MA212 Probability and Statistics

Chapter Five: SLLN and CLT

Wenwu GONG & Qianqian WANG
Department of Statistics and Data Science
Email: 12031299@mail.sustech.edu.cn
12032005@mail.sustech.edu.cn

November 22, 2021

1.1 Strong Law Large of Number

Strong Law Large of Number

Theorem **(Strong law of large numbers, SLLN)** *Let* $(X_j)_{j\geq 1}$ *be i.i.d. Let*

$$\mu = \mathbb{E}(X_j) \text{ and } \sigma^2 = \sigma^2_{X_j} < \infty.$$

Let $S_n = \sum_{i=1}^{n} X_i$. *Then*

$$\frac{S_n}{n} \to \mu \qquad a.s. \text{ and in } L^2.$$

Convergence in probability

Theorem （Bernoulli's law of large numbers） **Assuming n_A is the amount of event A in n trials, and $P(A) = p$. Then $\forall \varepsilon > 0$,**

$$\lim_{n\to\infty} P\{|\frac{n_A}{n} - p| \geq \varepsilon\} = 0$$

Theorem:(Chebyshev law of large numbers) Assuming $\{X_n\}$ is an independent random variable list and these R.V. have same mathematic expectation and variance.

$$\lim_{n\to\infty} P\{|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu| \geq \varepsilon\} = 0$$

Theorem:(Khinchine law of large numbers) **Assuming $\{X_n\}$ is** independent and identically distributed **R.V. list, and $E(X_1) \triangleq \mu$ exists. Then $\{X_n\}$ follows the law of large numbers. Thus $\forall \varepsilon > 0$,**

$$\lim_{n\to\infty} P\{|\frac{1}{n}\sum_{k=1}^{n} X_k - \mu| \geq \varepsilon\} = 0$$

Markov's Inequality: can be used to verify Convergence in probability (SLLN)!!!

1.2 Central Limit Theorem

Central Limit Theorem

Theorem (Central limit theorem) *Let* $(X_j)_{j\geq 1}$ *be i.i.d. with* $\mathbb{E}(X_j) = \mu$ *and* $Var(X_j) = \sigma^2$, $0 < \sigma^2 < \infty$. *Let*

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}.$$

Then $Y_n \xrightarrow{\mathcal{D}} Y$ *where* $\mathcal{L}(Y) = N(0,1)$.

Convergence in normal distribution

Theorem: Assuming $\{X_n\}$ is the independent and identically distributed R.V. list.

$$E(X_k) = \mu, D(X_k) = \sigma^2 > 0 \quad (k = 1, 2, \cdots)$$

Then $\{X_n\}$ follows central limit theorem. i.e., the standard R.V. (Zn) is

$$Z_n = \frac{\sum_{k=1}^{n} X_k - E(\sum_{k=1}^{n} X_k)}{\sqrt{D(\sum_{k=1}^{n} X_k)}} = \frac{\sum_{k=1}^{n} X_k - n\mu}{\sqrt{n}\,\sigma} \quad (n = 1, 2, \cdots)$$

And the distribution function $F_n(x)$ of the R.V. Zn for any x has

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} P\left\{ \frac{\sum_{k=1}^{n} X_k - n\mu}{\sqrt{n}\,\sigma} \leq x \right\} = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \Phi(x)$$

Note that: De Moivre-Laplace Central Limit Theorem

SLLN

- Suppose that $\{X_n\}$ are independent and identically distributed, $Var(X_n) = \sigma^2 < +\infty$. Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and ${S^2}_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$, prove that ${S^2}_n \xrightarrow{P} \sigma^2$. You should use many LLNs

CLT

- Prove that:

$$\lim_{n \longrightarrow +\infty} (1 + n + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!})e^{-n} = \frac{1}{2} \tag{1.1}$$

$\{X_n\} \overset{iid}{\sim} Possion(1)$, $P(Y_n \leq n) \Leftrightarrow P(\frac{Y_n - n}{\sqrt{n}} \leq 0)$

1. Let $\{X_k\}$ be a list of independent random variables, and its probability distribution satisfies:

$$P\left(X_k = \pm\sqrt{\ln k}\right) = \frac{1}{2}, \qquad k = 1,2,\ldots$$

Prove: $\{X_k\}$ follows the law of large numbers.

2. Let $\{X_n\}$ be a list of independent random variables, and its probability distribution satisfies:

$$P(X_n = 1) = p_n, \qquad P(X_n = 0) = 1 - p_n \quad n = 1,2,\ldots$$

Prove: $\{X_n\}$ follows the law of large numbers.

1. Assume that the lifespan of an electronic component follows the exponential distribution with mean value being 100 hours. There are 16 randomly picked components and suppose that their lifespans are independent. What is the probability that the sum of their lifespans is greater than 1920 hours?

2. There are 10000 elder people participating in a type of insurance. The premium is 200 yuan per year. If the insured person passes away within the insured year, the beneficiary will receive 10000 yuan. Assume that the probability of death is 0.017, what is the probability that the insurance company will suffer a deficit in a year?

3. Let $\overline{X}$ be the average of a sample of 16 independent normal random variables with mean 0 and variance 1. Determine c such that

$$P(|\overline{X}| < c) = .5$$

6. Show that if $T \sim t_n$, then $T^2 \sim F_{1,n}$.

8. Show that if X and Y are independent exponential random variables with $\lambda = 1$, then X/Y follows an F distribution. Also, identify the degrees of freedom.