MA212 Probability and Statistics

Chapter Three: Joint Distributions

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1.1 Basic concepts

Two-dimensional Random Variable

Suppose $X = X(\omega)$ and $Y = Y(\omega)$ are two Random Variables defined on Ω, then $(X, Y) = (X(\omega), Y(\omega))$ is called Two-dimensional Random Variable.

- (X, Y) defines on $(\Omega, \mathcal{A}, P)$ and maps to $(\Re^2, \mathcal{B}^2\ , P^{X*Y})$
- Recall intersection of Event $A = \{\omega : X(\omega) < x\}$ and $B = \{\omega : Y(\omega) < y\}$: $P(AB) = P(\{\omega : X(\omega) < x, Y(\omega) < y\})$

Cumulative Distribution Function

Cumulative Distribution Function defines on $(\Re^2, \mathcal{B}^2)$, a joint function.

- Cumulative Distribution Function : $F(x, y) = P(X \leq x, Y \leq y)$ $= P(\{\omega : X(\omega) < x, Y(\omega) < y\})$, calculate the probability P(AB) .
- For two-dimensional discrete random variable: $F(x_i, y_j) = P(X = x_i, Y = y_j) = p_{ij}$, Contingency Table
- For two-dimensional continuous random variable: $F(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) d_u d_v$, Joint Density Function .

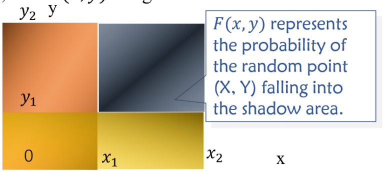
The property of distribution function $F(x, y)$

- For any x_0, $F(x_0, y)$ is the monotonic non-decreasing function of y.
  For any y_0, $F(x, y_0)$ is the monotonic non-decreasing function of x.
- $0 \le F(x, y) \le 1, and$

$$F(+\infty, +\infty) = 1, \quad F(-\infty, -\infty) = 0$$

$$F(-\infty, y) = 0, \quad F(x, -\infty) = 0 \quad (\forall x, y)$$

- $F(x, y) = F(x, y + 0)$, thus $F(x, y)$ is right continuous about y.
  $F(x, y) = F(x + 0, y)$, thus $F(x, y)$ is right continuous about x.



- $\forall x_1 < x_2, \quad y_1 < y_2$

$$F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1) \ge 0$$

$$= P\{x_1 < X \le x_2, \ \ y_1 < Y \le y_2\}$$

- N-dimensional random variables
  $F(x_1, x_2, \cdots, x_n) = P\{X_1 \le x_1, X_2 \le x_2, \cdots, X_n \le x_n\}$

1.2 Basic concepts (Cont'd)

Marginal Distribution Functions

The marginal distribution of random variables depends on their joint cumulative distribution $F(x, y)$.

- $F_X(x) = P(X \leq x) = P(X \leq x, Y < +\infty) = F(x, +\infty)$
- $F_Y(y) = P(Y \leq y) = P(X < +\infty, Y \leq y) = F(+\infty, y)$
- $f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y) d_y$, $x \in$ meaningful region, ?integral calculation
- For discrete R.V.s: Contingency Table

For two-dimensional discrete random variable:

- Multinomial distribution $P(X_1 = n_1, X_2 = n_2, \cdots, X_r = n_r) = \frac{n!}{n_1! n_2! \cdots n_r!} p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$ and $n_1 + n_2 + \cdots n_r = n$, the marginal is Binomial.

For two-dimensional continuous random variable:

- Bivariate Normal Density: if $(X, Y) \sim N(\mu_1, \mu_2, {\sigma_1}^2, {\sigma_2}^2, \rho)$, then $X \sim N(\mu_1, {\sigma_1}^2)$ and $Y \sim N(\mu_2, {\sigma_2}^2)$

Do more exercise!!!: probability calculation and marginal distribution

Independent Random Variables

- Event A and B are mutually independent $\Leftrightarrow$ P(AB) = P(A)P(B).
- R.V.s X and Y are independent: Note that Cumulative Distribution Function is a probability!!!

