MA212 Probability and Statistics

Chapter One: Probability

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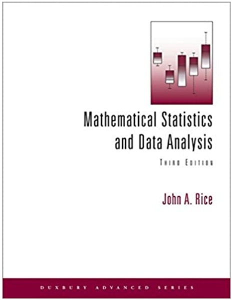
Some reminders of MA212 class

- Lecture Class:
  - Time: Tuesday 10:20 - 12:10 (every week), Thursday 10:20 - 12:10 (even week), attendance will be recorded!
  - Classroom: Lychee Hill, Teaching building 1, 201
    For International students: please join the lectures online through Tencent Course Classroom
  - Course Assessment:

The assessment consists of:

1. Attendance : 10% (**both lectures and tutorial sessions**)
2. Quiz: 10% (**in total 4 times**)
3. Assignment: 10% (**Nearly every week**)
4. Mid-term test: 20% (**Weekend** of **Week 8 or 9**)
5. Final examination: 50% (**Week 16 or Week 17**)

Course Material



Some reminders of MA212 class (Cont'd)

- Tutorial Class:
  - Time: Monday and Tuesday (every week), attendance will be recorded!
  - Classroom: Lychee Hill, Teaching building 2, 308
  - Arrangements: Recaps of lectures (about 15 minutes), Supplement exercises (more harder), some questions of your HW, Quiz (we will have 4 quiz in Week (4, 8, 12, 15))
- Homework:
  - Submitting site: Floor 5, Block 3, Wisdom Vally
  - Deadline Sun. 9:00
  - Attention!: Write your name and student ID on the cover page
  - Supplements: Assignments submitted late or made up will not be accepted in principle. But if you have an warrant for submitting late, please tell us before the deadline
  - More importantly, if you submit more than one pages, please use a stapler to bind them

Introduction of the course content

Two parts: Probability & Statistics

- Basic concepts of Probability
- Random variables
- Joint distributions (Calculation, Integration)
- Expected values (Calculation, Integration)
- Limit theorems (You need know how to use SLLN and CLT)
- Basic concepts of statistics and sample distribution (What can we learn from samples)
- Inference: Parametric estimation and Hypotheses Testing

Notifications!

Attendance

- From now on, we will record your attendance in lectures. So, please make sure that you are here!
- Your Tutorial attendance is recorded by your quiz.
- You should take away your corrected HW by Tuesday night.
- Your homework has been recorded, if you have missed the past two HWs, you have one chance to submitted until Sep. 26 Sun. 9am.

1.1 Basic concepts

Random Experiments

- Experiments can be repeated under the same condition
- All the possible outcomes are known before the experiment

Sample Space Ω

- A set including all sample points in the experiment
- All sample points are equivalent to all possible outcomes

Random Event

- A set of sample points which satisfy some specific conditions
- Subset of the Sample Space or composed of fundamental outcomes

Basic Operations of Random Event A and B

- Union: At least one of A, B happens
- Intersection: A, B happen simultaneously
- Complementary: Complementary event of A

1.1 Basic concepts (Cont'd)

The Null set and Completed set

If $A \cap B = \Phi$ (the null set)

$\Longleftrightarrow$ A, B cannot happen simultaneously (同时), i.e., A and B are contradictory (互斥的) or disjoint (不相交).

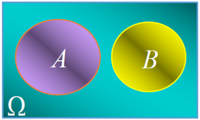
If $A \cup B = \Omega$ (the sample space) and $A \cap B = \Phi$

$\Longleftrightarrow$ One and only one of A, B must happen. A and B are complementary events (互补事件/对立事件/逆事件).

$$A = \Omega - B = \bar{B} = B^C$$

$$B = \Omega - A = \bar{A} = A^C$$

$A \cap B = \Phi$



$A \cup B = \Omega$ and $A \cap B = \Phi$

1.2 Operation law in events

Commutative law (交换律)

$$A \cup B = B \cup A, \qquad A \cap B = B \cap A$$

Associative law (结合律)

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive law (分配律)

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

De Morgan law (摩根律)

$$\overline{A \cup B} = \bar{A} \cap \bar{B}, \overline{A \cap B} = \bar{A} \cup \bar{B}$$

$$\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \bar{A}_k \quad , \quad \overline{\bigcap_{k=1}^{n} A_k} = \bigcup_{k=1}^{n} \bar{A}_k$$

1.3 Definition of Probability

Def: Let $\mathcal{A}$ be an event region in Ω, $\forall A \in \mathcal{A}$. If there exists corresponding $P(A)$, and

