Bayesian Statistics and Hierarchical Bayesian Modeling for Psychological Science

Lecture 09

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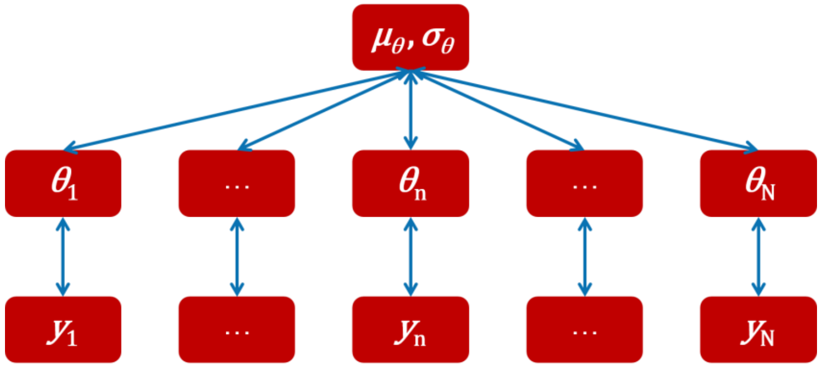
https://github.com/lei-zhang/BayesCog_Wien

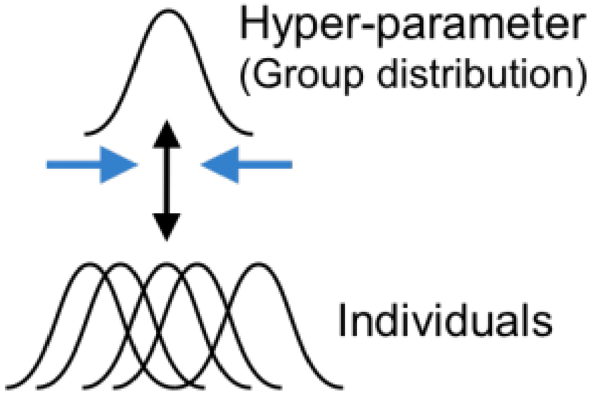
universität wien

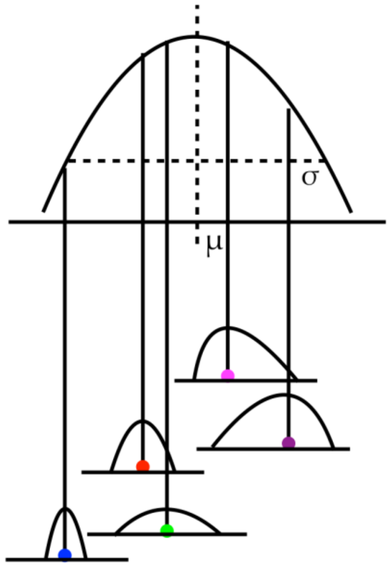
Fakultät für Psychologie

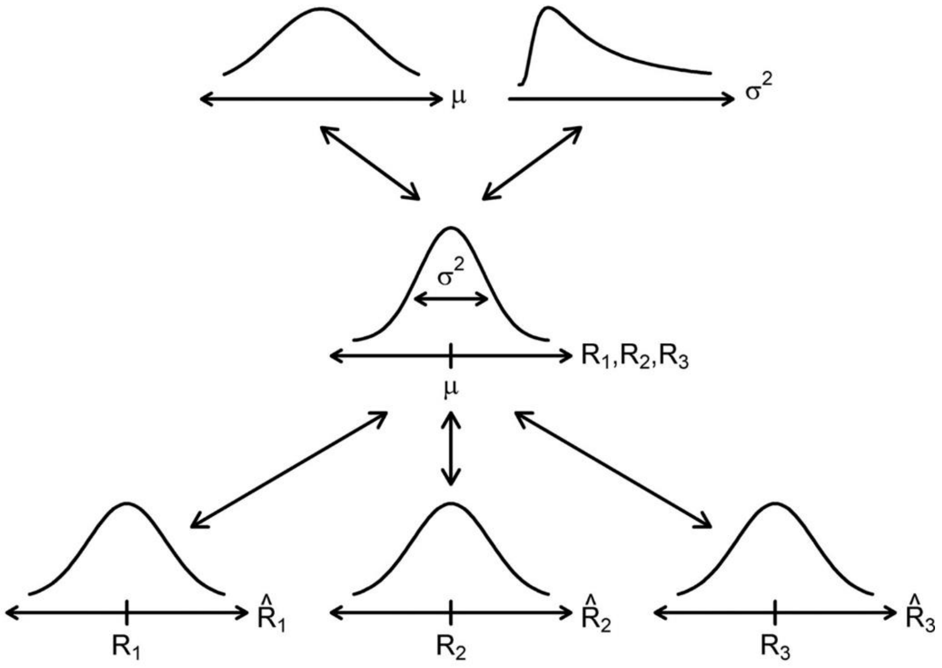
Hierarchical Structure



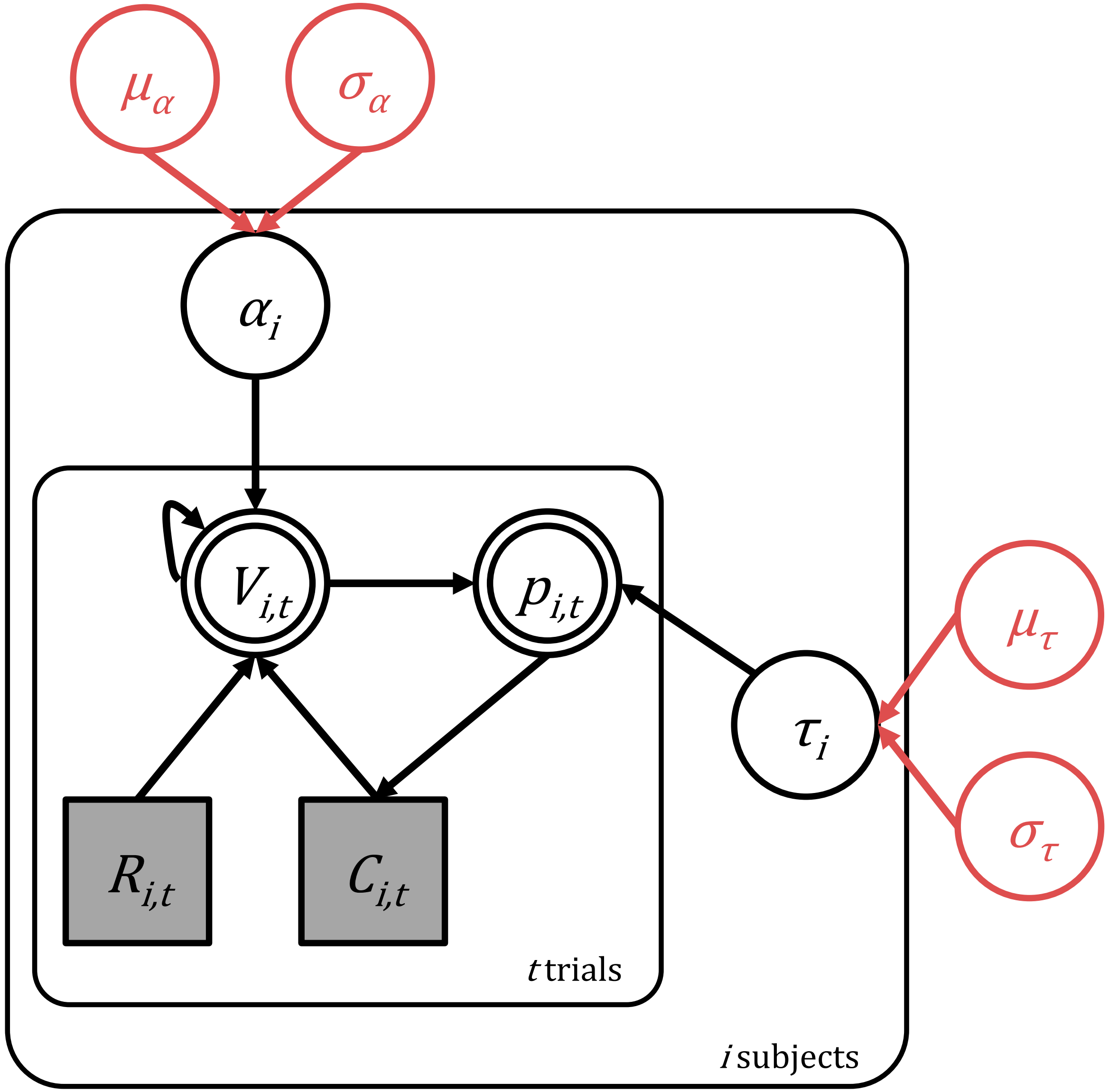
$$P(\Theta,\Phi \mid D) = \frac{P(D \mid \Theta,\Phi)P(\Theta,\Phi)}{P(D)} \propto P(D \mid \Theta)P(\Theta \mid \Phi)P(\Phi)$$





