Bayesian Statistics and Hierarchical Bayesian Modeling for Psychological Science

Lecture 03

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https://github.com/lei-zhang/BayesCog_Wien

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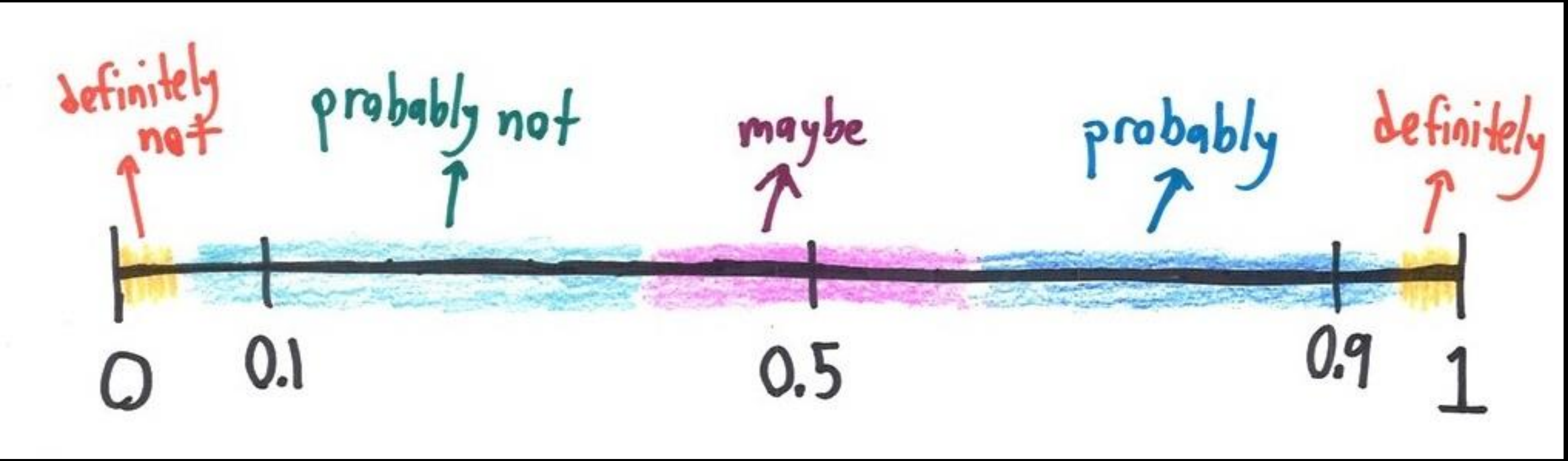
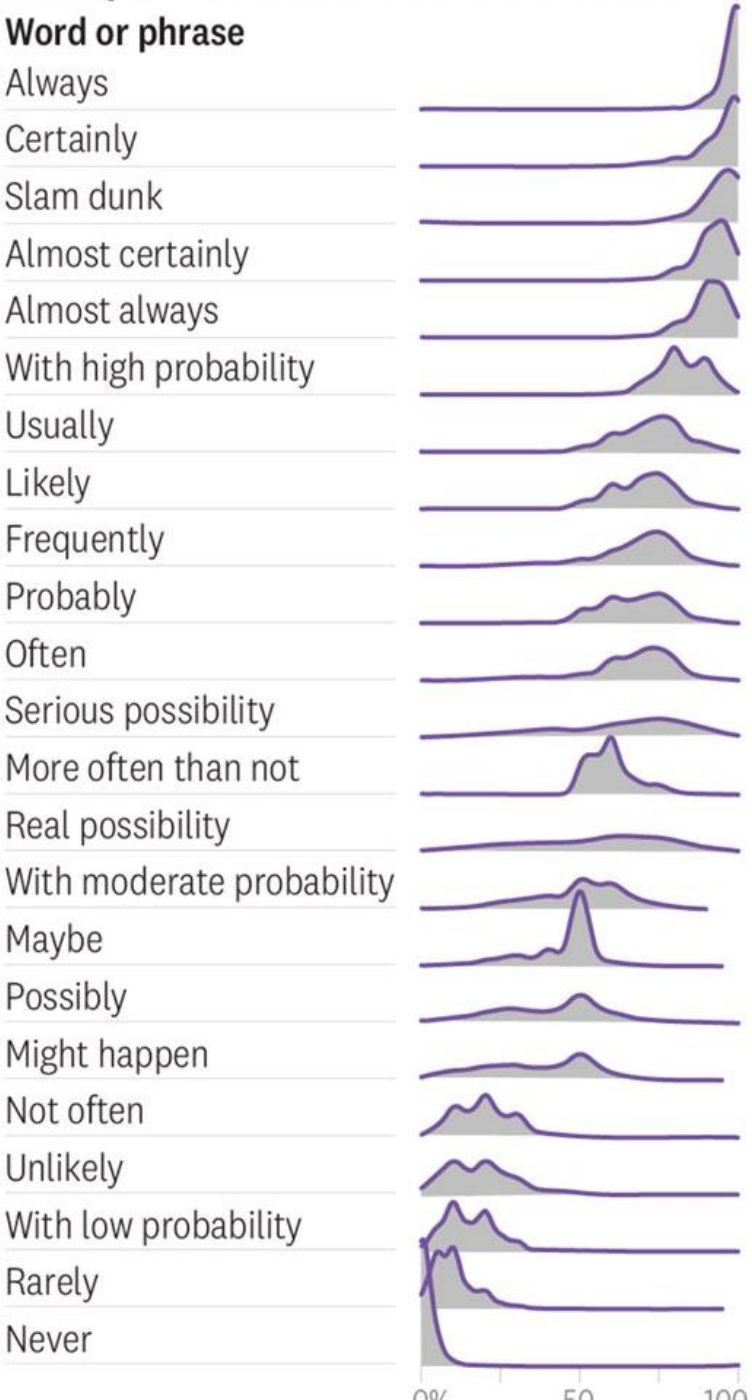
BASICS OF PROBABILITY



