520.445/645 – Audio Signal Processing
☞ Topic 3 ☜

Review of Digital Signal Processing

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What is Digital Signal Processing (DSP)?

Input signal → Analog-to-digital converter → Computer → Digital-to-analog converter → Output signal

- **Digital**
  - Method to represent a quantity, a phenomenon or an event
- **Signal**
  - A signal is a function of independent variable(s) that carry information
  - It is a detectable physical quantity (voltage, current, magnetic field, sound, photo, gesture) In this class, interested in sound and speech
- **Processing**
  - Transformation, analysis
  - Filtering, recognition, synthesis, coding, detection, estimation

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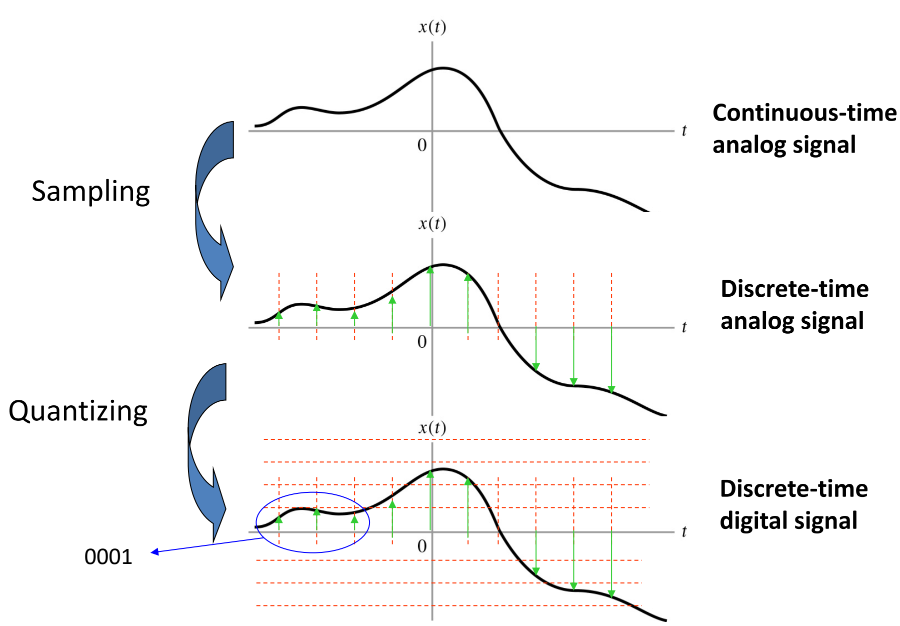
Digital processing of analog signals

- **A-to-D conversion:** bandwidth control, sampling and quantization
- **Computational processing:** implemented on computers or customized ICs with finite-precision arithmetic
  - **basic numerical processing:** add, subtract, multiply (scaling, amplification, attenuation), mute, …
  - **algorithmic numerical processing:** convolution or linear filtering, non-linear filtering, difference equations, DFT, …
- **D-to-A conversion:** re-quantification and filtering (or interpolation) for reconstruction



CONTINUOUS-TIME

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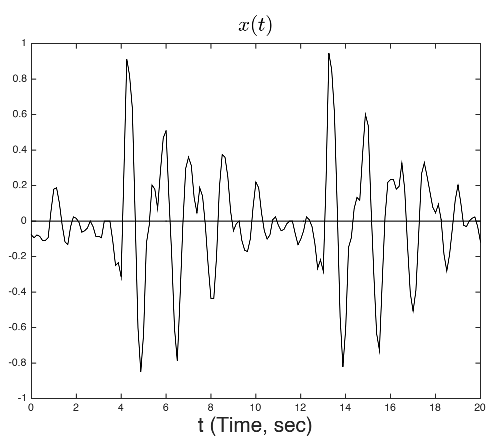
Continuous-time signal

A quantity of interest that changes over time:

- ○Quantity of interest $x(.)$ => sound pressure, $x(.) \in \mathbb{R}$
- ○Depends on independent variable t => time, $t \in \mathbb{R}$



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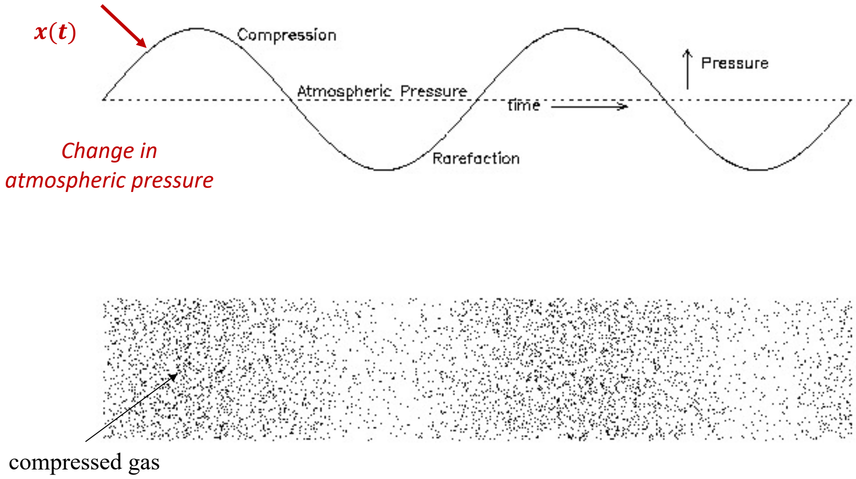
Sound as a signal

Time signal

$x(t)$



Change in atmospheric pressure

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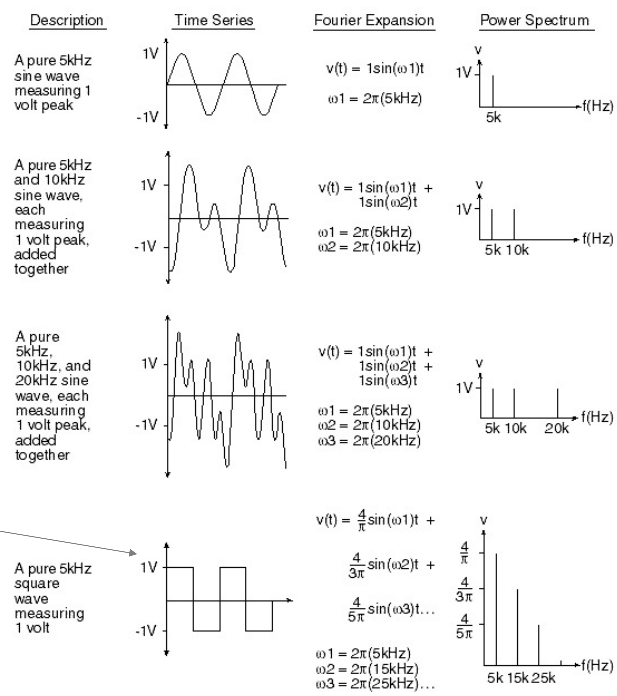
Signal decomposition

Sound is converted into an electrical signal



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Fourier series

- Each <u>periodic</u> signal can be expressed as linear combination of weighted exponentials at integer multiples of the fundamental:

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi kt}{T_0}}$$

where T is the signal period and c_k are the Fourier coefficients. They are defined as:

$$c_k = \int_0^{T_0} x_p(t) e^{-j\frac{2\pi kt}{T}} dt$$

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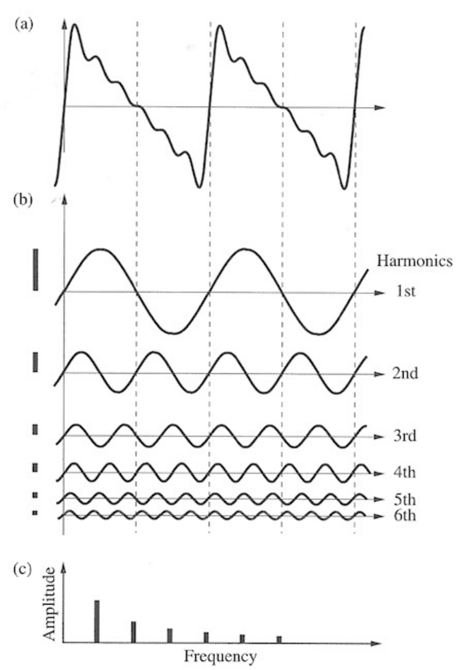
Fourier series example



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Fourier transform

- The principle of expressing time signals as sums of weighted sinusoids can be extended to aperiodic signals.
  - [Note: practically, we are talking about finite duration signals, so they cannot be periodic]
- The Fourier transform of a general signal is:

$$\boxed{X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt}$$

- The Fourier transform is an invertible operation

$$\boxed{x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df}$$

- No information is lost because this is a 1:1 reversible mapping

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Fourier transform

$$\boxed{X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt}$$

- The variable f has units of Hertz (i.e. cycles per second).
- Equivalently, we can define a new variable

$\Omega = 2\pi f$ in units of radians/second.

- So, we can write:

$$\boxed{X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt}$$

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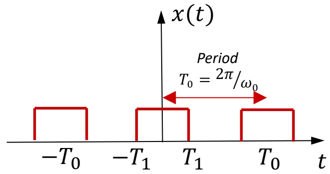
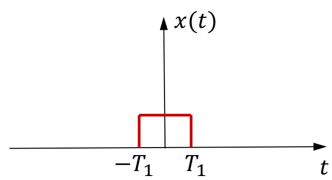
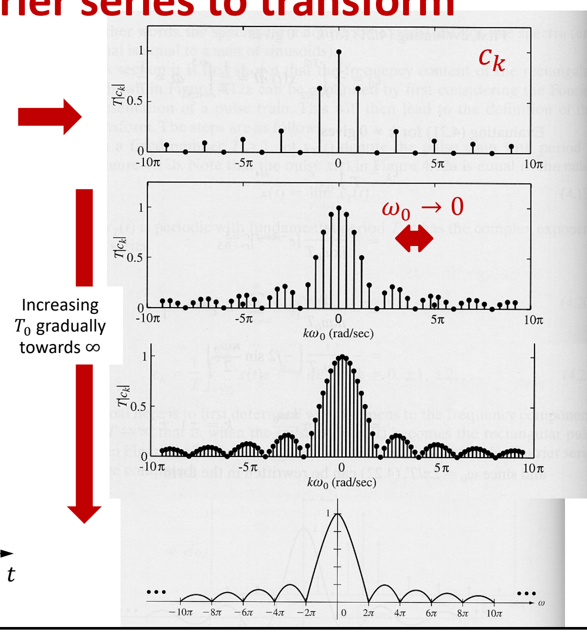
From Fourier series to transform



Let $T_0 \to \infty$





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Important

	Time-Domain	Frequency-Domain
Continuous-time Fourier Series CTFS	Continuous Periodic	Discrete Aperiodic
Continuous-time Fourier Transform CTFT	Continuous Aperiodic	Continuous Aperiodic

