Literature Review: Tensor Completion Based on Nuclear Norm

Tensor Completion for Estimating Missing Values in Visual Data

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December 29, 2021

Outline

Tensor Completion for Low Rank Data[1]

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Contributions: Extensions of low rank matrix completion

- Definition of the trace norm for tensors
- Algorithms for the low rank tensors completion
  - Simple low rank tensor completion (SiLRTC): employs a relaxation technique to separate the dependent relationships and uses the block coordinate descent method to achieve a globally optimal solution
  - High accuracy low rank tensor completion (HaLRTC): applies the alternating direction method of multipliers (ADMM) algorithm to solve low rank tensor estimation
  - Fast low rank tensor completion (FaLRTC): utilizes a smoothing scheme to transform the original nonsmooth problem into a smooth problem
- Comparison with different completion algorithms

[1] Ji Liu et al. "Tensor Completion for Estimating Missing Values in Visual Data". In: *IEEE Transactions on Pattern Analysis and Machine Intelligence* 35.1 (2013), pp. 208–220.

Abstract

- Tensor completion is formulated as a convex optimization problem[2] (Extension of matrix completion)
- Three algorithms (based on the trace norm) are proposed to estimate missing values in tensors of visual data (Any other algorithms?)
- The algorithms work even with a small amount of samples (Missing ratio is high) and it can propagate structure to fill larger missing regions
- FaLTRC and HaLRTC are more efficient than SiLRTC and between FaLRTC and HaLRTC the former is more efficient to obtain a low accuracy solution and the latter is preferred if a high-accuracy solution is desired
- Experiments show potential applications with different types of data: images and videos (traffic data ?)

[2] Ryota Tomioka, Kohei Hayashi, and Hisashi Kashima. "Estimation of low-rank tensors via convex optimization". In: *arXiv:1010.0789* (2010).

Background

- Missing value estimation in computer version and graphics: image in-painting, video decoding and video in-painting.
- Core problem: Build up the relationship between the known elements and the unknown ones. (Capture the global information in the data)
- How to do? Low rank approximation[3]: minimize the trace norm of matrices to approximate the rank of matrices[4].(Data is of low rank and matrix based)
- Gap: Low Rank Tensor Completion[5] based on the trace norm of tensor. (a convex but nondifferentiable optimization problem: interdependent matrix trace norm terms)

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[3] Benjamin Recht, Maryam Fazel, and Pablo A. Parrilo. "Guaranteed Minimum-Rank Solutions of Linear Matrix Equations via Nuclear Norm Minimization". In: *SIAM Review* 52.3 (2010), pp. 471–501. DOI: 10.1137/070697835.

[4] Jian-Feng Cai, Emmanuel J. Candès, and Zuowei Shen. "A Singular Value Thresholding Algorithm for Matrix Completion". In: *SIAM Journal on Optimization* 20.4 (2010), pp. 1956–1982. DOI: 10.1137/080738970.

[5] Ryota Tomioka, Kohei Hayashi, and Hisashi Kashima. "Estimation of low-rank tensors via convex optimization". In: *arXiv:1010.0789* (2010).

Notation

- Matrix X, Entries x_{ij}, Matrix SVD $X = U\Sigma V\top$
- $\Sigma(X)$ (vector) singular values of X in descending order, $\sigma_i(X)$ i-th largest singular value
- Frobenius norm $\|X\|_F := \left(\sum_{i,j} |x_{ij}|^2\right)^{\frac{1}{2}}$,
  Spectral norm $\|X\| := \sigma_1(X)$,
  Trace norm $\|X\|_{tr} := \sum_i \sigma_i(X)$
- Ω an index set for observed value
- n -mode tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_n}$. Its elements are denoted as $x_{i_1,\ldots,i_n}$, where $1 \le i_k \le I_k, 1 \le k \le n$, I_k is the size of k-th dimension.

