On scaling limits and Brownian interlacements by Alain-Sol Sznitman

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1 Overview

We will cover the following aspects:

- Random interlacement and occupation time
- Brownian interlacement and occupation time
- Scaling limits of occupation times under two different regime.
- Scaling limit via isomorphism theorem
- For $d = 3$, derive a continuous isomorphism theorem.

2 Random Interlacement

Firstly, we introduce continuous time random interlacement on $\mathbb{Z}^d$. Let $\widehat{W}^+$ be infinite sequences taking values in $\mathbb{Z}^d \times (0, \infty)$, i.e. as $(Z_n, \sigma_n)_{n \geq 0}$.
Let $\widehat{W}$, written as $(Z_n, \sigma_n)_{n \in \mathbb{Z}}$.
Let $\widehat{W}_K$ denote trajectories enter K.
Let $\widehat{W}^* = \widehat{W} \backslash$time shift, i.e. $W^* = \pi^*(W)$. Let $W_K^* = \pi^*(W_K)$.
Let X_t be the continuous random walk. i.e. $X_t(\widehat{w}) = Z_k(\widehat{w})$ for $\sigma_0(\widehat{w}) + \sigma_1(\widehat{w}) + \cdots + \sigma_{k-1}(\widehat{w}) \leq t \leq \sigma_0(\widehat{w}) + \cdots + \sigma_k(\widehat{w})$ for $\widehat{w} \in \widehat{W}^+$
We write H_U be the hitting time of U, $\widetilde{H}_U = \inf\{t > 0; X_t \in U, \text{ and for some } s \in (0, t), X_s \neq X_0\}$, T_U be the exit time of U.

Definition. 2.1 (Green function killed when exiting U).

$$g_U(x, y) := \frac{1}{2d} E_x \left[\int_0^{T_U} 1\{X_s = y\} ds \right] \tag{1}$$

when $U = \mathbb{Z}^d$, we drop the subscript.

We then define

Definition. 2.2 (Equilibrium measure and capacity). *For $K \subset\subset U$,*

$$\begin{aligned} e_{K,U}(x) &= P_x[\widetilde{H}_K > T_U]2d1_K(x) \\ cap_U(K) &= \sum_{x\in\mathbb{Z}^d} e_{K,U}(x) \end{aligned} \tag{2}$$

Note that the capacity is increase in K and decrease in U. Moreover, there is a useful identity

$$P_x(H_K < T_U) = \sum_{y\in\mathbb{Z}^d} g_U(x,y)e_{K,U}(y) \tag{3}$$

In our discussion, we always take $U = \mathbb{Z}^d$ and drop the subscript.
(There is also a continuous version once we have defined continuous version capacity and equilibrium measure.i.e.$Q_x(\widetilde{H}_K < \infty) = \int g(x,y)e(dy)$)

2.1 Random interlacement point process

Construct probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$, where $\bar{\Omega} := \{\omega = \sum_{i\geq 0}\delta_{(\widehat{w}_i^*,u_i)}\}$

Definition. 2.3 (continuous time random interlacements). *The continuous-time random interlacement PPP is on the space $\widehat{W}^* \times \mathbb{R}^+$. It's intensity measure has the form $\nu(d\hat{w}^*)du$. The restriction of ν on W_K^*, is equal to $\pi^* \circ \widehat{Q}_K$, the measure $\widehat{Q}_K$ is defined by*

- $\widehat{Q}_K(Z_0 = x) = e_K(x)$
- *when $e_K(x) > 0$, conditionally on $Z_0 = x$, $(Z_n)_{n\geq 0}, (Z_{-n})_{n\geq 0}, (\sigma_n)_{n\in\mathbb{Z}}$ are independent, respectively, SRW starting from x, SRW starting from x conditioned never return to K and iid doubly infinite exponential variables.*

Definition. 2.4 (Occupation times of random interlacement). *The random field of occupation times of random interlacements at level $u \geq 0$ is*

$$L_{x,u}(\omega) = \frac{1}{2d}\sum_{i\geq 0}\sum_{n\in\mathbb{Z}}\sigma_n(\widehat{w}_i)1\{Z_n(\widehat{w}_i) = x, u_i \leq u\} \text{ for } x \in \mathbb{Z}^d \text{ and } \omega \in \bar{\Omega} \tag{4}$$

Intuitively, $L_{x,u}$ is the time at x by trajectories with label at most u normalized by degree of x.