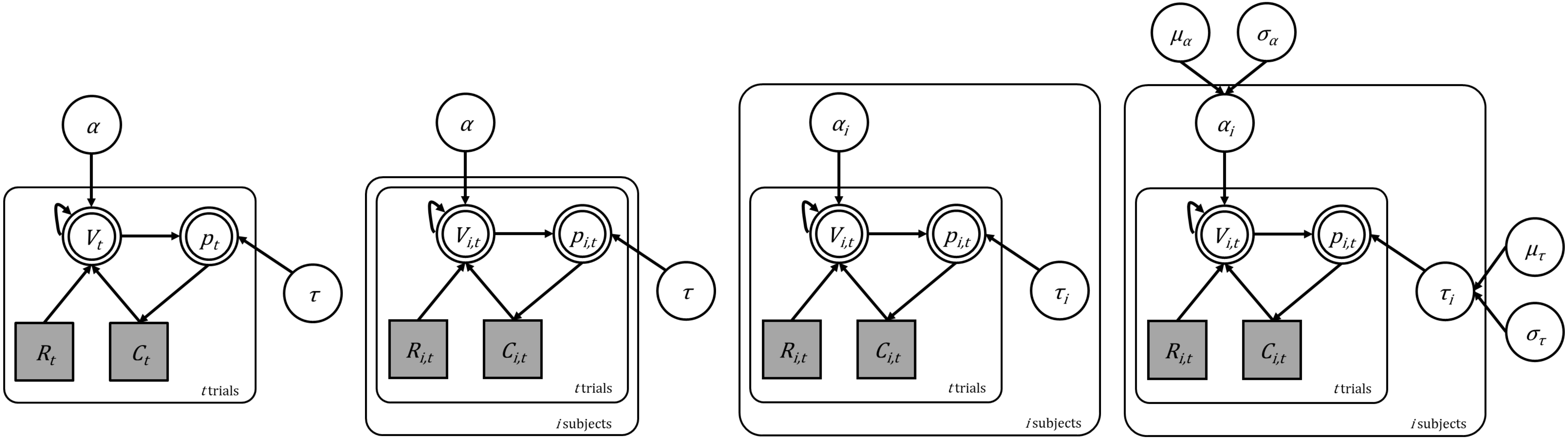


Hierarchical RL Model



HOW DID WE GET HERE?

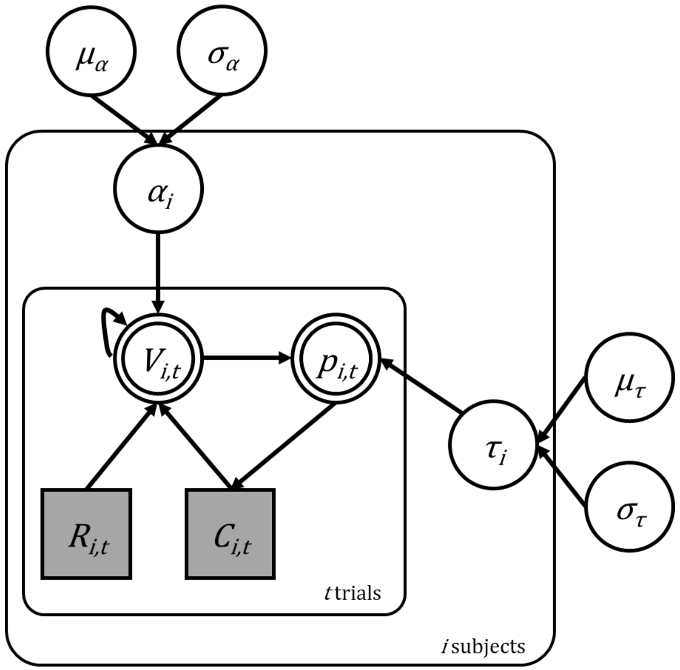
cognitive model
statistics
computing



The cognitive model *per se* is the same!

Implementing Hierarchical RL Model

cognitive model
statistics
computing



$$\mu_\alpha \sim Uniform(0,1)$$
$$\sigma_\alpha \sim halfCauchy(0,1)$$
$$\mu_\tau \sim Uniform(0,3)$$
$$\sigma_\tau \sim halfCauchy(0,3)$$
$$\alpha_i \sim Normal(\mu_\alpha, \sigma_\alpha)_{\mathcal{T}(0,1)}$$
$$\tau_i \sim Normal(\mu_\tau, \sigma_\tau)_{\mathcal{T}(0,3)}$$

$$p_{i,t}(C=A) = \frac{1}{1+e^{\tau_i(V_{i,t}(B)-V_{i,t}(A))}}$$

$$V_{i,t+1}^c = V_{i,t}^C + \alpha_i(R_{i,t} - V_{i,t}^C)$$

```
parameters {
  real<lower=0,upper=1> lr_mu;
  real<lower=0,upper=3> tau_mu;

  real<lower=0> lr_sd;
  real<lower=0> tau_sd;

  real<lower=0,upper=1> lr[nSubjects];
  real<lower=0,upper=3> tau[nSubjects];
}

model {
  lr_sd  ~ cauchy(0,1);
  tau_sd ~ cauchy(0,3);
  lr     ~ normal(lr_mu, lr_sd) ;
  tau    ~ normal(tau_mu, tau_sd) ;

  for (s in 1:nSubjects) {
    vector[2] v;
    real pe;
    v = initV;

    for (t in 1:nTrials) {
      choice[s,t] ~ categorical_logit( tau[s] * v );
      pe = reward[s,t] - v[choice[s,t]];
      v[choice[s,t]] = v[choice[s,t]] + lr[s] * pe;
    }
  }
}
```

Exercise XI

cognitive model
statistics
computing

```
…/06.reinforcement_learning/_scripts/reinforcement_learning_multi_parm_main.R
```

TASK: (1) complete the model (TIP: individual ~ group)

