MA212 Probability and Statistics

Chapter Four: Expectation and Variance

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Expectation of R.V.

Discrete: **Assuming the frequency function of r.v. X is $p(x)$, if**

$$\sum_{k=1}^{\infty} |x_k| p_k < +\infty ,$$

Then

$$\boxed{E(X) \triangleq \sum_{k=1}^{\infty} x_k p_k = \sum_{k=1}^{\infty} x_k P\{X = x_k\}}$$

Continuous: **Assuming the probability density function of r.v. X is $f(x)$, if**

$$\int_{-\infty}^{\infty} |x| f(x) dx < +\infty$$

Then

$$\boxed{E(X) \triangleq \int_{-\infty}^{\infty} x f(x) dx}$$

Expectation of R.V.'s function Y = g(X)

- **If X is discrete with frequency function $p(x)$,**

  when $\sum_{k=1}^{\infty} |g(x_k)| \cdot p_k < +\infty$, then

$$\boxed{E(Y) = E[g(X)] = \sum_{k=1}^{\infty} g(x_k) p_k}$$

- **If X is continuous with density function $f(x)$,**

  when $\int_{-\infty}^{\infty} |g(x)| f(x) dx < \infty$, then

$$\boxed{E(Y) = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx}$$

Expectation of R.V.s' function Z = g(X, Y)

- For the joint frequency function of X and Y is $p(i,j)$. If

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|g(x_i,y_j)|\,p_{ij} < +\infty, \quad \text{then}$$

$$E(Z) = E[g(X,Y)] \boxed{= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} g(x_i,y_j)\,p_{ij}}$$

- For the joint density of X and Y is $f(x,y)$. If

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(x,y)|\,f(x,y)dxdy < \infty, \quad \text{then}$$

$$E(Z) = E[g(X,Y)] \boxed{= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,f(x,y)dxdy}$$

Note: the formula can be generalized to high-dimensional random variables.

1.2 The properties of expectation

Properties

If $a \le X \le b \ (a.e)$, then $a \le E(X) \le b$

If c is constant, then $E(cX) = cE(X)$

If X、Y are r.v., then $E(X+Y) = E(X) + E(Y)$

If X, Y are independent, then $\boxed{E(XY) = E(X)E(Y)}$

Inferences

If $X = c \ (a.e)$, then $E(X) = c$

If $a_1, a_2, \cdots, a_n$ are constants, $X_1, X_2, \cdots, X_n$ are r.v., then $\boxed{E(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i E(X_i)}$

If $X_1, X_2, \cdots, X_n$ are independent, then

$$\boxed{E(X_1 X_2 \cdots X_n) = E(X_1)E(X_2)\cdots E(X_n)}$$

Markov's Inequality

Theorem: If X is a R.V. with $P(X \geq 0) = 1$ and for which E(X) exists, then

$$P\{X \geq t\} \leq \frac{E(X)}{t}.$$

Proof: We will prove this for the discrete case. the continuous case is entirely analogous.

$$E(X) = \sum_{x} xp(x) = \sum_{x<t} xp(x) + \sum_{x \geq t} xp(x)$$

All the terms in the sums are nonnegative because X takes on only nonnegative values. Thus

$$E(X) \geq \sum_{x \geq t} xp(x) \geq \sum_{x \geq t} tp(x) = tP\{X \geq t\}.$$

$$\longrightarrow \quad P\{X \geq t\} \leq \frac{E(X)}{t}.$$

Variance of R.V.

The calculation of variance is the expected value of $g(X) = (X - E(X))^2$.

$$\boxed{Var(X) \triangleq D(X) \triangleq E(X - E(X))^2}$$

- Assuming the frequency function of X is $p(x)$, then

$$\boxed{D(X) = \sum_{k=1}^{\infty} (x_k - E(X))^2 \cdot p_k}$$

- Assuming the probability density of X is, then

$$\boxed{D(X) = \int_{-\infty}^{\infty} (x - E(X))^2 f(x) dx}$$

More general, you can use the following equation to calculate variance

$$\boxed{D(X) = E(X - E(X))^2 = E(X^2) - [E(X)]^2}$$

1.5 The properties of variance

Properties

If $X = c(constant)$, then $D(X) = 0$

If c is constant, then $D(cX) = c^2 D(X)$

For independent r.v X, Y $\quad D(X+Y) = D(X) + D(Y)$

$$D(X-Y) = D(X) + D(Y)$$

If X and Y are not independent

$$D(X \pm Y) = D(X) + D(Y) \pm 2E[(X - E(X))(Y - E(Y))]$$

1.6 Summary of expectation and variances

Some common distributions

Distribution of X	$E(X)$	$D(X)$
$X \sim p(\lambda)$	λ	λ
$X \sim b(n, p)$	np	$np(1-p)$
$X \sim U(a, b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
$X \sim EXP(1/\theta)$	θ	θ^2
$X \sim N(\mu, \sigma^2)$	μ	σ^2

