in A\}$, we can see that

$$\begin{aligned}
\widetilde{M}_t &= \mathbb{P}(V_A|\mathcal{F}_t) \\
&= \mathbb{P}(V_A|\mathcal{F}_t)\mathbf{1}_{t<\tau} \\
&= \mathbb{P}(\gamma[0,\infty) \cap A = \phi|\mathcal{F}_t)\mathbf{1}_{t<\tau} \\
&= \mathbb{P}(g_t(\gamma[0,\infty)) \cap g_t(A) = \phi|\mathcal{F}_t)\mathbf{1}_{t<\tau} \\
&= \mathbb{P}(\gamma[0,\infty) \cap \{g_t(A) - U(t)\} = \phi)\mathbf{1}_{t<\tau} \\
&= \mathbb{P}(V_{g_t(A)-U(t)})\mathbf{1}_{t<\tau} \text{ (by conformal Markov property)}
\end{aligned}$$

And we have known that if M_t is another bounded martingale with $M_t \to \mathbf{1}_{V_A}$, then $M_t = \widetilde{M}_t$ almost surely. By Lemma 4.1, it is natural to guess

$$M_t = (\psi'_{g_t(A)-U(t)}(0))^{\alpha}$$

Note that



$$\Rightarrow \quad \psi_{g_t(A)-U_t}(z) = \widetilde{g}_t \circ \psi_A \circ g_t^{-1}(z + U_t)$$

therefore we let $\psi_t := \widetilde{g}_t \circ \psi_A \circ g_t^{-1}$, rewrite M_t as

$$M_t = (\psi'_t(U_t))^{\alpha}\mathbf{1}_{t<\tau}$$

Apply Itô's formula, when $\kappa = 8/3$ and $\alpha = 5/8$, we have that

$$\frac{dM_t}{\alpha M_t} = \frac{\psi''_t(U_t)}{\psi'_t(U_t)}\sqrt{\kappa}dB_t$$

Therefore M_t is a continuous local martingale. Note that it is bounded we then know that $M_{t\wedge\tau}$ is a martingale. Then the martingale convergence theorem implies that

$$M_{t\wedge\tau} \to M_\infty \text{ as } t \to \infty \text{ a.s.}$$

In particular, $0 \leq M_\infty \leq 1$, our goal is to show $M_\infty = \mathbf{1}_{V_A}$. We will accomplish this in two steps:

- Step 1: $M_{t\wedge\tau} \to 1$ on V_A a.s.
- Step 2: $M_{t\wedge\tau} \to 0$ on V_A^c a.s.

By scaling, we may assume that $\sup Im(w) : w \in A = 1$. For each $r > 2$, we let $\sigma_r := \inf\{t \geq 0 : Im(\gamma(t)) = r\}$. We note that $\sigma_r < \infty$ a.s. for all $r > 0$ since $SLE_{8/3}$ is transient.
Note that

$$\begin{aligned}\psi'_{g_t(A)-U_t}(0) &= \mathbb{P}_0[\widehat{B}[0,\infty) \cap \{g_t(A) - U_t\} = \phi] \\ &= \mathbb{P}_{U_t}[\widehat{B}[0,\infty) \cap \{g_t(A)\} = \phi] \\ &= \mathbb{P}_{\gamma(t)}[\widehat{B}[0,\infty) \cap A = \phi]\end{aligned}$$

In particular,

$$\psi'_{g_{\sigma_r}(A)-U_{\sigma_r}}(0) = \mathbb{P}_{\gamma(\sigma_r)}[\widehat{B}[0,\infty) \cap A = \phi]$$

. Now we proceed to show that

$$1 - \psi'_{g_{\sigma_r}(A)-U_{\sigma_r}}(0) = \mathbb{P}_{\gamma(\sigma_r)}[\widehat{B}[0,\infty) \cap A \neq \phi] \to 0$$

Note that by definition of Brownian excursion, we have

$$1 - \psi'_{g_{\sigma_r}(A)-U_{\sigma_r}}(0) = \lim_{\epsilon\to 0}\lim_{R\to\infty} \frac{\mathbb{P}_z[B[0,\tau_R] \subseteq \mathbb{H}\backslash\gamma[0,\sigma_r], B[0,\tau_R] \cap A \neq \phi]}{\mathbb{P}_z[B[0,\tau_R] \subseteq \mathbb{H}\backslash\gamma[0,\sigma_r]]}$$

where B is a complex Brownian motion, $z = \gamma(\sigma_r) + i\epsilon$ and

$$\tau_R = \inf\{t \geq 0 : Im(B_t) = R\}$$

Proof of step 1. Firstly, we use figure 1 to illustrate our estimate of the denominator in step 1,
We let $\eta = \inf\{t \geq 0 : B$ hits the boundary of the square$\}$.
For the denominator, we consider B hit the upper side of the square, which has probability larger than 1/4, by strong Markov property and Gambler's ruin formula, we have,

$$\mathbb{P}_z[B[0,\tau_R] \subseteq \mathbb{H}\backslash\gamma[0,\sigma_r]] \geq \frac{1}{4} \cdot \mathbb{P}(B[0,\eta] \cap \gamma[0,\sigma_r]) \cdot \frac{1}{R-r} \tag{13}$$