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DISCRETE-TIME

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Discrete-time signals

- A sequence of numbers
- Mathematical representation:
    - $X = \{\mathrm{x[n]}\}, \quad -\infty < n < \infty$
  - Sampled from an analog signal $x_a(t)$, at times nT:
    - $x[n] = x_a(nT), \quad -\infty < n < \infty$
- T is called the sampling period, and its reciprocal $F_s = {}^1/_T$ the sampling frequency

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Sampling rate

- Changing the sampling rate controls the sampling period (how often a sample is drawn from the signal)

$$F_S = 8000 \text{ Hz} \quad \leftrightarrow \quad T = 1/8000 = 125\ \mu\text{sec}$$
$$F_S = 10000 \text{ Hz} \quad \leftrightarrow \quad T = 1/10000 = 100\ \mu\text{sec}$$
$$F_S = 16000 \text{ Hz} \quad \leftrightarrow \quad T = 1/16000 = 62.5\ \mu\text{sec}$$
$$F_S = 20000 \text{Hz} \quad \leftrightarrow \quad T = 1/20000 = 50\ \mu\text{sec}$$

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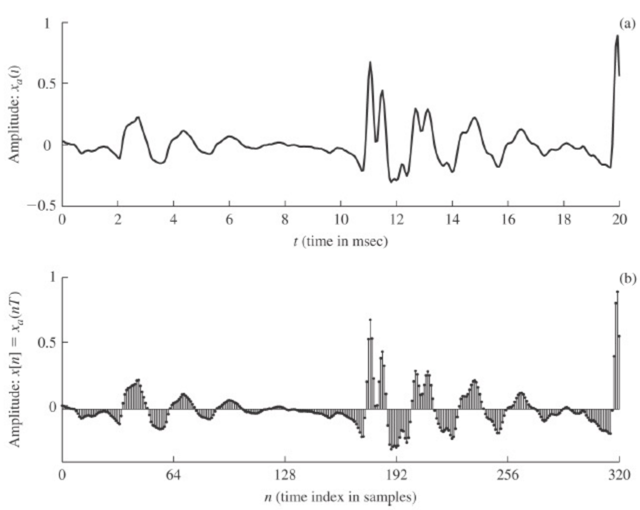
Discrete-time signals



plot( );

stem( );

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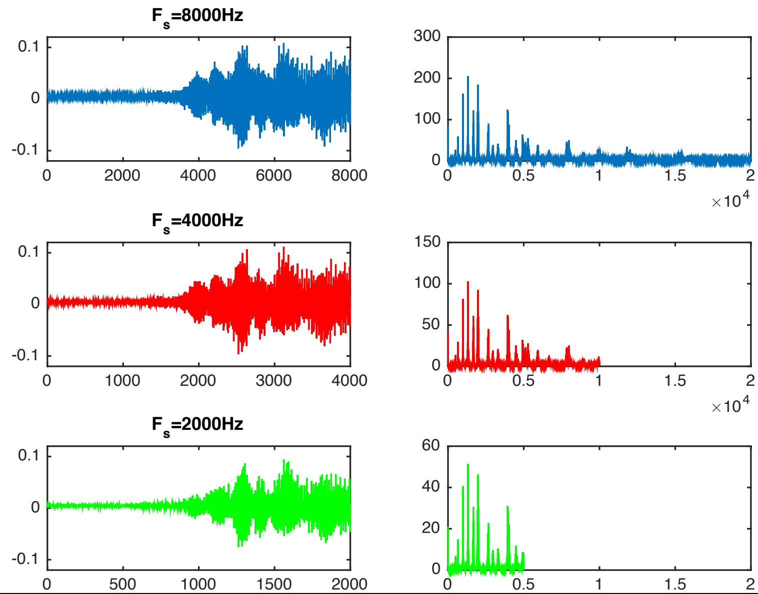
Changing the sampling rate



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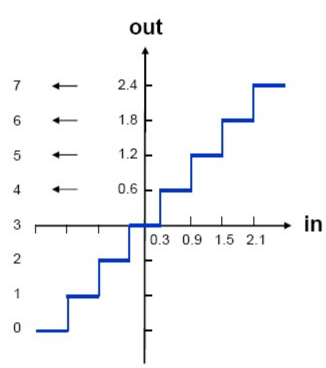
Quantization

- $x[n]$ can be quantized to one of a finite set of values which is then represented digitally in bits, hence a truly digital signal.



A 3-bit uniform quantizer

Quantization:

- Transforming a continuously-valued input into a representation that assumes one of a finite set of values
- The finite set of output values is indexed; e.g., the value 1.8 has an index of 6 (or 110) in binary representation
- Storage or transmission uses binary representation; a quantization table is needed

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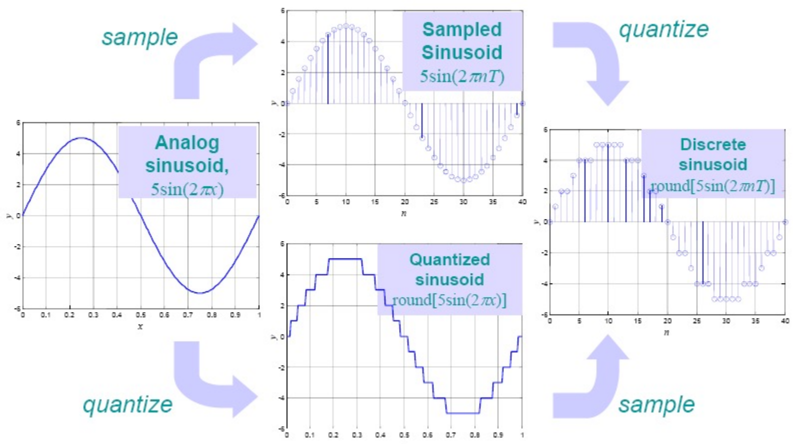
Discrete signals

- Confusion: discrete-time vs. discrete signals!

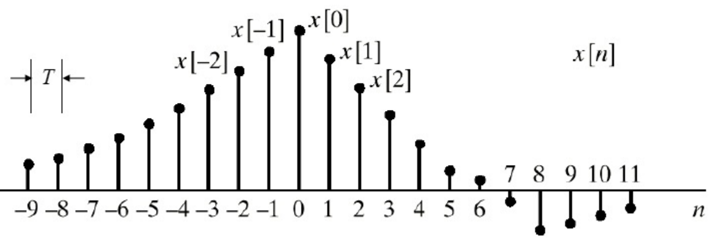


Issues with discrete signals

- What sampling rate is appropriate?
  - 6.4 kHz (telephone bandwidth)
  - 8 kHz (extended telephone BW)
  - 10 kHz (extended bandwidth)
  - 16 kHz (hi-fidelity speech)
  - 44.1 KHz (CD quality music)
- How many quantization levels are necessary at each bit rate (bits/sample)
  - 16, 12, 8.. => ultimately determines the S/N ratio of the signal
  - Speech coding/compression is concerned with answering this question in an optimal manner

Discrete-time signals

- Discrete-time signals are sequences



- $x[n]$ denotes the “sequence value at ‘time’ n”

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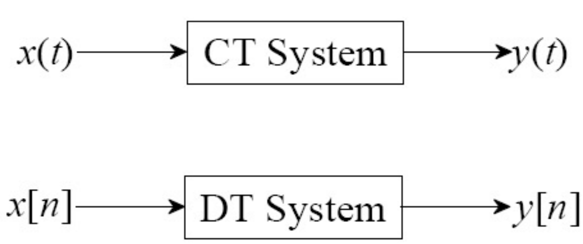
Signal processing

- Transform digital signal into more desirable form
- A system is an entity that manipulates one or more signals to accomplish a function, thereby yielding new signals.



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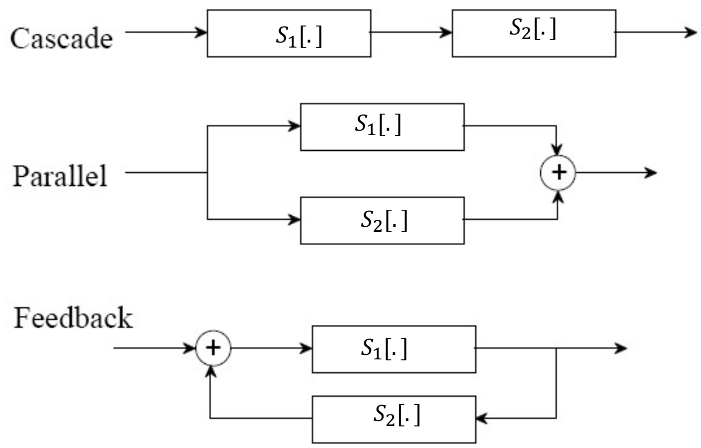
System interconnections



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System properties

- System properties apply to both continuous-time and discrete-time systems
  - Causality
  - Linearity
  - Time invariance
  - Invertibility

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System properties

- Causality
  - A system is said to be causal if the present value of the output signal depends only on the present or past values of the input signal.

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System properties

- Stability
  - A system is said to be BIBO stable (i.e. **B**ounded-**I**nput, **B**ounded-**O**utput stable) if a bounded-input yields a bounded output.

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System properties

- Linearity
  – A system is said to be linear in terms of the system input x(t) and the system output y(t) if it satisfies the following two properties of superposition and homogeneity.

Superposition :

$$x_1(t) \rightarrow \blacksquare \rightarrow y_1(t) \qquad x_2(t) \rightarrow \blacksquare \rightarrow y_2(t)$$

$$\Rightarrow \quad x_1(t)+x_2(t) \rightarrow \blacksquare \rightarrow y_1(t)+y_2(t)$$

Homogeneity :

$$x_1(t) \rightarrow \blacksquare \rightarrow y_1(t) \quad \Rightarrow \quad ax_1(t) \rightarrow \blacksquare \rightarrow ay_1(t)$$

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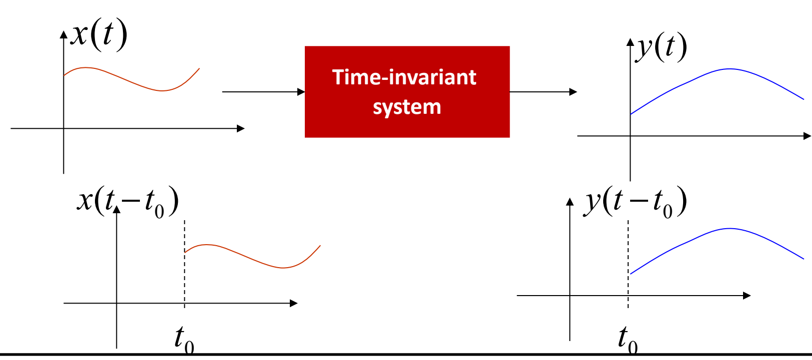
System properties

- Time invariance
  – A system is said to be time invariant if a time delay or time advance of the input signal leads to an identical time shift in the output signal.