to respondents' estimate of likelihood



Probability

…assigning numbers to a set of possibilities

Properties (Kolmogorov, 1956)

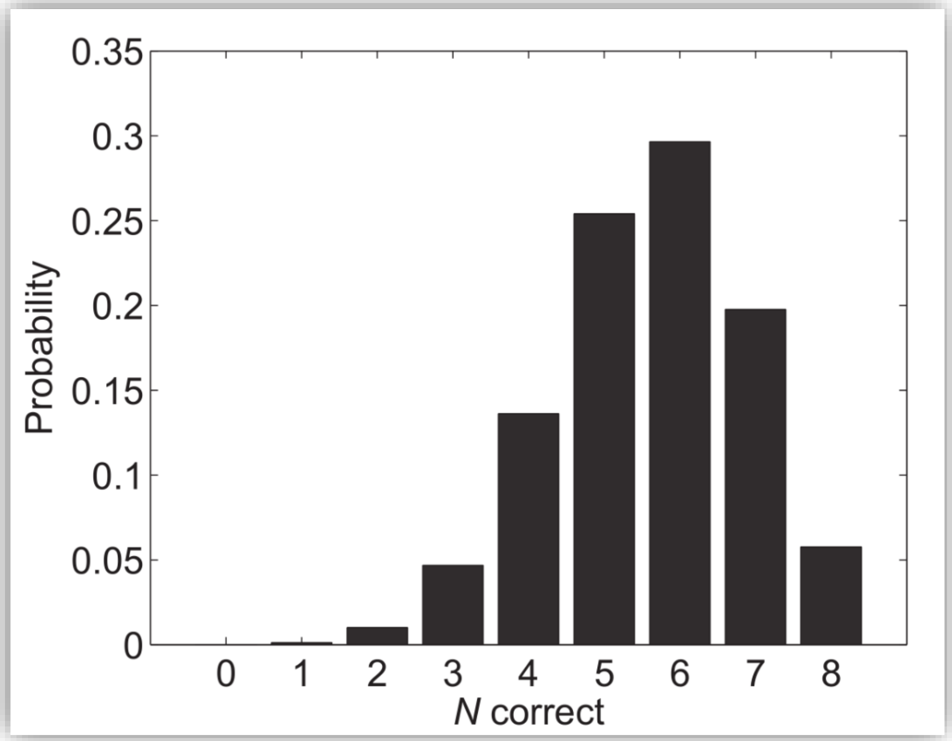
- $p \in [0,1]$
- $\Sigma p = 1$

Probability are used to express uncertainty.

Probability Functions

discrete events – we talk about mass

Run a test and record each student's correct responses

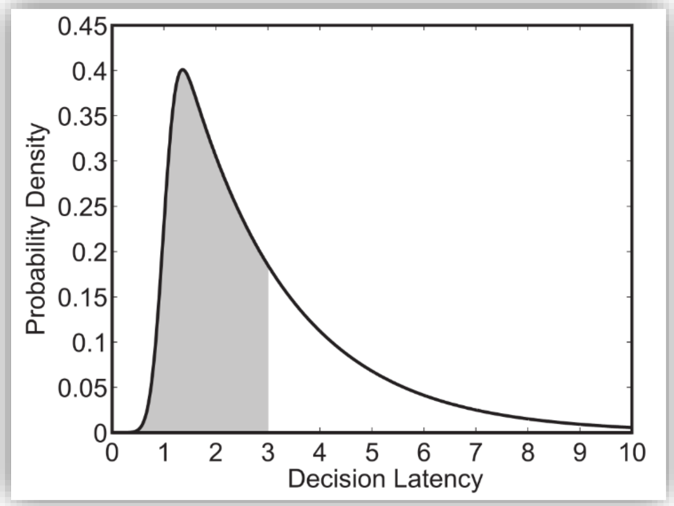


Lewandowsky & Farrell (2011)

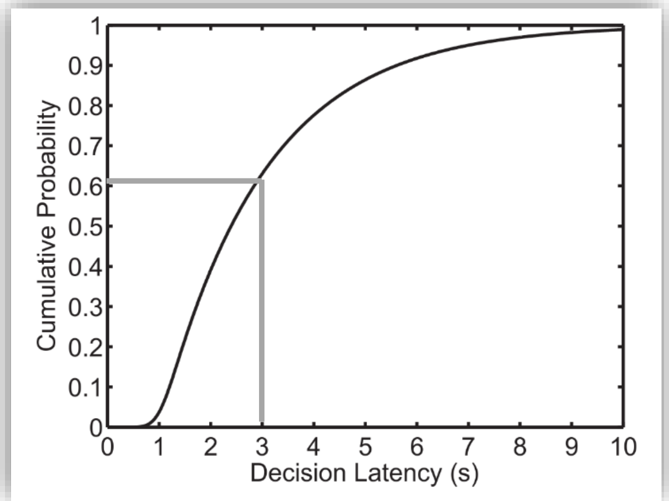
Probability Functions

continuous events – we talk about density

probability density function
(PDF)



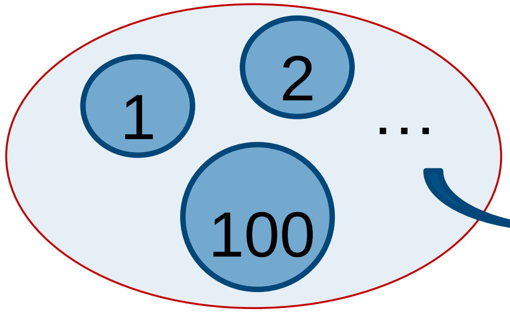
cumulative distribution function
(CDF)