Notation

'Shrinkage' operator $\mathbf{D}_\tau(X)$ is defined as

$$\mathbf{D}_\tau(X) := U\mathbf{D}_\tau(\Sigma)V^\top = U\Sigma_\tau V^\top$$

where $\mathbf{D}_\tau(\Sigma) = \Sigma_\tau = \operatorname{diag}\left(\max\left(\sigma_i - \tau, 0\right)\right) = \operatorname{diag}\left(\{\sigma_i - \tau\}_+\right)$.
Soft-thresholding rule is applied to the singular values of matrix X.
'Truncate' operator $\mathbf{T}_\tau(X)$ is defined as

$$\mathbf{T}_\tau(X) = U\Sigma_{\bar{\tau}}V^\top,$$

where $\Sigma_{\bar{\tau}} = \operatorname{diag}\left(\min\left(\sigma_i, \tau\right)\right)$ and $X = \mathbf{T}_\tau(X) + \mathbf{D}_\tau(X)$.
Unfold a tensor into a matrix

$$\operatorname{unfold}_i(\mathcal{X}) := \mathcal{X}_{(i)} := X_{(i)} \in \mathbb{R}^{I_i \times (I_1 \ldots I_{i-1} I_{i+1} \ldots I_n)}$$

The inverse of the unfolding operation is written

$$fold_i\left(\operatorname{unfold}_i(\mathcal{X})\right) = \mathcal{X}.$$

Tensor Completion

- Define the trace norm for the general tensor case:

$$\|\mathcal{X}\|_{tr} := \sum_{i=1}^{n} \alpha_i \left\|\mathcal{X}_{(i)}\right\|_{tr} \quad \textit{trace norm}$$

  where α_is are constants satisfying $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. In essence, the trace norm of a tensor is a convex combination of the trace norms of all matrices unfolded ($\mathcal{X}_{(i)}$) along each mode.

- The tensor completion optimization problem can be written as

$$\begin{aligned} &\min_{\mathcal{X}} : \textstyle\sum_{i=1}^{n} \alpha_i \left\|\mathcal{X}_{(i)}\right\|_{tr} \\ &s.t. : \mathcal{X}_{\Omega} = \mathcal{T}_{\Omega} \ (noiseless) \end{aligned} \qquad \textit{nonsmooth convex problem}$$

  where $\mathcal{X}, \mathcal{T}$ are n-mode tensors with identical size in each mode. Difficult: the matrices share the same entries and cannot be optimized independently (multiple dependent nonsmooth terms)

Matrix Completion

The low rank matrix completion (nonconvex optimization):

$$\min_X : \operatorname{rank}(X) \quad s.t. : X_\Omega = M_\Omega(noiseless), \quad Nonconvex$$

where $X, M \in \mathbb{R}^{p\times q}$. The missing elements of X are determined such that the rank of the matrix X is as small as possible.
This leads to the following trace norm convex optimization problem (tightest convex envelop) for matrix completion:

$$\min_X : \|X\|_{tr}\ (trace\ norm\ , subgradient) \quad s.t. : X_\Omega = M_\Omega. \quad Convex$$

The trace norm minimization problem can be solved by Block coordinate descent iterative method[6], singular value thresholding algorithm[7], alternating direction method of multipliers (ADMM)[8]

[6] Shiqian Ma, Donald Goldfarb, and Lifeng Chen. "Fixed point and Bregman iterative methods for matrix rank minimization". In: *Mathematical Programming* 128 (2011), pp. 321–353. DOI: 10.1007/s10107-009-0306-5.

[7] Jian-Feng Cai, Emmanuel J. Candès, and Zuowei Shen. "A Singular Value Thresholding Algorithm for Matrix Completion". In: *SIAM Journal on Optimization* 20.4 (2010), pp. 1956–1982. DOI: 10.1137/080738970.

[8] Zhouchen Lin, Minming Chen, and Yi Ma. "The Augmented Lagrange Multiplier Method for Exact Recovery of Corrupted Low-Rank Matrices". In: *arXiv:1009.5055v3* (2010).

Baseline Methods for Tensor Completion[9]

- Tucker model: tensor factorization

$$\min_{\mathcal{X},C,U_1,\ldots,U_n} : \frac{1}{2}\|\mathcal{X}-C\times_1 U_1\times_2 U_2\times_3\cdots\times_n U_n\|_F^2 \quad s.t.: \mathcal{X}_\Omega = \mathcal{T}_\Omega$$

- Parafac model: Parafac model-based decomposition

$$\min_{\mathcal{X},U_1,U_2,\ldots,U_n} : \frac{1}{2}\|\mathcal{X} - U_1\circ U_2\circ\cdots\circ U_n\|_F^2 \quad s.t.: \mathcal{X}_\Omega = \mathcal{T}_\Omega$$