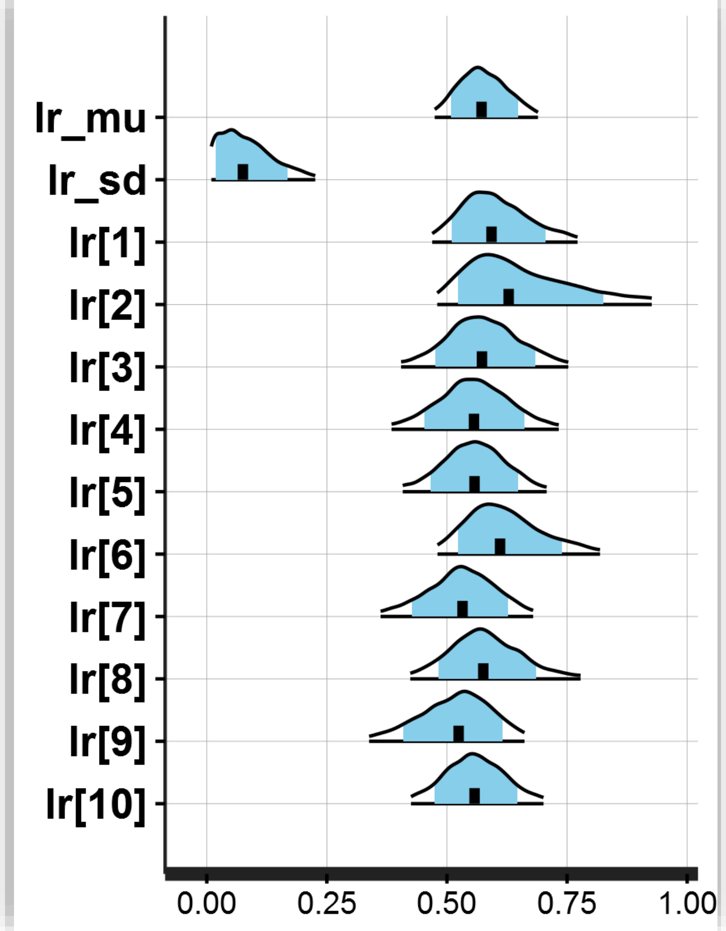
(2) fit the hierarchical RL model

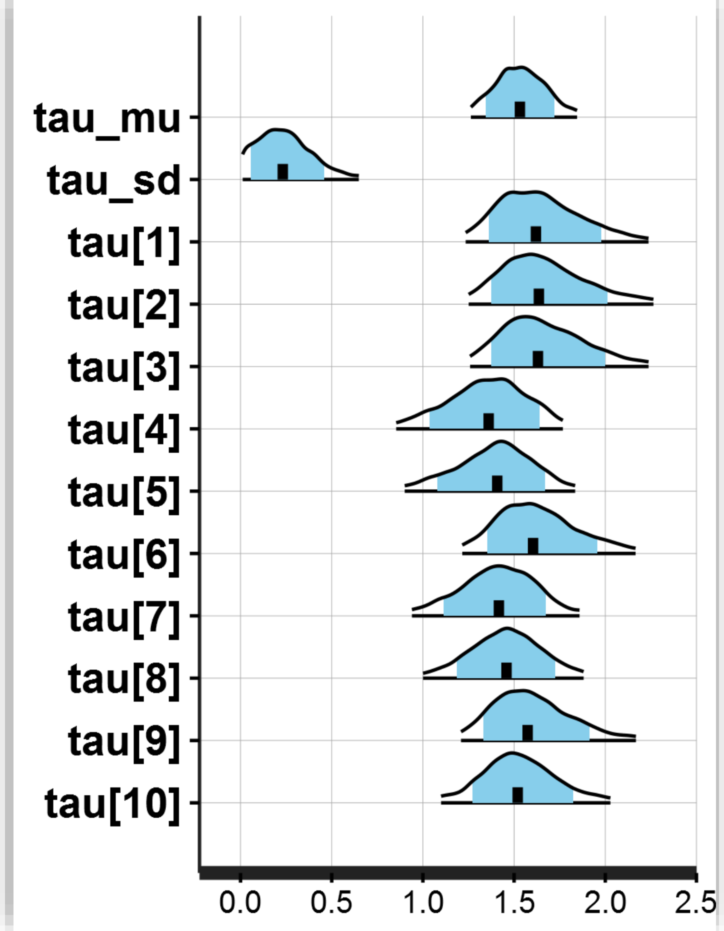
```
> source('_scripts/reinforcement_learning_multi_parm_main.R')

> fit_rl3 <- run_rl_mp( modelType ='hrch' )
```

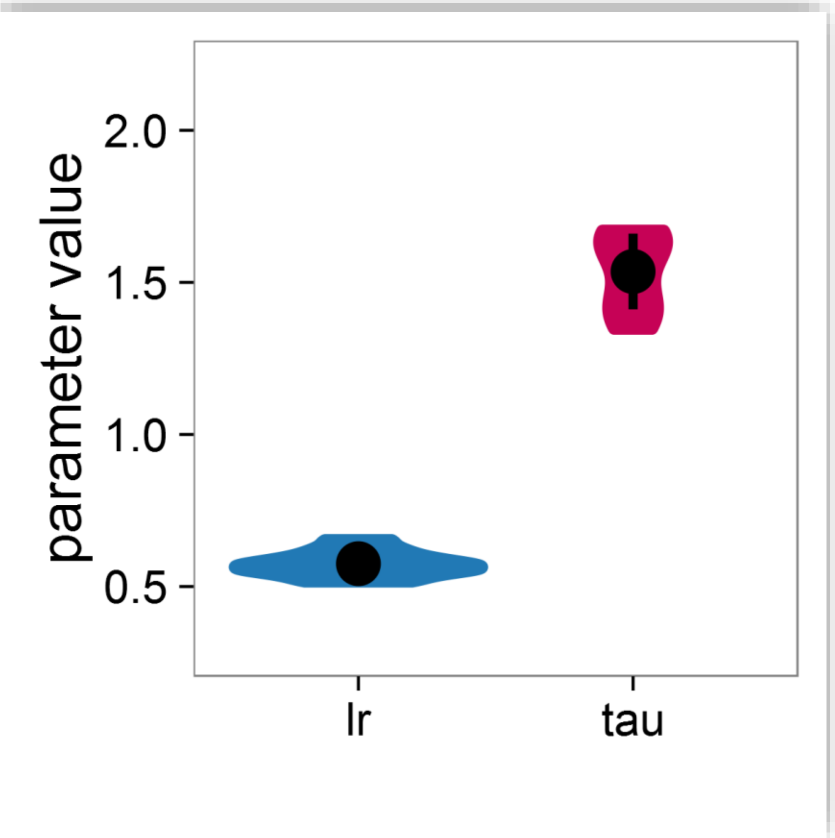
```
In addition: Warning messages:
1: There were 97 divergent transitions after warmup. Increasing adapt_delta above 0.8 may help. See
http://mc-stan.org/misc/warnings.html#divergent-transitions-after-warmup
2: Examine the pairs() plot to diagnose sampling problems
```

Hierarchical Fitting*

cognitive model
statistics
computing





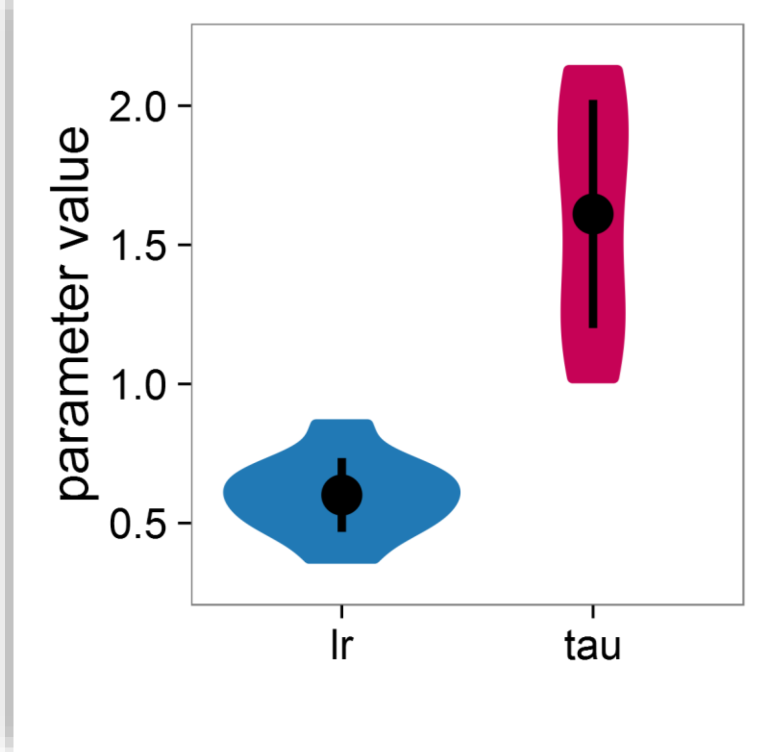


*: adapt_delta=0.999, max_treedepth=100

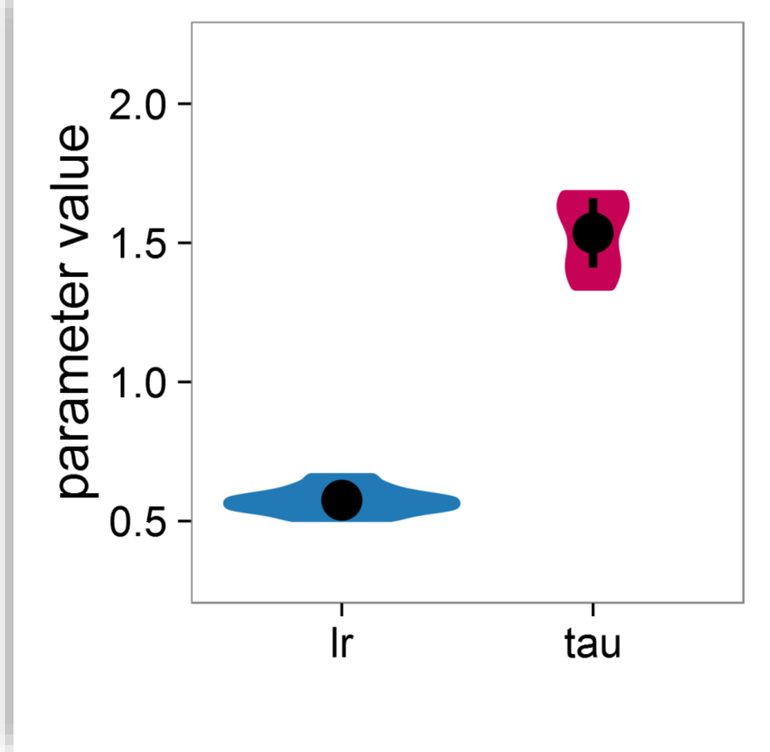
Comparing with True Parameters

cognitive model
statistics
computing

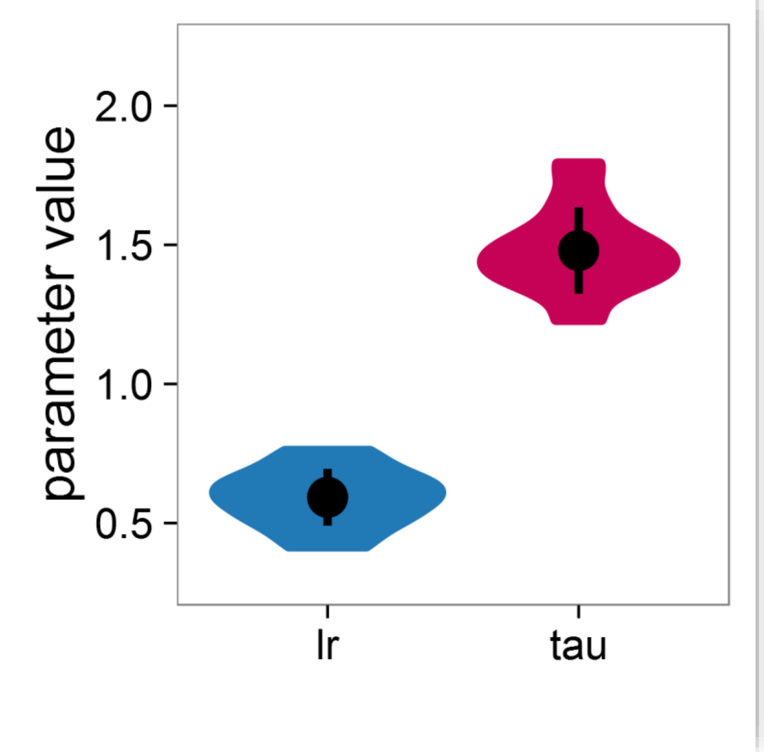
Posterior Means (indv)