Assuming

$$(X,Y) \sim F(x,y), X \sim F_X(x), Y \sim F_Y(y)$$

If $\forall\ x, y \in (-\infty, \infty)$,then

$$\boxed{P\{X \le x, Y \le y\} = P\{X \le x\} \cdot P\{Y \le y\}}$$

Thus

$$\boxed{F(x,y) = F_X(x) \cdot F_Y(y)}$$

R.V X, Y **are independent.**

Independent Random Variables

Assuming the **frequency function** of (X, Y) is

$$P\{X = x_i, Y = y_j\} = p_{ij} \quad (i, j = 1, 2, \cdots)$$

X and Y are independent, which is equivalent to that $\forall i, j = 1,2 \ldots$ has

$$\boxed{P\{X = x_i, Y = y_j\} = P\{X = x_i\} \cdot P\{Y = y_j\}}$$

Assuming (X, Y) is a continuous r.v, and **density function**

$$(X, Y) \sim f(x, y)$$

$$X \sim f_X(x), \quad Y \sim f_Y(y)$$

If X and Y are independent, then

On the continuous point of $f(x, y)$, $f_X(x)$, $f_Y(y)$ has

$$\boxed{f(x, y) = f_X(x) f_Y(y)}$$

Conditional Distribution

Assuming the **joint frequency function** of (X, Y) is

$$P\{X = x_i, Y = y_j\} = p_{ij} \quad (i, j = 1,2, \dots)$$

For a fixed j, if $P\{Y = y_j\} = p_{\cdot j} > 0$, thus

$$\boldsymbol{P\{X = x_i | Y = y_j\} = \frac{P_{ij}}{p_{\cdot j}}} \quad (i = 1,2, \dots)$$

is the conditional frequency function of $r.v\ X$ under the condition of $Y = y_j$.

Assuming the **joint probability density** of (X, Y) is $f(x, y)$. If for a fixed y, the marginal density of (X, Y) for Y is $f_Y(y) > 0$, then

$$\frac{f(x, y)}{f_Y(y)} \triangleq f_{X|Y}(x|y) \quad (-\infty < \boldsymbol{x} < \infty)$$

is the conditional density of X under the condition of $Y = y$. And

$$\boldsymbol{F_{X|Y}(x|y) \triangleq \int_{-\infty}^{x} f_{X|Y}(u|y) du} \quad (-\infty < \boldsymbol{x} < \infty)$$

is the conditional distribution (function) of X under the condition of $Y = y$.

2.1 Bayesian Inference: Application of Conditional Distribution

Bayesian Inference

Given θ ***(Frequency Inference),*** *then* $r.v.\ X \sim b(n, \theta)$

$$f_{X|\Theta}(x|\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}, \qquad x = 0,1,\dots,n$$

For Θ is a $r.v.$ ***(Bayesian Inference)***, such as U(0,1) (prior) $\quad f_\Theta(\theta) = 1, \qquad 0 \le \theta \le 1$
we have the joint distribution of R.V.s Θ and X:

$$f_{\Theta,X}(\theta, x) = f_{X|\Theta}(x|\theta) f_\Theta(\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad x = 0,1,\dots,n \quad 0 \le \theta \le 1$$

The marginal density of X:

$$f_X(x) = \int_0^1 \binom{n}{x} \theta^x (1-\theta)^{n-x} d\theta \ = \frac{1}{n+1}, \qquad x = 0,1,\dots,n$$

Given $X = x$, then the conditional density (posterior) of Θ

$$f_{\Theta|X}(\theta|x) = \frac{f_{\Theta,X}(\theta,x)}{f_X(x)} = (n+1)\binom{n}{x} \theta^x (1-\theta)^{n-x}$$

$$f_{\Theta|X}(\theta|x) = \frac{\Gamma(n+2)}{\Gamma(x+1)\Gamma(n-x+1)} \theta^x (1-\theta)^{n-x} \sim Beta(x+1, n-x+1)$$

Distribution of Random Variables' Function

Generally: Assmuing $z = g(x, y)$ is a bivariate function, how to find out the **distribution of $r.v. Z = g(X, Y)$?** Given joint density function