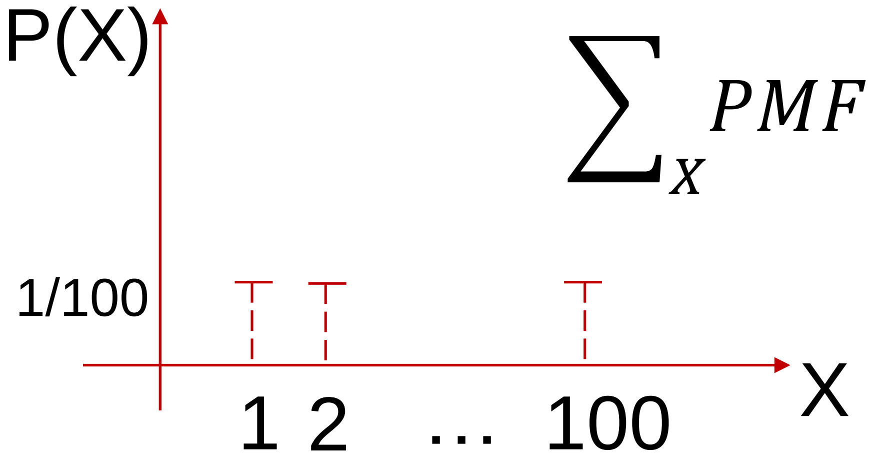
Lewandowsky & Farrell (2011)