Posterior Means (hrch)*



True Parameters



*: adapt_delta=0.999, max_treedepth=100

Group-level Parameters

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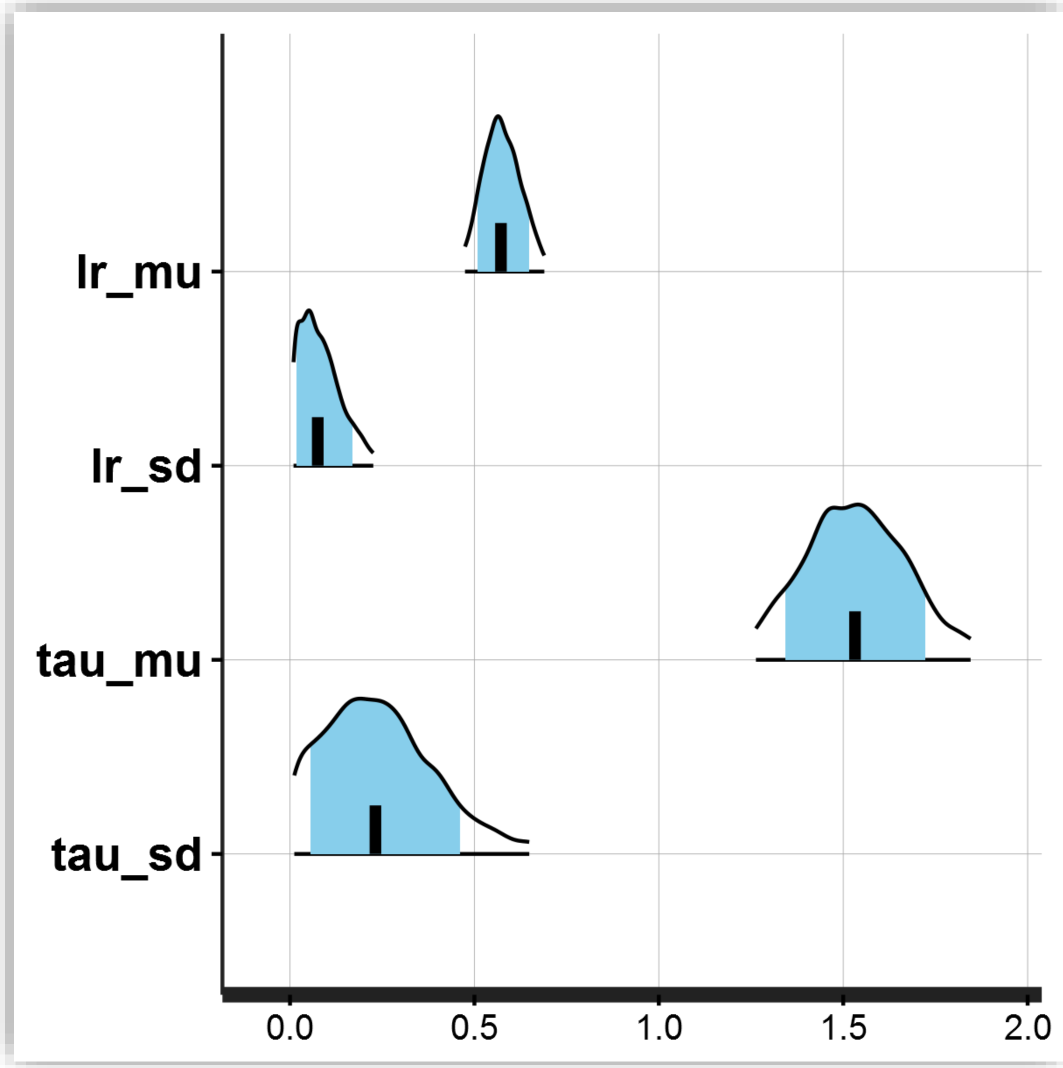
statistics
computing

True group parameters

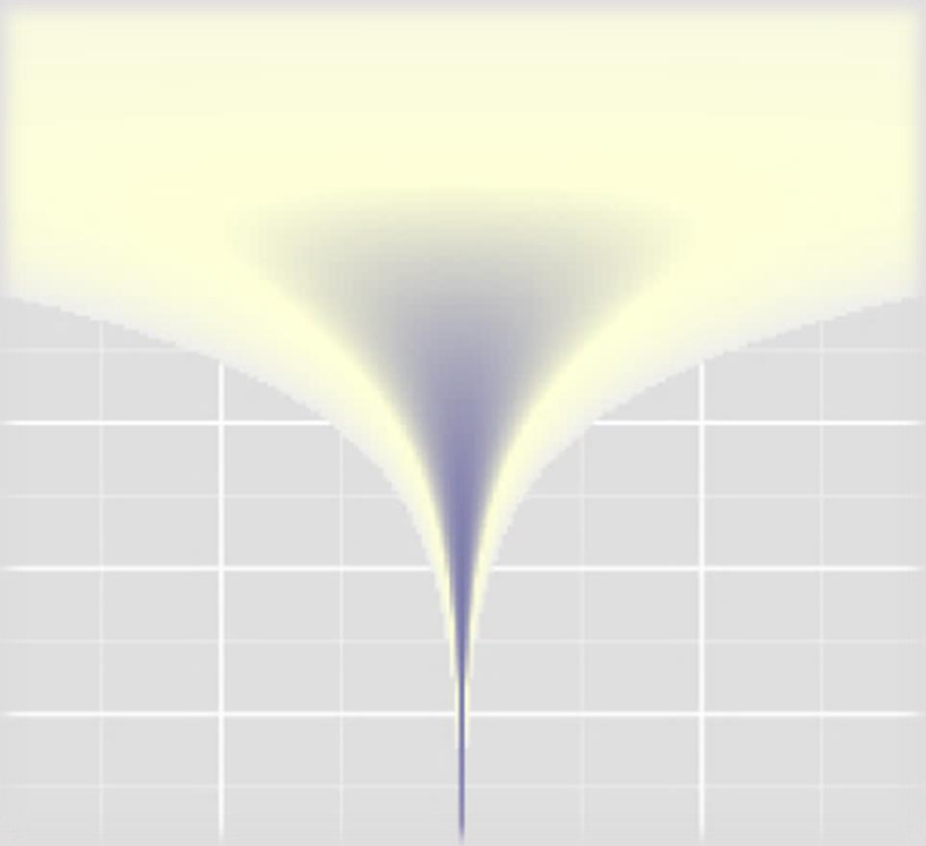
```
lr  = rnorm(10, mean=0.6, sd=0.12)
tau = rnorm(10, mean=1.5, sd=0.2)
```

