Computational complexity of wPM-S

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May, 2022

In this document we study the computational complexity of the weighted Pressure Matching with Subband-Domain formulation (wPM-S) algorithm proposed in [1]. In particular, we focus on the number of Floating Point Operations (FLOPs) required to compute the subband filters. Other computational aspects, as memory allocation and memory displacements, are not considered in this document. The algorithm has been implemented using C language, and the FLOPs counts in this document are based on this specific implementation. The Basic Linear Algebra Subprograms (BLAS) [2] included in Intel MKL v.2017.0.31 [3] and the Linear Algebra Package (LAPACK) v.3.7.0 [4] are used for all algebra-related operations. For DFT computations, the Fastest Fourier Transform in the West (FFTW) v.3.3.5 [5] is used. We consider that one FLOP is one of the following operations in real arithmetic: addition, subtraction, multiplication, and division. Also, we consider that 2 FLOPs are required for complex addition or subtraction, and 6 FLOPs for complex multiplication or division. Furthermore, we assume that a N-point DFT with complex-valued input and output requires $5N\log_2 N$ FLOPs, while half the number of FLOPs are required when either the input or the output are real-valued [6]. The number of FLOPs required by the routines of BLAS/LAPACK can be found in [7]. Nonetheless, we consider that other procedures that only involve memory displacements, e.g., truncation or decimation of signals, do not require any FLOPs.

1 Formulation for wPM-S

Let us define L as the number of loudspeakers of the system, M as the number of control points, and h_{ml} as the Room Impulse Response (RIR) between the l-th loudspeaker and the m-th control point, which is modelled as a Finite Impulse Response (FIR) of length I_{h}. Also, let us consider a GDFT filter bank with K subbands, resampling factor $R \leq K$, and prototype filter $p(n)$. We can define the cascade impulse response in the m-th control point and the k-th subband as

$$x_{m,k}(n) = \sum_{l=0}^{L-1} h_{ml,k}(n) * g_{l,k}(n) = \sum_{l=0}^{L-1} \sum_{r=0}^{I_{\mathrm{g},k}-1} h_{ml,k}(n-r) g_{l,k}(r) \tag{1}$$

where $h_{ml,k}$ is the subband component of h_{ml} in the k-th subband, which is obtained using the subband decomposition propose in [1, 8]. Then, we can define a $M(I_{\mathrm{h},k}+I_{\mathrm{g},k}-1)\times 1$ vector with the cascade impulse responses in all time instants, all control points, and subband k as

$$\mathbf{x}_k = \mathbf{H}_k \, \mathbf{g}_k, \tag{2}$$

where $\mathbf{g}_k$ is a $LI_{\mathrm{g},k}\times 1$ vector of the filters for all loudspeakers, all time instants, and subband k, which is given by

$$\mathbf{g}_k = \left[\mathbf{g}_{0,k}^T \; \cdots \; \mathbf{g}_{I_{\mathrm{g},k}-1,k}^T\right]^T, \tag{3}$$

in which $\mathbf{g}_{n,k} = \left[g_{0,k}(n) \ \dots \ g_{L-1,k}(n)\right]^T$. Matrix $\mathbf{H}_k$ in (2) is a $M\,(I_{\mathrm{h},k}+I_{\mathrm{g},k}-1)\times LI_{\mathrm{g},k}$ block-toeplitz matrix containing the subband components of the RIRs in all time instants, all loudspeakers, all control points, and subband k, which is defined as

$$\mathbf{H}_k = \begin{bmatrix} \mathbf{H}_{0,k}^T & \dots & \mathbf{H}_{I_{\mathrm{h},k}-1,k}^T & \mathbf{0}_{L\times M} & \dots & \mathbf{0}_{L\times M} \\ \mathbf{0}_{L\times M} & & & & & \\ \vdots & & & \mathit{Toeplitz} & & \\ \mathbf{0}_{L\times M} & & & & & \end{bmatrix}^T, \tag{4}$$

where

$$\mathbf{H}_{n,k} = \begin{bmatrix} h_{00,k}(n) & \dots & h_{0(L-1),k}(n) \\ \vdots & \ddots & \vdots \\ h_{(M-1)0,k}(n) & \dots & h_{(M-1)(L-1),k}(n) \end{bmatrix}^T. \tag{5}$$