① Non negative （非负性）： $P(A) \geq 0$ $(\forall A \in \mathcal{A})$

② Normality （规范性）： $P(\Omega) = 1$

③ Countable Additivity （可列可加性）: for any disjoint pairwise event $\{A_k\}_{k=1}^{\infty}$

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$$

Then $P(A)$ is Probability of Event A. $(\Omega, \mathcal{A}, P)$ is probability space

1.4 Basic property of Probability

Basic property of Probability

- The Null Set and Completed Set

$$\mathrm{P}(\emptyset) = 0 \text{ and } \mathrm{P}(\Omega) = 1$$

- Finite Additivity for pairwise disjoint events (More general, Countable additivity)

$$\mathrm{P}(\cup_{k=1}^{n} \mathrm{A}_k) = \sum_{k=1}^{n} \mathrm{P}(\mathrm{A}_k) \text{ or } \mathrm{P}(\cup_{k=1}^{\infty} \mathrm{A}_k) = \sum_{k=1}^{\infty} \mathrm{P}(\mathrm{A}_k)$$

- Addition Law for any event A and B

$$\mathrm{P}(\mathrm{A} \cup \mathrm{B}) = \mathrm{P}(\mathrm{A}) + \mathrm{P}(\mathrm{B}) - \mathrm{P}(\mathrm{AB})$$

For n events，we have

+ All

- two

$$P(A_1 \cup A_2 \cup A_3 \dots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} P(A_i A_j)$$

+ three

$$+ \sum_{1 \le i < j < k \le n} P(A_i A_j A_k)$$

- four

$$- \sum_{1 \le i < j < k < l \le n} P(A_i A_j A_k A_l)$$

$$+ \cdots$$

$$+ (-1)^{n-1} P(A_1 A_2 \dots A_n)$$

加奇减偶

"+" for odd number
"–" for even number

1.6 Probability computation

Classical probability

- The probability of the possible outcomes are equally likely, such as Coin thrown and tossing dice

$$P(\omega_1) = P(\omega_2) = \ldots P(\omega_n)$$

- Addition Principle and Multiplication Principle
- Permutation

$$A_n^k = n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

- Combination

$$C_n^k = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{A_n^k}{A_k^k} = \frac{n!}{k!(n-k)!}$$

1.6 Probability computation

Geometric Probability

- The probability of event A is not related with position, but the area of A (equally likely)

$$\mathrm{P(A)} = \frac{\text{Area of A}}{\text{Area of } \Omega}$$

2.1 Extra Exercises 1

There are events A, B and C, please write the following events by basic operations

- A, B and C all happen or not happen;
- A, B and C happen at least one event;
- A, B and C happen at most two events.

The following equations true or false, please state your reasons

- $$A - (B - C) = (A - B) \cup C$$
- $$\text{If } AB = \emptyset \text{ and } C \subset A, \text{ then } BC = \emptyset$$
- $$(A \cup B) - B = A$$

2.1 Answers1

Events: A、B、C， **please write the following events by basic operations：**

1）All of A、B、C ***happens or not happens 都发生或都不发生；***

2） A、B、C ***happens at least one 至少发生一个***

3） A、B、C ***happens at most two events 至多发生两个***

The following equations true or false, please state your reasons ：

1） $A-(B-C)=(A-B)\cup C$ ***2）If*** $AB=\Phi$ ***and*** $C\subset A$ ***, then*** $BC=\Phi$

3） $(A\cup B)-B=A$

Please state the complementary events：

1）A=“Toss two coins, both heads” 掷两枚硬币，皆为正面

2）B=”Process four parts, at least one qualified product “ 加工四个零件，至少有一个合格品

Ans : $(A\cap B\cap C)\cup(A^c\cap A^c\cap B^c)$ and $(A\cup B\cup C)$ and $(A\cap B\cap C)^c$

Ans : FALSE TURE FALSE

Classical probability

- There are n people sit around a round table randomly, find the probability that A and B sit next to each other;
- There are 5 people in a dormitory. What is the probability that at least two of them have their birthdays in the same month?
- There are 3 people, and everyone is assigned to 5 rooms with the same probability. Find the probability of three people being assigned to different rooms.

Geometric Probability

- Take any two points in the line segment of length a and divide it into three segments, and find the probability that they can form a triangle.

$$\text{Ans}: \frac{1}{4}$$

Classical probability

*1)*There are n people sit around a round table randomly, find the probability that A and B sit next to each other

***n* 个人随机地围一圆桌而坐，求甲、乙两人相邻而坐的概率**

2) There are 5 people in a dormitory. What is the probability that at least two of them have their birthdays in the same month?