Theorem. 2.1 (Isomorphism). *The law of $(L_{x,u} + \frac{1}{2}\varphi_x^2)_{x\in\mathbb{Z}^d}$ under product measure $\bar{P} \times P^g$ equals the law of $\left(\frac{1}{2}(\varphi_x + \sqrt{2u})^2\right)_{x\in\mathbb{Z}^d}$ under P^g*

Some more notations:

the operation :: stands for Wick product, i.e. $:YZ := YZ - E[YZ]$
Let $|\cdot|$ and $|\cdot|_\infty$ denote the Euclidean and supremum norm respectively.
Let $g(\cdot,\cdot)$ denote the Green function on $\mathbb{Z}^d$, set $g(x) = g(x,0)$.
Let $G(y)$ denote the continuous Green's function on $\mathbb{R}^d$, which has the form

$$G(y) = \frac{1}{(2\pi)^{d/2}}\Gamma(\frac{d}{2}-1)|y|^{2-d} \tag{5}$$

For any integer $N \geq 1$, define the rescaled lattice

$$\mathbb{L}_N = \frac{1}{N}\mathbb{Z}^d \tag{6}$$

and define the inner product on $\mathbb{L}_N$ by

$$\langle f,h\rangle_{\mathbb{L}_N} = \frac{1}{N^d}\sum_{y\in\mathbb{L}_N} f(y)h(y) \tag{7}$$

Rescale the green function:

$$g_N(y,y') = \frac{1}{d}N^{d-2}g(Ny,Ny') \tag{8}$$

We then yield,

$$\lim_N \sup_{|y|\geq\gamma} |g_N(y) - G(y)| = 0 \tag{9}$$

Introduce the rescaled operator,

$$G_N f(y) = \frac{1}{N^d}\sum_{y'\in\mathbb{L}_N} g_N(y,y')f(y') \tag{10}$$

We often consider function f with 'compact' support.
The inner product in $\mathbb{R}^d$,

$$\langle f,h\rangle = \int_{\mathbb{R}^d} f(y)h(y)dy \tag{11}$$

Define continuous linear operator

$$Gf(y) := \int G(y,y')f(y')dy' \tag{12}$$

$G_N V$ represents the composition of multiplication by V with operator G_N.
Law of discrete GFF by P^g
continuum GFF by P^G which is a law on $S'(\mathbb{R}^d)$

3 Brownian interlacements

Similar to discrete case, W^+ be the subspace of $C(\mathbb{R}_+, \mathbb{R}^d)$ continuous path tending to infinity,
and by W the subspace of $C(\mathbb{R}, \mathbb{R}^d)$ tending to infinity at plus and minus infinity times.
X_t be canonical process and $\mathcal{W}$ the sigma algebra generated by canonical process,
θ_t be shift operators.
W^* be the equivalence class of trajectories modulo time shifts. We then naturally introduce $\mathcal{W}^*$ the largest sigma algebra generated by π^*, which maps trajectories to their equivalence classes.
r.v. Z on W is invariant iff $Z \circ \theta_t = Z$ for all t. Then determines a unique r.v. $Z = Z^* \circ \pi^*$.
W_K be the trajectories enter K. and W_K^* be its image under π^*.
P_y be the Wiener measure starting at y and P_y^B be the law of Brownian motion starting at y conditioned on never to enter B. We can define P_y^B when $y \in \partial B$ by weak limits of P_z^B as $z \to y$.
We let $p_t(y, y')$ be the transition density of BM. Recall that in applied stochastic process, we have learned

$$G(y, y') = \int_0^\infty p_t(y, y')dt \tag{13}$$

The equilibrium measure e_K is the unique radon measure satisfying $Q_x(\widetilde{H}_K < \infty) = \int G(x, y)e(dy)$ where Q_x is the Wiener measure under this circumstance.

- Supported on the boundary of K
- total mass denoted by $cap(K)$
- $P_y(L_K > 0, L_K \in dt, X_{L_K} \in dz) = p_t(y, z)e_K(dz)dt$

We then introduce the following measure on W_B^0.

$$\begin{aligned} &Q_B\left[(X_{-t})_{t\geq 0} \in A', X_0 \in dy, (X_t)_{t\geq 0} \in A\right] \\ &\quad = e_B(dy)P_y^B[A']P_y[A] \text{ for } A, A' \in \mathcal{W}^+ \end{aligned} \tag{14}$$

Now we proceed to define ν on W^*, however, the consistency in continuous case is not as obvious as discrete cases because we may encounter 'bad' domains. So it is reasonable to define firstly on good domains such as balls and then define on general domains.

Lemma. 3.1. *Assume that B and B' are closed balls, and B lies in the interior of B', then*

$$\theta_{H_B} \circ (1_{H_B<\infty} Q_{B'}) = Q_B \tag{15}$$

(Explain it.)Intuitively, it describe such a process: We first sample a point on B' according to equilibrium measure $e_{B'}$, and then hits a point B on the boundary of B. To prove this fact, we can show there characteristics are equal. i.e. $E^{\cdot}[\int_{\mathbb{R}} f(s) \cdot X_s ds]$.
We then unambiguously define for any compact K and any closed ball $B \supset K$,

$$\theta_{H_K} \circ (1_{H_K<\infty} Q_B) = Q_K \tag{16}$$

Theorem. 3.1. *There exists a unique $\sigma - finite$ measure ν on $(W^*, \mathcal{W}^*)$ such that for each compact subset K of $\mathbb{R}^d$*

$$1_{W_K^*} \nu = \pi^* \circ Q_K \tag{17}$$

The proof of theorem 2.2 follows directly from Kolmogorov's extension theorem and the previous lemma.

proof of 3.1 Consistency of measure. We only need to show

$$E^{\theta_{H_B} \circ (1_{H_B<\infty} Q_{B'})} \left[exp\left(i \int_{\mathbb{R}} f(v) X_v dv \right) \right] = E^{Q_B} \left[exp\left(i \int_{\mathbb{R}} f(v) X_v dv \right) \right]$$