Analysis:
$$F_Z(z) = P\{Z \le z\} = P\{g(X,Y) \le z\}$$
$$= \iint_{g(x,y)\le z} f(x,y)dxdy$$
$$= \cdots = \int_{-\infty}^{z} f_Z(u)du \qquad \therefore \quad Z \sim f_Z(z)$$

- **Such as the distribution function of $Z = X + Y$ is**

$$F_Z(z) = P\{Z \le z\} = P\{X + Y \le z\}$$
$$\therefore \quad f_Z(z) = \int_{-\infty}^{\infty} f(z-y, y)dy = \int_{-\infty}^{\infty} f(x, z-x)dx$$

If X *and* Y are independent, then the density function of $Z = X + y$ is

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)dy$$
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx$$

- **Others distribution functions: $Z = g(X, Y) = XY$ or X/Y ...**

Order Statistic

The distribution of $max(X, Y)$ and $min(X, Y)$

- Let $X \sim F_X(x), Y \sim F_Y(y)$, and X, Y are independent

$$
\begin{aligned}
F_{\max}(z) &= P\{\max(X,Y) \le z\} = P\{X \le z, Y \le z\} \\
&= P\{X \le z\} \cdot P\{Y \le z\} \\
&= F_X(z) \cdot F_Y(z)
\end{aligned}
$$

$$
\begin{aligned}
F_{\min}(z) &= P\{\min(X,Y) \le z\} = 1 - P\{\min(X,Y) > z\} \\
&= 1 - P\{X > z, Y > z\} \\
&= 1 - P\{X > z\} \cdot P\{Y > z\} = 1 - [1 - P\{X \le z\}] \cdot [1 - P\{Y \le z\}] \\
&= 1 - [1 - F_X(z)] \cdot [1 - F_Y(z)]
\end{aligned}
$$

- When $X_1, X_2, \cdots, X_n$ are independent & distributed at $F(x)$

$$F_{\max}(z) = F^n(z) \qquad F_{\min}(z) = 1 - [1 - F(z)]^n$$

3.1 Extra Exercises 1

Joint and Marginal distribution

- Discrete. Both R.V.s X_1, X_2 have PMF: $P(X_1 = -1) = 0.25$, $P(X_1 = 0) = 0.5$ and $P(X_1 = 1) = 0.25$. If $P(X_1X_2 = 0) = 1$, try to calculate $P(X_1 = X_2)$.
- Discrete. Indep. R.V.s X, Y have joint distribution:

X\Y	y1	y2	y3
x1	a	1/9	c
x2	1/9	b	1/3

  The value of a,b,c. Contingency Table!
- Continuous. Let X and Y have the joint density function:

$$f(x, y) = \begin{cases} k, & 0 \le x^2 \le y \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

  (1)$k, P(X > 0.5)$; (2) $f_X(x)$ and $f_Y(y)$; (3) $f_{X|Y}(x|y)$.

Supplementary Questions

1. Toss a coin three times. Let X denote the number of heads in the results. Let Y denote the absolute value of the difference between the number of heads and the number of tails. What is the frequency function of (X,Y)?

2. Suppose that the distribution for X has P (X = -1) = P(X=0) = P(X=1) = 1/3. Let $Y=X^2$, find the joint and marginal frequency functions of (X, Y).

3. Suppose that the r.v. Y follows the exponential distribution Exp(1),
$$X_k = \begin{cases} 0, & \text{when } Y \le k, \\ 1, & \text{when } Y > k, \end{cases} \quad k = 1,2$$
Find the joint and the marginal frequency functions of (X_1, X_2).

5. (Buffon's Needle Problem) A needle of length L is dropped randomly on a plane ruled with parallel lines that are a distance D apart, where $D \geq L$. Show that the probability that the needle comes to rest crossing a line is $2L/(\pi D)$. Explain how this gives a mechanical means of estimating the value of π.

6. A point is chosen randomly in the interior of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Find the marginal densities of the x and y coordinates of the point.