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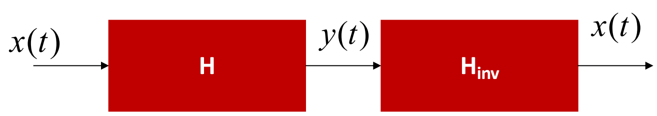
System properties

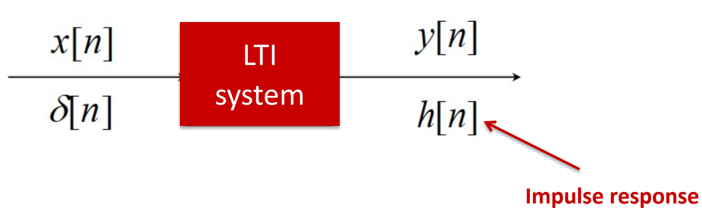
- Invertibility
  - A system is said to be Invertible if the input of the system can be recovered from the output.



LTI systems

- A system that is <u>BOTH</u> linear and time-invariant is said to be LTI.
- The output can be determined via a convolution sum

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]}$$



LTI systems

- Easiest to understand/manipulate
- Powerful processing capabilities
- Characterized completely by their response to unit sample, h[n], via convolution relationship
- Basis for linear filtering
- Used as models for speech production (source convolved with system)

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Eigen property of LTI systems

- Consider the complex exponential: $x[n] = e^{j\omega_0 n}$
- The output of a stable LTI system is

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{j\omega_0 (n-k)} \\
&= \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k}\right) e^{j\omega_0 n} \\
&= H(\omega_0)e^{j\omega_0 n}
\end{aligned}
$$

  - i.e A complex exponential input to an LTI system results in the same complex exponential at the output scaled by $H(\omega_0)$.
  - Therefore, the complex exponentials are called eigenfunctions of LTI systems, and $H(\omega_0)$ is the associated eigenvalue.

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Transform representations

- Z-transform
- Discrete-time Fourier transform
- Discrete Fourier transform

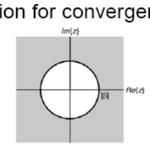
Z-transform

- The Z-transform is a general mapping from time-domain to another domain, defined by: $$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$
  - X(z) is an infinite power (geometric) series in the complex variable, $z = re^{j\omega}$
    - so the sum does not always converge.

- ***X(z)*** **converges (is finite) only for certain values of *z*:**

$$\sum_{n=-\infty}^{\infty} |x[n]||z^{-n}| < \infty$$ - sufficient condition for convergence

- **region of convergence: $R_1 < |z| < R_2$**



Rational Functions

- We are often interested in a special class of functions of the form:

$$X(z) = \frac{P(z)}{Q(z)}$$

  - The roots of the Polynomial P(z) are called *zeros* (values of z for which P(z)=0).
  - The roots of the Polynomial Q(z) are called *poles* (values of z for which Q(z)=0).
    - ✓The zeros of X(z) are values for which X(z)=0, and poles are values for which X(z) goes to infinity.

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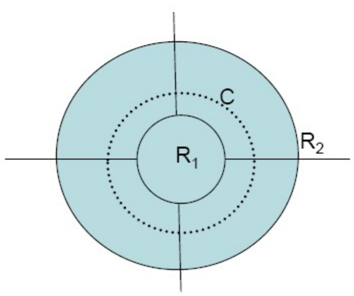
Inverse Z-transform

$$x[n] = \frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz$$

where C is a closed contour that encircles the origin of the z-plane and lies inside the region of convergence



for *X(z)* rational, can use a partial fraction expansion for finding inverse transforms

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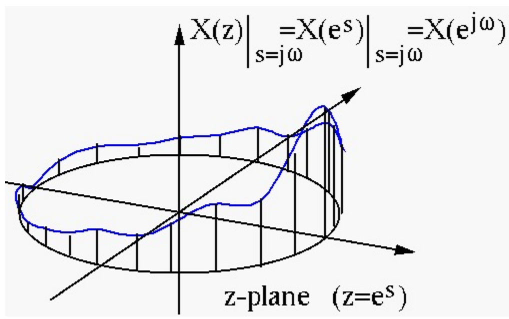
Discrete-Time Fourier Transform (DTFT)

- If the ROC includes the unit circle, then

$$X(z)\big|_{z=e^{j\omega}} = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(\omega)$$

Abuse of notation/ Used interchangeably

  – is called the discrete-time Fourier transform (DTFT) of x[n]



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Discrete-Time Fourier Transform (DTFT)

- If the ROC includes the unit circle, then

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Abuse of notation/ Used interchangeably

  – is called the discrete-time Fourier transform (DTFT) of x[n]

- The 'synthesis equation' is defined as:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

$X(\omega)$: periodic with period 2π

- The pair of equation is known as the discrete-time Fourier transform pair representation of a sequence.

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Discrete-Time Fourier Transform (DTFT)

- Note that $X(\omega)$ is an infinite sum
  - It may not converge
  - We need $|X(\omega)| < \infty \quad \forall \omega$
  - $|X(\omega)| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega}\right| \le \sum_{n=-\infty}^{\infty} |x[n]||e^{-jn\omega}| \le \sum_{n=-\infty}^{\infty} |x[n]| < \infty$
  - This inequality $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ is called absolute summability
  - It is sufficient condition for existence of DTFT

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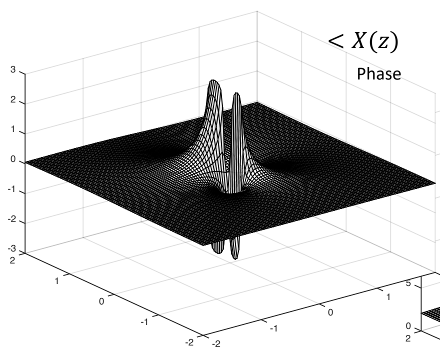
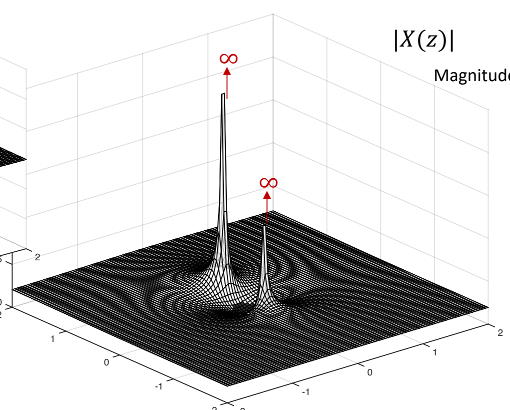
From Z-transform to Fourier transform

$$\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \quad \xrightarrow{z} \quad \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} \quad , |z| > \frac{1}{2}$$





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From Z-transform to Fourier transform

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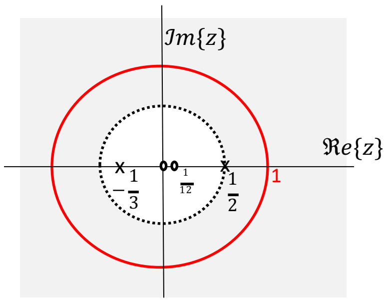

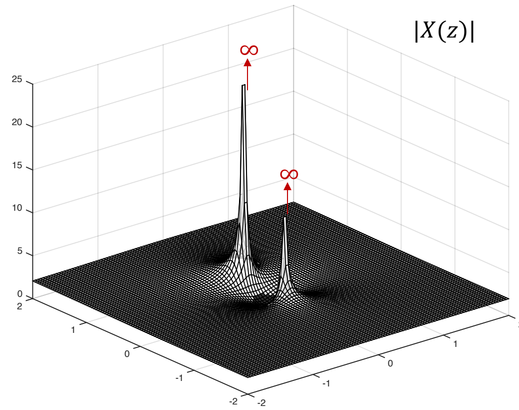

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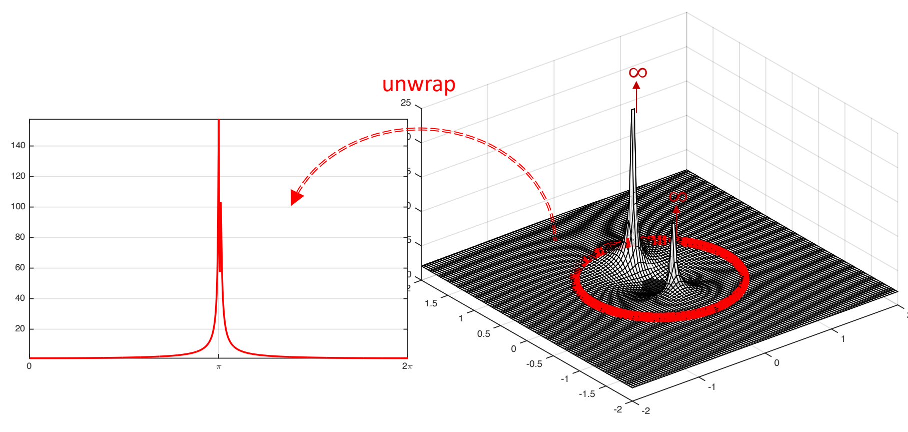
From Z-transform to Fourier transform

$$\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \quad \xrightarrow{F} \quad \frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{1}{1+\frac{1}{3}e^{-j\omega}}$$



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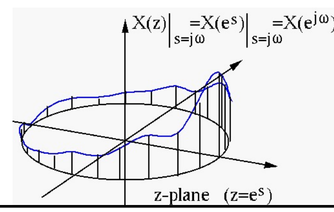
DTFT properties

- Polar Form
$$\begin{aligned} X(\omega) &= X_r(\omega) + jX_i(\omega) \\ &= |X(\omega)| e^{j<X(\omega)} \end{aligned}$$

- Periodicity
  - The Fourier transform is periodic, with period 2π:
$$X(\omega + 2\pi) = X(\omega)$$
  - So, it is sufficient to represent a discrete-time signal over the frequency range $[-\pi,\pi]$.



Discrete Fourier Transform

- The Fourier transform of a discrete-time signal is a continuous function of frequency. In practice, digital computers cannot work with continuous frequency. So, we need to sample the Fourier transform.