- SVD: consider the tensor as multiple matrices and force the unfolding matrix along each mode of the tensor to be low rank as follows:

$$\min_{\mathcal{X},M_1,M_2,\ldots,M_n} : \frac{1}{2}\sum_{i=1}^{n}\left\|\mathcal{X}_{(i)} - M_i\right\|_F^2 \quad (\textit{additional matrices } M_i)$$

$$\text{s.t.} \quad : \mathcal{X}_\Omega = \mathcal{T}_\Omega, \quad \operatorname{rank}(M_i) \le r_i \quad i = 1,\ldots,n,$$

where $M_i \in \mathbb{R}^{I_i\times(\prod_{k\neq i} I_k)}$ and $\mathcal{T},\mathcal{X} \in \mathbb{R}^{I_1\times\cdots\times I_n}$.

[9] Tamara G. Kolda and Brett W. Bader. "Tensor Decompositions and Applications". In: *SIAM REVIEW* 51.3 (2009), pp. 455–500.

Singular value thresholding (SVT) algorithm[10]

Goal: SVT algorithm is used to approximately minimize the trace norm of a matrix under convex constraints

Problem:

$$\min \|\boldsymbol{X}\|_{tr} \quad s.t.: \boldsymbol{X}_\Omega = \boldsymbol{M}_\Omega$$

$$\Leftrightarrow \min_{X \in \mathbb{R}^{p \times q}} : \frac{1}{2}\|\boldsymbol{X} - \boldsymbol{M}\|_F^2 + \tau \|\boldsymbol{X}\|_{tr}$$

Then the Singular Value Shrinkage Operator $\mathbf{D}_\tau(\boldsymbol{Y})$ obeys:

$$\mathcal{D}_\tau(\boldsymbol{Y}) = \arg\min_{\boldsymbol{X}} \frac{1}{2}\|\boldsymbol{X} - \boldsymbol{Y}\|_F^2 + \tau\|\boldsymbol{X}\|_{tr}$$

where Y is the initial matrix and need to update.

Algorithm: Fix $\tau > 0$ and a sequence $\{\delta_k\}$ of positive step sizes. Starting with $\boldsymbol{Y}^0$, the algorithm inductively defines

$$\begin{aligned} \boldsymbol{X}^k &= \mathbf{D}_\tau(\boldsymbol{Y}) = \text{shrink}\left(\boldsymbol{Y}^{k-1}, \tau\right) \\ \boldsymbol{Y}^k &= \boldsymbol{Y}^{k-1} + \delta_k \left(\boldsymbol{M} - \boldsymbol{X}^k\right)_\Omega \end{aligned}$$

until stopping criterion $\left\|\left(\boldsymbol{M} - \boldsymbol{X}^k\right)_\Omega\right\|_F / \|\boldsymbol{M}_\Omega\|_F \leq \epsilon$ is reached.

[10] Jian-Feng Cai, Emmanuel J. Candès, and Zuowei Shen. "A Singular Value Thresholding Algorithm for Matrix Completion". In: *SIAM Journal on Optimization* 20.4 (2010), pp. 1956–1982. DOI: 10.1137/080738970.

Alternating direction method of multipliers (ADMM)[11]

Goal: ADMM algorithm has the superior convergence property to solve Robust PCA problem (matrix completion).

Problem:

$$\min f(X), \quad s.t.: h(X) = 0$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^m$. One may define the augmented Lagrangian function:

$$L(X, Y, \mu) = f(X) + \langle Y, h(X) \rangle + \frac{\mu}{2} \|h(X)\|_F^2$$

where μ is a positive scalar.

Algorithm:

$$\begin{aligned}
&solve\ X_{k+1} = \arg\min_X L\left(X, Y_k, \mu_k\right) \\
&Y_{k+1} = Y_k + \mu_k h\left(X_{k+1}\right) \\
&updata\ \mu_k\ to\ \mu_{k+1}
\end{aligned}$$

[11] Zhouchen Lin, Minming Chen, and Yi Ma. "The Augmented Lagrange Multiplier Method for Exact Recovery of Corrupted Low-Rank Matrices". In: *arXiv:1009.5055v3* (2010).