一间宿舍有 5 个人，求他们之间至少有两个人的生日在同一月份的概率

3) There are three people, and everyone is assigned to 5 rooms with the same probability. Find the probability of three people being assigned to different rooms

有三个人，每个人都以相同的概率分配到 5 个房间中，求三个人分配到不同房间的概率

Geometric Probability

Take any two points in the line segment of length a and divide it into three segments, and find the probability that they can form a triangle

在长度为 a 的线段内任取两点将其分为三段，求他们可以构成一个三角形的概率。

$$\text{Ans}: \frac{(n-2)!*2}{(n-1)!} = \frac{2}{n-2}\ ,\ 1-\frac{12*11*10*9*8}{12^5} = \frac{89}{144}\ ,\ \frac{5*4*3}{5^3} = \frac{12}{25}$$

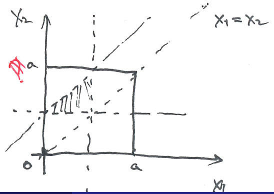
2.2 Answers2 (Cont'd)

Suppose the random point x_1, x_2 satisfying $0 < x_1 < x_2 < a$, this is our Sample Space Ω. Then the Event A (the segments can form a triangle) can be denoted as

$$\begin{cases} x_1 + x_2 - x_1 & > a - x_2 \\ x_1 + a - x_2 & > x_2 - x_1 \\ x_2 - x_1 + a - x_2 & > x_1 \end{cases}$$

Which is equivalent to

$$\begin{cases} x_1 < \frac{a}{2} \\ x_2 > \frac{a}{2} \\ x_2 - x_1 < \frac{a}{2} \end{cases}$$



2. Two six-sided dice are thrown sequentially, and the face values that come up are recorded.
   a. List the sample space.
   b. List the elements that make up the following events: (1) $A =$ the sum of the two values is at least 5, (2) $B =$ the value of the first die is higher than the value of the second, (3) $C =$ the first value is 4.
   c. List the elements of the following events: (1) $A \cap C$, (2) $B \cup C$, (3) $A \cap (B \cup C)$.
3. An urn contains three red balls, two green balls, and one white ball. Three balls are drawn without replacement from the urn, and the colors are noted in sequence. List the sample space. Define events A, B, and C as you wish and find their unions and intersections.
4. Draw Venn diagrams to illustrate De Morgan's laws:
$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$
5. Let A and B be arbitrary events. Let C be the event that either A occurs or B occurs, but not both. Express C in terms of A and B using any of the basic operations of union, intersection, and complement.
6. Verify the following extension of the addition rule (a) by an appropriate Venn diagram and (b) by a formal argument using the axioms of probability and the propositions in this chapter.
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$$
$$- P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$
7. Prove Bonferroni's inequality:
$$P(A \cap B) \geq P(A) + P(B) - 1$$
8. Prove that
$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

3.1 Conditional Probability

Definition of Conditional Probability

Let A and B be two events with $P(B) > 0$, the conditional probability of event A happening given event B happens.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)}$$

- Note that the expression $A|B$ is not an Event! Just a notation.
- Conditional Probability is a kind of probability.

Multiplication law

For any events $P(A) > 0$ and $P(B) > 0$,

$$P(AB) = P(A|B)P(B) = P(B|A)P(A)$$

- Recall that Addition Law for any events: $P(A \cup B) = P(A) + P(B) - P(AB)$

Law of Total Probability

Partition（分划） of the sample space:

Ω be the sample space, if events $B_1, B_2, \dots, B_n$ meet the requirements:

1. $B_1, B_2, \dots, B_n$ are **pairwise disjoint**, i.e., $B_iB_j = \Phi\ (i \neq j, i, j = 1, \dots, n)$.
2. $B_1 \cup B_2 \dots \cup B_n = \Omega$

We call $B_1, B_2, \dots, B_n$ a Partition of the sample space Ω.

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

Bayes' Rule

Note that Prior Probabilities $P(B_i)$ and Posterior Probabilities $P(B_i|A)$

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{n} P(A|B_j)P(B_j)}$$

Independence

Assuming A and B are two events, if

$$P(AB) = P(A)P(B)$$

then A and B are said to be Independent Events.

- Note that disjoint $AB = \emptyset$
- Pairwise independent and mutually independent

Conditional probability

- Suppose there are 4 unqualified products out of 10 products. Take any two of them, if one is known to be unqualified, find the probability that the other is also unqualified. Sample space, Events, Probability
- Glass cups are sold in full boxes, each containing 20 pieces, assuming that the probability of each box containing 0, 1, and 2 defective products is 0.8, 0.1 and 0.1, respectively. The customer orders a box of glasses, and the salesperson picks one box at random when buying, and the customer randomly checks the 4 glasses in the box. If there is no defective product, then buy the box of glasses, otherwise return it. Try to find:
  - The probability that the customer buys the box; What is the meaning of buying box?
  - The probability that there is indeed no defective product in a box bought by the customer.