- For sequences of length N, we define a discrete-Fourier transform (DFT) as:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \qquad 0 \le k \le N-1$$
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} \qquad 0 \le n \le N-1$$

Discrete Fourier Transform

- DFT of a finite length sequence (length N) is basically a discrete sequence X[k] of length N which correspond to X(z) {or X(ω)} evaluated at N equally-spaced points on the unit circle.

$$X[k]=X(e^{j\frac{2\pi k}{N}})=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi kn}{N}}\ ,\ \ k=0,1,\ldots,N-1$$

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Important

	Time-Domain	**Frequency-Domain**
Continuous-time Fourier Series (CTFS) or (FS)	Continuous Periodic	Discrete Aperiodic
Continuous-time Fourier Transform (CTFT)	Continuous Aperiodic	Continuous Aperiodic
Discrete-Time Fourier Transform (DTFT)	Discrete Aperiodic	Continuous Periodic
Discrete Fourier Transform DFT	Discrete Periodic	Discrete Periodic

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SAMPLING

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The sampling process

- Sampling is equivalent to taking snapshots of $x(t)$ every T_s seconds
- Each snapshot is called a *sample*
- T_s is the sampling interval, i.e., the time interval between each sample (second/sample)
  - Typically regularly spaced samples, though not necessary
- The sequence of samples is given by

$$x[n] = x(nT_s), \quad n \in I$$

  - $F_S = \dfrac{1}{T_s}$ is the sampling frequency or rate (units of samples/second = hertz)

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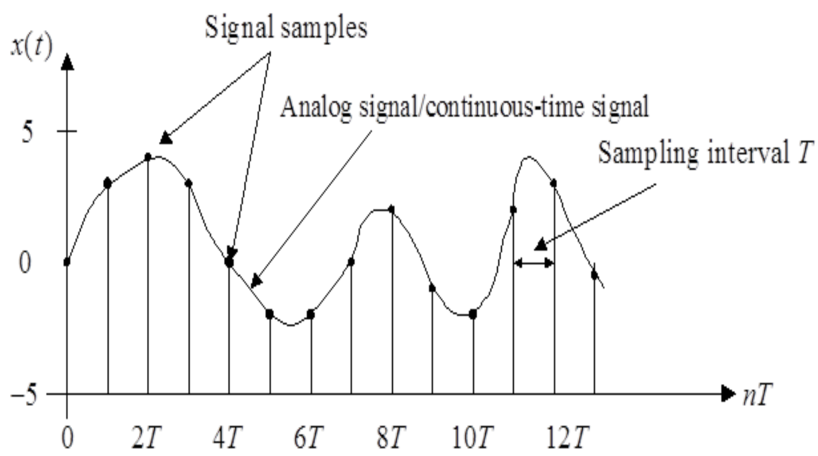

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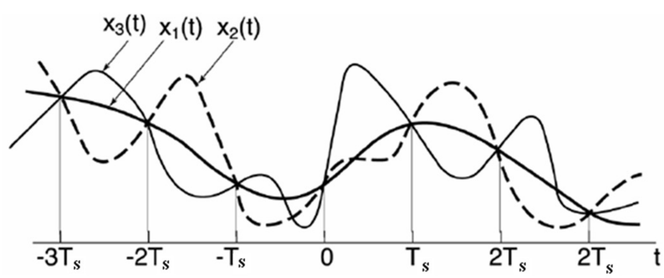
Sampling Interval Selection



- Given a set of sampling points, under what conditions can we uniquely reconstruct the original CT signals from which those samples came?
- Need to consider the frequency content of the CT signal being sampled → Fourier Transform

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Sampling CT signals

- Obtain $x[n]$ from $x_c(t)$ in 2 steps:
  1. Sample $x_c(t)$ at uniform intervals T_s to obtain $x_s(t)$
  2. Convert sampled signal $x_s(t)$ into discrete-time signal $x[n]$

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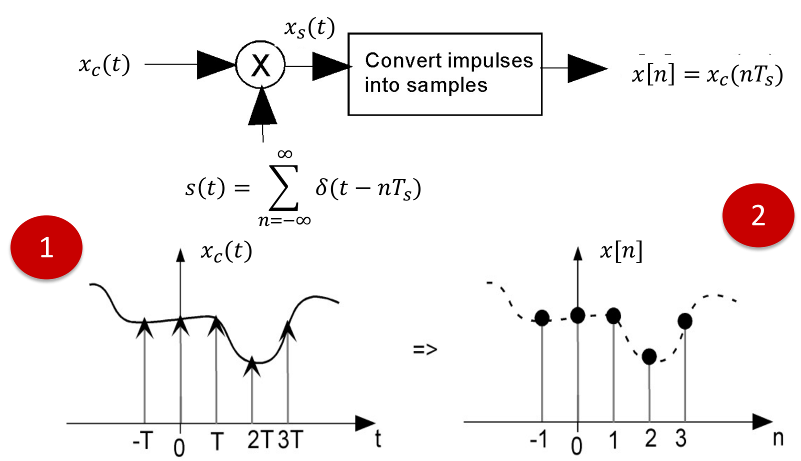
Sampling CT signals



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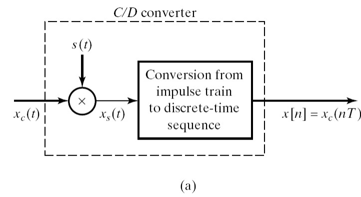
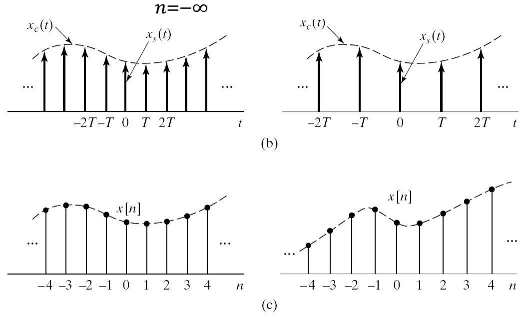
Sampling CT signal



$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



Two sampling rates

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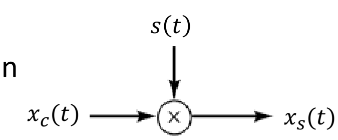
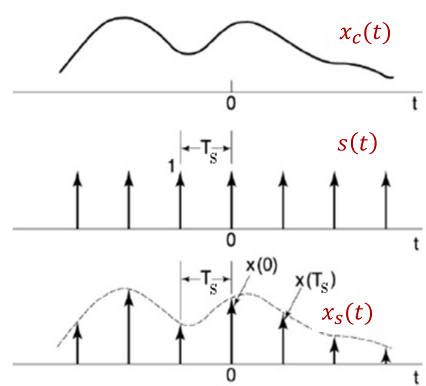
Step 1: Impulse sampling

- Multiplying x(t) by a sampling function

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



$$\begin{aligned} x_s(t) &= x_c(t).s(t) \\ &= \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT_s) \\ &= \sum_{n=-\infty}^{\infty} x_c(nT_s)\delta(t - nT_s) \end{aligned}$$



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Step 1: Impulse sampling

- Multiplication in time is convolution in Freq $x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT_s)\delta(t - nT_s)$

$$
\begin{aligned}
x_c(t) &\mapsto X_c(\Omega)\\
\sum_{n=-\infty}^{\infty} \delta(t - nT_s) &\mapsto \frac{1}{T_s}\sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)\\
x_c(t).\sum_{n=-\infty}^{\infty} \delta(t - nT_s) &\mapsto X_c(\Omega) * \frac{1}{T_s}\sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)\\
&= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c(\Omega - k\Omega_s)
\end{aligned}
$$

The Fourier transform of a delta comb is also a delta comb

Sampling Freq: $\Omega_s = 2\pi f_s = 2\pi \frac{1}{T_s}$

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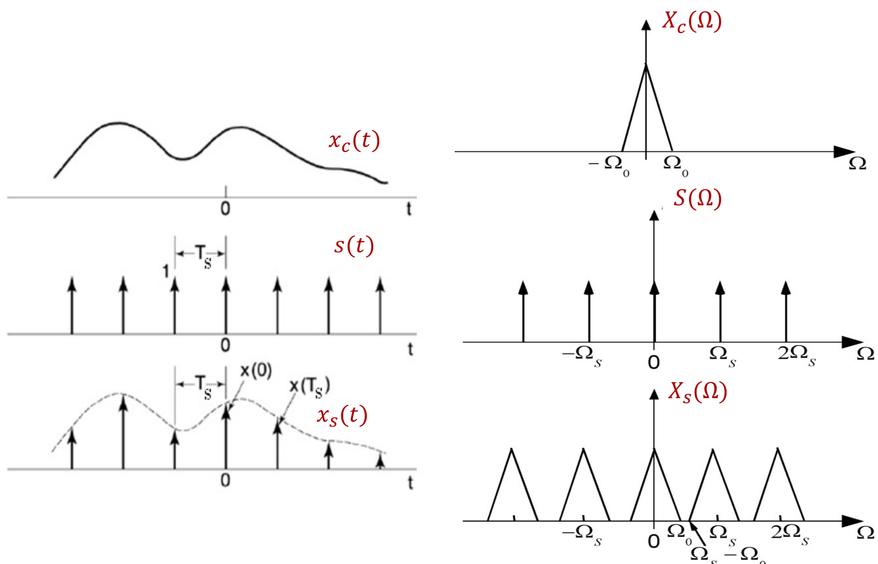
Step 1: Impulse sampling



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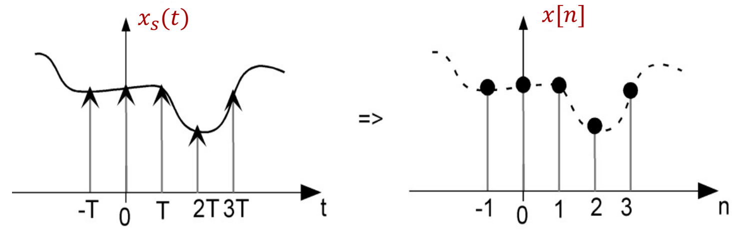
Step 2: Discrete samples

- Step 2 involves conversion into discrete samples



Step 2: Discrete samples

- We know that:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT_s)\delta(t - nT_s)$$ *Sampled signal*

$$x[n] = x_c(nT_s)$$ *Discrete-time signal*

- In freq. domain

$$\delta(t - t_0) \mapsto e^{-j\Omega t_0}$$

$$\delta(t - nT_s) \mapsto e^{-j\Omega nT_s}$$

$$x_c(nT_s)\delta(t - nT_s) \mapsto x_c(nT_s)e^{-j\Omega nT_s}$$

$$\underbrace{\sum_{n=-\infty}^{\infty} x_c(nT_s)\delta(t - nT_s)}_{x_s(t)} \mapsto \underbrace{\sum_{n=-\infty}^{\infty} x_c(nT_s)e^{-j\Omega nT_s} \mapsto \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega nT_s}}_{X_s(\Omega)}$$

$$\boxed{X(\omega) = X_s(\Omega)|_{\Omega = \frac{\omega}{T_s}}}$$

$$x[n] \mapsto X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

C/D in Fourier Domain

- We have: Step1:

$$X_s(\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c(\Omega - k\Omega_s)$$

Step 2:

$$X(\omega) = X_s\,(\Omega)\Big|_{\Omega=\frac{\omega}{T_s}} = X_s\left(\frac{\omega}{T_s}\right)$$

- Therefore:

$$X(\omega) = X_s\left(\frac{\omega}{T_s}\right) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T_s} - k\Omega_s\right) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T_s} - k\frac{2\pi}{T_s}\right)$$

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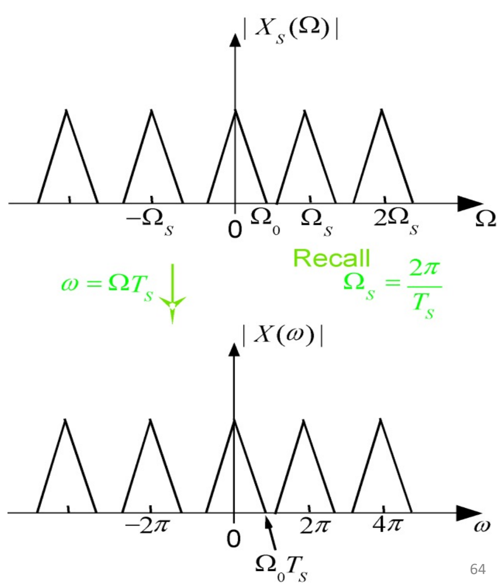
In other words…

- Make infinite number of duplicates of the spectrum of the continuous signal
- Scale the frequency axis by multiplying it by in order to obtain new axis
  - Makes periodic with a period (since is discrete)



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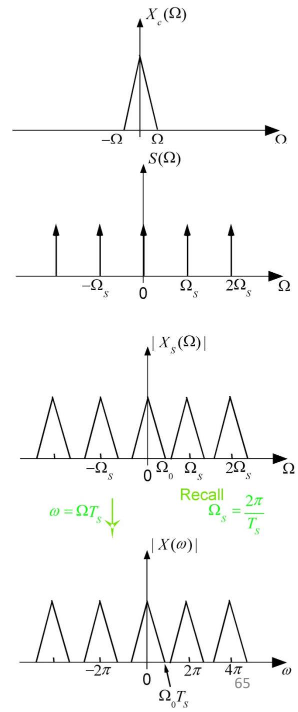
All together..