For the numerator, we first state Beurling estimate,

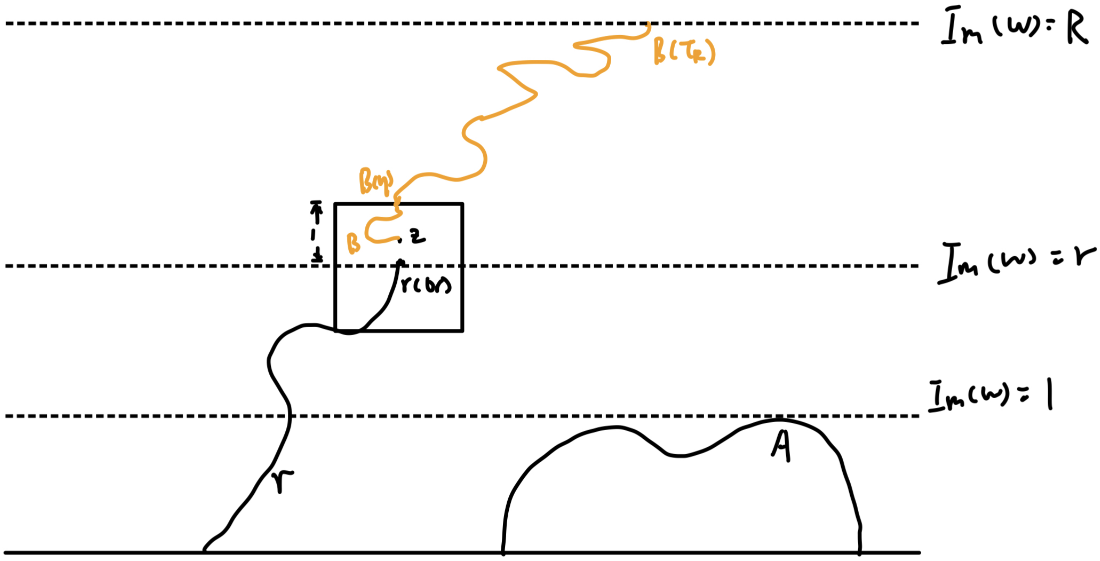


Figure 1: Estimate of denominator in step 1

Theorem. 4.1 (Beurling Estimate)**.** *There exists constant $c > 0$ such that the following is true. Suppose that B is a complex Brownian motion and $A \subseteq \overline{\mathbb{D}}$ is connected with $0 \in A$ and $A \cap \partial\mathbb{D} \neq \phi$, then*

$$\mathbb{P}_z(B[0,\tau] \cap A = \phi) \leq c|z|^{1/2}$$

where $\tau = \{t \geq 0 : B_t \notin \mathbb{D}\}$.

We then use figure 2 to illustrate our estimate for the numerator, By strong Markov property, we now estimate the green line:

$$\begin{aligned}
\mathbb{P}_{B\eta}[B[0,\tau_R] \cap A \neq \phi, B[0,\tau_R] \cap (\gamma(0,\sigma_r) \cup \mathbb{R}) = \phi] &\leq \mathbb{P}_{B_\eta}[B[0,\tau_R] \cap \mathbb{R} = \phi] \\
&\leq \mathbb{P}_{B_\eta}[B[0,\tau_R] \cap (\mathbb{R} \cap B(0,f(r))) = \phi] \\
&\leq \mathbb{P}_{B_\eta}[\frac{1}{f(r)}B[0,\tau_{f(r)}] \cap (\mathbb{R} \cap B(0,1)) = \phi] \\
&= \mathbb{P}_{B_\eta/f(r)}[B[0,\tau_{f(r)}/(f(r))^2] \cap (R \cap B(0,1)) = \phi] \\
&= \mathbb{P}_{B_\eta/f(r)}[B[0,\tau_1] \cap (R \cap B(0,1)) = \phi] \\
&\leq C\left(\frac{B(\eta)}{f(r)}\right)^2 \text{ (Beurling estimate)}
\end{aligned}$$

Here, we let $B(\eta) = o(f(r))$ and $f(r) = o(R)$, then the probability is $o(1)$. Then we use strong Markov property and Gambler's ruin formula again.

$$\text{numerator} \leq \mathbb{P}(B[0,\eta] \cap \gamma[0,\sigma_r]) \cdot o(1) \cdot \frac{1}{R} \tag{14}$$

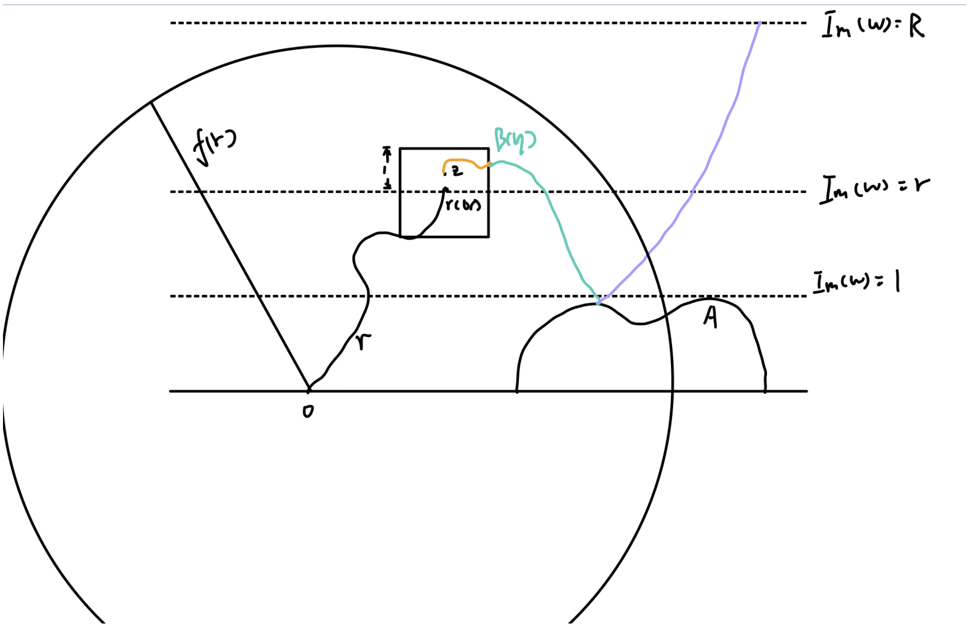


Figure 2: Estimate of numerator in step 1

Combining 13 and 14 yields the desired result. □

Proof of step 2. We now show $M_{t\wedge\tau} \to 0$ on V_A^c as $t \to \infty$.
We suppose that the upper boundary of A is smooth enough and can be