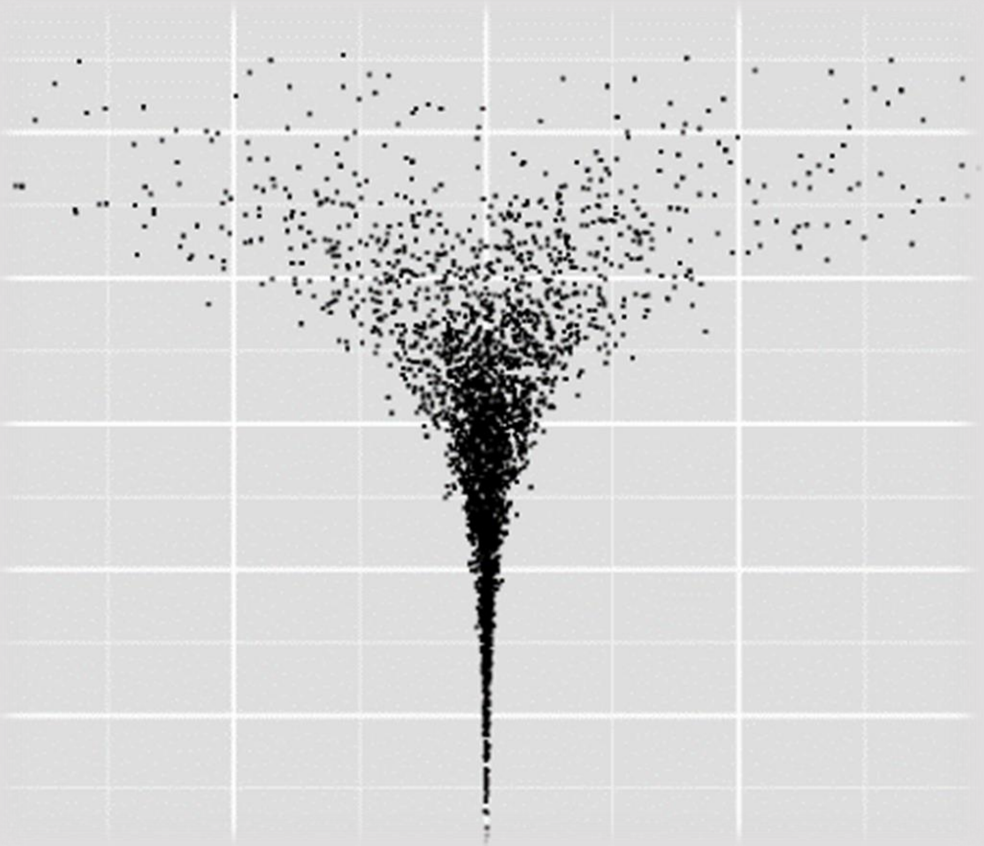
Estimated group parameters

	mean	2.5%	25%	50%	75%	97.5%
lr_mu	0.58	0.47	0.54	0.57	0.61	0.69
lr_sd	0.09	0.01	0.04	0.08	0.12	0.23
tau_mu	1.54	1.26	1.43	1.53	1.63	1.85
tau_sd	0.25	0.01	0.13	0.23	0.34	0.65



OPTIMIZING STAN CODES





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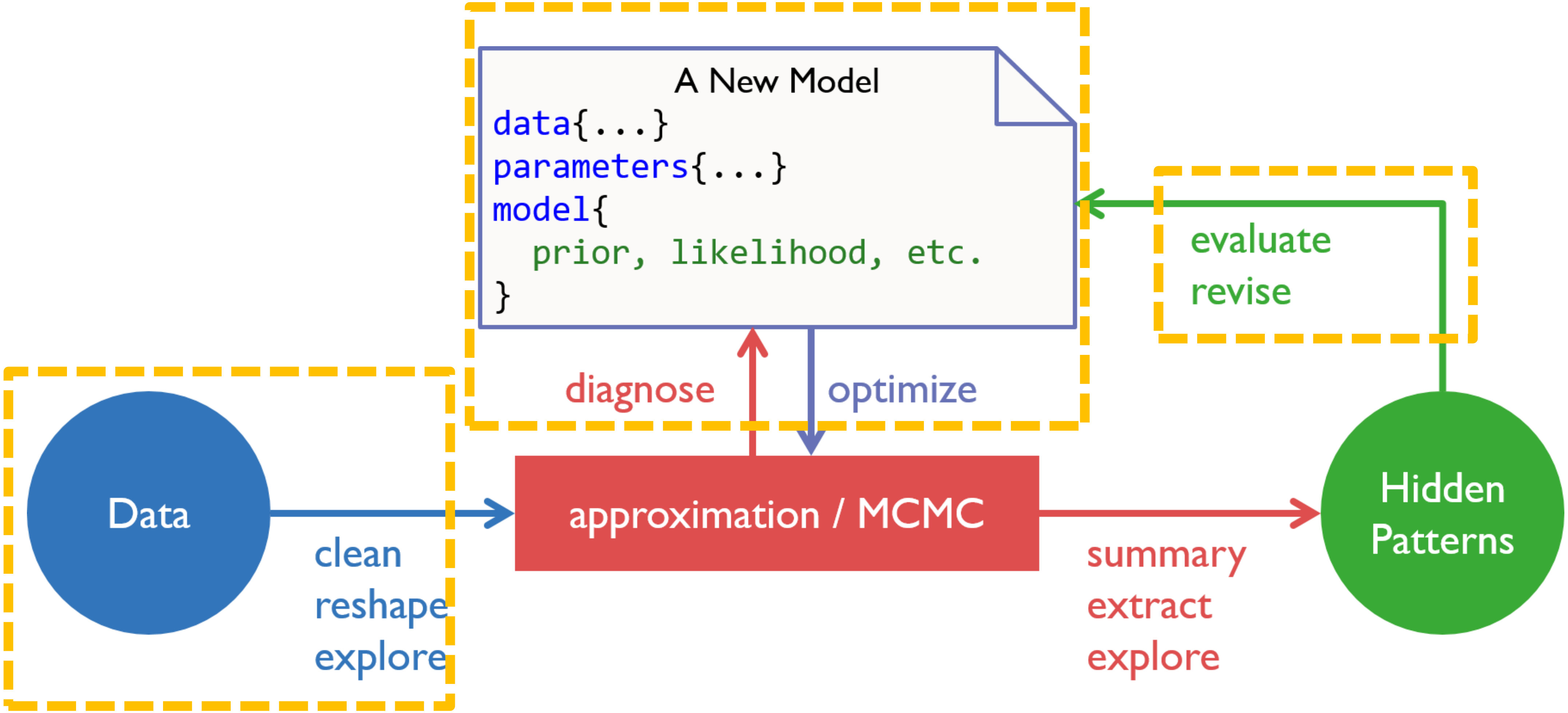
statistics
computing



Optimizing Stan Code

Preprocess data

run as many calculations as you can outside Stan

Specify a proper model

follow literature, supervision, experience, etc.

Vectorizing

vectorize Stan code whenever you can

Reparameterizing

reparameterize target parameter to simple distributions

Preprocess Data

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statistics
computing

$$\overline{\text{height}} = \alpha + \beta 1 * \text{weight} + \beta 2 * \text{weight}^2$$

```
d$weight_sq <- d$weight^2
```

```
data {
  int<lower=0> N;
  vector<lower=0>[N] height;
  vector<lower=0>[N] weight;
  vector<lower=0>[N] weight_sq;
}
```

Specify a Proper Model

- Visualize your data
- Follow Literatures
- Start from simple and then build complexities
- Simulate data and run model recovery

A New Model

```
data{...}
parameters{...}
model{
  prior, likelihood, etc.
}
```

Vectorization

cognitive model
statistics
computing

```
model {
  for (n in 1:N) {
    flip[n] ~ bernoulli(theta);
  }
}
```

```
model {
  flip ~ bernoulli(theta);
}
```

```
parameters {
  ...
  real<lower=0,upper=1> lr[nSubjects];
  real<lower=0,upper=3> tau[nSubjects];
}

model {
  ...
  lr      ~ normal(lr_mu, lr_sd) ;
  tau     ~ normal(tau_mu, tau_sd) ;
  ...
}
```

```
model {
  vector[N] mu;
  for (i in 1:N) {
    mu[i] = alpha + beta * weight[i];
    height[i] ~ normal(mu[i], sigma)
  }
}
```

```
model {
  vector[N] mu;
  mu = alpha + beta * weight;
  height ~ normal(mu, sigma);
}
```

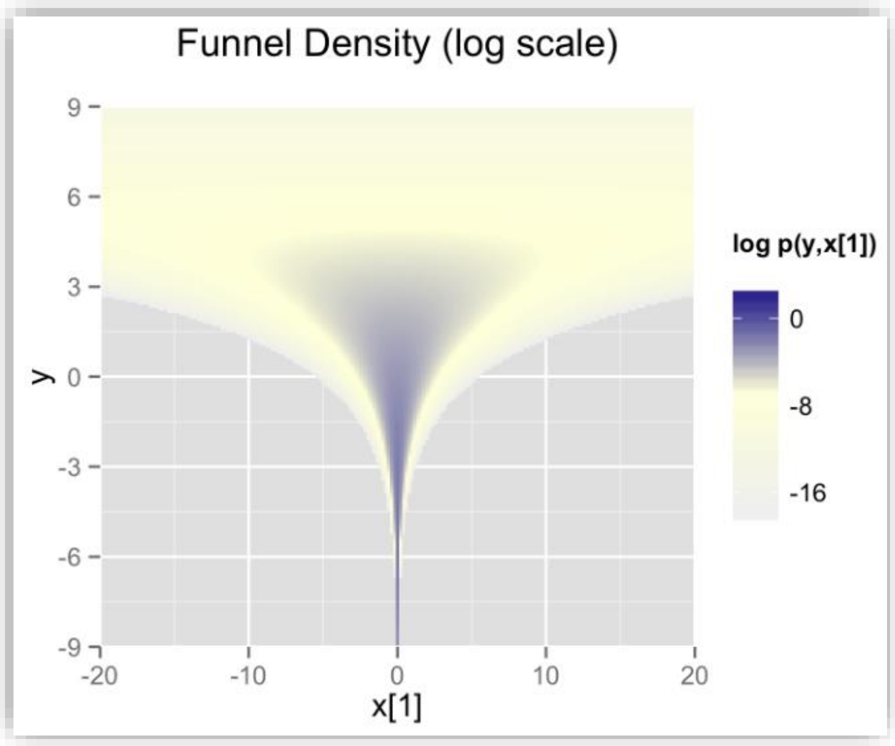
```
model {
  height ~ normal(alpha + beta * weight, sigma);
}
```