- Step1: $X_s(\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c(\Omega - k\Omega_s)$

  Step 2: $X(\omega) = X_s(\Omega)\Big|_{\Omega=\frac{\omega}{T_s}} = X_s\left(\frac{\omega}{T_s}\right)$

- Therefore:

$$X(\omega) = X_s\left(\frac{\omega}{T_s}\right) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T_s} - k\Omega_s\right) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T_s} - k\frac{2\pi}{T_s}\right)$$



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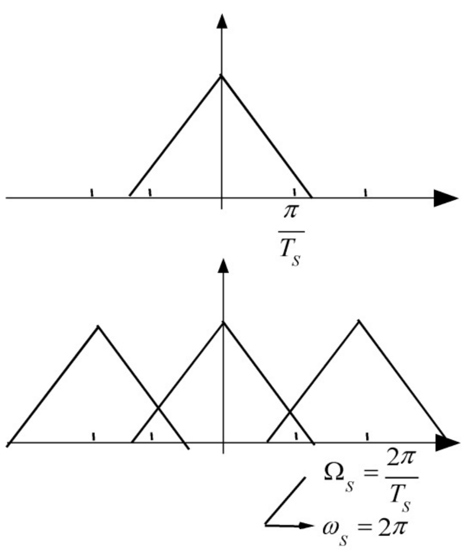
Aliasing



- Largest bandwidth allowed is π

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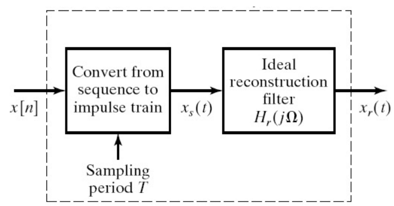
D/C Conversion

- The reconstruction of continuous-time signal involves 2 steps:
  - **Step1:** convert samples of into a sequence of impulses:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$$

  - **Step2:** filter $x_s(t)$ with ideal low pass filter (reconstruction filter) with frequency response:



$$H_r(\Omega) = \begin{cases} T_s & , \quad |\Omega| \le \dfrac{\pi}{T_s} \\ 0 & , \quad |\Omega| > \dfrac{\pi}{T_s} \end{cases}$$

or

$$h_r(t) = \frac{\sin\left(\frac{\pi t}{T_s}\right)}{\frac{\pi t}{T_s}}$$

$$X_c(\Omega) = H_r(\Omega)X(\Omega T_s)$$

$$\begin{aligned} x_c(t) &= x_s(t) * h_r(t) \\ &= \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT_s) \end{aligned}$$

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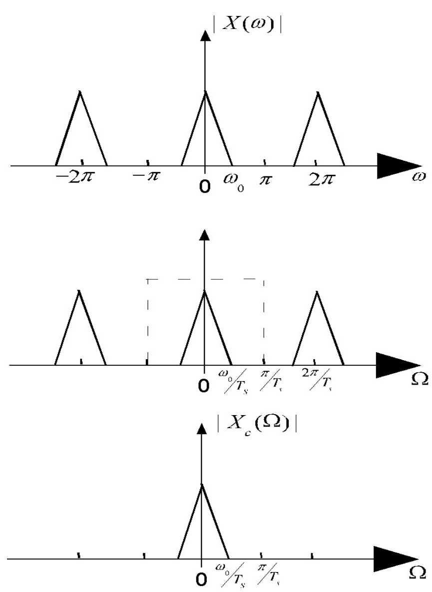
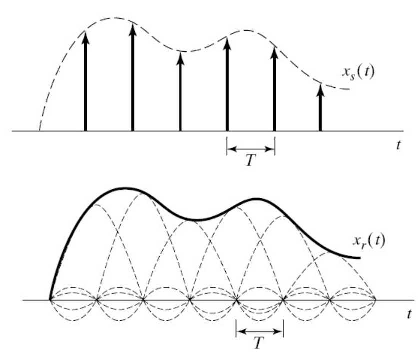
D/C Conversion





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Nyquist Sampling Theorem

- Let $x(t)$ be a band-limited signal such that

$$X(\Omega) = 0, \qquad |\Omega| > \Omega_M$$

Then, $x(t)$ is uniquely determined by its samples $x(nt), n \in \mathbb{Z}$ if

$$\boldsymbol{\Omega_s > 2\Omega_M}$$

where $\Omega_s = \frac{2\pi}{T_s}$ is the sampling frequency

and Ω_M is called the Nyquist frequency

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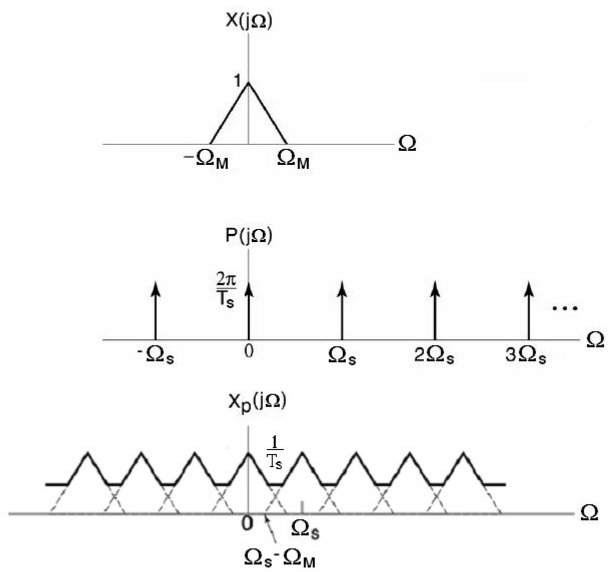
Aliasing Effect



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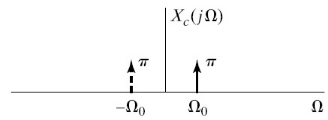
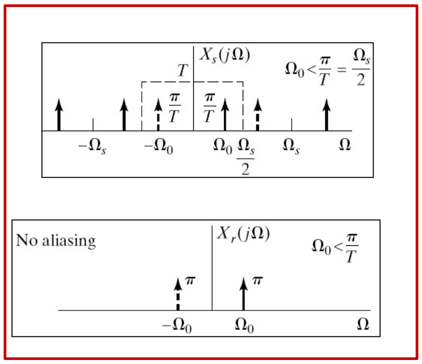
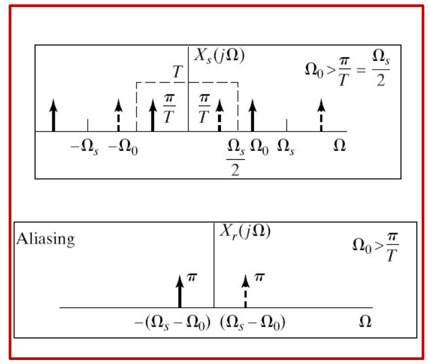
Aliasing example: cos signal

- Suppose $x_c(t) = \cos(\Omega_0 t)$





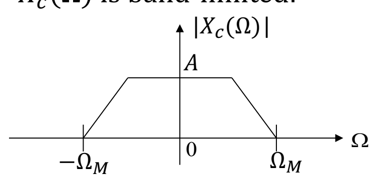
No Aliasing



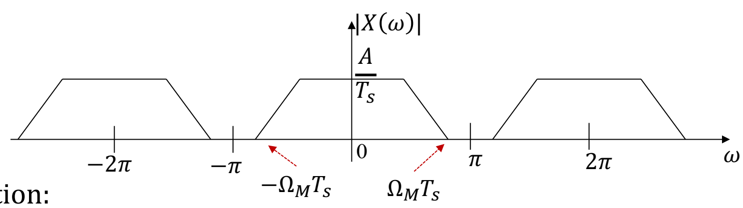
Aliasing

Oversampling

- Suppose that $x_c(t) \leftrightarrow X_c(\Omega)$ is band-limited:



- Then if T_s is sufficiently small, $X(\omega)$ appears as:

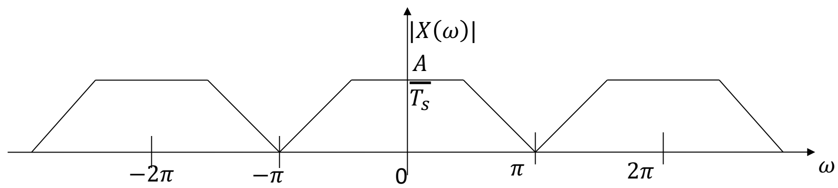


- Condition:

$\boxed{\Omega_s > 2\Omega_M}$ or $\boxed{\Omega_M T_s < \pi}$

Critically sampling

Critically sampled: $\boxed{\Omega_s = 2\Omega_M}$ or $\boxed{\Omega_M T_s = \pi}$



- According to the Sampling Theorem, in general the signal cannot be reconstructed from samples at the rate $T_s = \frac{\pi}{\Omega_M}$
- This is because of errors will occur if $X_c(\Omega_M) \neq 0$, the folded frequencies will add at $\omega = \pi$

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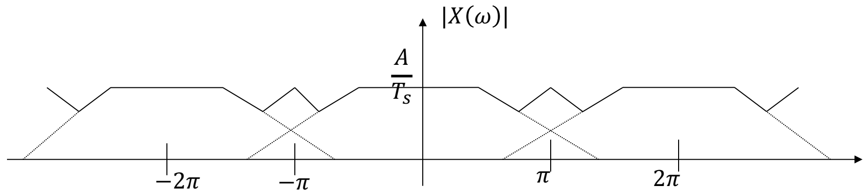
Undersampling (aliasing)

If sampling theorem condition is **not** satisfied $\boxed{\Omega_s = 2\Omega_M}$ or $\boxed{\Omega_M T_s = \pi}$



- The frequencies are **folded** and **summed**
  - This **changes** the shape of the spectrum.
  - The process is not reversible.
- If a signal is not **strictly band-limited**, sampling can still be done at twice the **effective band-limited**.