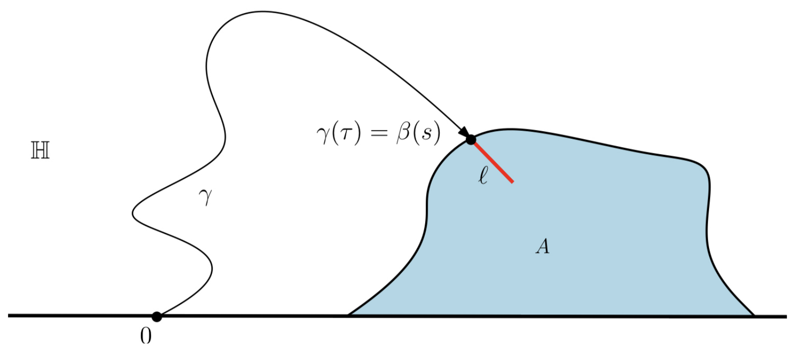


Figure 3: Illustration of Step 2, from J. Miller

parametrized by $\{\beta(s) : 0 \leq s \leq 1\}$. For each $M \in \mathbb{N}$, we let

$$\tau_M = \inf\{t \geq 0 : |\gamma(t) - \beta(s)| = 1/M\}$$

.
Since β is smooth, we can choose $\delta > 0$ so that

$$l = [\beta(s), \beta(s) + \delta\vec{n}]$$

Note that there exists $p_0 > 0$ such that a Brownian motion starting from any point on l has probability at least p_0 of hitting $\mathbb{R} \cup \gamma[0, \tau_M]$ for the first time on the right side of $\gamma[0, \tau_M]$. The same is true for left side. This implies

$$L_{\tau_M} := g_{\tau_M}(l) - U_{\tau_M}$$

is in a sector in $\mathbb{H}$. By the conformal invariance of Brownian motion, it's not hard to see that the probability a brownian excursion starting from $\gamma(\tau_M)$ hits A tends to 1 as $M \to \infty$. $\square$

5 Critical Site Percolation on Hexagonal Lattice and Locality of SLE_6

5.1 Exploration path

It has been known that on hexagonal lattice, the critical value $p_c^{site} = 1/2$, which means each face of the lattice is black with probability $1/2$. See figure 4 There exists a unique interface from x to y with black hexagons on its left

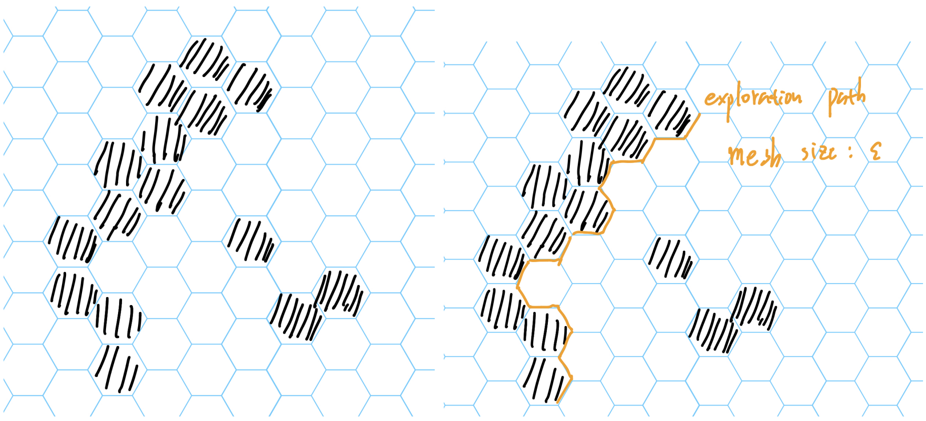


Figure 4: Percolation on hexagon lattice and exploration path

side and white on its left side. See figure 4. We denote this path by γ_ϵ where ϵ is the mesh size of the hexagons.

Theorem. 5.1 (Convergence of γ_ϵ). *There exists a γ, s.t.*

$$\gamma_\epsilon \to \gamma \text{ in law}$$

We omit the proof here, later we will show that $\gamma \stackrel{d}{=} SLE_6$. We call γ the scaling limit of exploration path.
It was conjectured(proved by Smirnov) that the limit γ is conformally invariant, which means that if D is a domain in $\mathbb{H}$ and $\widetilde{D}$ is the image of D under conformal map ψ, then $\psi(\gamma)$ is equal in distribution to the scaling

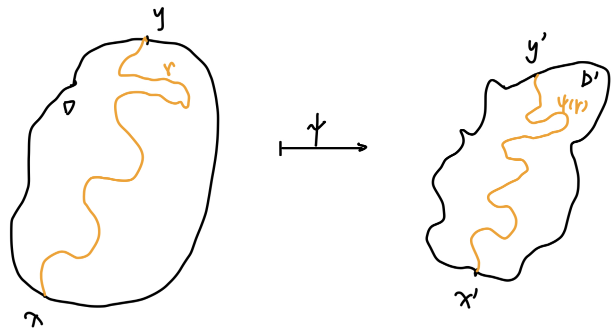


Figure 5: Conformal invariance of the scaling limit

limit on $\widetilde{D}$ from $\widetilde{x} = \psi(x)$ to $\widetilde{y} = \psi(y)$ which we denote by γ'. See figure 5. Moreover, γ satisfies domain Markov property. We want to find a κ such that SLE_κ satisfies domain Markov property and is conformally invariant.