Nonsmooth convex optimization

Problem Definition:

$$Minimize_{x \in \mathbb{R}^n} \quad f(\boldsymbol{x}) \quad \textit{nonsmooth convex problem}$$

where f is convex but not always differentiable.

- Subgradient methods yield ε-accuracy in $O\left(\frac{1}{\varepsilon^2}\right)$ $iterations$
- Nesterov's smoothing: if f is smooth, then accelerated GD yields ε -accuracy in $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$ $iterations$
  - approximate the nonsmooth objective by a smooth function
  - solve the smooth problem and use its solution to approximate the original problem

Smooth approximation: Convex function f is called (α, β) smoothable if, for any $\mu > 0, \exists$ convex function f_μ s.t.

- Approximation accuracy: $f_\mu(\boldsymbol{x}) \leq f(\boldsymbol{x}) \leq f_\mu(\boldsymbol{x}) + \beta\mu, \forall \boldsymbol{x}$
- Smoothness: $f_\mu \ is \ \frac{\alpha}{\mu}$

  where μ trade-off between approximation accuracy and smoothness

f_μ is a $\frac{1}{\mu}$ smooth approximation of f with parameters (α, β).

Nonsmooth optimization

Approximation methods: Moreau envelope and Conjugation

- The Moreau envelope (or Moreau-Yosida regularization) of a convex function f with parameter $\mu > 0$ is defined as

$$M_{\mu f}(\boldsymbol{x}) := \inf_{\boldsymbol{z}} \left\{ f(\boldsymbol{z}) + \frac{1}{2\mu} \|\boldsymbol{x} - \boldsymbol{z}\|_2^2 \right\}$$

  where $M_{\mu f}$ is a smoothed or regularized form of f.
  Minimizing f and minimizing $M_{\mu f}$ are equivalent.

- Suppose $f = g^*$, namely, $f(\boldsymbol{x}) = \sup_{\boldsymbol{z}} \{\langle \boldsymbol{z}, \boldsymbol{x} \rangle - g(\boldsymbol{z})\}$ One can build a smooth approximation of f by adding a strongly convex component to its dual, namely, $f_\mu(\boldsymbol{x}) = \sup_{\boldsymbol{z}} \{\langle \boldsymbol{z}, \boldsymbol{x} \rangle - g(\boldsymbol{z}) - \mu d(\boldsymbol{z})\} = (g + \mu d)^*(\boldsymbol{x})$ for some 1-strongly convex and continuous function $d \geq 0$ (called proximity function).

Examples: l_1-norm (Huber function approximate)
trace norm (Research problem)

Equivalent convex optimization problem

- Recall that: tensor completion optimization problem

$$\begin{aligned} &\min_{\mathcal{X}} : \sum_{i=1}^{n} \alpha_i \left\| \mathcal{X}_{(i)} \right\|_{tr} \\ &s.t. : \mathcal{X}_\Omega = \mathcal{T}_\Omega \ (noiseless) \end{aligned} \qquad nonsmooth\ convex\ problem$$

- Introduce additional matrices $M_1, \ldots, M_n$ to obtain equivalent formulation (Omitted) and relax the $M_i = \mathcal{X}_{(i)}$ by $\left\| M_i - \mathcal{X}_{(i)} \right\|_F^2 \leq d_i$ (interdependent terms have been split and can be solved independently):

$$\begin{aligned} &\min_{\mathcal{X}, M_i} : \sum_{i=1}^{n} \alpha_i \left\| M_i \right\|_{tr} \\ &s.t. : \left\| \mathcal{X}_{(i)} - M_i \right\|_F^2 \leq d_i \ for \ i = 1, \ldots, n \quad and \quad \mathcal{X}_\Omega = \mathcal{T}_\Omega \end{aligned}$$

  $d_i (> 0)$ is a threshold that could be defined by the user.