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Sampling rate conversion

- i.e. Resampling in the digital domain
- **Case 1:** reducing by an integer
  - Goal: Reduce sampling rate by factor of M
    - (instead of 10000 samples per second, use only 5000 samples per second)

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Downsampling

- Goal: Reduce sampling rate by factor of M
  - New sampling rate: $T' = MT_s$
  - Resampled signal $x_d[n] = x_c(nT') = x_c(MT_s) = x[nM]$ ($x_d[.]$ for decimated)
  - So, we can reduce the rate by taking every Mth sample of $x[n]$
    - System called <u>downsampler</u>

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n]$$

  - Dowsampling often results in aliasing

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Downsampling

- Recall:

$$X(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T_s} - k\frac{2\pi}{T_s}\right)$$

Similarly:

$$X_d(\omega) = \frac{1}{MT_s} \sum_{l=-\infty}^{\infty} X_c\left(\frac{\omega}{MT_s} - l\frac{2\pi}{MT_s}\right)$$

- We can show that:

$$\boxed{X_d(\omega) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(\frac{\omega}{M} - k\frac{2\pi}{M}\right)}$$

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Downsampling

$$\boxed{X_d(\omega) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(\frac{\omega}{M} - k\frac{2\pi}{M}\right)}$$

- The equation above means:
  - Expand frequency axis by factor of M
  - Make M copies within interval of 2π
  - Divide spectrum by M

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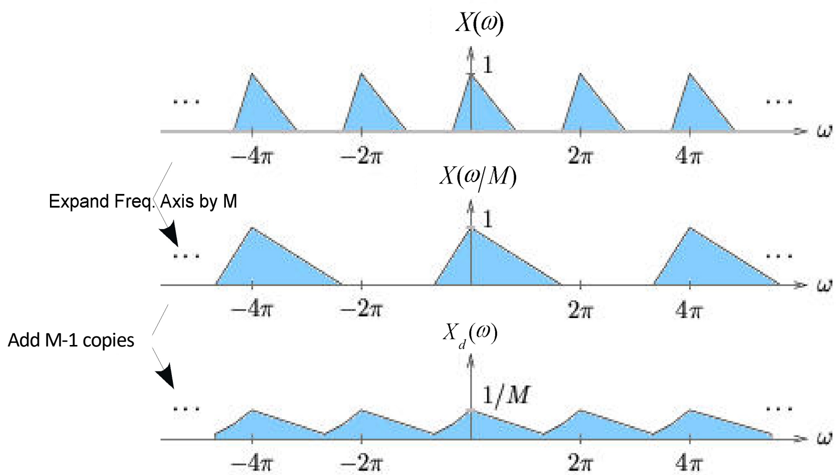
Downsampling (Case M=2)



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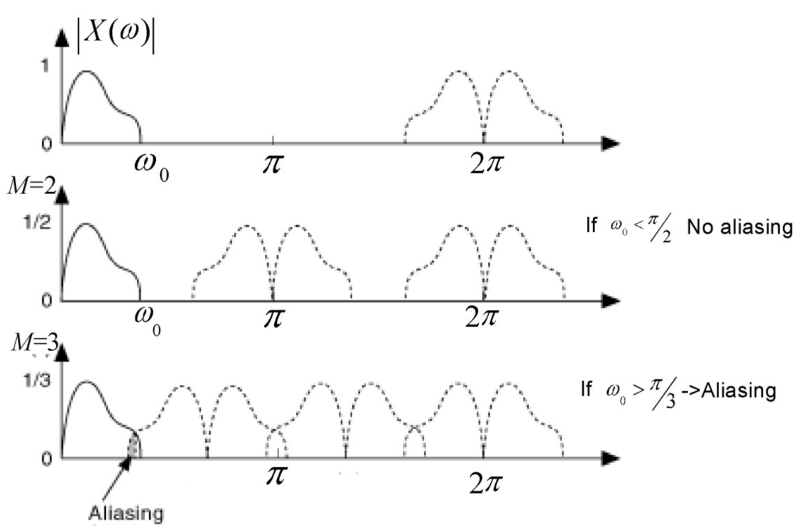
Another example



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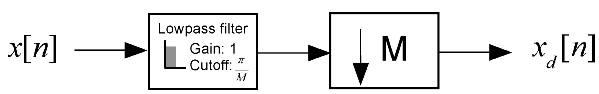
Dealing with aliasing

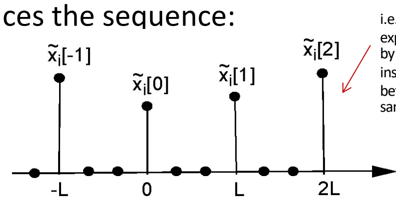
- In order to prevent aliasing, x[n] should have its highest frequency limited to $\frac{\pi}{M}$
  - should be filtered by a lowpass filter with cutoff $\frac{\pi}{M}$
- The cascade of low-pass filter and downsampler is called a decimator



Case 2: Increasing rate by integer

- Steps:
  - Increase by factor L
  - New sampling rate: $T' = \frac{T_s}{L}$
  - So, $x_i[n] = x_c\left(n\frac{T_s}{L}\right)$ (interpolator)
  - Samples of $x_i[n]$ for values of n that are integer multiple of L: $x_i[nL] = x[n]$
  - So, an upsampler produces the sequence:

$$\breve{x}_i[n] = \begin{cases} x\left[\frac{n}{L}\right] & , \quad n = 0, \pm L, \pm 2L, \dots \\ 0 & , \quad otherwise \end{cases}$$



i.e. Upsampler expands time scale by factor of L by inserting L-1 zeros between each sample of x[n]

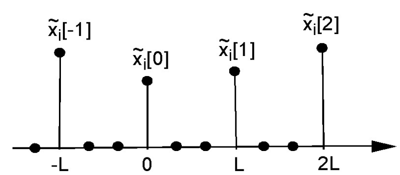
Upsampling in Frequency domain

$$
\begin{aligned}
\breve{X}_i(\omega) &= \sum_{n=-\infty}^{\infty} \breve{x}_i[n]\, e^{-jn\omega} \\
&= \sum_{n=-\infty}^{\infty} x[n]\, e^{-jnL\omega} \\
&= X(L\omega)
\end{aligned}
$$

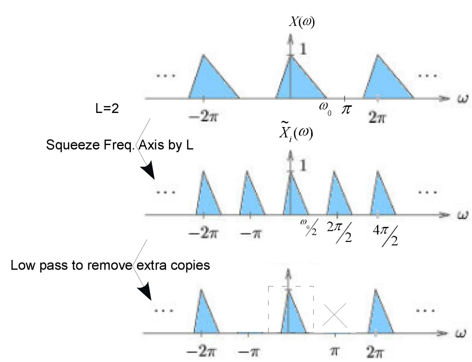
- Upsampling is simply a scaling in frequency

Upsampling (cont.)

- Now.. We need to fill the zeros

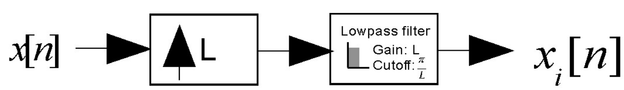


  – Done by filtering $\breve{x}_i[n]$ with a lowpass with cutoff $\frac{\pi}{L}$ and gain L



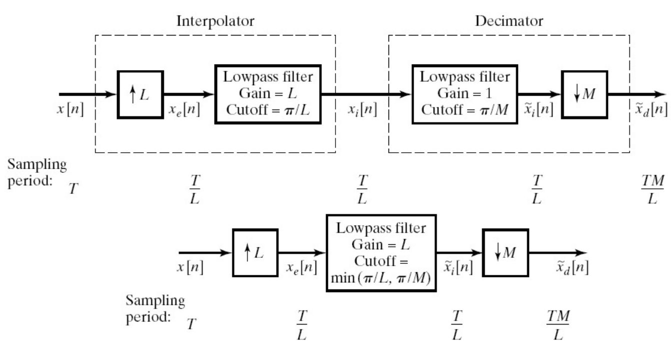
Interpolator

- Cascade of upsampler and lowpass filter is called interpolator



Sampling rate conversion by rational factor

- If we have a rational factor (say 2.5), it corresponds to a change of sampling rate of 5/2; i.e. $^L/_M$
  - Downsampling by M can create aliasing
  - Upsampling by L is safe
  - Start with upsampling, then downsampling



DISCRETE-TIME FILTERS

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Digital Filters

- A digital filter is a discrete-time linear time-invariant system that performs a filtering operation
  - **Filtering**: refers to extracting a desired signal from noise

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Discrete-time Filters

- Filters are a special case of LTI systems. They are defined by frequency response:
- It has two parts:

$$H(\omega) = \underbrace{|H(\omega)|}_{\text{Magnitude response}} \underbrace{e^{j\varphi(\omega)}}_{\text{Phase response}}$$

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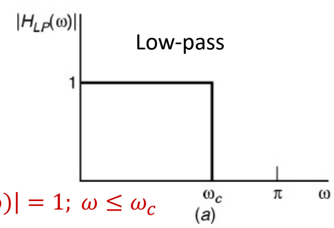
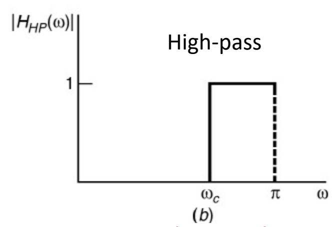
Ideal Magnitude Response

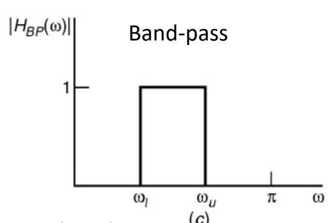
Recall Eigenfunction property of LTI systems



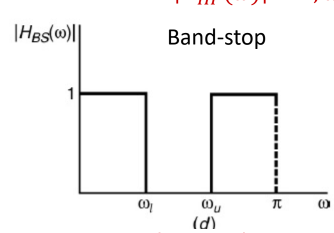
$|H_{LP}(\omega)| = 1;\ \omega \le \omega_c$



$|H_{HP}(\omega)| = 1; \omega_c \le \omega < \pi$



$|H_{BP}(\omega)| = 1; \omega_L \le \omega \le \omega_U$



$|H_{BS}(\omega)| = 0; \omega_L \le \omega \le \omega_U$

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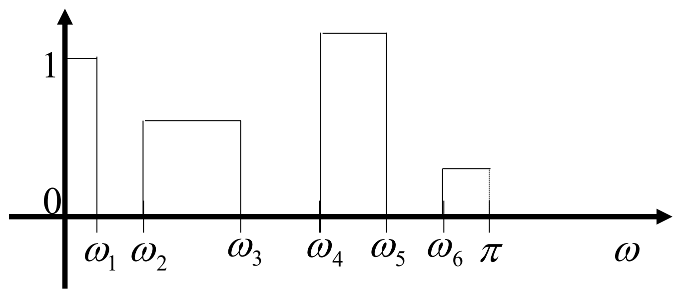
Multiband filters