5.2 Locality property

5.2.1 Locality property of γ

In special situation that we consider the percolation exploration on $\mathbb{H}$, it can be formulated as follows. Suppose that D is a simply connected domain in $\mathbb{H}$ with 0 on its boundary. Then a percolation exploration on D with black (resp. white) boundary conditions on $\mathbb{R}^- \cap \partial D$ (resp. $\mathbb{R}^+ \cap \partial D$), run up until hitting $\partial D \backslash \partial \mathbb{H}$, has the same distribution as a percolation exploration in all of $\mathbb{H}$ with black (resp. white) boundary conditions on $\mathbb{R}^-$ (resp. $\mathbb{R}^+$), also stopped upon hitting $\partial D \backslash \partial \mathbb{H}$. See figure 6

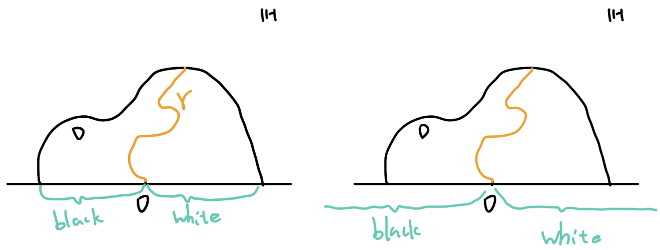


Figure 6: Locality of percolation exploration

5.2.2 Locality of SLE_6

Therefore, the SLE_κ should satisfy an analogous property. That is, we want to figure out for which value of κ the following is true. Suppose that $D \subseteq \mathbb{H}$ is a simply connected domain with 0 on its boundary. Let γ be an SLE_κ in $\mathbb{H}$ from 0 to ∞ and consider γ stopped upon hitting $\partial D \backslash \mathbb{H}$. Then we want that γ has the same law as an SLE_κ in D starting from 0 stopped at the analogous time. Formally, if ψ: $D \to \mathbb{H}$ is a conformal transformation with $\psi(0) = 0$, then we want that $\psi(\gamma)$ is an SLE_κ in $\mathbb{H}$. This is the so-called "locality property". We then state the theorem for SLE_6.

Theorem. 5.2. *If γ is an SLE_6 curve, then $\psi(\gamma)$ is an SLE_6 (up until hitting $\psi(\partial D \backslash \mathbb{H})$ and considered modulo a time-change).*

Proof of 5.2. We have known that,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \; g_0(z) = z$$

Let $\widetilde{A_t} = \psi(A_t)$, $\widetilde{a}(t) = hcap(\widetilde{A_t})$. For $\widetilde{g}_t(z)$, we have that

$$\partial_t \widetilde{g}_t(z) = \frac{\partial_t \widetilde{a}(t)}{\widetilde{g}_t(z) - \widetilde{U}_t}, \; \widetilde{g}_0(z) = z$$

where $\widetilde{U}_t = \psi_t(U_t)$. Also,

$$\widetilde{a}(t) = \int_0^t 2(\psi_s'(U_s))^2 ds$$

We define $\psi_t = \widetilde{g}_t \circ \psi \circ g_t^{-1}$ and use figure 7 to explain this map. Next,

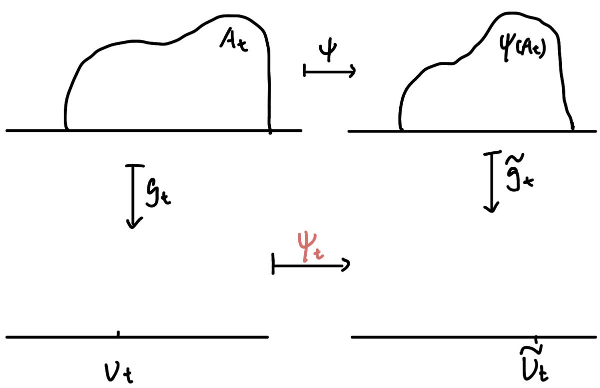


Figure 7: Representation of ψ_t

we give a proposition of ψ_t,

Proposition. 5.1. *The maps ψ_t satisfy*

$$\partial_t\psi_t(z) = 2\left(\frac{(\psi_t'(U_t))^2}{\psi_t(z)-\psi_t(U_t)} - \psi_t'(z)\frac{1}{z-U_t}\right)$$

Moreover, at $z = U_t$, we have

$$\partial_t\psi_t(U_t) = \lim_{z\to U_t}\partial_t\psi_t(z) = -3\psi_t''(U_t)$$

Proof of 5.1. We have,

$$\begin{aligned}\partial_t\psi_t(z) &= (\partial_t\widetilde{g}_t)(\psi(g_t^{-1}(z))) + \widetilde{g}_t'(\psi(g_t^{-1}(z)))\psi'(g_t^{-1}(z))\partial_t(g_t^{-1}(z)) \\ &= \frac{2(\psi_t'(U_t))^2}{\psi_t(z)-\psi_t(U_t)} + \widetilde{g}_t'(\psi(g_t^{-1}(z)))\psi'(g_t^{-1}(z))\partial_t(g_t^{-1}(z))\end{aligned} \tag{15}$$

Moreover, notice that

$$\begin{aligned}0 = \partial_t(g_t(g_t^{-1}(z))) &= (\partial_t g_t)(g_t^{-1}(z)) + g_t'(g_t^{-1}(z))\partial_t(g_t^{-1}(z)) \\ &= \frac{2}{z-U_t} + g_t'(g_t^{-1}(z))\partial_t(g_t^{-1}(z))\end{aligned}$$

which means that

$$\partial_t(g_t^{-1}(z)) = -\frac{2}{z-U_t}\cdot(g_t^{-1})'(z)$$

Plug into 15, we have proved the first assertion. For the second assertion, we only need to use Taylor's expansion and the desired results follows. □

Note that $U_t = \sqrt{\kappa}B_t$, apply Itô's formula,

$$\begin{aligned}d\widetilde{U}_t &= d\psi_t(U_t) \\ &= (\partial_t\psi_t(U_t) + \frac{\kappa}{2}\psi_t''(U_t))dt + \sqrt{\kappa}\psi_t'(U_t)dB_t \\ &= (-3\psi_t''(U_t) + \frac{\kappa}{2}\psi_t''(U_t))dt + \sqrt{\kappa}\psi_t'(U_t)dB_t \text{ (by Proposition 5.1)} \\ &= \frac{\kappa-6}{2}\psi_t''(U_t)dt + \sqrt{\kappa}\psi_t'(U_t)dB_t.\end{aligned}$$