- This convex but nondifferentiable optimization problem can be converted to an equivalent formulation for certain positive values of β_is :

$$\min_{\mathcal{X}, M_i} : \sum_{i=1}^{n} \alpha_i \left\| M_i \right\|_* + \frac{\beta_i}{2} \left\| \mathcal{X}_{(i)} - M_i \right\|_F^2 \quad s.t. : \mathcal{X}_\Omega = \mathcal{T}_\Omega$$

Algorithm : block coordinate descent framework

- Computing $\mathcal{X}$. The optimal $\mathcal{X}$ with all other variables fixed is given by solving the following subproblem:

$$\begin{array}{c} \min_{\mathcal{X}} : \sum_{i=1}^{n} \frac{\beta_i}{2} \left\| M_i - \mathcal{X}_{(i)} \right\|_F^2 \\ \text{s.t.: } \mathcal{X}_{\Omega} = \mathcal{T}_{\Omega} \end{array}$$

Then the solution is given by

$$\mathcal{X}_{i_1,\ldots,i_n} = \begin{array}{ll} \left(\frac{\sum_i \beta_i \text{fold}_i(M_i)}{\sum_i \beta_i} \right) & (\mathrm{i}_1, \ldots, \mathrm{i}_\mathrm{n}) \notin \Omega; \\ \mathcal{T}_{i_1,\ldots,i_n} & (\mathrm{i}_1, \ldots, \mathrm{i}_\mathrm{n}) \in \Omega \end{array}$$

- Computing M_i. M_i is the optimal solution of the following problem:

$$\min_{M_i} : \frac{\beta_i}{2} \left\| M_i - \mathcal{X}_{(i)} \right\|_F^2 + \alpha_i \left\| M_i \right\|_{tr} \equiv \frac{1}{2} \left\| M_i - \mathcal{X}_{(i)} \right\|_F^2 + \frac{\alpha_i}{\beta_i} \left\| M_i \right\|_{tr}$$

Singular value thresholding (SVT) algorithm !!!

ADMM framework

- Tensor completion optimization problem can be equivalent to

$$\min_{\mathcal{X},\mathcal{M}_1,\ldots,\mathcal{M}_n} : \sum_{i=1}^{n} \alpha_i \left\|\mathcal{M}_{i(i)}\right\|_{tr}$$
$$s.t. : \mathcal{X}_\Omega = \mathcal{T}_\Omega \; and \; \mathcal{X} = \mathcal{M}_i, \;\; i = 1, \ldots, n.$$

by replacing the dummy matrices M_i s by their tensor versions

- Define the augmented Lagrangian function as follows:

$$\begin{aligned} L_\rho\left(\mathcal{X}, \mathcal{M}_1, \ldots, \mathcal{M}_n, \mathcal{Y}_1, \ldots, \mathcal{Y}_n\right) = & \sum_{i=1}^{n} \alpha_i \left\|\mathcal{M}_{i(i)}\right\|_{tr} \\ & + \langle \mathcal{X} - \mathcal{M}_i, \mathcal{Y}_i \rangle \\ & + \frac{\rho}{2} \left\|\mathcal{M}_i - \mathcal{X}\right\|_F^2 \end{aligned}$$

Then update $\mathcal{M}_i$s, $\mathcal{X}$, and $\mathcal{Y}_i$s as follows:

- $\left\{\mathcal{M}_1^{k+1}, \ldots, \mathcal{M}_n^{k+1}\right\} = \operatorname{argmin}_{\mathcal{M}_1,\ldots,\mathcal{M}_n} :$ $L_\rho\left(\mathcal{X}^k, \mathcal{M}_1, \ldots, \mathcal{M}_n, \mathcal{Y}_1^{k+1}, \ldots, \mathcal{Y}_n^{k+1}\right)$
- $\mathcal{X}^{k+1} = \arg\min_{\mathcal{X}\in Q} :$ $L_\rho\left(\mathcal{X}, \mathcal{M}_1^{k+1}, \ldots, \mathcal{M}_n^{k+1}, \mathcal{Y}_1^{k}, \ldots, \mathcal{Y}_n^{k+1}\right)$
- $\mathcal{Y}_i^{k+1} = \mathcal{Y}_i^k - \rho\left(\mathcal{M}_i^{k+1} - \mathcal{X}^{k+1}\right)$

Algorithm illustration

Algorithm 4. HaLRTC: High Accuracy Low Rank Tensor Completion

Input: $\mathcal{X}$ with $\mathcal{X}_\Omega = \mathcal{T}_\Omega$, ρ, and K

Output: $\mathcal{X}$

1: Set $\mathcal{X}_\Omega = \mathcal{T}_\Omega$ and $\mathcal{X}_{\bar{\Omega}} = 0$.