It suffices to show that for any continuous compactly supported function $f : \mathbb{R} \to \mathbb{R}^d$

$$\begin{aligned} &\int e_{B'}(dy') E_{y'} \left[H_B < \infty, exp\left(i \int_0^{H_B} f(v - H_B) \cdot X_v dv \right) E_{X_{H_B}} \left[exp\left(i \int_0^{\infty} f(v) \cdot X_v dv \right) \right] \right. \\ &\left. E_{y'}^{B'} \left[exp\left(i \int_0^{\infty} f(-v - H_B) \cdot X_v dv \right) \right] \right] = \int e_B(dy) E_y^B \left[exp\left(i \int_0^{\infty} f(-v) \cdot X_v dv \right) \right] \\ &E_y \left[exp(i \int_0^{\infty} f(v) \cdot X_v dv) \right] \text{ for } y' \in \partial B' \end{aligned} \tag{18}$$

The LHS of the previous identity is exactly $E^{\theta_{H_B} \circ (1_{H_B<\infty} Q_{B'})} \left[exp\left(i \int_{\mathbb{R}} f(v) X_v dv \right) \right]$.
We then proceed to prove 18.
We denote by $P_{y,y'}^t$ be the Brownian Bridge measure from y to y' during time period t and $E_{y,y'}^t$ be the corresponding expectation. under P_y, conditionally on $L_{B'} = t$ and $X_{L_{B'=y'}}$, the law of $(X_{L_{B'}+v})_{v \geq 0}$ and $(X_v)_{0 \leq v \leq t}$ are independent, respectively distributed as $P_{y'}^{B'}$ and $P_{y,y'}^t$ and $(L_{B'}, X_{L_{B'}})$ has law $p_t(y, y') e_{B'}(dy') 1_{t>0} dt$.

Then for $y \in B'$,

$$
\begin{aligned}
&E_y\left(exp(i\int_0^\infty f(-v)\cdot X_v dv), H_B=\infty\right)\\
&=\int_0^\infty dt\int e_{B'}(dy')E_{y'}^{B'}\left[exp(i\int_0^\infty f(-v-t)\cdot X_v dv)\right]\\
&\times E_{y,y'}^t\left[exp(i\int_0^t f(-v)\cdot X_v dv), T_{B^c}>t\right]p_t(y,y')\\
&=\int_0^\infty\int e_{B'}(dy')F'(y',t)E_{y',y}^t\left[exp(i\int_0^t f(v-t)\cdot X_v dv), T_{B^c}>t\right]p_t(y',y)
\end{aligned}
\tag{19}
$$

where $F'(y',t)=E_{y'}^{B'}\left[exp(i\int_0^\infty f(-v-t)\cdot X_v dv)\right]$ and we reverse the position of y and y' in the last step according to the properties of Brownian bridge.

We then introduce

$$
F(z)=\psi(z)E_z\left[exp(i\int_0^\infty f(v)\cdot X_v dv)\right]
\tag{20}
$$

Here ψ is a continuous compactly supported $[0,1]$ valued function which equals 1 on a neighborhood of B. Pick $\epsilon>0$ and introduce the finite measure

$$
\begin{aligned}
m_\epsilon(dy)&=\frac{1}{\epsilon}\int dzF(z)P_z[H_B<\epsilon, X_\epsilon\in dy]\\
&=\frac{1}{\epsilon}\int dzF(z)P_{z,y}^\epsilon[H_B<\epsilon]p_\epsilon(z,y)
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&\int dzF(z)\frac{1}{\epsilon}E_z\left[H_B<\epsilon, E_{X_\epsilon}\left[H_B=\infty, \exp\left(i\int_0^\infty f(-v)\cdot X_v dv\right)\right]\right]\\
&=\int dzF(z)\frac{1}{\epsilon}E_z\left[H_B<\epsilon, P_{X_\epsilon}\left[H_B=\infty\right]E_{X_\epsilon}^B\left[\exp\left(i\int_0^\infty f(-v)\cdot X_v dv\right)\right]\right]\\
&=\int dzF(z)\frac{1}{\epsilon}E_z\left[H_B<\epsilon, H_B\circ\theta_\epsilon=\infty, E_{X_\epsilon}^B\left[\exp\left(i\int_0^\infty f(-v)\cdot X_v dv\right)\right]\right]\\
&=\int dzF(z)\frac{1}{\epsilon}E_z\left[0<L_B<\epsilon, E_{X_\epsilon}^B\left[\exp\left(i\int_0^\infty f(-v)\cdot X_v dv\right)\right]\right]
\end{aligned}
\tag{22}
$$

The last decomposition of Brownian motion starting at z at the last visit of B, the above expression equals

$$
\begin{aligned}
&\int dzF(z)\frac{1}{\epsilon}\int_0^\epsilon dsp_s(z,y)\int e_B(dy)E_y^B\left[E_{X_{\epsilon-s}}^B\left[\exp\left(i\int_0^\infty f(-v)\cdot X_v dv\right)\right]\right]\\
&=\int e_B(dy)\frac{1}{\epsilon}\int_0^\epsilon ds\left(\int p_s(y,z)F(z)dz\right)E_y^B\left[E_{X_{\epsilon-s}}^B\left[\exp\left(i\int_0^\infty f(-v)\cdot X_v dv\right)\right]\right]
\end{aligned}
\tag{23}
$$