We now let

$$\sigma(t) = \inf\{u \geq 0 : \int_0^u (\psi_s'(U_s))^2 ds = t\}$$

Then

$$\partial_t\widetilde{g}_{\sigma(t)}(z) = \frac{2}{\widetilde{g}_{\sigma(t)} - \widetilde{U}_{\sigma(t)}}$$

Also, if we let $\widetilde{U}_t^* = \widetilde{U}_{\sigma(t)}$, then we have that

$$\widetilde{U}_t^* = \frac{\kappa-6}{2}\frac{\psi_{\sigma(t)}''(U_{\sigma(t)})}{(\psi_{\sigma(t)}'(U_{\sigma(t)}))^2}dt + \sqrt{\kappa}d\widetilde{B}_t$$

where $\widetilde{B}_t = \int_0^{\sigma(t)}\psi_s'(U_s)dB_s$ is a standard Brownian motion by Lévy's characterization. We have proved $(\widetilde{A}_{\sigma(t)})$ is equal in distribution to the family of hulls of SLE_6. The proof is completed. □

5.3 Cardy-Smirnov's formula

Theorem. 5.3 (Smirnov)**.** *Suppose that D is a simply connected domain and γ is an SLE_6 in D. Consider points o, a, c, x on ∂D. Let ψ be the conformal transformation from D to Δ, a equilateral triangular domain $o'a'c'$. and let o', a', c', x' be the image of o, a, c, x under ψ. Then we have that*

$$\mathbb{P}(\gamma \text{ hits } \widehat{cx} \text{ before hitting } \widehat{ax}) = \frac{|c'x'|}{|c'a'|}$$

For an illustration, see figure 8,

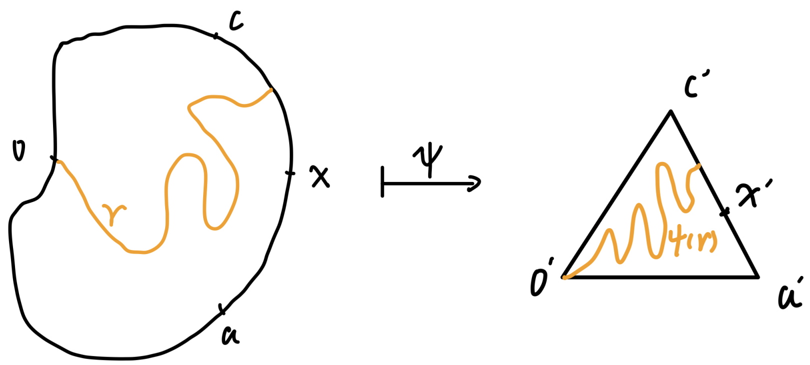


Figure 8: Illustration of theorem 5.3

Corollary. 5.1. *The hitting point to $c'a'$ has the uniform distribution.*

5.4 Convergence to SLE_6

Theorem. 5.4. *If $\gamma_\epsilon \to \gamma$, then $\gamma \overset{d}{=} SLE_6$*

Proof Sketch of theorem 5.4. Roughly, it can be proved in the following two steps:

- For every subsequence $\{\gamma_{\epsilon_n}\}$, there exists a further subsequence $\{\gamma_{\epsilon_{n_k}}\}$ which converge in law.
- The limiting curve of $\{\gamma_{\epsilon_{n_k}}\}$ is SLE_6.

The first step follows from tightness. To prove tightness, we need to show they are uniformly bounded and equicontinuous.
The proof of step 2 is similar to Cardy's formula. □

6 Gaussian Free Field and SLE_4

In this section, we mainly introduce GFF and its relation to SLE_4.

6.1 The discrete GFF

6.1.1 Notations

When f is a function from $\mathbb{Z}^d$ to $\mathbb{R}$, we define $\bar{f}$ to be the "average" of its neighbors. In other words,

$$\bar{f}(x) = \frac{1}{2d} \sum_{y:y\sim x} f(y)$$

Definition. 6.1 (Discrete Laplacian (**may not be standard**))**.** *We define the discrete Laplacian Δf of f to be the function*

$$\Delta f(x) := \bar{f}(x) - f(x)$$

Definition. 6.2 (An alternative definition of discrete Laplacian)**.** *In most cases, we define the discrete Laplacian to be*

$$\Delta f := \sum_{y:y\sim x} (f(y) - f(x))$$

Remark. 6.1. *These two definitions are essentially the same because they only differ by a multiplicative constant.*

When D is a subset of $\mathbb{Z}^d$, we define its **discrete boundary** to be $\{x \in \mathbb{Z}^d : d(x, D) = 1\}$. Moreover, let $E_{\overline{D}}$ to be the set of edges of $\mathbb{Z}^d$ such that at least one end-point of the edge is in D. We denote by $\mathcal{F}_{(D)}$ the set of functions from $\mathbb{Z}^d$ to $\mathbb{R}$ which equal to 0 outside of D.
When F is a function from $\overline{D}$ into $\mathbb{R}$, for $e = (xy) \in E_{\overline{D}}$, we define the "gradient" to be $|\nabla F(e)| = |F(x) - F(y)|$. Finally, when D is finite, we define the **Dirichlet Energy** of function F to be

$$\mathcal{E}_D(F) := \sum_{e\in E_{\overline{D}}} |\nabla F(e)|^2$$

We use a figure 9 to illustrate the notation,

6.1.2 Two definitions of DGFF

Definition. 6.3 (Discrete GFF via its density function)**.** *The discrete GFF in D with Dirichlet boundary conditions (zero boundary condition, abbreviated as b.c.) on ∂D is the centered Gaussian vector $(\Gamma(x))_{x\in D}$ whose density function on $\mathbb{R}^D$ at $(\gamma(x))_{x\in D}$ is a constant multiple of*

$$exp\left(-\frac{1}{2} \times \frac{\mathcal{E}_D(F)}{2d}\right) = exp\left(-\frac{1}{2} \times \frac{1}{2d} \sum_{e\in E_{\overline{D}}} |\nabla\gamma(e)|^2\right)$$

with the convention that $\gamma = 0$ on ∂D.