2: **for** $k = 0$ to K **do**

3: **for** $i = 1$ to n **do**

4: $$\mathcal{M}_i = \text{fold}_i\left[\mathbf{D}_{\frac{\alpha_i}{\rho}}\left(\mathcal{X}_{(i)} + \frac{1}{\rho}\mathcal{Y}_{i(i)}\right)\right]$$

5: **end for**

6: $$\mathcal{X}_\Omega = \frac{1}{n}\left(\sum_{i=1}^{n}\mathcal{M}_i - \frac{1}{\rho}\mathcal{Y}_i\right)_{\bar{\Omega}}$$

7: $$\mathcal{Y}_i = \mathcal{Y}_i - \rho(\mathcal{M}_i - \mathcal{X})$$

8: **end for**

Nesterov's smoothing framework

Tensor completion optimization problem can be equivalent to

$$\min_{\mathcal{X}} f_\mu(\mathcal{X}) := \min_{\mathcal{X}} \sum_{i=1}^{n} \max_{\|\mathcal{Y}_{i(i)}\|\leq 1} : \alpha_i \langle \mathcal{X}, \mathcal{Y}_i \rangle - \frac{\mu_i}{2} \|\mathcal{Y}\|_F^2$$
$$s.t. : \mathcal{X}_\Omega = \mathcal{T}_\Omega$$

by introducing n dual variables $\mathcal{Y}_1, \ldots, \mathcal{Y}_n \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_n}$ and n positive constants $\mu_1, \ldots, \mu_n$.

$$\mathcal{Z}^{k+1} = \mathcal{Z}^k - \frac{\theta^{k+1}}{L^k} \nabla f_\mu \left(\mathcal{W}^{k+1}\right),$$
$$\left(\nabla f_\mu \left(\mathcal{W}^{k+1}\right)\right)_{i_1,\ldots,i_n} = \left(\sum_i \frac{(\alpha_i)^2}{\mu_i} \mathbf{T}_{\frac{\mu_i}{\alpha_i}} \left(\mathcal{W}_{(i)}^{k+1}\right)\right)_\Omega$$

Algorithm illustration

Algorithm 3. FaLRTC: Fast Low Rank Tensor Completion

Input: $c \in (0,1)$, $\mathcal{X}$ with $\mathcal{X}_\Omega = \mathcal{T}_\Omega$, K, μ_is, and L.

Output: $\mathcal{X}$

```
Initialize Z = W = X, L' = L, and B = 0
for k = 0 to K do
    while true do
        θ = L/(2L') (1 + sqrt(1 + 4L'B));
        W = (θ/L)/(B+θ/L) Z + B/(B+θ/L) X;
        if f_μ(X) ≤ f_μ(W) − ||∇f_μ(W)||_F^2 / 2L' then
            break;
        end if
        X' = W − ∇f_μ(W)/L';
        if f_μ(X') ≤ f_μ(W) − ||∇f_μ(W)||_F^2 / 2L' then
            X = X';
            break;
        end if
        L' = L'/c;
    end while
    L = L';
    Z = Z − (θ/L) ∇f_μ(W);
    B = B + θ/L;
end for
```

$$\theta = \frac{L}{2L'}\left(1 + \sqrt{1 + 4L'B}\right); \qquad \mathcal{W} = \frac{\theta/L}{B+\theta/L}\mathcal{Z} + \frac{B}{B+\theta/L}\mathcal{X};$$

$$f_\mu(\mathcal{X}) \le f_\mu(\mathcal{W}) - \|\nabla f_\mu(\mathcal{W})\|_F^2/2L'$$

$$\mathcal{X}' = \mathcal{W} - \nabla f_\mu(\mathcal{W})/L'; \qquad L' = L'/c;$$

$$L = L'; \quad \mathcal{Z} = \mathcal{Z} - \frac{\theta}{L}\nabla f_\mu(\mathcal{W}); \quad B = B + \frac{\theta}{L};$$