Reparameterization

Neal's Funnel

$$p(y,x) = \text{Normal}(y|0,3) \times \prod_{n=1}^{9} \text{Normal}(x_n|0,\exp(y/2))$$

```
parameters {
  real y;
  vector[9] x;
}
model {
  y ~ normal(0,3);
  x ~ normal(0,exp(y/2));
}
```



Non-centered Reparameterization*

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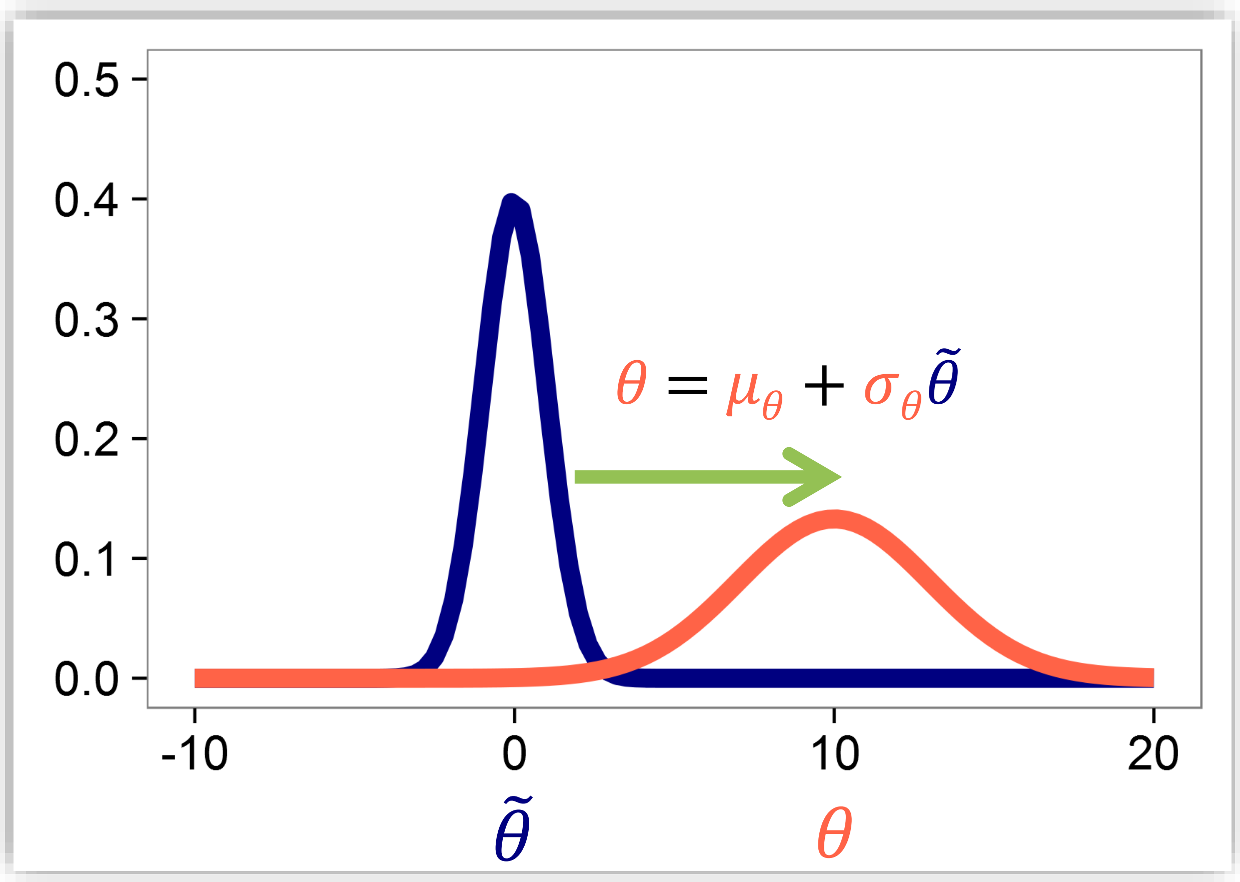
$$\theta \sim Normal(\mu_\theta, \sigma_\theta)$$

$$\tilde{\theta} \sim Normal(0, 1)$$

$$\theta = \mu_\theta + \sigma_\theta \tilde{\theta}$$

Stan likes **simple** distributions!



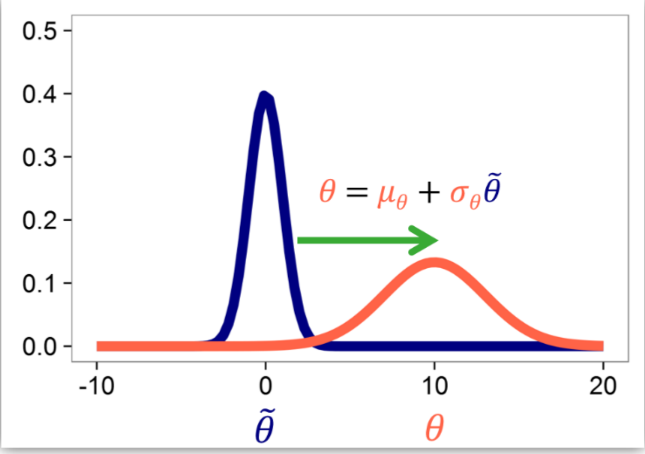
*:Stan Development Team (2016), aka, Matt Trick

Reparameterization

Neal's Funnel

$$p(y, x) = \text{Normal}(y|0, 3) \times \prod_{n=1}^{9} \text{Normal}(x_n|0, \exp(y/2))$$

```
parameters {
  real y;
  vector[9] x;
}
model {
  y ~ normal(0,3);
  x ~ normal(0,exp(y/2));
}
```



→

```
parameters {
  real y_raw;
  vector[9] x_raw;
}
transformed parameters {
  real y;
  vector[9] x;

  y = 3.0 * y_raw;
  x = exp(y/2) * x_raw;
}
model {
  y_raw ~ normal(0,1);
  x_raw ~ normal(0,1);
}
```

Stan Sampling Parameters

parameter	description	constraint	default
iterations	number of MCMC samples (per chain)	int, > 0	2000
delta: δ	target Metropolis acceptance rate	$\delta \in [0,1]$	0.80
stepsize: ε	initial HMC step size	real, $\varepsilon > 0$	2.0
max_treedepth: L	maximum HMC steps per iteration	int, $L > 0$	10

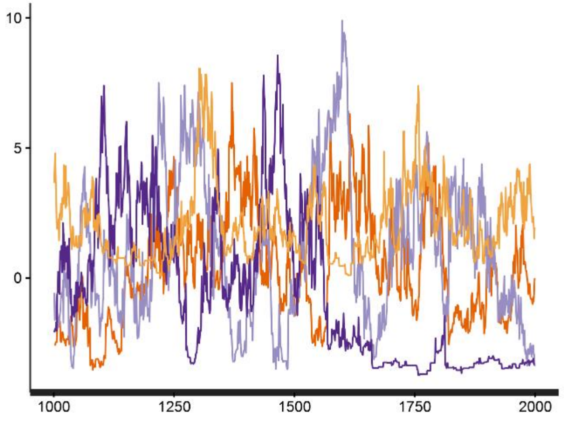
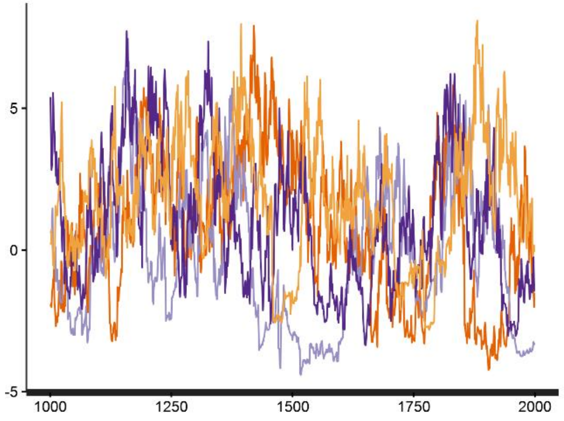
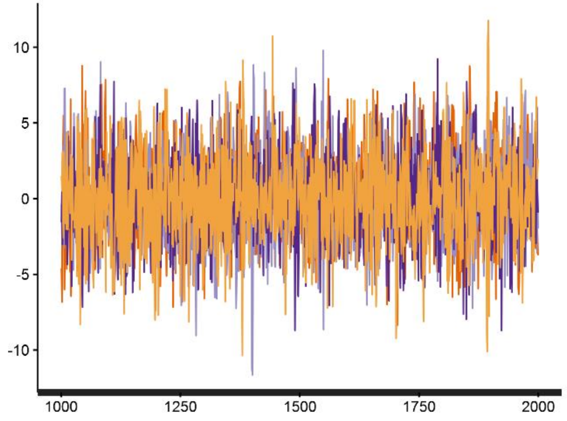
Typical adjustments