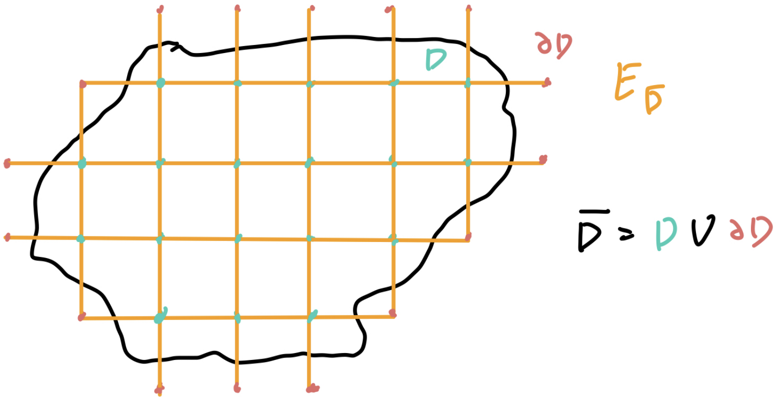


Figure 9: Illustration of Notations in DGFF

Remark. 6.2. *It is well-defined since $\mathcal{E}_D(F)$ is positive defined.*

Now we ask what is the conditional distribution of $\Gamma(x)$ given $\{\Gamma(y); y \neq x\}$. A slight inspection of the density function shows that the conditional distribution of $\Gamma(x)$ given $(\Gamma(y))_{y \in D\setminus\{x\}} = (h(y))_{y\in D\setminus\{x\}}$ has a density which is proportional to

$$exp\left(-\frac{1}{2\times 2d}\sum_{y:y\sim x}|\gamma_x - h(y)|^2\right)$$

We then notice this equals to

$$exp\left(-\frac{1}{2}(\gamma_x - \overline{h}(x))^2\right)$$

times normalizing constant. In other word, this conditional law is $\mathcal{N}(\overline{h}(x), 1)$ and $\Gamma(x) - \overline{\Gamma}(x)$ is independent of $(\Gamma(x))_{x\in D\setminus\{x\}}$. Also, it implies that the $(\Gamma(x))_{x\in D}$ have zero mean and are Gaussian.
We then compute covariance function of Γ, we denote this covariance function by

$$\Sigma(x,y) = \Sigma_x(y) := \mathbb{E}[\Gamma(x)\Gamma(y)]$$

For $x \neq y$ both in D,

$$\begin{aligned}\Sigma_x(y) = \mathbb{E}[\Gamma(x)\Gamma(y)] &= \mathbb{E}[\Gamma(x)\overline{\Gamma}(y)] + \mathbb{E}[\Gamma(x)(\Gamma(y) - \overline{\Gamma}(y))]\\ &= \frac{1}{2d}\sum_{z:z\sim y}\mathbb{E}[\Gamma(x)\Gamma(z)] = \overline{\Sigma_x}(y)\end{aligned}$$

For $x \in D$,

$$\begin{aligned}\Sigma_x(x) &= \mathbb{E}[(\Gamma(x))^2] = \mathbb{E}[\Gamma(x)\overline{\Gamma}(x)] + \mathbb{E}[\Gamma(x)(\Gamma(x) - \overline{\Gamma}(x))] \\ &= \frac{1}{2d}\sum_{z:z\sim x}\mathbb{E}[\Gamma(x)\Gamma(z)] + \mathbb{E}[(\Gamma(x) - \overline{\Gamma}(x))^2] \\ &+ \mathbb{E}[\overline{\Gamma}(x)(\Gamma(x) - \overline{\Gamma}(x))] \\ &= \overline{\Sigma_x}(x) + 1\end{aligned}$$

In other words, we have

$$\Delta\Sigma_x(y) = -\mathbf{1}_{\{y=x\}}$$

Definition. 6.4 (Green's function)**.** *We define the Green's function G_D in D to be the function defined on $D \times D$ by*

$$G_D(x,y) := E_x\left[\sum_{k=0}^{\tau-1} \mathbf{1}_{\{X_k=y\}}\right]$$

where $\{X_k; k \geq 0\}$ is a SRW on $\mathbb{Z}^d$, and we denote the law and the expectation by P_x and E_x respectively when it is started at x.

The Green's function is symmetric, i.e. $G_D(x,y) = G_D(y,x)$,

Proof.

$$\begin{aligned}G_D(x,y) &= E_x[\sum_{k\geq 0}\mathbf{1}_{\{X_k=y,k<\tau\}}] = \sum_{k\geq 0} P_x(X_k = y, k < \tau) \\ &= \sum_{k\geq 0}\#\{\text{paths } x \to y \text{ in } k \text{ steps within } D\}(1/2d)^k\end{aligned}$$

which is symmetric in x and y. □

Proposition. 6.1 (The inverse of $-\Delta_D$)**.** *The Green's function G_D is the inverse of $-\Delta_D$, and it is equal to Σ.*

Proof is easy. After noting this fact, we then give our second definition of Green's function,

Definition. 6.5 (Discrete GFF via the covariance function)**.** *The discrete Gaussian Free Field in D with Dirichlet boundary conditions on ∂D is the centered Gaussian process $(\Gamma(x))_{x\in D}$ with covariance function $G_D(x,y)$ on $D \times D$*

6.2 Continuum Gaussian Free Field

6.2.1 One possible definition by scaling limit

For $D \in \mathbb{C}$, We consider the discrete GFF on rescaled lattice $\delta\mathbb{Z}^d$, denoted by $(\widetilde{\Gamma}(x))_{x\in D_\delta}$, where D_δ is defined by the union of the $\delta-$square which intersects D. Then as $\delta \to 0$, we want the process converge to a limit. However, in this way, we cannot have our desired continuum GFF. This is because of the fact

$$\Sigma_\delta(x,x) = \mathbb{E}(\widetilde{\Gamma}_\delta(x))^2 = G_{D_\delta}(x,x) \to \infty$$

Therefore, we are not supposed to define the continuum GFF in this manner.