Now, let us define $d_{m,k}$ as the target response of length $I_{\mathrm{d},k} = I_{\mathrm{h},k}+I_{\mathrm{g},k}-1$ for the m-th control point and the k-th subband. Also, let us define a $MI_{\mathrm{d},k}\times 1$ vector containing the target for all time instants, all control points, and subband k as

$$\mathbf{d}_k = \left[\mathbf{d}_{0,k}^T \quad \dots \quad \mathbf{d}_{I_{\mathrm{d},k}-1,k}^T\right]^T, \tag{6}$$

where $\mathbf{d}_{n,k} = \left[d_{0,k}(n) \quad \dots \quad d_{M-1,k}(n)\right]^T$. Using the formulation previously described, the optimal filters in the k-th subband for wPM-S [1] are given by

$$\mathbf{g}_{\mathrm{opt},k} = \arg\min_{\mathbf{g}_k} \left\{ \left\|\mathbf{H}_k\mathbf{g}_k - \mathbf{d}_k\right\|^2 + \beta_k \left\|\mathbf{g}_k\right\|^2 \right\}, \tag{7}$$

where $\beta_k > 0$ is the regularization factor for the k-th subband. The optimal solution for (7) is given by [1]

$$\mathbf{g}_{\mathrm{opt},k} = \left(\mathbf{H}_k^H\mathbf{H}_k + \beta_k\mathbf{I}_{LI_{\mathrm{g},k}}\right)^{-1}\mathbf{H}_k^H\mathbf{d}_k. \tag{8}$$

2 Exact solver

Now, we focus on the computational complexity required to compute the subband filters of length $I_{\mathrm{g},k}$ for wPM-S using a exact solver based on the Cholesky decomposition, assuming that the subband components $h_{ml,k}$ and $d_{m,k}$ are already known. The subband filters in the k-th subband for wPM-S are computed as

$$\mathbf{g}_{\mathrm{opt},k} = \left(\mathbf{H}_k^H\mathbf{H}_k + \beta_k\mathbf{I}_{LI_{\mathrm{g},k}}\right)^{-1}\mathbf{H}_k^H\mathbf{d}_k, \tag{9}$$

or alternatively as

$$\mathbf{g}_{\mathrm{opt},k} = \mathbf{R}_k^{-1}\mathbf{r}_k, \tag{10}$$

where $\mathbf{R}_k = \mathbf{H}_k^H\mathbf{H}_k + \beta_k\mathbf{I}_{LI_{\mathrm{g},k}}$ and $\mathbf{r}_k = \mathbf{H}_k^H\mathbf{d}_k$. Since $\mathbf{R}_k$ and $\mathbf{r}_k$ are a correlation matrix and a correlation vector, respectively, they can be computed using Fast Fourier Transforms (FFTs) of size $N = I_{\mathrm{d},k}$. Now, using the Cholesky decomposition [9], we can express matrix $\mathbf{R}_k$ as

$$\mathbf{R}_k = \mathbf{L}_k\mathbf{L}_k^H \tag{11}$$

where $\mathbf{L}_k$ is a lower-triangular matrix. Then, we can write the optimal solution as

$$\mathbf{g}_{\mathrm{opt},k} = \left(\mathbf{L}_k\mathbf{L}_k^H\right)^{-1}\mathbf{r}_k = \left(\mathbf{L}_k^H\right)^{-1}\mathbf{L}_k^{-1}\mathbf{r}_k. \tag{12}$$

