EN.520.612 HW3

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for $\pi_k = P(z_k = 1)$, $z = \{z_1, \dots, z_k\}$

$$P(x_n|z) = \prod_{k=1}^{K} \mathcal{N}(x_n|\mu_k, \Sigma_k)^{z_k}$$

$\because P(A \cap B) = P(B|A)P(A)$

$\therefore P(x_n, z) = P(x_n|z)P(z)$

$\because P(x_n|z) = \mathcal{N}(x_n|\mu_3, \Sigma_3)$

$$\therefore P(x_n) = \left[\sum_{k=1}^{K} P(x_n|z)P(z)\right] = \left[\sum_{k=1}^{K} \pi_k \cdot \mathcal{N}(x_n|\mu_k, \Sigma_k)\right]$$

$$p(x) = \left[\prod_{n=1}^{N} P(x_n)\right] = \left[\prod_{n=1}^{N} \sum_{k=1}^{K} \pi_k \cdot \mathcal{N}(x_n|\mu_k, \Sigma_k)\right]$$

$$\ln p(x) = \sum_{n=1}^{N} \ln \sum_{k=1}^{K} \pi_k \cdot \mathcal{N}(x_n|\mu_k, \Sigma_k)$$

$$\ln p(x, z|\theta^*) = \sum_{n=1}^{N} \sum_{k=1}^{K} z_{nk}\left[\ln \pi_k + \ln \mathcal{N}(x_n|\mu_k, \Sigma_k)\right]$$

$$Q(\theta^*, \theta) = \sum_{n=1}^{N} \sum_{k=1}^{K} r(z_{nk})\left[\ln \pi_k + \ln \mathcal{N}(x_n|\mu_k, \Sigma_k)\right]$$

$$\theta^* = \arg\max Q(\theta^*, \theta)$$

$$Q(\theta^*, \theta) = \sum_{n=1}^{N} \sum_{k=1}^{K} r(z_{nk})\left[\ln \pi_k + \ln \mathcal{N}(x_n|\mu_k, \Sigma_k)\right] - \lambda\left(\sum_{k=1}^{K} \pi_k - 1\right)$$

$$\frac{dQ(\theta^*, \theta)}{d\pi_k} = \sum_{n=1}^{K} \frac{r(z_{nk})}{\pi_k} - \lambda = 0$$

$$\sum_{n=1}^{N} r(z_{nk}) = \pi_k \lambda$$

$$\sum_{k=1}^{K} \sum_{n=1}^{N} r(z_{nk}) = \sum_{k=1}^{K} \pi_k \lambda$$

$$\therefore \sum_{k=1}^{K} \pi_k = 1 \quad , \quad \lambda = N \quad \text{and} \quad \pi_k = \frac{\sum_{n=1}^{N} r(z_{nk})}{\lambda} = \frac{\sum_{n=1}^{N} r(z_{nk})}{N}$$

$$\mu_k^* = \frac{\sum_{n=1}^{N} r(z_{nk}) x_n}{\sum_{n=1}^{N} r(z_{nk})}$$

$$\Sigma_k^* = \frac{\sum_{n=1}^{N} r(z_{nk})(x_n - \mu_k)(x_n - \mu_k)^T}{\sum_{n=1}^{N} r(z_{nk})}$$