6.2.2 Definitions of continuum GFF

Throughout this part, D will be an open set of $\mathbb{R}^d$, and satisfying the following properties:

- if $d = 2$, then $D \neq \mathbb{R}^2$.
- if $\partial D \neq \phi$, then all boundary point $z \in \partial D$ are regular, meaning that for B a d-dimensional Brownian motion started from z, we have $\inf\{t > 0, B_t \notin D\} = 0$ almost surely. It's not very restrictive, for instance, it will be satisfied by any domain D with a smooth boundary.

These two requirements is to ensure that the Green's function in D is finite.

Definition. 6.6 (Continuum GFF)**.** *We say that the process $(\Gamma(\mu))_{\mu\in\mathcal{M}}$ is a Gaussian Free Field in D if it is a centered Gaussian process with covariance function*

$$\Sigma(\mu,\nu) := \int_{D\times D} G_D(x,y)d\mu(x)d\nu(x)$$

where $\mathcal{M}$ is the vector space of signed measures with compact support.

Definition. 6.7 (GFF with non-constant boundary conditions)**.** *We say that $\hat{\Gamma}$ is a GFF in D with boundary conditions given by H if $\hat{\Gamma} = H + \Gamma$, where Γ is a Dirichlet GFF in D. The equation $\hat{\Gamma} = H + \Gamma$ should be understand in the sense that*

$$\hat{\Gamma}(\mu) = \int H\mu(dx) + \Gamma(\mu)$$

6.2.3 Green's functions on D

We have known that the Green's functions on $\mathbb{R}^d$ is

$$H_y(x) = \begin{cases} \frac{1}{2\pi} log\left(\frac{1}{|x-y|}\right), \, d = 2 \\ \frac{1}{a_d}|x-y|^{2-d}, \, d \geq 3 \end{cases}$$

where a_d is the $d-1$ surface measure of the unit ball in $\mathbb{R}^d$, for Green's function in D, we let

$$h_{y,D}(x) = E^x[H_y(B_\tau)]$$

where τ is the first time that Brownian motion goes outside of D and it is the unique solution to the Dirichlet problem in D with boundary conditions H_y on ∂D.

Definition. 6.8 (Green's function). *For $x \neq y$ in D, we set*

$$G_D(x,y) := H_y(x) - h_{y,D}(x)$$

For $d = 2$, the Green's function has the conformal invariance property, which we state formally below.

Proposition. 6.2 (Conformal Invariance of Green's function). *The Green's function is conformally invariant. That is, if $D, \widetilde{D} \subseteq \mathbb{C}$ are domains and $\phi : D \to \widetilde{D}$ is a conformal transformation, then*

$$G_D(x,y) = G_{\widetilde{D}}(\phi(x), \phi_(y))$$

Proof of 6.2. Let $\psi(x,y) := G_D(x,y) - G_{\widetilde{D}}(\phi(x), \phi_(y))$, it's easy to verify that $\psi(x,y)$ is harmonic in x and y, then it follow by the fact that $\psi(x,y)$ vanishes on ∂D. $\square$

6.3 SLE$_4$ and GFF

Theorem. 6.1. *Let γ be the SLE_4 from -1 to 1 in $\mathbb{D}$, let Γ^+_λ be the GFF with b.c. λ on u^+ and let Γ^-_λ be the GFF with b.c. $-\lambda$ on u^-. Moreover, we require the two process are independent. For any f, we set*

$$\widetilde{\Gamma}(f) := \Gamma^+(f\boldsymbol{1}_{u^+}) + \Gamma^-(f\boldsymbol{1}_{u^-})$$

Then $\{\widetilde{\Gamma}\}$ is a GFF with b.c. $\lambda\boldsymbol{1}_{\partial D^+} - \lambda\boldsymbol{1}_{\partial D^-}$

For a illustration of 6.1, see figure 10,

Theorem. 6.2. *Let D be a triangular domain, $\{\Gamma(x)\}$ is the DGFF on D_δ, of which the upper side of the boundary equals λ and the lower side the boundary equals $-\lambda$. Moreover, for a point on the edge of D_δ, we let the value of this point to be the linear interpolation of the endpoints of the edge. We then define the curve γ_δ to be the unique path from a_δ to b_δ on the left side of which is positive while the right side is negative.*
Then we have that

$$\gamma_\delta \xrightarrow{d} \gamma(SLE_4)$$

For a illustration of 6.2, see figure 11,

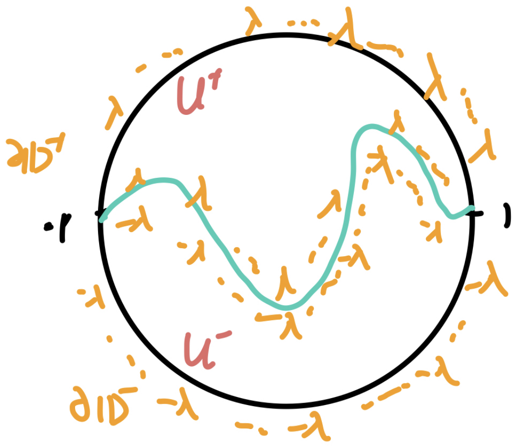


Figure 10: Illustration of 6.1

Theorem. 6.3. *Let $\lambda = \pi/2$. Let γ be an SLE_4 in $\mathbb{H}$ from 0 to ∞, (g_t) be its Loewner evolution with driving function $U_t = \sqrt{\kappa}B_t = 2B_t$, and $f_t = g_t - U_t$. Let $W \subseteq \mathbb{H}$ open and let $\tau := \inf\{t \geq 0 : \gamma(t) \in W\}$. Let h be a GFF on $\mathbb{H}$ and $\eta = \lambda - \frac{2\lambda}{\pi} arg(\cdot)$. Then we have that*

$$h \circ f_{t\wedge\tau} + \eta \circ f_{t\wedge\tau} \stackrel{d}{=} h + \eta$$

where the left and right sides are restricted to W.