Operation		**Routine**	**Calls**	**FLOPs/call**
Comp. $\mathbf{R}_k$	DFT of $h_{ml,k} \rightarrow \bar{\mathbf{H}}_{f,k}$	FFTW/`dft`	$\frac{1}{2}KML$	$5N\log_2 N$
	$\bar{\mathbf{H}}_{f,k}^H \bar{\mathbf{H}}_{f,k} + \beta \mathbf{I}_L \rightarrow \bar{\mathbf{R}}_{f,k}$	BLAS/`zdotc`	$\frac{1}{4}KNL(L+1)$	$8M$
	IDFT of $\bar{\mathbf{R}}_{f,k} \rightarrow \mathbf{R}_{\mathrm{e},k}$	FFTW/`dft`	$\frac{KL(L+1)}{4}$	$5N\log_2 N$
	$\left[\mathbf{R}_{\mathrm{e},k}\right]_{\left(0:LI_{\mathrm{g},k}-1,0:LI_{\mathrm{g},k}-1\right)}$ $\downarrow$ $\mathbf{R}_k$	-	-	-
Comp. $\mathbf{r}_k$	DFT of $d_{m,k} \rightarrow \bar{\mathbf{d}}_{f,k}$	FFTW/`dft`	$\frac{1}{2}KM$	$5N\log_2 N$
	$\bar{\mathbf{H}}_{f,k}^H \bar{\mathbf{d}}_{f,k} \rightarrow \bar{\mathbf{r}}_{f,k}$	BLAS/`zdotc`	$\frac{1}{2}KNL$	$8M$
	IDFT of $\bar{\mathbf{r}}_{f,k} \rightarrow \mathbf{r}_{\mathrm{e},k}$	FFTW/`dft`	$\frac{1}{2}KL$	$5N\log_2 N$
	$\left[\mathbf{r}_{\mathrm{e},k}\right]_{0:LI_{\mathrm{g},k}-1} \rightarrow \mathbf{r}_k$	-	-	-
Solve with Chol. dec. $\mathbf{R}_k \setminus \mathbf{r}_k \rightarrow \mathbf{g}_{\mathrm{opt},k}$		LAPACK/`zpotrf`	$\frac{1}{2}K$	$\frac{4}{3}(LI_{\mathrm{g},k})^3+$ $3(LI_{\mathrm{g},k})^2+$ $\frac{5}{3}(LI_{\mathrm{g},k})$
		LAPACK/`zpotrs`	$\frac{1}{2}K$	$8\,(LI_{\mathrm{g}})^2$

Table 1: Operations required to compute the subband filters for wPM-S using an exact solver based on the Cholesky decomposition and considering a system with L loudspeakers, M control points, subband filters of length $I_{\mathrm{g},k}$, and targets of length $I_{\mathrm{d},k} = I_{\mathrm{g},k}+I_{\mathrm{h},k}-1$. The size for the DFTs/IDFTs is $N = I_{\mathrm{d},k}$. We assume that $h_{ml,k}$ and $d_{m,k}$ are known.

Making use of the previous expression, and by defining the auxiliary vector $\mathbf{y}_k = \mathbf{L}_k^{-1}\mathbf{r}_k$, the optimal solution $\mathbf{g}_{\mathrm{opt},k}$ can be computed by solving the two following systems of linear equations:

$$\mathbf{L}_k \mathbf{y}_k = \mathbf{r}_k, \tag{13}$$

$$\mathbf{L}_k^H \mathbf{g}_{\mathrm{opt},k} = \mathbf{y}_k. \tag{14}$$

The previous systems can be solved using forward and backward substitution [9], respectively, because $\mathbf{L}_k$ is a lower-triangular matrix.

We summarize in Table 1 the steps required to compute the optimal subband filters for wPM-S using the exact solver based on the Cholesky decomposition. For each step, we show the associated routine, the number of calls to the routine, and the number of FLOPs required to execute each routine. The DFTs/IDFTs in Table 1 are of size $N = I_{\mathrm{d},k}$. Also, it is important to highlight that the subband filters are only computed for $K/2$ subbands due to the hermitian symmetry of the GDFT filter bank. Let us point out that the names of the routines of FFTW are shortened in Table 1 for the sake of simplicity, e.g., `fftw_execute_dft` is presented as `dft`. This naming criteria is used from now on. From Table 1, the number of FLOPs required by the exact solver to compute the filters for wPM-S is:

$$N_{\mathrm{flop}} = \sum_{k=0}^{K/2-1} \left(\left[\frac{4}{3}L^3\right] I_{\mathrm{g},k}^3 + \left[11L^2\right] I_{\mathrm{g},k}^2 + \left[5\left(ML+M+\frac{1}{2}L^2+\frac{3}{2}L\right)\right] I_{\mathrm{d},k}\log_2 I_{\mathrm{d},k} + \left[4ML^2+12ML\right] I_{\mathrm{d},k} + \left[\frac{5}{3}L\right] I_{\mathrm{g},k} \right). \tag{15}$$