Covariance of R.V

Definition: If the variance of both X and Y exists, then

$$\boxed{\mathrm{Cov}(X,Y) \triangleq E[(X-E(X))(Y-E(Y))]}$$

$Cov(X,Y)$ is the Covariance of X and Y.

X, Y are independent $\Rightarrow$ $\mathrm{Cov}(X,Y)=0$

$\mathrm{Cov}(X,Y)\neq 0$ $\Rightarrow$ X, Y are not independent

$\Rightarrow$ X, Y must have a relation

1.8 The properties of Covariance

Properties

- If X, Y are independent, then $\mathrm{Cov}(X,Y)=0$
- $\mathrm{Cov}(X,Y)=\mathrm{Cov}(Y,X)$
- $\boxed{\mathrm{Cov}(X,Y)=E(XY)-E(X)E(Y)}$
- $$\begin{aligned} D(X+Y) &= D(X)+D(Y)+2E[(X-E(X))(Y-E(Y))] \\ &= D(X)+D(Y)+2\mathrm{Cov}(X,Y) \end{aligned}$$
- For any constant a, b
  $$\mathrm{Cov}(aX,bY)=E[(aX-E(aX))(bY-E(bY))]$$
- $\mathrm{Cov}(X_1+X_2,Y)=\mathrm{Cov}(X_1,Y)+\mathrm{Cov}(X_2,Y)$

Conditional Expectation

- When $X = x$, the conditional expectation definition of Y is

$$E(Y \mid X = x) = \sum_y y p_{Y|X}(y \mid x) \qquad \text{(Discrete)}$$

$$E(Y \mid X = x) = \int y f_{Y|X}(y \mid x) dy \qquad \text{(Continuous)}$$

- Generally, the conditional expectation of function $h(Y)$ is

$$E[h(Y) \mid X = x] = \sum_y h(y) p_{Y|X}(y \mid x) \qquad \text{(Discrete)}$$

$$E[h(Y) \mid X = x] = \int h(y) f_{Y|X}(y \mid x) dy \qquad \text{(Continuous)}$$

- **Theorem** $\boxed{E(Y) = E[E(Y \mid X)].}$

$$\boxed{D(Y) = D[E(Y \mid X)] + E[D(Y \mid X)].}$$

2.1 Extra Exercises 1

Expexctation calculation

- The CDF of R.V. X is:

$$F(x) = \begin{cases} \exp(x)/2, & x < 0 \\ 1/2, & 0 \leq x < 1 \\ 1 - 1/2 \exp(-\frac{1}{2}(x-1)), & x \geq 1 \end{cases}$$

  Try to calculate E(X).
- R.V. X has $E(X) = Var(X) = \lambda$. If $E(X-1)(X-2) = 1$, what is λ?

Expexctation and Probability

For non-negative R.V. X and $E(X) < +\infty$, try to prove: (Changeable of summation and integration)

- Discrete: $E(X) = \sum_{k=1}^{+\infty} P(X \geq k)$;
- Continuous: $E(X) = \int_0^{+\infty} P(X > x) d_x$.

6. Let X be a continuous random variable with probability density function $f(x) = 2x, 0 \le x \le 1$.

a. Find $E(X)$.

b. Let $Y = X^2$. Find the probability mass function of Y and use it to find $E(Y)$.

c. Use Theorem A in Section 4.1.1 to find $E(X^2)$ and compare to your answer in part (b).

d. Find $\mathrm{Var}(X)$ according to the definition of variance given in Section 4.2. Also find $\mathrm{Var}(X)$ by using Theorem B of Section 4.2.

15. Suppose that two lotteries each have n possible numbers and the same payoff. In terms of expected gain, is it better to buy two tickets from one of the lotteries or one from each?