Model comparison

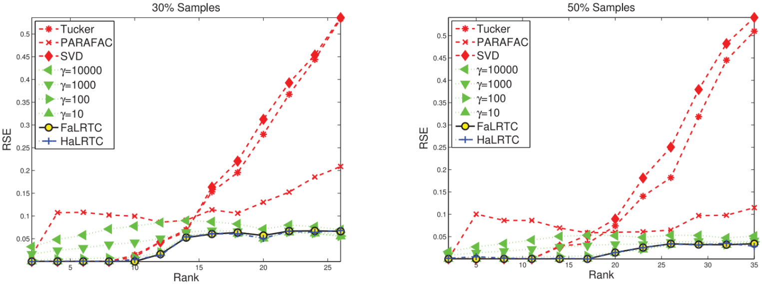


Figure 1: The RSE comparison on the synthetic data

Tensor completion and matrix completion

RSE Comparison (10^{-4}), Size $20 \times 20 \times 20$

Samples	MC1	MC2	MC3	TC
25%	1663	1782	1685	34
40%	247	258	241	2
60%	0	0	0	0

RSE Comparison (10^{-4}), Size $20 \times 20 \times 20 \times 20$

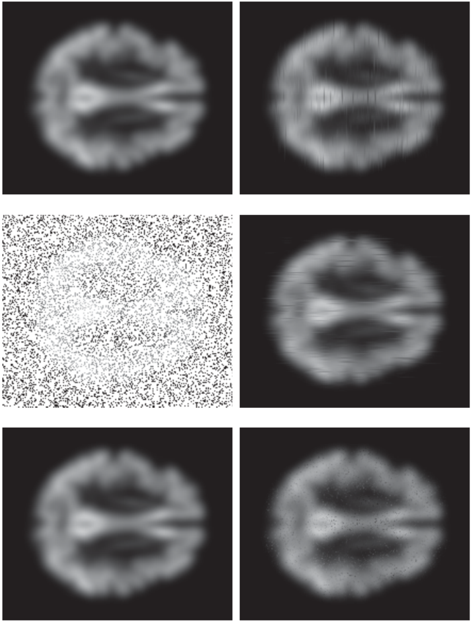
Samples	MC1	MC2	MC3	MC4	TC
20%	1875	1763	2011	1804	3
40%	92	102	97	88	0
60%	0	1	0	0	0

RSE Comparison (10^{-4}), Size $20 \times 20 \times 20 \times 20 \times 20$

Samples	MC1	MC2	MC3	MC4	MC5	TC
15%	1874	1830	1663	1502	1688	50
40%	125	119	131	114	136	0
60%	0	0	0	0	0	0

Figure 2: The RSE comparison on the synthetic data (MC_i represents the unfold tensor along the i th mode into a matrix structure $\mathcal{T}_{(i)}$)

Tensor completion and matrix completion (MRI Data)



Efficiency Comparison

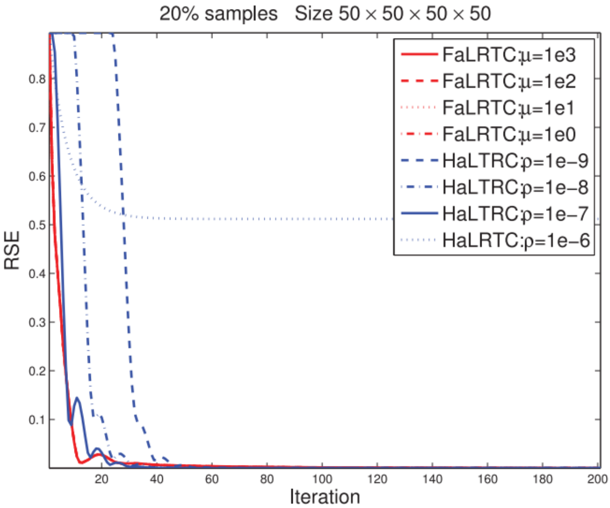


Figure 3: The RSE curves of the FaLRTC algorithm and the HaLRTC algorithm in terms of the number of iterations and the computation time

Missing data in images

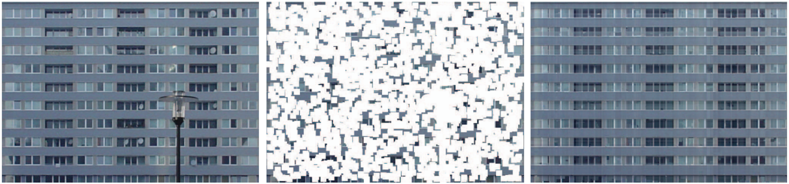


Figure 4: Facade in-painting. The left image is the original image; we select the lamp and satellite dishes, together with a large set of randomly positioned squares, as the missing parts, shown in white in the middle image; the right image is the result of the proposed LTRC algorithm.