Note that for $y \in \partial B$,then P_y^B a.s., as $u \to 0$, $P_{X_u}^B$ converges weakly to P_y^B, by DCT, the above quantity tends to

$$\int e_B(dy)E_y\left[\exp\left(i\int_0^\infty f(v)\cdot X_v dv\right)\right]E_y^B\left[\exp\left(i\int_0^\infty f(-v)\cdot X_v dv\right)\right] \tag{24}$$

The last line of 19 after integration with respect to $m_\epsilon(dy)$ yields

$$\begin{aligned}
&\int e_{B'}(dy')\int_0^\infty \frac{dt}{\epsilon}F'(y',t)\int dzdyF(z) \\
&\times E_{y',y}^t\left[\exp\left(i\int_0^t f(v-t)\cdot X_v\right), T_{B^c}>t\right]p_t(y',y)P_{y,z}^\epsilon[H_B<\epsilon]p_\epsilon(y,z) \\
&=\int e_{B'}(dy')\int_0^\infty \frac{dt}{\epsilon}F'(y',t)E_{y'}\left[\exp\left(i\int_0^t f(v-t)\cdot X_v\right), t<H_B<t+\epsilon, F(X_{t+\epsilon})\right] \\
&=\int e_{B'}(dy')E_{y'}\left[H_B<\infty, \frac{1}{\epsilon}\int_{(H_B-\epsilon)^+}^{H_B}\exp\left(i\int_0^t f(v-t)\cdot X_v\right)F(X_{t+\epsilon})F'(y',t)dt\right]
\end{aligned} \tag{25}$$

applying DCT as $\epsilon \to 0$, the above quantity tends to

$$\begin{aligned}
\int e_{B'}(dy')E_{y'}&\left[H_B<\infty, \exp\left(i\int_0^{H_B} f(v-H_B)\cdot X_v dv\right)\right] \\
&\times E_{X_{H_B}}\left[\exp\left(i\int_0^\infty f(v)\cdot X_v dv\right)F'(y',H_B)\right]
\end{aligned} \tag{26}$$

We have proved 18 the target identity. $\square$

Similar to discrete case, we define the canonical space for the Brownian interlacement point process,

$$\begin{aligned}
\Omega := \Bigg\{ &\omega = \sum_{i\geq 0}\delta_{(w_i^*,\alpha_i)}, \text{ with } (w_i^*,\alpha_i)\in W^*\times[0,\infty) \text{ and} \\
&\omega(W_K^*\times[0,\alpha])<\infty, \text{ for any compact subset } K \text{ of } \mathbb{R}^d \text{ and } \alpha\geq 0\Bigg\}
\end{aligned} \tag{27}$$

Endow Ω with the $\sigma-$algebra $\mathcal{A}$ generated by maps $\omega \to \omega(B)$, for $B \in \mathcal{W}^*\otimes\mathbb{R}_+$, $\mathbb{P}$ the law on $(\Omega,\mathcal{A})$ of the PPP with intensity measure $\nu\otimes d\alpha$,

Proposition. 3.1. *$\mathbb{P}$ is invariant under the following transformations on Ω:*

$$\omega = \sum_{i\geq 0}\delta_{(w_i^*,\alpha_i)} = \sum_{i\geq 0}\delta_{(\check{w}_i^*,\alpha_i)} \tag{28}$$

$$\omega = \sum_{i \geq 0} \delta_{(w_i^*, \alpha_i)} = \sum_{i \geq 0} \delta_{(w_i^* - y, \alpha_i)} \tag{29}$$

$$\omega = \sum_{i \geq 0} \delta_{(w_i^*, \alpha_i)} = \sum_{i \geq 0} \delta_{(R w_i^*, \alpha_i)} \tag{30}$$

$$\omega = \sum_{i \geq 0} \delta_{(w_i^*, \alpha_i)} = \sum_{i \geq 0} \delta_{(s_\lambda(w_i^*), \lambda^{2-d} \alpha_i)} \tag{31}$$

Brownian interlacement at level α with radius r.

$$\mathcal{I}_r^\alpha(\omega) = \bigcup_{i \geq 0 : \alpha_i \leq \alpha} \bigcup_{s \in \mathbb{R}} B(w_i(s), r) \tag{32}$$

where $\pi^*(w_i) = w_i^*$. When $r = 0$, it is called the Brownian fabric.
Let Σ be the set of closed subsets of $\mathbb{R}^d$, let $\mathcal{F}$ be the sigma algebra generated by $\{F \in \Sigma : F \cap K = \phi\}$, where K varies over compact subsets of $\mathbb{R}^d x$,