7. Find the joint and marginal densities corresponding to the cdf

$$F(x, y) = (1-e^{-\alpha x})(1-e^{-\beta y}), \qquad x \geq 0, \qquad y \geq 0, \qquad \alpha > 0, \qquad \beta > 0$$

8. Let X and Y have the joint density

$$f(x, y) = \frac{6}{7}(x + y)^2, \qquad 0 \leq x \leq 1, \qquad 0 \leq y \leq 1$$

a. By integrating over the appropriate regions, find (i) $P(X > Y)$, (ii) $P(X + Y \leq 1)$, (iii) $P(X \leq \frac{1}{2})$.

b. Find the marginal densities of X and Y.

c. Find the two conditional densities.

Supplementary Questions

1. Suppose that the two-dimensional continuous random variable (X, Y) has joint distribution function

$$F(x,y)=\begin{cases} k(1-e^{-x})(1-e^{-y}), & x>0, y>0, \\ 0, & \text{otherwise} \end{cases}$$

   Find the marginal density functions and compute $P(1<X<3, 1<Y<2)$.

2. Suppose that the two-dimensional continuous random variable (X, Y) has joint density function

$$f(x,y)=\begin{cases} x+y, & 0<x,y<1, \\ 0, & \text{otherwise,} \end{cases}$$

   a) Find the marginal density functions;
   b) Compute P (X > Y);
   c) Compute P (X < 0.5).

1. The joint frequency function of two discrete random variables, X and Y, is given in the following table:

y \ x	1	2	3	4
1	.10	.05	.02	.02
2	.05	.20	.05	.02
3	.02	.05	.20	.04
4	.02	.02	.04	.10

9. Suppose that (X, Y) is uniformly distributed over the region defined by $0 \le y \le 1 - x^2$ and $-1 \le x \le 1$.

a. Find the marginal densities of X and Y.
b. Find the two conditional densities.

19. Suppose that two components have independent exponentially distributed lifetimes, T_1 and T_2, with parameters α and β, respectively. Find (a) $P(T_1 > T_2)$ and (b) $P(T_1 > 2T_2)$.

3.2 Homework1

1. Let P be a point within a triangle of ABC. Let Q be a point on the bottom edge BC of the triangle. What is the probability that line PQ and line AB intersect?

2. There are five balls in a bag, among which 2 are white and 3 are black. Now pick two balls:
   a) with replacement;
   b) without replacement.

   Let X and Y denote the number of white balls picked. Find the joint and marginal frequency functions of (X,Y). Are X and Y independent?

3. Randomly choose a point on a circle which is centered at the origin with radius R. Let (X,Y) denote the coordinate of the point which has the following density function.

$$f(x,y)=\begin{cases} c, & x^2+y^2\leq R^2, \\ 0, & \text{otherwise} , \end{cases}$$

   a) Find c and the marginal density functions of X and Y;
   b) Check if X and Y are independent.

2. Suppose X follows U [0,1]. Given X=x (0<x<1), Y is uniformly distributed on (0, x).
   a) Compute the joint density function of (X,Y);
   b) Compute the marginal density function of Y
   c) Compute P(X+Y>1)

Conditional distribution and Independence

- Discrete. The joint PMF of (X,Y) is:

$$P(X=1, Y=1) = P(X=2, Y=1) = 1/8,$$
$$P(X=1, Y=2) = 1/4,\ P(X=2, Y=2) = 1/2$$

  (1)Calculate the conditional PMF of X when $Y = 1, 2$. (2) Are they indep.?

- Continuous. Let X and Y have the joint density function:

$$f(x,y) = \begin{cases} \frac{21}{4}x^2 y, & x^2 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

  Calculate $P(Y \leq 0.75 | X = 0.5)$.

14. Suppose that

$$f(x, y) = xe^{-x(y+1)}, \qquad 0 \le x < \infty, \qquad 0 \le y < \infty$$

a. Find the marginal densities of X and Y. Are X and Y independent?
b. Find the conditional densities of X and Y.

15. Suppose that X and Y have the joint density function

$$f(x, y) = c\sqrt{1 - x^2 - y^2}, \qquad x^2 + y^2 \le 1$$

a. Find c.