- Increase iterations
- Increase delta
- Decrease stepsize
- Might have to increase max_treedepth

```
funnel_fit2 <- stan("_scripts/funnel.stan",
       iter = 4000,
       control = list(adapt_delta = 0.999,
                      stepsize = 1.0,
                      max_treedepth = 20))
```

Stan Development Team (2016)

Neal's Funnel: Comparing Performance

	direct model	adjusted direct model	reparameterized model
Rhat (*y*)	1.22	1.1	1.0
n_eff (*y*)	18	42	3886
runtime*	48.50 sec	50.76 sec	50.12 sec
n_eff (*y*) / runtime	0.37 / sec	0.82 / sec	77.53 / sec
n_divergent	53	0	0
traceplot (*y*)			

*: 2 cores in parallel, including compiling time

How about Bounded Parameters?

$$\tilde{\theta} \sim Normal(0,1) \qquad\qquad \tilde{\theta} \sim Normal(0,1)$$

$$\theta = \mu_\theta + \sigma_\theta \tilde{\theta} \quad \longrightarrow \quad \theta = Probit^{-1}(\mu_\theta + \sigma_\theta \tilde{\theta})$$

$$\theta \in (-\infty, +\infty) \qquad\qquad \theta \in [0,1]$$

constraint	**reparameterization**
$\theta \in (-\infty, +\infty)$	$\theta = \mu_\theta + \sigma_\theta \tilde{\theta}$
$\theta \in [0, N]$	$\theta = Probit^{-1}(\mu_\theta + \sigma_\theta \tilde{\theta}) \times N$
$\theta \in [M, N]$	$\theta = Probit^{-1}(\mu_\theta + \sigma_\theta \tilde{\theta}) \times (N - M) + M$
$\theta \in (0, +\infty)$	$\theta = exp(\mu_\theta + \sigma_\theta \tilde{\theta})$

* $Probit^{-1}$: Normal cumulative distribution function (normcdf)

Apply to Our Hierarchical RL Model

cognitive model
statistics
computing

```
parameters {
  real<lower=0,upper=1> lr_mu;
  real<lower=0,upper=3> tau_mu;

  real<lower=0> lr_sd;
  real<lower=0> tau_sd;

  real<lower=0,upper=1> lr[nSubjects];
  real<lower=0,upper=3> tau[nSubjects];
}
```

```
parameters {
  # group-level parameters
  real lr_mu_raw;
  real tau_mu_raw;
  real<lower=0> lr_sd_raw;
  real<lower=0> tau_sd_raw;

  # subject-level raw parameters
  vector[nSubjects] lr_raw;
  vector[nSubjects] tau_raw;
}

transformed parameters {
  vector<lower=0,upper=1>[nSubjects] lr;
  vector<lower=0,upper=3>[nSubjects] tau;

  for (s in 1:nSubjects) {
    lr[s]  = Phi_approx( lr_mu_raw  + lr_sd_raw * lr_raw[s] );
    tau[s] = Phi_approx( tau_mu_raw + tau_sd_raw * tau_raw[s] ) * 3;
  }
}
```

Apply to Our Hierarchical RL Model

```
model {
  lr_sd  ~ cauchy(0,1);
  tau_sd ~ cauchy(0,3);
  lr     ~ normal(lr_mu, lr_sd) ;
  tau    ~ normal(tau_mu, tau_sd) ;

  for (s in 1:nSubjects) {
    vector[2] v;
    real pe;
    v = initV;

    for (t in 1:nTrials) {
      choice[s,t] ~ categorical_logit( tau[s] * v );
      pe = reward[s,t] - v[choice[s,t]];
      v[choice[s,t]] = v[choice[s,t]] + lr[s] * pe;
    }
  }
}
```

```
model {
  lr_mu_raw  ~ normal(0,1);
  tau_mu_raw ~ normal(0,1);
  lr_sd_raw  ~ cauchy(0,3);
  tau_sd_raw ~ cauchy(0,3);

  lr_raw  ~ normal(0,1);
  tau_raw ~ normal(0,1);

  for (s in 1:nSubjects) {
    ...
```

```
generated quantities {
  real<lower=0,upper=1> lr_mu;
  real<lower=0,upper=3> tau_mu;

  lr_mu  = Phi_approx(lr_mu_raw);
  tau_mu = Phi_approx(tau_mu_raw) * 3;
}
```

Exercise XII

cognitive model
statistics
computing

```
…/07.optm_rl/_scripts/reinforcement_learning_hrch_main.R
```