Proposition. 3.2. *($\alpha \geq 0, r \geq 0, y \in \mathbb{R}^d$, R linear isometry, $\lambda > 0$)define Q_r^α to be determined by the identity*

$$Q_r^\alpha(F \in \Sigma, F \cap K = \phi) = \mathbb{P}[\mathcal{I}_r^\alpha \cap K = \phi] = e^{-\alpha cap(K_r)} \tag{33}$$

where K_r is the $r-$neighborhood of K. Moreover, the law of $\mathcal{I}_r^\alpha$ is invariant under translation and isometry. $\lambda \mathcal{I}_r^\alpha$ has the same law as $\mathcal{I}_{\lambda r}^{\lambda^{2-d}\alpha}$.
$\mathcal{I}_\alpha^0$ is a.s. connected when $d = 3$, disconnected when $d \geq 4$ and $\alpha > 0$

We then define the occupation-time measure of Brownian interlacements at level $\alpha \geq 0$, for $A \in \mathbb{R}^d$, here $A \subset B$ for some closed ball B,

$$\mathcal{L}_\alpha(\omega)(A) = \sum_{i \geq 0 : \alpha_i \leq \alpha} \int_{\mathbb{R}} 1_{w_i(s) \in A} ds = \langle \omega, f_A \otimes 1_{[0,\alpha]} \rangle \tag{34}$$

where $f_A(w^*) = \int_{\mathbb{R}} 1_A(w(s))ds$

$$\begin{aligned} \mathbb{E}\left[\mathcal{L}_\alpha(\omega)(A)\right] &= \mathbb{E}[\langle \omega, f_A \otimes 1_{[0,\alpha]} \rangle] \\ &= \alpha \langle \nu, f_A \rangle \\ &= \alpha E_{e_B}\left[\int_0^\infty 1_A(X_s) ds\right] \\ &= \alpha \int e_B(dy) G(y, y') 1_A(y') dy' \\ &= \alpha |A| \end{aligned} \tag{35}$$

The intensity measure of $\mathcal{L}_\alpha$ is αdy and the support of $\mathcal{L}_\alpha$ is $\mathcal{I}_0^\alpha$

Proposition. 3.3 (Laplace transform of $\mathcal{L}_\alpha$). *V bounded, measurable, and compactly supported*

$$|G|V||_{L^\infty(\mathbb{R}^d)} < 1 \tag{36}$$

then $I - GV$ is a bounded invertable operator on $L^\infty(\mathbb{R}^d)$ and for any $\alpha \geq 0$,

$$\mathbb{E}[\exp\{\langle \mathcal{L}_\alpha, V\rangle\}] = \exp\{\alpha\langle V, (I - GV)^{-1}1\rangle\} \tag{37}$$

where the notation $\langle\rangle$ is the integral.

4 Scaling limits of occupation times

In this section, we show that $\mathcal{L}^N$ converges in distribution to the occupation time measure of Brownian interlacement in the constant intensity regime and $\widehat{\mathcal{L}}^N$ converges to Gaussian free field.
where $\mathcal{L}^N = \frac{1}{dN^2}\sum_{x\in\mathbb{Z}^d} L_{x,u_N}\delta_{\frac{x}{N}}$, (Here we better review the definition of local time)
and $\widehat{\mathcal{L}}^N = \sqrt{\frac{d}{2N^{2-d}u_N}}(\widehat{\mathcal{L}}^N - \bar{\mathbb{E}}[\widehat{\mathcal{L}}^N])$
Constant regime: $u_N = d\alpha N^{2-d}$ for some $\alpha > 0$
High intensity regime: $N^{d-2}u_N \to \infty$
To show our definition is reasonable, we can compute the expectation,

$$\bar{\mathbb{E}}[\mathcal{L}^N([0,1]^d)] = \frac{1}{dN^2} \times u_N \times N^d \tag{38}$$

In constant regime, $= \alpha$ and in high intensity regime, $\to \infty$

Proposition. 4.1. *For V on $\mathbb{R}^d$ with support in $C_M = [-M, M]^d$.*

$$\sup_{N\geq 1} \|G_N|V|\|_{L^\infty(\mathbb{L}_N)} \leq c_0(M)\|V\|_{L^\infty(\mathbb{R}^d)} \tag{39}$$

$$\|G_N|V|\|_{L^\infty(\mathbb{L}_N)} \leq c_0(M)\|V\|_{L^\infty(C_M\cap\mathbb{L}_N)} \tag{40}$$

$$\lim_N \langle V, (G_N V)^{n-1}1\rangle_{\mathbb{L}_N} = \langle V, (GV)^{n-1}1\rangle \tag{41}$$

Proof. Assume $\|V\|_{L^\infty(C_M\cap\mathbb{L}_N)} = 1$. By a classical harmonicity argument we know that

$$\sup_{y\in\mathbb{L}_N} |G_N|V|(y)| = \sup_{y\in C_M\cap\mathbb{L}_N} |G_N|V|(y)| \tag{42}$$

Then it follows from the definition of the operator G_N that

$$\sup_{y\in C_M\cap\mathbb{L}_N} |G_N|V|(y)| \leq \frac{c}{N^2}\left(1 + \sum_{0<|x|_\infty\leq 2MN} \frac{1}{|x|^{d-2}}\right) \leq c' M^2 \tag{43}$$