Another example

Discrete



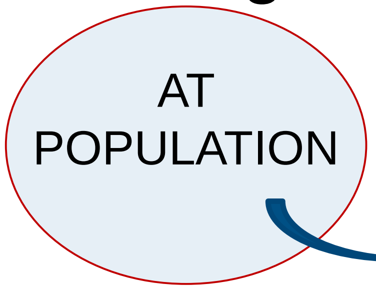
X	P(X)
1	1/100
2	1/100
…	…
100	1/100



$$\sum_X PMF(X) = 1$$

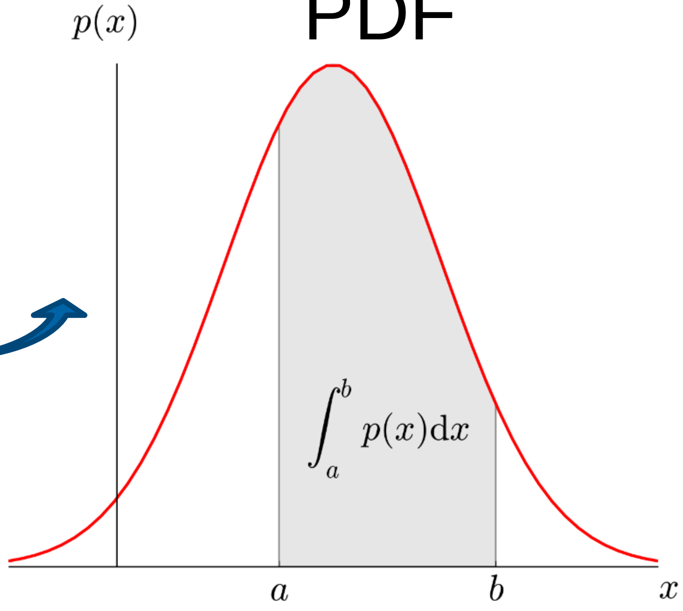
Continuous

Height



X	P(X)
1.8 m	0

PDF



P given by the area

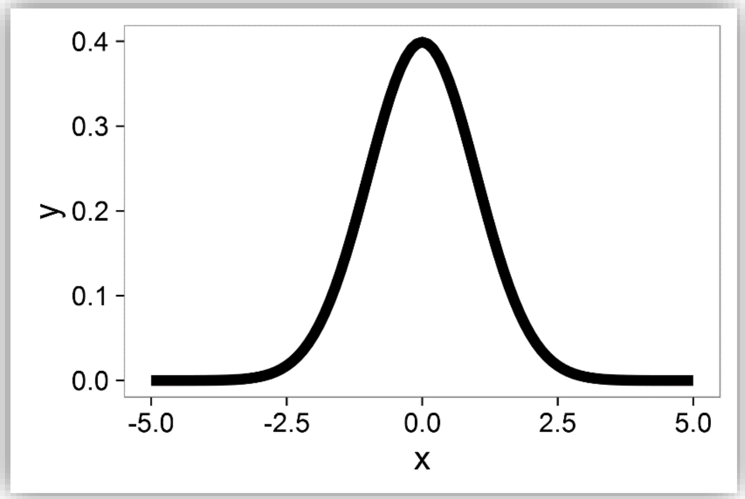
$$1.75 \le X \le 1.85$$

Playing with Probability Functions in R

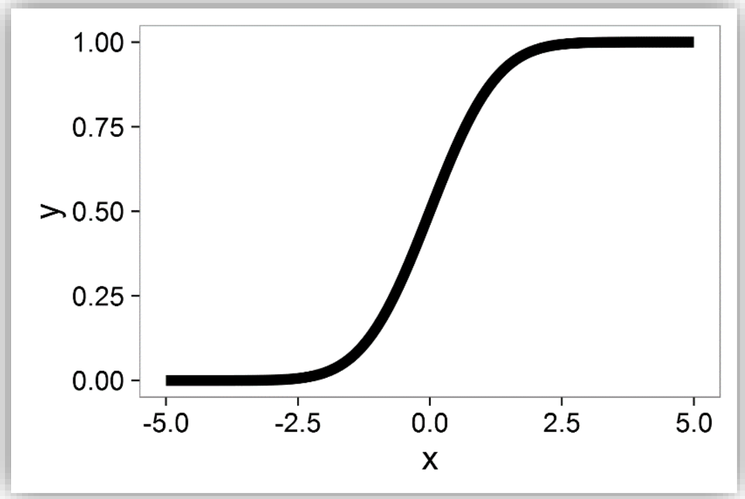

```
dnorm() - PDF
pnorm() - CDF
qnorm() - quantile, inverse cdf
rnorm() - random number generator
```

Example: Normal(0,1)