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b. Sketch the joint density.
c. Find $P(X^2 + Y^2) \le \frac{1}{2}$.
d. Find the marginal densities of X and Y. Are X and Y independent random variables?
e. Find the conditional densities.

Supplementary Questions

1. Suppose that there is a stick of length d. Select one point randomly and cut the stick into two pieces. Then randomly pick one piece up and randomly divide it into two pieces. Find the probability that the three pieces of sticks can form a triangle.

3. Suppose that the joint density function of (X, Y) is

$$f(x,y) = \begin{cases} e^{-y}, & 0 < x < y, \\ 0, & \text{otherwise} \end{cases}$$

a) Find the marginal density functions and check if they are independent.

b) Find the conditional density function $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

43. Let U_1 and U_2 be independent and uniform on [0, 1]. Find and sketch the density function of $S = U_1 + U_2$.

50. Suppose that X and Y are independent discrete random variables and each assumes the values 0, 1, and 2 with probability $\frac{1}{3}$ each. Find the frequency function of $X + Y$.

51. Let X and Y have the joint density function $f(x, y)$, and let $Z = XY$. Show that the density function of Z is

$$f_Z(z) = \int_{-\infty}^{\infty} f\left(y, \frac{z}{y}\right) \frac{1}{|y|}\, dy$$

52. Find the density of the quotient of two independent uniform random variables.

57. Suppose that Y_1 and Y_2 follow a bivariate normal distribution with parameters $\mu_{Y_1} = \mu_{Y_2} = 0, \sigma_{Y_1}^2 = 1, \sigma_{Y_2}^2 = 2$, and $\rho = 1/\sqrt{2}$. Find a linear transformation $x_1 = a_{11}y_1 + a_{12}y_2, x_2 = a_{21}y_1 + a_{22}y_2$ such that x_1 and x_2 are independent standard normal random variables. (*Hint:* See Example C of Section 3.6.2.)

3.5 Extra Exercises 3

Distribution of Random Variables' Function

Let X and Y have the joint density function:

$$f(x,y) = \begin{cases} e^{-(x+y)}, & x > 0,\ y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Calculate the density functions of Z: (1) $Z = (X+Y)/2$; (2)$Z = Y - X$. (Note that we only have the conventional formula, we can solve this problem by using probability calculation

Order Statistic

Let X and Y have the following PMF:

X	-1	0	1
P	1/4	1/2	1/4

Y	0	1
P	1/2	1/2

If $P(XY = 0) = 1$, calculate the PMF of $Z = \max\{X, Y\}$.

Distribution of Random Variables' Function

Let X, Y are indep. and have the density function:

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

(1) Calculate the joint density functions of $U = X + Y$ and $V = X/X + Y$;
(2) Are they independent? The essence of these problems is to calculate conditional density function under new joint density

Supplementary Questions

1. Suppose that X and Y are independent random variables, and both follow normal distribution N (0, 1). Let U = X + Y, V = X - Y.
   a) Compute the joint density function of (U, V) and the marginal density functions;
   b) Check if U and V are independent or not.

2. Suppose that the joint density function of (X, Y) is

$$f(x,y) = \begin{cases} 1, & 0 < x < 1, 0 < y < 2x, \\ 0, & \text{otherwise}, \end{cases}$$

   a) Find the marginal density functions;
   b) Find the density function of Z = 2X - Y;
   c) Compute P(Y<1/2|X<1/2).

72. Let $X_1, X_2, \ldots, X_n$ be independent continuous random variables each with cumulative distribution function F. Show that the joint cdf of $X_{(1)}$ and $X_{(n)}$ is

$$F(x, y) = F^n(y) - [F(y) - F(x)]^n, \qquad x \le y$$

15

Supplementary Questions

1. Choose two numbers from 1, 2, 3. Let X denote the first number chosen and Y the second number. Let Z = max (X, Y), find the joint frequency functions for (X, Y) and (X, Z), and compute all corresponding marginal frequency functions.

2. Suppose that X and Y are two independent r.v.s and follow the Normal Distribution N(0, 1). Let Z = min (X, Y), find the density function of Z.