Now it suffices to show that for any $W \in C_c(\mathbb{R}^d)$,

$$\lim_N \sup_{y\in C_M\cap\mathbb{L}_N} |(G_N VW)(y) - (GVW)(y)| = 0 \tag{44}$$

Then it follows immediately,

$$sup_{y\in C_M\cap\mathbb{L}_N}|(G_NV)^{n-1}1(\cdot)-(GV)^{n-1}1(\cdot)|\to 0 \text{ for each } n\geq 1 \tag{45}$$

The claim now follows. We now prove 44. Write $F=VW$, therefore F is continous on $\mathbb{R}^d$ with support in C_M. We partition $|G_NF(y)-GF(y)|$ into two parts, i.e. $|y'|\leq\gamma$ and $\gamma\leq|y'|\leq 2M$. The first part in L^∞ norm of F while the second part follows by Uniform continuity and Riemann sum approximation. □

We now state our 1st main theorem,

Theorem. 4.1 (constant intensity regime)**.** *Under constant intensity regime,* $u_N=d\alpha N^{2-d}$,
$\mathcal{L}^N$ *converges in distribution to* $\mathcal{L}_\alpha$ *as* $N\to\infty$

Proof of 4.1. It suffices to show for any continuous compactly supported function V on $\mathbb{R}^d$

$$\langle\mathcal{L}^N,V\rangle \text{ converge in distribution to } \langle\mathcal{L}_\alpha,V\rangle \tag{46}$$

w.l.o.g. we assume that V has continuous support in C_M and $\|V\|_{L^\infty(\mathbb{R}^d)}\leq 1$, by Theorem 2.1 in [39], we have for $|z|<\frac{1}{c_0}$,

$$\begin{aligned}\bar{\mathbb{E}}\text{exp}\{z\langle\mathcal{L}^N,V\rangle\}&=\exp\{\alpha\langle zV,(I-zG_NV)^{-1}1\rangle_{\mathbb{L}_N}\}\\&=\exp\{\alpha\sum_{n\geq1}z^n\langle V,(G_NV)^{n-1}1\rangle_{\mathbb{L}_N}\}\end{aligned}\tag{47}$$

In particular, we have shown that

$$\sup_{N\geq1}\bar{\mathbb{E}}[cosh(r\langle\mathcal{L}^N,V\rangle)]<\infty$$

which implies tightness of $\langle\mathcal{L}^N,V\rangle$ and the uniform integrability of $e^{z\langle\mathcal{L}^N,V\rangle}$. Then for any $|z|<c_0^{-1}$, we have

$$\begin{aligned}E[e^{zU}]&=\lim_k\bar{\mathbb{E}}[e^{z\langle\mathcal{L}^{N_k},V\rangle}]\\&=\lim_k\exp\{\alpha\sum_{n\geq1}z^n\langle V,(G_{N_k}V)^{n-1}1\rangle_{\mathbb{L}_N}\}\\&=\exp\{\alpha\sum_{n\geq1}z^n\langle V,(GV)^{n-1}1\rangle_{\mathbb{L}_N}\}\\&=\mathbb{E}[e^{z\langle\mathcal{L}_\alpha,V\rangle}]\end{aligned}\tag{48}$$

The merit of using Laplace transform here is that we can get tightness. □

We then turn to high intensity regime. Now we consider the convergence of $\widehat{\mathcal{L}}^N=\sqrt{\frac{d}{2N^{2-d}u_N}}(\widehat{\mathcal{L}}^N-\bar{\mathbb{E}}[\widehat{\mathcal{L}}^N]$

Theorem. 4.2. *Under high intensity regime, i.e.* $N^{d-2}u_N \to \infty$. *When* $V \in C_c(\mathbb{R}^d)$, *then, as* $N \to \infty$,

$$\langle \widehat{\mathcal{L}}^N, V\rangle \xrightarrow{d} N(0, \sigma_V^2) \tag{49}$$

where $\sigma_V^2 = E(V,V) = \int V(y)G(y-y')V(y')dydy'$

Proof. w.l.o.g. assume that $\|V\|_{L^\infty(\mathbb{R}^d)} \le 1$. Further, write in shorthand $a_N = (\frac{2}{d}N^{d-2}u_N)^{\frac{1}{2}}$

$$\begin{aligned} \bar{\mathbb{E}}[e^{z\langle \widehat{\mathcal{L}}^N, V\rangle}] &= \bar{\mathbb{E}}\left[\exp\left\{\frac{z}{a_N}(\langle \mathcal{L}^N, V\rangle - \frac{u_N}{d}N^{d-2}\langle V, 1\rangle_{\mathbb{L}_N})\right\}\right] \\ &= \exp\left\{\frac{u_N}{d}N^{d-2}\sum_{n\ge 2}\frac{z^n}{a_N^n}\langle V, (G_N V)^{n-1}1\rangle_{\mathbb{L}_N}\right\} \end{aligned} \tag{50}$$

As $N \to \infty$, the $n = 2$ is only thing that left.

$$\bar{\mathbb{E}}[e^{z\langle \widehat{\mathcal{L}}^N, V\rangle}] \to e^{\frac{z^2}{2}\langle V, GV\rangle} \tag{51}$$