For an illustration of 6.3, see figure 12,

Before we proceed to prove theorem 6.3, firstly we state some facts which is useful.

- The function η is harmonic in $\mathbb{H}$ with boundary conditions given by $-\lambda$ on $\mathbb{R}_-$ and λ on $\mathbb{R}^+$.
- The function $\eta \circ f_{t\wedge\tau}$ is harmonic in $\mathbb{H}\backslash\gamma[0, t\wedge\tau]$ with boundary conditions on $-\lambda$ on $\mathbb{R}^-$ and left side of $\gamma[0, t\wedge\tau]$, λ on $\mathbb{R}^+$ and right side of $\gamma[0, t\wedge\tau]$.

Proof of theorem 6.3. We only need to show that for $\phi \in C_0^\infty(W)$,

$$(h \circ f_{t\wedge\tau} + \eta \circ f_{t\wedge\tau}, \phi) \stackrel{d}{=} (h + \eta, \phi)$$

In other words, we need to show that the LHS is a $N(m_0(\phi), \sigma_0^2(\phi))$ random variable where

$$m_0(\phi) = (\eta, \phi) \text{ and } \sigma_0^2(\phi) = \int\int \phi(x)G_{\mathbb{H}}(x, y)\phi(y)dxdy$$

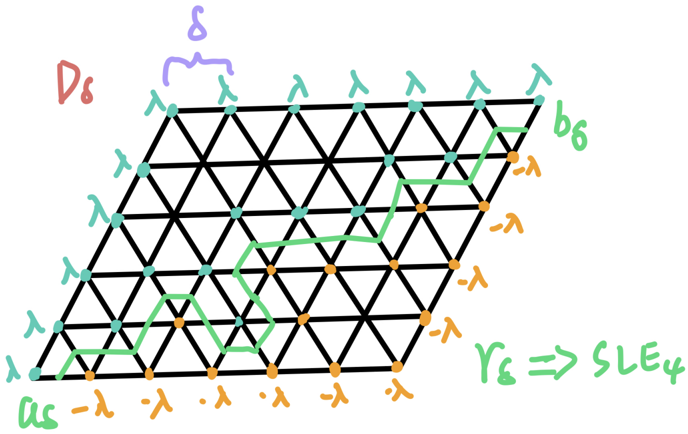


Figure 11: Illustration of 6.2

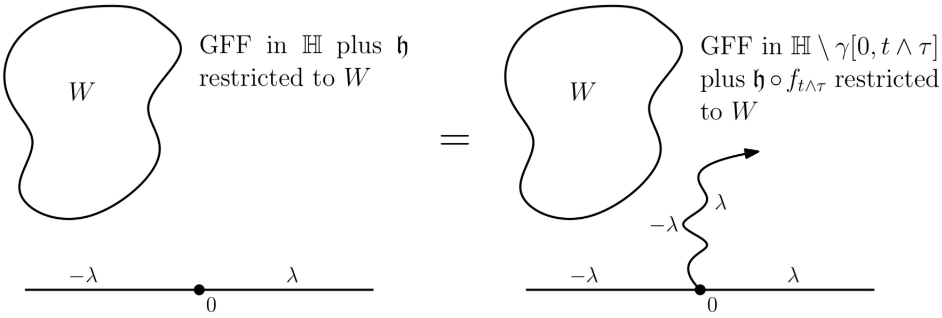


Figure 12: Illustration of 6.3, from J. Miller

This is equivalent to showing that

$$\mathbb{E}[\exp[i\theta(h \circ f_{t\wedge\tau} + \eta \circ f_{t\wedge\tau}, \phi)]] = \exp[i\theta m_0(\phi) - \frac{\theta^2}{2}\sigma_0^2(\phi)]$$

Let $(\mathcal{F}_t)$ be the filtration generated by U_t. Then we have that

$$\begin{aligned}&\mathbb{E}[\exp[i\theta(h \circ f_{t\wedge\tau} + \eta \circ f_{t\wedge\tau}, \phi)]|\mathcal{F}_{t\wedge\tau}]\\ &= \mathbb{E}[\exp[i\theta(h \circ f_{t\wedge\tau}, \phi)]|\mathcal{F}_{t\wedge\tau}]\exp(i\theta m_{t\wedge\tau}(\phi))\end{aligned}$$

where $m_t(\phi) = (\eta \circ f_t, \phi)$. Note that the conditional law of $h \circ f_{t\wedge\tau}$ given $\mathcal{F}_{t\wedge\tau}$ is that of a GFF on $\mathbb{H}\setminus\gamma[0, t \wedge \tau]$, then the previous formula equals to

$$\exp(i\theta m_{t\wedge\tau}(\phi) - \frac{\theta^2}{2}\sigma_{t\wedge\tau}^2(\phi))$$

where

$$\sigma_t^2(\phi) = \int \int \phi(x) G_t(x,y) \phi(y) dx dy$$
$$G_t(x,y) = G_{\mathbb{H}}(g_t(x), g_t(y)) \text{ (by conformal invariance 6.2)}$$

Then it turns out that we need to show that $m_t(\phi)$ is a martingale with quadratic variation process

$$\langle m_\cdot(\phi) \rangle_t = \sigma_0^2(\phi) - \sigma_t^2(\phi)$$

After noticing that

$$\begin{aligned} \eta \circ f_t(z) &= \lambda - \frac{2\lambda}{\pi} \arg(f_t(z)) \\ &= \lambda - \frac{2\lambda}{\pi} \text{Im} \log(g_t(z) - U_t) \end{aligned}$$

Then a simple application of Itô's formula shows that $\kappa = 4$ and direct computations proves our claim. □