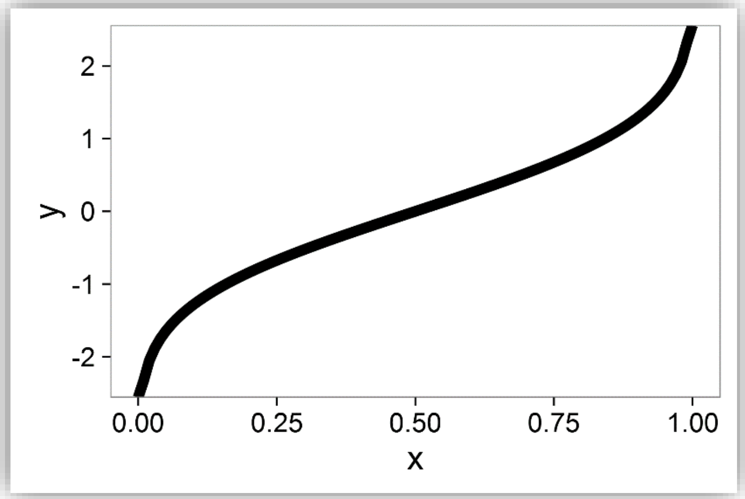
`dnorm()`



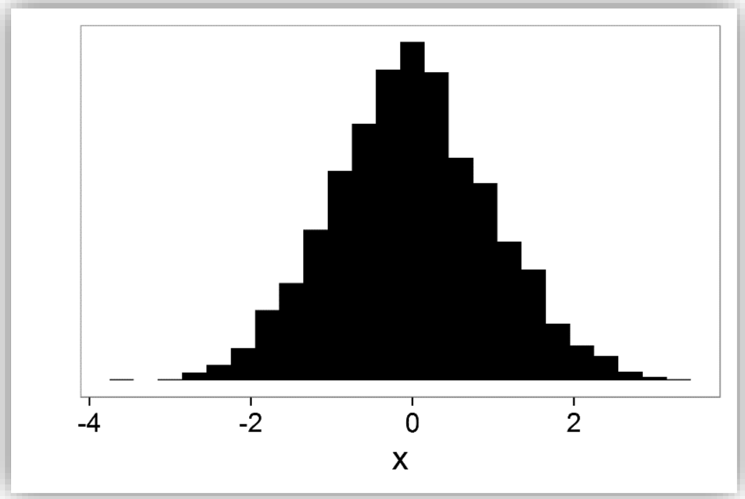
`pnorm()`



`qnorm()`



`rnorm()`



Joint Probability and Conditional Probability

Joint Probability

$p(A, B) = p(B, A)$

- e.g., $p(\text{raining, cold})$: $p(\text{raining})$ AND $p(\text{cold})$

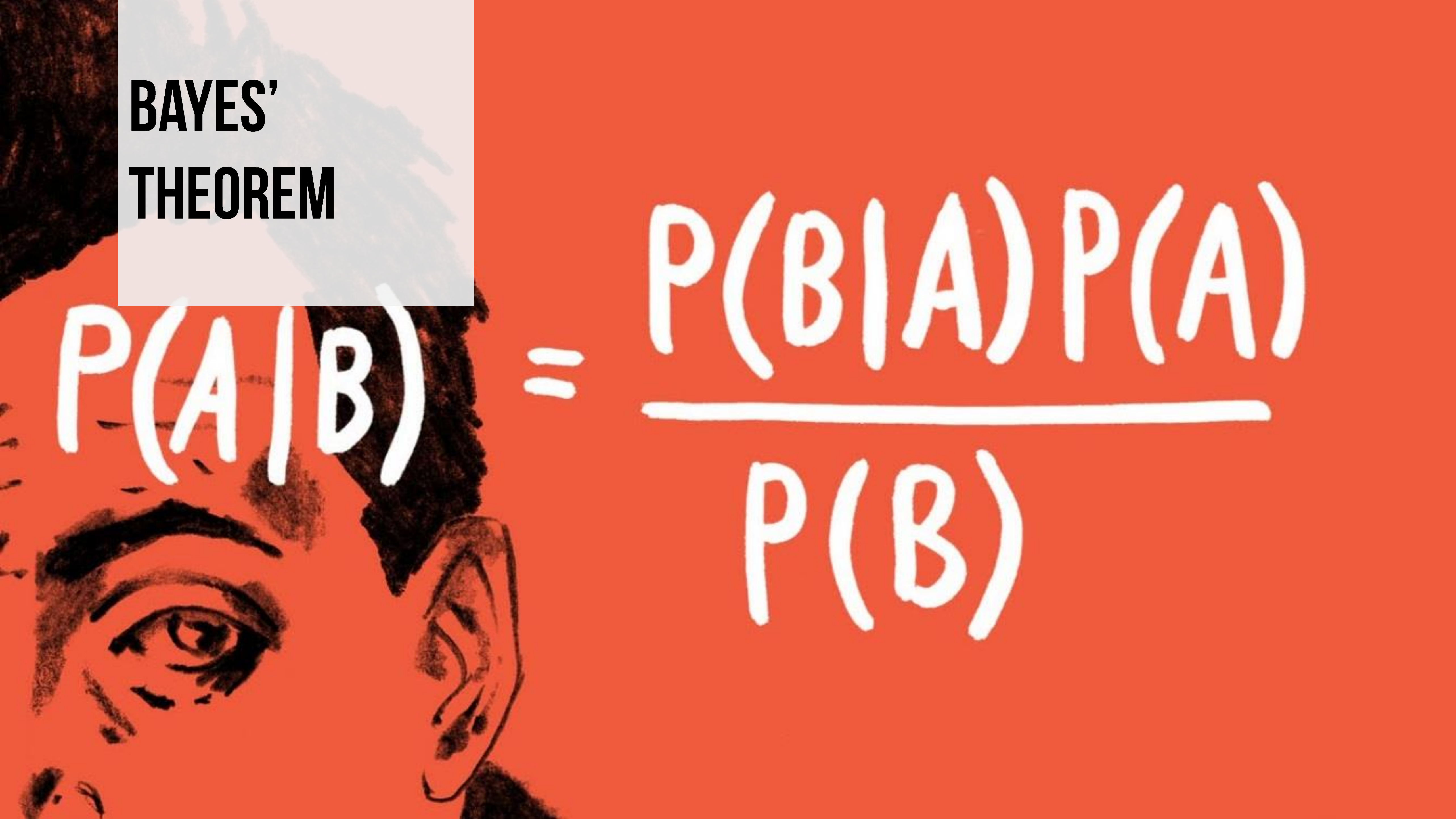
Conditional Probability

$p(A|B)$ – 'p of A given B' – event B is fixed, not uncertainty