□

As a corollary,

Corollary. 4.1. *Under high intensity regime,*

$$\widehat{\mathcal{L}}^N \text{ converges in law to } P^G \text{ as } N \to \infty \tag{52}$$

Proof. We use a continuous compact support truncate function χ taking values in $[0,1]$ which equals 1 on unit box. Then let $\chi_L(\cdot) = \chi(\frac{\cdot}{L})$.
Consider $V_L := V\chi_L$, by DCT we have $E(V_L, V_L) \to E(V,V)$. It remains to show

$$\lim_{L\to\infty}\sup_{N\ge 1}\bar{\mathbb{E}}[\langle \widehat{\mathcal{L}}^N, V - V_L\rangle^2] = 0 \tag{53}$$

□

5 Scaling limits via the isomorphism theorem

In this section, we give a new illustration of 4.2 under sufficiently high intensity regime.
First we introduce the random signed measure on $\mathbb{R}^d$

$$\Phi^N = \frac{1}{\sqrt{d}N^{\frac{d}{2}+1}}\sum_{x\in\mathbb{Z}^d}\varphi_x\delta_{\frac{x}{N}} \tag{54}$$

By discrete version of isomorphism theorem, we have

$$
\begin{aligned}
&\frac{1}{2}\sum_{z\in\mathbb{Z}^d} V(\frac{x}{N})(\varphi_x^2 - g(0)) + \sum_{x\in\mathbb{Z}^d} V(\frac{x}{N})(L_{x,u_N} - u_N) \\
&\stackrel{d}{=} \frac{1}{2}\sum_{z\in\mathbb{Z}^d} V(\frac{x}{N})(\varphi_x^2 - g(0)) + \sqrt{2u_N}\sum_{x\in\mathbb{Z}^d} V(\frac{x}{N})\varphi_x
\end{aligned}
\tag{55}
$$

this identity can be written in the following form,

$$
\begin{aligned}
&\frac{1}{2}\sum_{z\in\mathbb{Z}^d} V(\frac{x}{N})(\varphi_x^2 - g(0)) + \sqrt{2du_N}N^{\frac{d}{2}+1}\langle\widehat{\mathcal{L}}^N, V\rangle \\
&\stackrel{d}{=} \frac{1}{2}\sum_{z\in\mathbb{Z}^d} V(\frac{x}{N})(\varphi_x^2 - g(0)) + \sqrt{2du_N}N^{\frac{d}{2}+1}\langle\Phi^N, V\rangle
\end{aligned}
\tag{56}
$$

Goal: we wish to divide by $\sqrt{2du_N}N^{d/2+1}$ and make the first term on both sides converge to zero in probability, the desired results follows by applying converging together lemma.
In order to do this, we first introduce the following lemma

Lemma. 5.1. *$\langle\Phi^N, V\rangle$ is a centered Gaussian variable with variance $\langle V, G_N V\rangle_{\mathbb{L}_N}$ $\langle\Phi^N, V\rangle$ converges in distribution to a centered Gaussian variable with variance $\int V(y)G(y-y')V(y')dydy'$*

$$
\frac{1}{b_N}E^{P^g}\left[\left(\sum_{x\in\mathbb{Z}^d} V(\frac{x}{N})(\varphi_x^2 - g(0))\right)^2\right]
\tag{57}
$$

converges to a positive limit. where

$$
b_N = \begin{cases} N^4 & \text{when } d = 3 \\ N^4\log N & \text{when } d = 4 \\ N^d & \text{when } d \geq 5 \end{cases}
\tag{58}
$$

Proof. The first two statement follows directly by direct computation.
To prove the third claim, we take $d = 3$ as an example. We only need to show that

$$
E^{P^g}[(\varphi_x^2 - g(0))(\varphi_{x'}^2 - g(0))] = 2g^2(x, x'), \text{ for } x, x' \in \mathbb{Z}^d
\tag{59}
$$

therefore,

$$\frac{1}{b_N}E^{P^g}\left[\left(\sum_{x\in\mathbb{Z}^d}V(\frac{x}{N})(\varphi_x^2-g(0))\right)^2\right] = \frac{2}{b_N}\sum_{x,x'\in\mathbb{Z}^d}V(\frac{x}{N})g^2(x,x')V(\frac{x'}{N})$$
$$= \frac{18}{N^6}\sum_{y,y'\in\mathbb{L}_N}V(y)g_N^2(y-y')V(y')$$
$$\to 18\int V(y)G^2(y-y')V(y')dydy'$$
$$= \int_{\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d} dsdtdzdz'\left(\int_{\mathbb{R}^d} dyV(y)p_{s/2}(y,z)p_{t/2}(y,z')\right)^2 \tag{60}$$

$d=4$ and $d\geq 5$ similarly. □

Therefore, when $u_N\frac{N^{d+2}}{b_N}\to\infty$, we have the desired result.