21. A random square has a side length that is a uniform [0, 1] random variable. Find the expected area of the square.

30. Find $E[1/(X + 1)]$, where X is a Poisson random variable.

31. Let X be uniformly distributed on the interval $[1,2]$. Find $E(1/X)$. Is $E(1/X) = 1/E(X)$?

Supplementary Questions:

1. Suppose that the density function of a random variable is

$$f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x \geq 0 \end{cases}$$

Find the expectation of

(1) $Y = 2X$

(2) $Y = e^{-2x}$

2. Suppose that the joint density function of random variable (X, Y) is

$$f(x) = \begin{cases} 12y^2, & 0 < y < x < 1 \\ x, & \text{Otherwise} \end{cases}$$

Compute $E(X), E(Y), E(XY), E(X^2 + Y^2)$

Expexctation calculation for functions

- R.V. $X_i \overset{\text{iid}}{\sim} U(0,\theta)$, $i = 1, \cdots, n$. Let $Y = \mathsf{max}\{X_1, X_2, \cdots, X_n\}$, $Z = \mathsf{min}\{X_1, X_2, \cdots, X_n\}$, try to calculate $E(Y), E(Z)$. (Distribution of order statistics)

Conditional expexctation

Let X, Y are $\overset{\text{iid}}{\sim} \text{Exp}(\lambda)$. We define

$$Z = \begin{cases} 3X + 1, & X \geq Y \\ 6Y, & X < Y \end{cases}$$

Try to calculate $E(Z)$. (Full expectation formula)

49. Two independent measurements, X and Y, are taken of a quantity μ. $E(X) = E(Y) = \mu$, but σ_X and σ_Y are unequal. The two measurements are combined by means of a weighted average to give

$$Z = \alpha X + (1 - \alpha)Y$$

where α is a scalar and $0 \leq \alpha \leq 1$.

a. Show that $E(Z) = \mu$.
b. Find α in terms of σ_X and σ_Y to minimize $\mathrm{Var}(Z)$.
c. Under what circumstances is it better to use the average $(X + Y)/2$ than either X or Y alone?

50. Suppose that X_i, where $i = 1, \ldots, n$, are independent random variables with $E(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$. Let $\overline{X} = n^{-1} \sum_{i=1}^{n} X_i$. Show that $E(\overline{X}) = \mu$ and $\mathrm{Var}(\overline{X}) = \sigma^2/n$.

54. Let X, Y, and Z be uncorrelated random variables with variances σ_X^2, σ_Y^2, and σ_Z^2, respectively. Let

$$U = Z + X$$
$$V = Z + Y$$

Find $\text{Cov}(U, V)$ and ρ_{UV}.

55. Let $T = \sum_{k=1}^{n} kX_k$, where the X_k are independent random variables with means μ and variances σ^2. Find $E(T)$ and $\text{Var}(T)$.

60. Let Y have a density that is symmetric about zero, and let $X = SY$, where S is an independent random variable taking on the values $+1$ and -1 with probability $\frac{1}{2}$ each. Show that $\text{Cov}(X, Y) = 0$, but that X and Y are not independent.

Supplementary Questions:

1. Suppose that X and Y are independent random variables. $E(X) = 3,\ E(Y) = 1,\ D(X) = 4,\ D(Y) = 9$. If $Z = 5X - 2Y + 15$, compute $E(Z),\ D(Z)$.

2. Suppose that $X_i(i = 1,2,3,4)$ are mutually independent to each other. $(X_i) = 2i,\ D(X_i) = 5 - i$. If $Z = 2X_1 - X_2 + 3X_3 - 0.5X_4$, compute $E(Z)$ and $D(Z)$.

P171: 54、60 and the extra question

Extra: Assume that the density function for X is

$$f(x) = \frac{1}{2}e^{-|x|}, -\infty < x < \infty$$

(1) Compute $E(X)$ and $D(X)$.
(2)Are X and $|X|$ independent or not? State your reason.
(3)Are X and $|X|$ correlated or not? State your reason

Supplementary Questions

1. Suppose that the joint density function of (X, Y) is

$$f(x,y) = \begin{cases} \dfrac{x+y}{8}, & 0 \le x \le 2, 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Compute $E(X)$, $E(Y)$, $Cov(X,Y)$, ρ_{XY}, $D(X+Y)$.

2. X and Y are independent random variables which both follow the normal distribution $N(\mu, \sigma^2)$. If $Z = \alpha X + \beta Y, W = \alpha X - \beta Y$, compute $Cov(Z, W)$ and ρ_{ZW}.