$p(A,B) = p(A|B)\,p(B)$

- e.g., $p(\text{raining, cold}) = p(\text{raining}|\text{cold})\,p(\text{cold})$

BAYES' THEOREM



$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Bayes' theorem

$$p(A,B) = p(B,A)$$

$$p(A,B) = p(A|B)\,p(B)$$

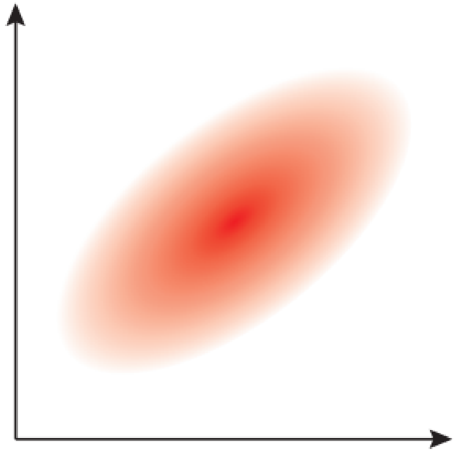
$$p(B,A) = p(B|A)\,p(A)$$

$$p(A|B)\,p(B) = p(B|A)\,p(A)$$

$$\boxed{p(A \mid B) = \frac{p(B \mid A)\,p(A)}{p(B)}}$$

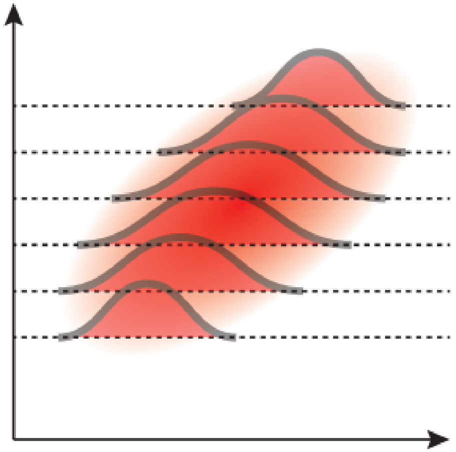