6 The special case of dimension 3

We now investigate what happens in dimension 3 at constant regime.
We denote by H the Gaussian space which is the L^2 closure of $\{\langle\Phi,f\rangle; f\in S(\mathbb{R}^d)\}$.
define Wick product

$$:YZ:=YZ-E^{P^G}[YZ] \tag{61}$$

And for $Y=\langle\Phi,f\rangle$, $Z=\langle\Phi,h\rangle$, one has the identity,

$$\begin{aligned} E^{P^G}[:Y^2::Z^2:] &= 2E^{P^G}[Y,Z]^2 \\ &= 2\langle f,Gh\rangle^2 \end{aligned} \tag{62}$$

When $d=3$, the Green function $G(y,y')$ is locally square integrable. For $V\in C_c(\mathbb{R}^3)$. We can prove

$$\int V(y):\Phi_y^2:dy = \lim_{\epsilon\to 0}\int V(y):\Phi_{y,\epsilon}^2:dy \text{ in } L^2(P^G) \tag{63}$$

where $\Phi_{y,\epsilon}=\langle\Phi,\rho_{y,\epsilon}$, and $\rho_{y,\epsilon}$ is the smoothing kernel and is radially symmetric and supported in $B(0,1)$.
Moreover, we can define

$$\int V(y)\Phi_y dy = \lim_{\epsilon\to 0}\int V(y)\Phi_{y,\epsilon}dy \text{ in } L^2(P^G) \tag{64}$$

We then rewrite the discrete isomorphism theorem in the following form, under $P^g\times\bar{\mathbb{P}}$,

$$\frac{1}{2}\sum_{x\in\mathbb{Z}^3}V(\frac{x}{N}):\phi_x^2:+3N^2\langle\mathcal{L}^N,V\rangle \overset{law}{=} \frac{1}{2}\sum_{x\in\mathbb{Z}^3)}V(\frac{x}{N}):(\varphi_x+\sqrt{2u_N})^2: \tag{65}$$

It then remains to prove the following theorem,

Theorem. 6.1 *($d = 3$, $V \in C_c(\mathbb{R}^3)$). For $\alpha \geq 0$ and $u_N = 3\alpha/N$, as $N \to \infty$*

$$\frac{1}{3N^2} \sum_{x \in \mathbb{Z}^3} V(\frac{x}{N}) : (\varphi_x + \sqrt{2u_N})^2 : \tag{66}$$

converges in law to

$$\int V(y) : (\Phi_y + \sqrt{2\alpha})^2 : dy \tag{67}$$

where the last term can be defined as

$$\int V(y) : \Phi_y^2 : dy + 2\sqrt{\alpha} \int V(y) \Phi_y dy + 2\alpha \int V(y) dy \tag{68}$$

Then one has the distributional identity under $P^G \otimes \mathbb{P}$

$$\frac{1}{2} \int V(y) : \Phi_y^2 : dy + \langle \mathcal{L}_\alpha, V \rangle \stackrel{law}{=} \frac{1}{2} \int V(y) : (\Phi_y + \sqrt{2\alpha})^2 : dy \tag{69}$$

Proof sketch. Let

$$\varphi_{y,N} = \sqrt{\frac{N}{d}} \varphi_{Ny} \text{ for y in } \mathbb{L}_N$$

The first term of 66 can be rewritten as

$$\frac{1}{N^3} \sum_{y \in \mathbb{L}_N} V(y) : (\varphi_{y,N} + \sqrt{2\alpha})^2 :$$

We choose $\epsilon(N)$ not too small s.t.

$$\epsilon(N) \to 0 \text{ and } N\epsilon^3(N) \to \infty \tag{70}$$

We establish the following three facts:

$$\lim_N \| \int V(y) : (\Phi_y + \sqrt{2\alpha})^2 : dy - \int V(y) : (\Phi_{y,\epsilon} + \sqrt{2\alpha})^2 : dy \|_{L^2(P^G)} = 0 \tag{71}$$

This is true by definition.

$$\lim_N \| \int V(y) : (\Phi_{y,\epsilon} + \sqrt{2\alpha})^2 : dy - \frac{1}{N^3} \sum_{y \in \mathbb{L}_N} V(y) : (\Phi_{y,\epsilon} + \sqrt{2\alpha})^2 : \|_{L^1(P^G)} = 0 \tag{72}$$

And for all sufficiently small z, we have

$$\begin{aligned} \lim_N \Bigg(& E^{P^G} \left[\exp \left\{ \frac{z}{N^3} \sum_{y \in \mathbb{L}_N} V(y) : (\Phi_{y,\epsilon} + \sqrt{2\alpha}) : \right\} \right] \\ & E^{P^G} \left[\exp \left\{ \frac{z}{N^3} \sum_{y \in \mathbb{L}_N} V(y) : (\varphi_{y,N} + \sqrt{2\alpha}) : \right\} \right] \Bigg) = 0 \end{aligned} \tag{73}$$

The rest follows by standard method using tightness by extracting weak convergent subsequence. $\square$

Corollary. 6.1. *Under $P^G \otimes \mathbb{P}$ one has the identity in law on $S'(\mathbb{R}^3)$,*

$$\frac{1}{2} : \Phi^2 : + \mathcal{L}_\alpha \overset{law}{=} \frac{1}{2} : (\Phi + \sqrt{2\alpha})^2 : \tag{74}$$

Remark. 6.1. *This is because $C_c(\mathbb{R}^3)$ and $S(\mathbb{R}^3)$ differ a little. Some estimation on the moment need to be done to justify this fact.*