- You can combine multiple simple filters into a multi-band filter



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Ideal filters

- The phase response $< H(\omega)$ of an ideal filter is linear

$$< H(\omega) = -t_0 \omega$$

Linear phase is often crucial for sound processing applications

  – The group delay function (derivative of phase function) is a constant

$$\tau(\omega) = -\frac{d < H(\omega)}{d\omega} = t_0$$

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Difference Equation

- The filters of interest often satisfy a difference equation that relates the input and output:

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{l=0}^{M} b_l x[n-l]$$

- Evaluating the z-transform of this equation:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} a_k z^{-l}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

Rational function in z^{-1}

Canonical form showing poles and zeros

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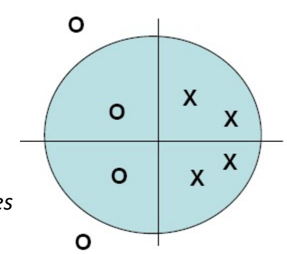
Difference Equation

- In general, we can factorize H(z)

$$H(z) = \frac{A \prod_{r=1}^{M} (1 - c_r z^{-1})}{A \prod_{k=1}^{M} (1 - d_k z^{-1})}$$



M zeroes, N poles

– This system is:
  - Causal ($h[n] = 0, n < 0$) if Region of Convergence: $|z| > R_1$
  - Stable if ROC contains unit circle; so $R_1 < 1$

  $\Rightarrow$ Stable and causal system if all poles inside unit circle

  $\Rightarrow$ If all zeros (in addition to poles) are inside unit circle, system is minimum-phase
    - Guarantees system and its inverse are causal and stable
    - Has minimum group delay

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Types of filters

$$y[n] = \sum_{l=0}^{M} b_l x[n-l] - \sum_{k=1}^{N} a_k y[n-k]$$

If current output depends on past/present inputs <u>and</u> outputs
$(N, M \neq 0)$

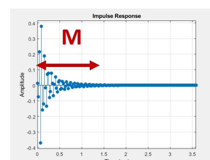
IIR system
(**I**nfinite **I**mpulse **R**esp)

System is not guaranteed to be stable (recursion/feedback)

If current output depends only on past/present inputs
$(N = 0)$

FIR system
(**F**inite **I**mpulse **R**esp)

$$h[n] = \begin{cases} b_n & , \quad 0 \leq n \leq M \\ 0 & , \quad otherwise \end{cases}$$



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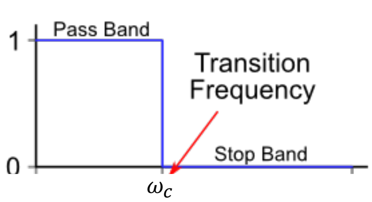
FIR filter design

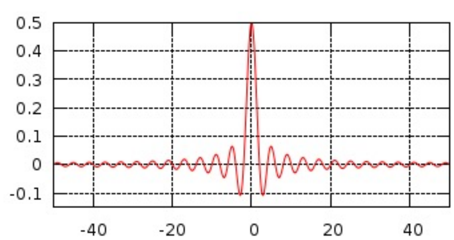
$$y[n] = \sum_{l=0}^{M} b_l x[n-l]$$

Non-recursive equation

ideal lowpass filter



In time-domain



Sinc function $h[n] = \dfrac{\sin(2\pi\omega_c n)}{\pi n}$

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FIR filter design

- **Goal:** Design a FIR filter that corresponds to sinc function impulse response
  - Option: Truncate $h[n]$ to some finite-duration M
    - Corresponds to multiplying $h[n]$ by a 'window'

☞ Called: Windowing design method
  - Convenient but not optimal

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FIR Window design

- There are 2 issues with window design methods:
  1. Need to truncate infinite impulse response
     - Multiply by a window / convolution by frequency response
     - Choice of window & frequency response?
  2. The impulse response (sinc function) is non-casual
     - Introduce a shift in impulse response
     - Resulting filter is causal

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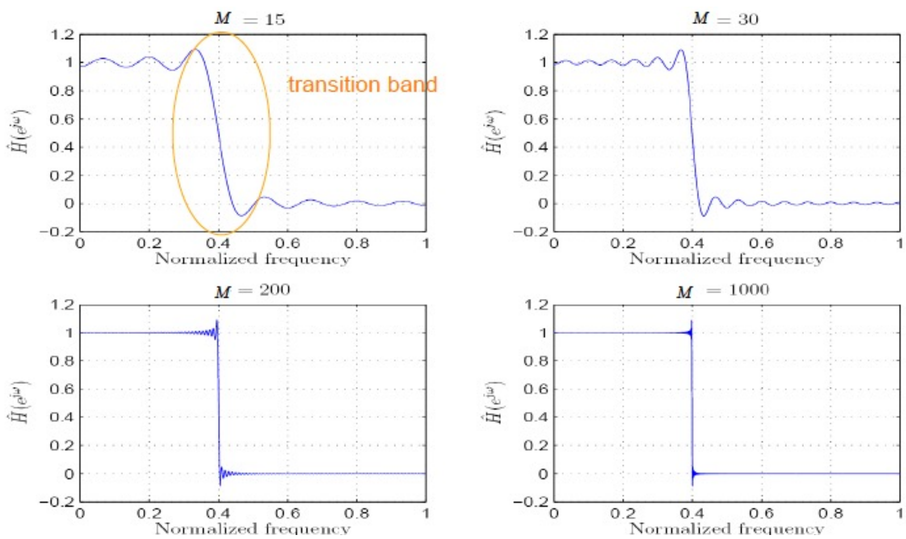
Truncated Low-pass FIR



Transition band gets narrower, but ripples remain

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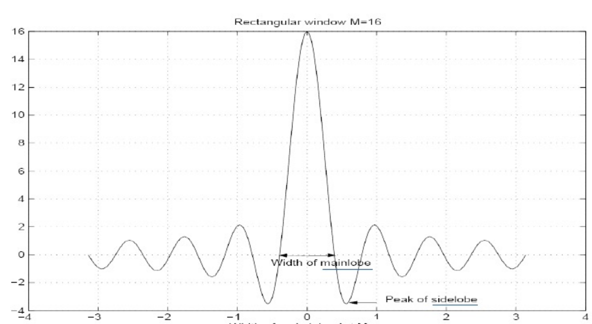
Window design method

- Ripples are caused by rectangular window
- Note: Fourier transform of rectangular window



- Recall: $h[n]$ .w[n] $\Longleftrightarrow$ $H(\omega) * W(\omega)$

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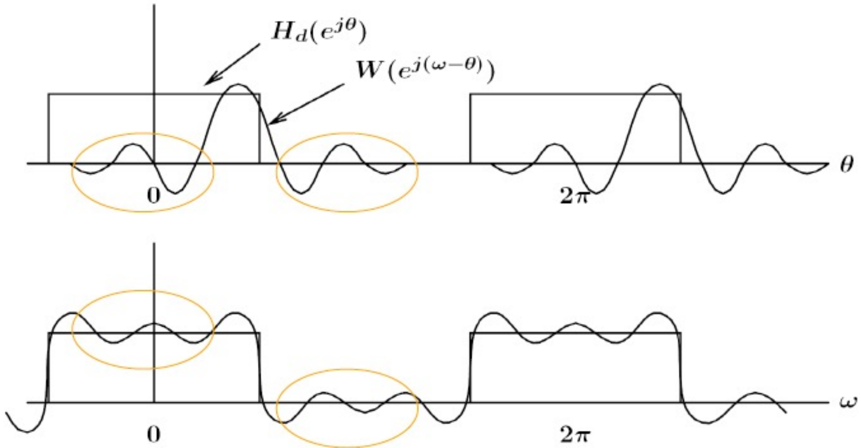
Truncated filter in Fourier domain



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FIR filter design

- Choice of window: Ideally, want:
  - M small (impulse response as short as possible)
    - ✓Fewest number of computations possible
  - $W(\omega)$ as close as possible to a dirac delta function
    - ✓Match as close as possible ideal low-pass filter

◆ These 2 requirements are at odds!

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FIR filter design

- Choice of window:
  - Use windows with no abrupt discontinuity in their time-domain response and consequently low side-lobes in their frequency response.
  - In this case, the reduced ripple comes at the expense of a wider transition region but can be compensated for by increasing the length of the filter.

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Common windows

1. Rectangular $w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & otherwise \end{cases}$

2. Bartlett $w[n] = 1 - \dfrac{2|n - M/2|}{M}$

3. Blackman $w[n] = 0.42 - 0.5\cos\left(\dfrac{2\pi n}{M}\right) + 0.08\cos\left(\dfrac{4\pi n}{M}\right)$

4. Hamming $w[n] = 0.54 - 0.46\cos\left(\dfrac{2\pi n}{M}\right)$

5. Hanning $w[n] = 0.5 - 0.5\cos\left(\dfrac{2\pi n}{M}\right)$

6. Kaiser $w[n] = \dfrac{I_0\left\{\beta\sqrt{1-((n-M/2)/(M/2))^2}\right\}}{I_0\{\beta\}}$

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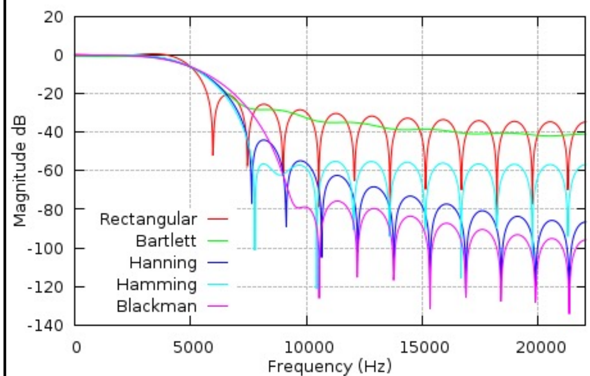
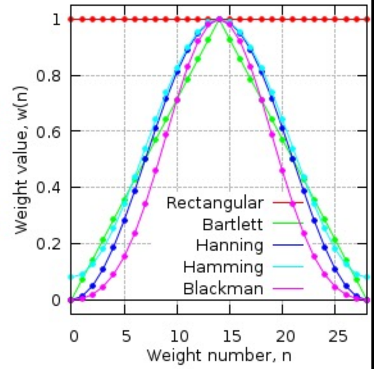
Choice of window





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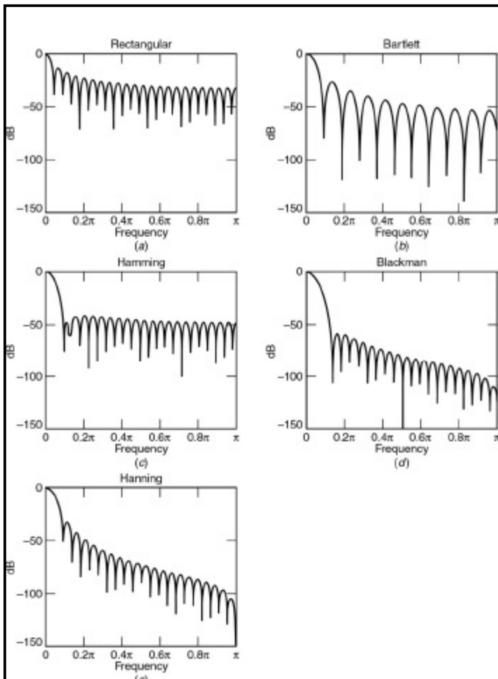