$$p(A \mid B) = \frac{p(B \mid A)\,p(A)}{\sum_{A^*} p(B \mid A^*)\,p(A^*)}$$

A^* is a variable that takes on all possible values joint distribution



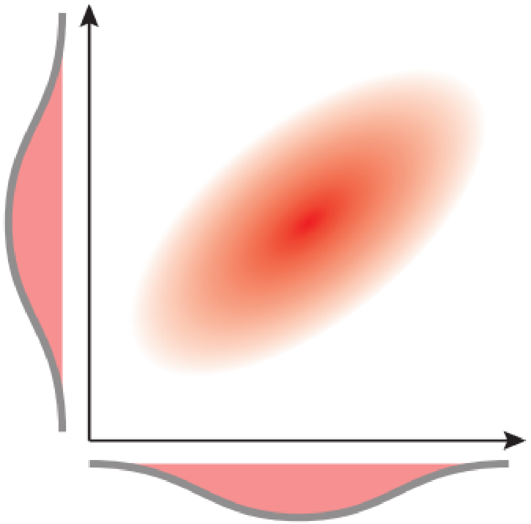
The “co-distribution” of x and y.

conditional distribution



The probability distribution of x, given that we know the value of y.

mariginal distribution



The density of x- (or y-) values, without knowing the other’s value.