Missing data in video

Figure 5: Video completion. The left image (one frame of the video) is the original; we randomly select pixels for removal, shown in white in the middle image; the right image is the result of the proposed LTRC algorithm.

Missing data in traffic data[12] (spatiotemporal)

- Application of HaLRTC algorithm
- Traffic data analysis: reshape tensor How to do ?

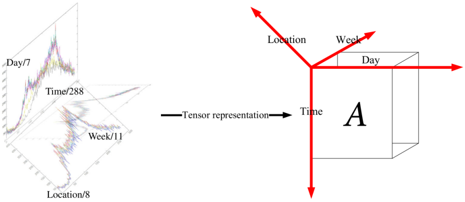


Figure 6: Tensor representation can capture the hidden weekly, daily, spatial correlation of traffic flow data.

- Correlation analysis of traffic data (similarity coefficient) to determine the parameters of tensor completion
- Two experiments design: all sensors are missing, impute missing data for local location independently

[12] Bin Ran et al. "Tensor based missing traffic data completion with spatial–temporal correlation". In: *Physica A: Statistical Mechanics and its Applications* 446 (2016), pp. 54–63.

Visual data recovery[13]

Contributions: Tensor train rank along with spatial and temporal regularization (Novel Model), Three solver subproblem Proximal Alternating Minimization (PAM) algorithm, Convergence analysis

- Tensor property: Sparsity in spatial and temporal mode (High order tensor transformer: KA)
- PAM algorithm with three subproblems are used to solve TT-Framelet model (Tensor decomposition and regularization)
- A detailed induction of PAM solver
- A novel tensor completion model (TT-framelet) by simultaneously exploiting the global low-rankness and local smoothness of visual data
- Use low-rank matrix factorization to characterize the global low-rankness; framelet and total variation regularization to enhance the local smoothness PAM solver
- Algorithm convergence analysis

[13] Jing-Hua Yang et al. "Tensor train rank minimization with hybrid smoothness regularization for visual data recovery". In: *Applied Mathematical Modelling* 81 (2020), pp. 711–726.

Missing data in traffic data[15] (spatiotemporal)

- Truncated Nuclear Norm New formulation for tensor rank approximation (Same as Tensor Train Rank[14])

$$\min_{\mathrm{M},\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3} \sum_{k=1}^{3} \alpha_k \left\|\mathcal{X}_{k(k)}\right\|_{r_k,*}$$
$$s.t. \begin{cases} \mathcal{X}_k = \mathcal{M}, k = 1,2,3, \\ \mathcal{P}_\Omega(\mathcal{M}) = \mathcal{P}_\Omega(\mathcal{Y}), \end{cases}$$

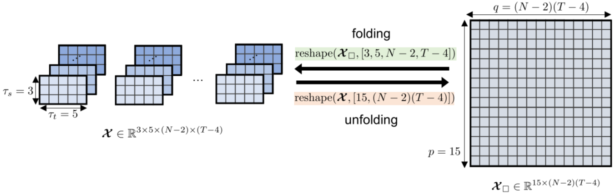
- LRTC-TNN (Extension of matrix form)
  `https://nbviewer.org/github/xinychen/transdim/blob/master/experiments/Imputation-LRTC-TNN.ipynb`
- Gaps: How to reshape traffic data (Traffic data analysis)? Roughness prior, How to tune the hyper-parameters (θ)

[14] Jing-Hua Yang et al. "Tensor train rank minimization with hybrid smoothness regularization for visual data recovery". In: *Applied Mathematical Modelling* 81 (2020), pp. 711–726.

[15] Xinyu Chen, Jinming Yang, and Lijun Sun. "A nonconvex low-rank tensor completion model for spatiotemporal traffic data imputation". In: *Transportation Research Part C: Emerging Technologies* 117 (2020), p. 102673.

Missing data in Traffic Speed Estimation[16] (spatiotemporal)

- Truncated Nuclear Norm New formulation for tensor rank approximation in traffic speed estimation
- Spatiotemporal Hankel tensor transformation



[16] Xudong Wang et al. *Low-Rank Hankel Tensor Completion for Traffic Speed Estimation*. 2021. arXiv: 2105.11335.

Thank You!