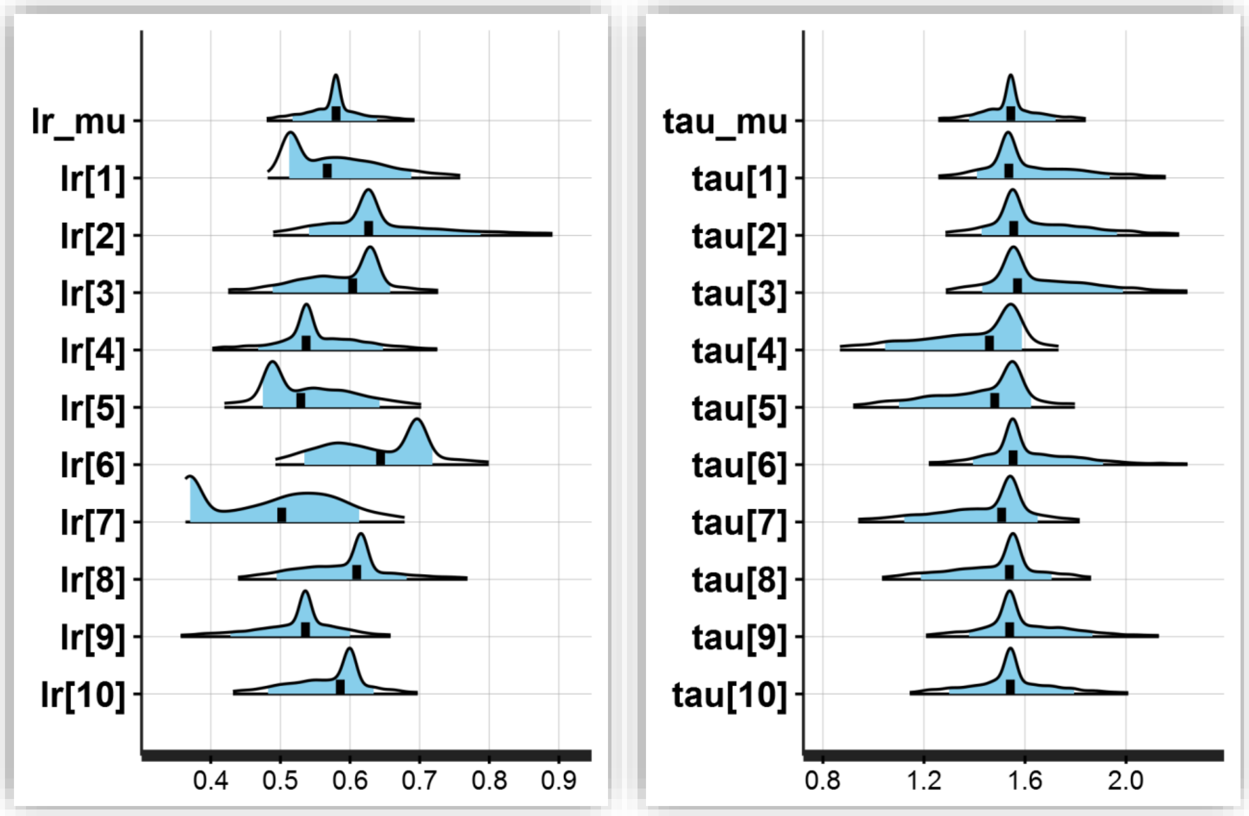
TASK: (1) Complete the Matt Trick

(2) fit the optimized hierarchical RL model

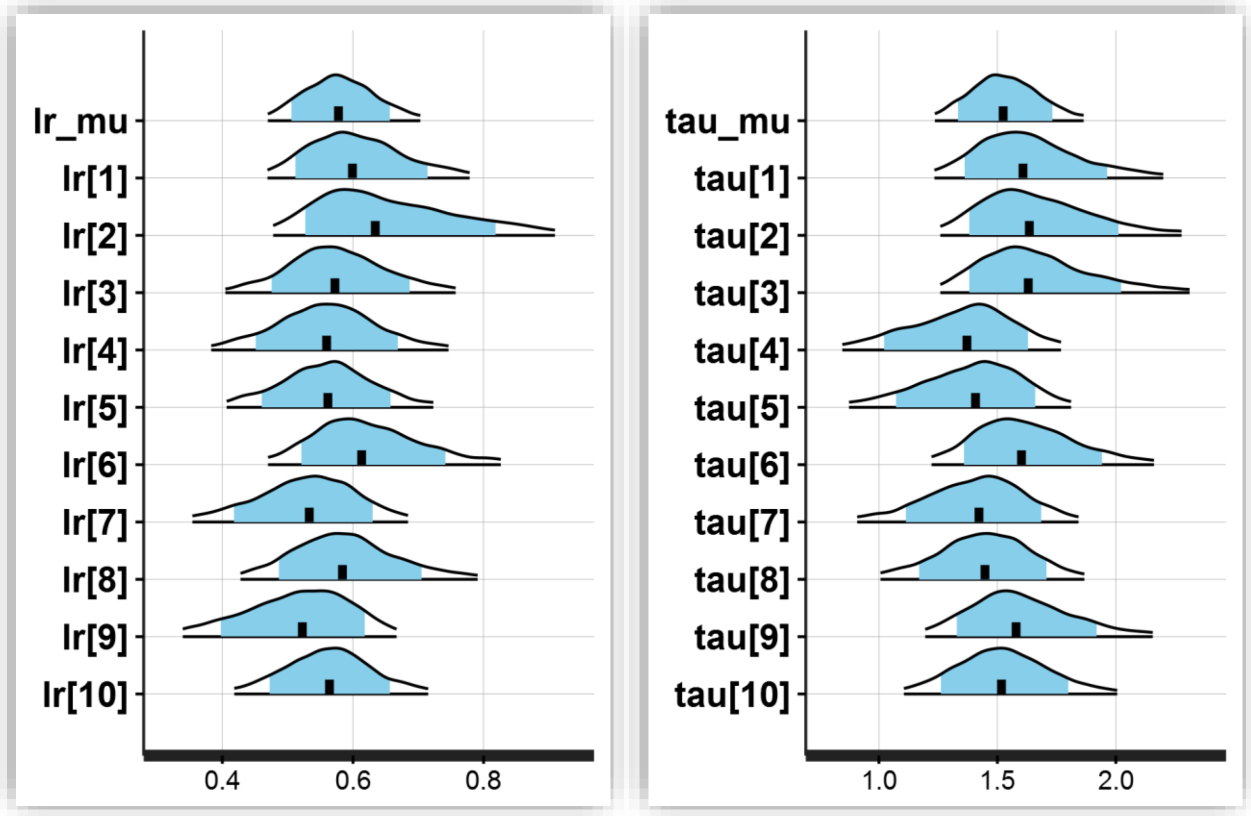
```
> source('_scripts/reinforcement_learning_hrch_main.R')

> fit_rl4 <- run_rl_mp2(optimized = TRUE)
```

Hierarchical Fitting – Optimized

Posterior Means (hrch)



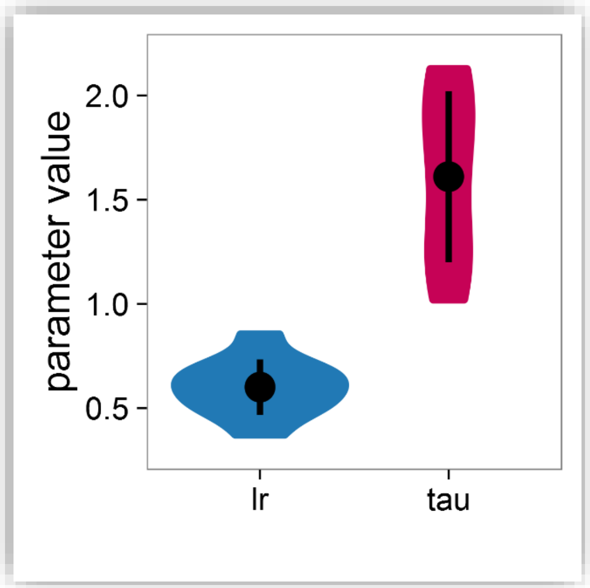
Posterior Means (hrch + optm)



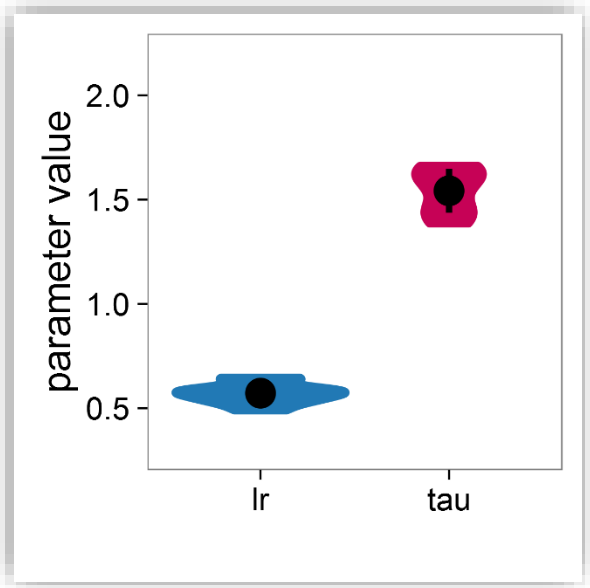
Comparing with True Parameters

cognitive model
statistics
computing

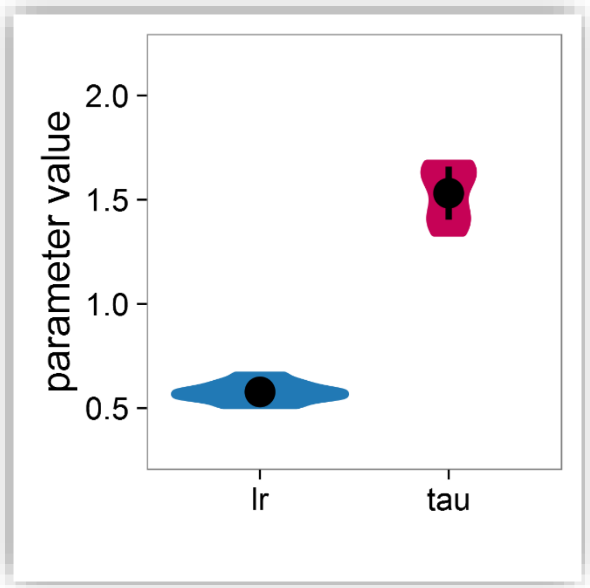
Posterior Means (indv)



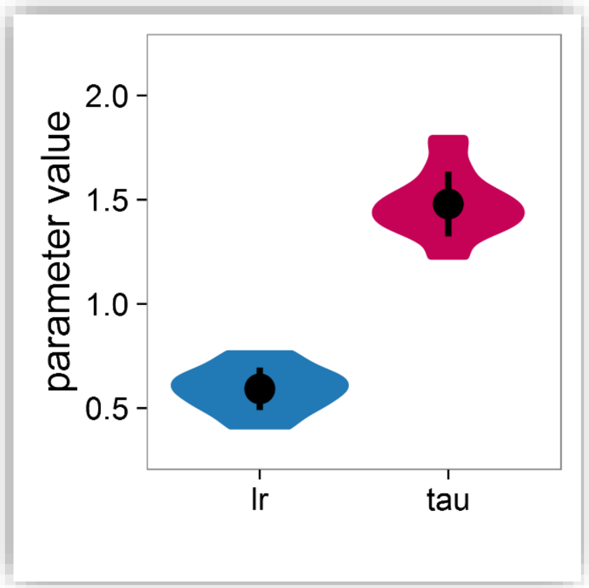
Posterior Means (hrch)

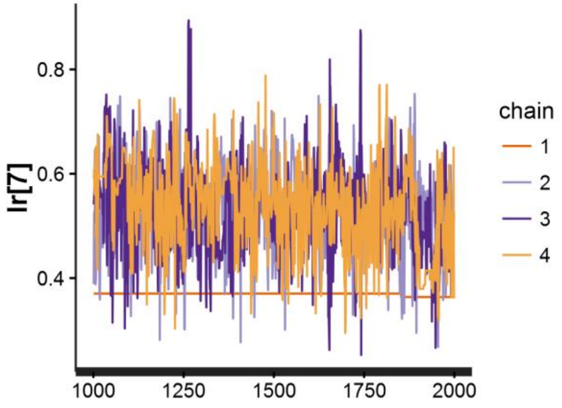


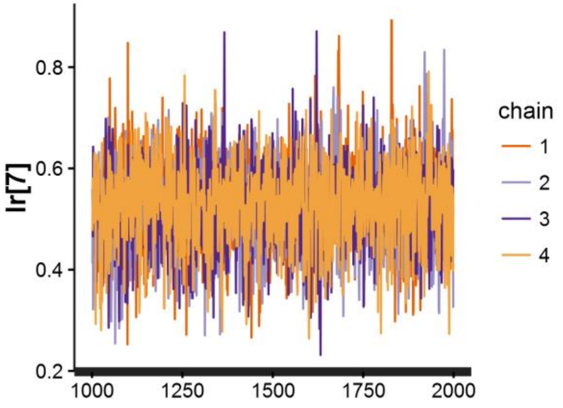
Posterior Means (hrch+optm)



True Parameters







Posterior Predictive Check

cognitive model
statistics
computing