3.2 Answers1

Conditional Probability ***条件概率计算***

Suppose there are 4 unqualified products out of 10 products. Take any two of them, ff one is known to be unqualified, find the probability that the other is also unqualified

设 10 件产品中有 4 件不合格品，从中任取两件，已知其中一件是不合格品，求另一件也是不合格品概率。

Glass cups are sold in full boxes, each containing 20 pieces, assuming that the probability of each box containing 0, 1, and 2 defective products is 0.8, 0.1 and 0.1, respectively. The customer pre-orders a box of glasses, and the salesperson picks one box at random when buying, and the customer randomly checks the 4 glasses in the box. If there is no defective product, then buy the box of glasses, otherwise return it. Try to find: (1) The probability that the customer buys the box; (2) The probability that there is indeed no defective product in a box bought by the customer.

玻璃杯整箱出售，每箱 20 只，假设各箱含 0、1、2 只残次品的概率分别为 0.8，0.1 和 0.1。顾客预购一箱玻璃杯，在购买时售货员随意取一箱，而顾客随机查看该箱中 4 只玻璃杯，若无残次品，则买下该箱玻璃杯，否则退回。试求：(1)顾客买下该箱的概率; (2)在顾客买下的一箱中，确实没有残次品的概率

$$\text{Ans}: \quad P(B|A) = \frac{P(AB)}{P(A)} = \frac{\frac{\binom{4}{2}}{\binom{10}{2}}}{\frac{\binom{4}{1}\binom{6}{1}+\binom{4}{2}}{\binom{10}{2}}} = \frac{1}{5}$$

A = 'There is one unqualified', B = 'Both of them are unqualified'

$$\text{Ans}: \quad P(A) = \frac{448}{475}, \quad P(B|A) = \frac{95}{112}$$

A = 'The custom buys the box', B = 'There is no defective product in the box'

Independece

- P_1 and P_2 shot the same target once independently, and their hit rates are 0.8 and 0.7 respectively. If we known that the target has been hit, find out the probability of P_1 hit the target.
- Suppose $P(A) = 0.4$, $P(A \cup B) = 0.9$, find out $P(B)$ under the following conditions:
  - A and B are independent;
  - A and B are disjoint;
  - $A \subset B$.
- Suppose 5 people independent of each other cast a vote. If each of them has a 50% chance of voting Yes, find out the probability that at least 2 person voted Yes. The event of at least 2 person voted Yes

3.2 Answers2

- A = 'Target has been hit', B = 'P_1 hits the target'

$$\text{Ans}: \quad P(B|A) = \frac{P(AB)}{P(A)} = \frac{0.8}{1 - 0.2 * 0.3} = \frac{40}{47}$$

- By the addition law $P(A \cup B) = P(A) + P(B) - P(A \cap B)$:

$$\begin{aligned}
\text{Ans}: \quad P(B) &= P(A)P(B) + P(A \cup B) - P(A) \Rightarrow P(B) = \frac{5}{6}, \\
P(B) &= P(A \cup B) - P(A) = 0.5, \text{since } P(AB) = 0 \\
P(B) &= P(A \cup B) = 0.9
\end{aligned}$$

- A = 'The person votes Yes', B = 'At least 2 person voted Yes'

$$\begin{aligned}
\text{Ans}: \quad P(A) &= 0.5, \quad P(B^c) = \binom{5}{0} * P(A)^5 + \binom{5}{1} * P(A)^1 * (1 - P(A))^4, \\
P(B) &= 1 - P(B^c) = 1 - 0.5^5 - 5 * 0.5^5 = 0.8125
\end{aligned}$$

3.3 Homework

12. In a game of poker, five players are each dealt 5 cards from a 52-card deck. How many ways are there to deal the cards?

29. A poker player is dealt three spades and two hearts. He discards the two hearts and draws two more cards. What is the probability that he draws two more spades?

46. Urn A has three red balls and two white balls, and urn B has two red balls and five white balls. A fair coin is tossed. If it lands heads up, a ball is drawn from urn A; otherwise, a ball is drawn from urn B.

a. What is the probability that a red ball is drawn?

b. If a red ball is drawn, what is the probability that the coin landed heads up?

Supplementary Question:

1. Choose 2r shoes from n pairs of shoes with different sizes (2r<n). Compute the probabilities of the following events.

a) There are no matching pair among the 2r shoes.
b) There are exact ONE matching pair among the 2r shoes.
c) Three are r matching pairs among the 2r shoes.