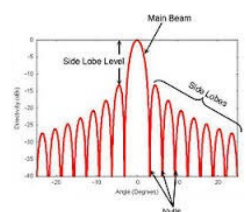

Window	Mainlobe Width	Sidelobe Attenuation
Rectangular	$4\pi / M$	-13 dB
Bartlett	$8\pi/M$	-27 dB
Hanning	$8\pi/M$	-32 dB
Hamming	$8\pi/M$	-43 dB
Blackman	$12\pi/M$	-58 dB

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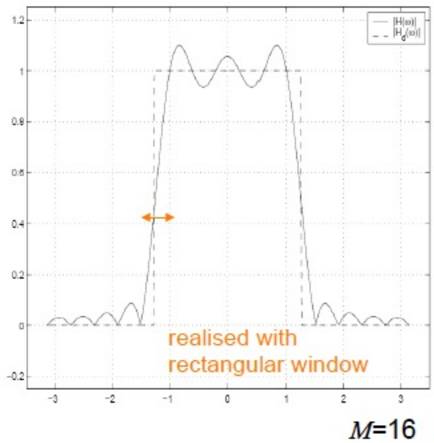
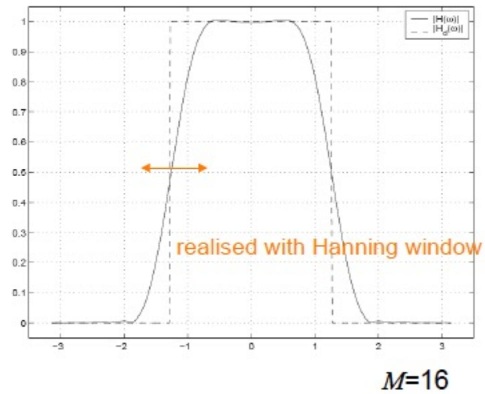
Filter realized with different windows





Note: There are much less ripples for the Hanning window but the transition width has increased

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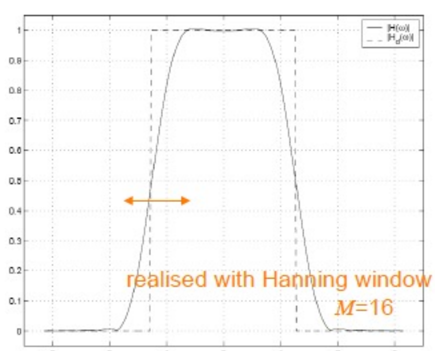
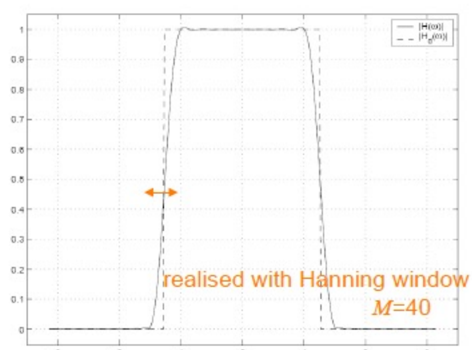
Filter realized with Hanning window





Transition width can be improved by increasing the size of the Hanning window to M = 40

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Window characteristics

- Fundamental trade-off between main-lobe width and side-lobe amplitude
- As window becomes smoother, peak side-lobe decreases, but the main-lobe width increases.
- Need to increase window length to achieve same transition bandwidth.

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FIR Filters in MATLAB

- MATLAB FIR1: window-based standard lowpass, bandpass, highpass, bandstop, and multiband configurations
- MATLAB FIR2: Designs a FIR digital filter any arbitrary frequency response
- MATLAB FIRPM: Designs a FIR digital filter with arbitrary constraints by finding optimal filter between desired and achieved response
- The function Freqz: Evaluates the Filter Frequency Response

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IIR filter design

- Impulse invariant transformation – match the analog impulse response by sampling; resulting frequency response is aliased version of analog frequency response
- Bilinear transformation – use a transformation to map an analog filter to a digital filter by warping the analog frequency scale (0 to ∞) to the digital frequency scale (0 to π); use frequency pre-warping to preserve critical frequencies of transformation (i.e., filter cutoff frequencies)

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IIR Filters

- IIR filter issues:
  – efficient implementations in terms of computations
  – can approximate any desired magnitude response with arbitrarily small error
  – non-linear phase => time dispersion of waveform
- IIR design methods
  – *Butterworth* designs-maximally flat amplitude
  – *Chebyshev* designs-equi-ripple in either passband **or** Stopband
  – *Elliptic* designs-equi-ripple in both passband **and** stopband

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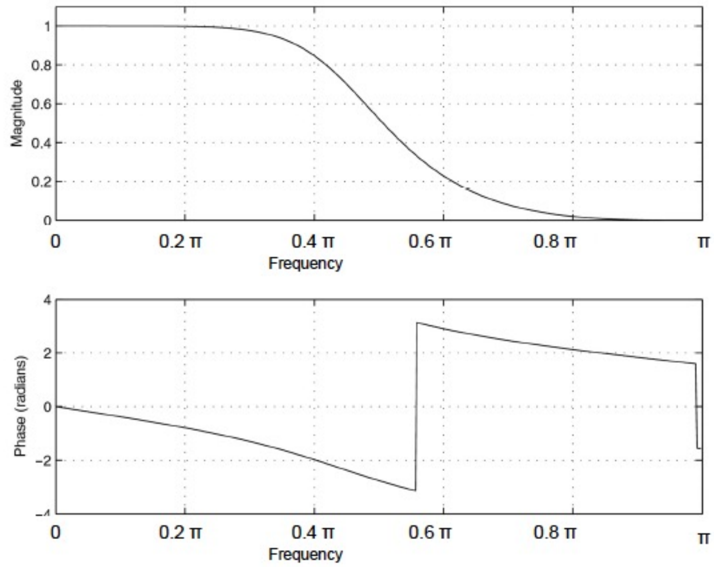
Butterworth design



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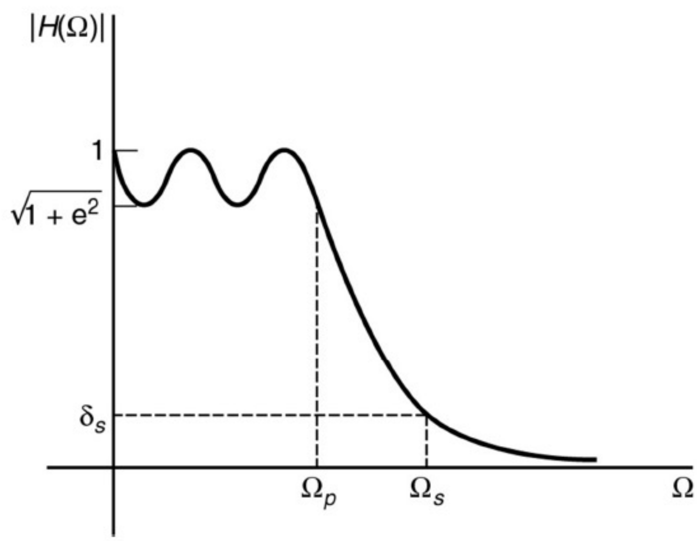
Chebyshev Type I Design



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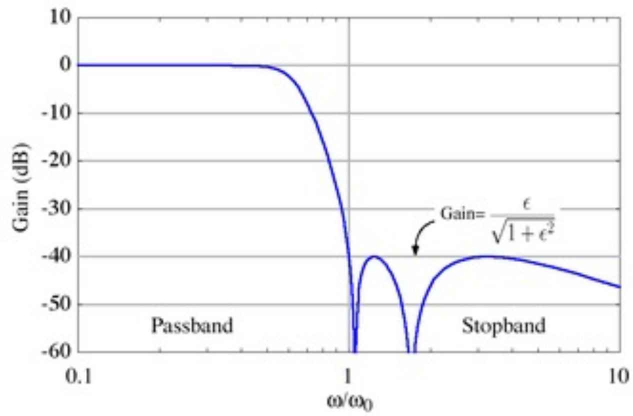
Chebyshev II Design



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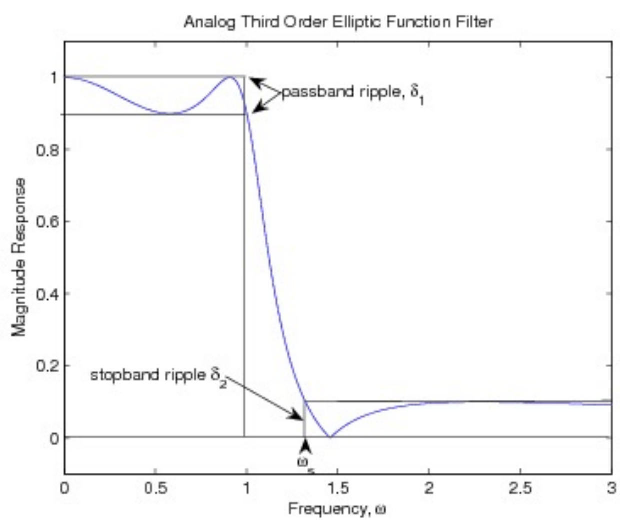
Elliptic Filter Design



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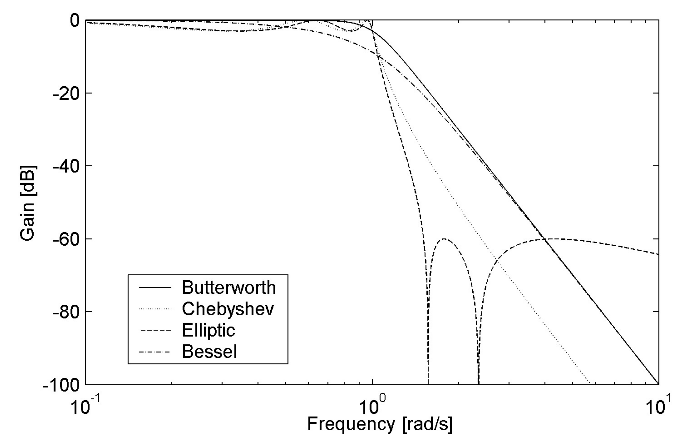
Which filter?

Choice of filter is a tradeoff between simplicity, sharp transitions, oscillations (ripples in passband and stopband)



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Example: MATLAB

- use ellip to design elliptic filter
  - [B,A]=ellip(N,Rp,Rs,Wn)
  - B=numerator polynomial—N+1 coefficients
  - A=denominator polynomial—N+1 coefficients
  - N=order of polynomial for both numerator and denominator
  - Rp=maximum in-band (passband) approximation error (dB)
  - Rs=out-of-band (stopband) ripple (dB)
  - Wp=end of passband (normalized radian frequency)
- use filter to generate impulse response
  - y=filter(B,A,x)
  - y=filter impulse response
  - x=filter input (impulse)
- use zplane to generate pole-zero plot
  - zplane(B,A)

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