Credit: [Eric Ma]

First Example

disease

X

symptoms

Y

		0	1
Y	0	0.5	0.1
	1	0.1	0.3

Joint probability : $P(X = 0, Y = 1) = 0.1$

$$\sum_{x,y} P(X = x, Y = y) = 1$$

Marginal probability :

$$P(Y = 1) = 0.1 + 0.3 = 0.4$$

$$P(X = 0) = 0.1 + 0.5 = 0.6$$

$$P(X = x) = \sum_{y} P(X = x, Y = y)$$

Second Example

Row	Column			Marginal
	...	c	...	
$\vdots$		$\vdots$		
r	$\cdots$	$p(r, c) = p(r\|c)\, p(c)$	$\cdots$	$p(r) = \sum_{c^*} p(r\|c^*)\, p(c^*)$
$\vdots$		$\vdots$		
Marginal		$p(c)$		

Kruschke (2015)

Second Example

	Hair color				
Eye color	**Black**	**Brunette**	**Red**	**Blond**	**Marginal (Eye color)**
Brown	0.11	0.20	0.04	0.01	0.37
Blue	0.03	0.14	0.03	0.16	0.36
Hazel	0.03	0.09	0.02	0.02	0.16
Green	0.01	0.05	0.02	0.03	0.11
Marginal (hair color)	0.18	0.48	0.12	0.21	1.0

$$p(A \mid B) = \frac{p(B \mid A)\, p(A)}{\sum_{A^*} p(B \mid A^*)\, p(A^*)}$$

Kruschke (2015)

Exercise IV

Suppose that in the general population, the probability of having a rare disease is 1/1000. We denote the true presence or absence of the disease as the value of a parameter, ϑ, that can have the value ϑ = ☹ if disease is present in a person, or the value ϑ = ☺ if the disease is absent. The base rate of the disease is therefore denoted $p(\vartheta =$ ☹ $) = 0.001$.

Suppose(1): a test for the disease that has a 99% hit rate: $p(T = + | \vartheta =$ ☹ $) = 0.99$

Suppose(2): the test has a false alarm rate of 5%: $p(T = + | \vartheta =$ ☺ $) = 0.05$

Q: Suppose we sample a person at random from the population, administer the test, and it comes up positive. What is the posterior probability that the person has the disease?

Kruschke (2015)

Exercise VI

Q: What is the posterior probability that the person has the disease?

→ $p(\vartheta = ☹ \mid T = +)$

Kruschke (2015)

Exercise VI

Test result	Disease		Marginal (test result)
	$\theta = \ddot\frown$ (present)	$\theta = \ddot\smile$ (absent)	
$T = +$	$p(+\|\ddot\frown)\,p(\ddot\frown)$ $= 0.99 \cdot 0.001$	$p(+\|\ddot\smile)\,p(\ddot\smile)$ $= 0.05 \cdot (1 - 0.001)$	$\sum_\theta p(+\|\theta)\,p(\theta)$
$T = -$	$p(-\|\ddot\frown)\,p(\ddot\frown)$ $= (1 - 0.99) \cdot 0.001$	$p(-\|\ddot\smile)\,p(\ddot\smile)$ $= (1 - 0.05) \cdot (1 - 0.001)$	$\sum_\theta p(-\|\theta)\,p(\theta)$
Marginal (disease)	$p(\ddot\frown) = 0.001$	$p(\ddot\smile) = 1 - 0.001$	1.0

$$
\begin{aligned}
p(\theta = \ddot\frown \,|\, T = +) &= \frac{p(T = + | \theta = \ddot\frown)\, p(\theta = \ddot\frown)}{\sum_\theta p(T = + | \theta)\, p(\theta)} \\
&= \frac{0.99 \cdot 0.001}{0.99 \cdot 0.001 + 0.05 \cdot (1 - 0.001)} \\
&= 0.019
\end{aligned}
$$