3 Superfast solver

Now, we study the computational complexity required to compute the subband filters of length $I_{\mathrm{g},k}$ for wPM-S using the superfast solver proposed in [10]. Let us denote $D_{m,k}(f)$ and $H_{ml,k}(f)$ as the Fourier Transforms of $d_{m,k}$ and $h_{ml,k}$, respectively. Also, let us define $\bar{\mathbf{d}}_{f,k}$ as the $M\times 1$ vector of $D_{ml,k}(f)$ for all control points and $\bar{\mathbf{H}}_{f,k}$ as the $M\times L$ matrix of $H_{ml,k}(f)$ for all loudspeakers and all control points, i.e.,

$$\bar{\mathbf{d}}_{f,k} = \begin{bmatrix} D_{0,k}(f) & \dots & D_{(M-1),k}(f) \end{bmatrix}^T, \tag{16}$$

$$\bar{\mathbf{H}}_{f,k} = \begin{bmatrix} H_{00,k}(f) & \dots & H_{0(L-1),k}(f) \\ \vdots & \ddots & \vdots \\ H_{(M-1)0,k}(f) & \dots & H_{(M-1)(L-1),k}(f) \end{bmatrix}. \tag{17}$$

Furthermore, let us define $\bar{\mathbf{d}}_k$ as a $MI_{\mathrm{d},k}\times 1$ vector containing $\bar{\mathbf{d}}_{f,k}$ for a set of $I_{\mathrm{d},k}$ control frequencies and $\bar{\mathbf{H}}_k$ as a $MI_{\mathrm{d},k}\times LI_{\mathrm{d},k}$ block-diagonal matrix containing $\bar{\mathbf{H}}_{f,k}$ for the same set of $I_{\mathrm{d},k}$ control frequencies, i.e.,

$$\bar{\mathbf{d}}_k = \begin{bmatrix} \bar{\mathbf{d}}^T_{f_0,k} & \dots & \bar{\mathbf{d}}^T_{f_{I_{\mathrm{d},k}-1},k} \end{bmatrix}^T, \tag{18}$$

$$\bar{\mathbf{H}}_k = \mathrm{diag}\left\{ \bar{\mathbf{H}}_{f_0,k} \quad \dots \quad \bar{\mathbf{H}}_{f_{I_{\mathrm{d},k}-1},k} \right\}, \tag{19}$$

where f_v is the frequency for the v-th frequency bin of a $I_{\mathrm{d},k}$-point Discrete Fourier Transform (DFT). Also, let us define $\bar{\mathbf{H}}^{\dagger}_k$ as the regularized right-pseudoinverse of $\bar{\mathbf{H}}_k$, i.e.,

$$\bar{\mathbf{H}}^{\dagger}_k = \bar{\mathbf{H}}^H_k \left(\bar{\mathbf{H}}_k \bar{\mathbf{H}}^H_k + \beta_k \, \mathbf{I}_{MI_{\mathrm{d},k}} \right)^{-1}, \tag{20}$$

which is a block-diagonal matrix formed by diagonal blocks of size $L\times M$. Moreover, let us define $\mathbf{F}_k$ as the $LI_{\mathrm{d},k}\times LI_{\mathrm{d},k}$ matrix used to compute the $I_{\mathrm{d},k}$-point DFT of a set of L impulse responses, which is defined as

$$\mathbf{F}_k = \begin{bmatrix} \Phi^{(0,0)}_{I_{\mathrm{d},k}} \, \mathbf{I}_L & \dots & \Phi^{(0,I_{\mathrm{d},k}-1)}_{I_{\mathrm{d},k}} \, \mathbf{I}_L \\ \vdots & \ddots & \vdots \\ \Phi^{(I_{\mathrm{d},k}-1,0)}_{I_{\mathrm{d},k}} \, \mathbf{I}_L & \dots & \Phi^{(I_{\mathrm{d},k}-1,I_{\mathrm{d},k}-1)}_{I_{\mathrm{d},k}} \, \mathbf{I}_L \end{bmatrix}, \tag{21}$$

where $\Phi^{(k,n)}_{I_{\mathrm{d},k}} = \frac{1}{\sqrt{I_{\mathrm{d},k}}} e^{-j\frac{2\pi kn}{I_{\mathrm{d},k}}}$. Also, let us define $\mathbf{T}_k$ as a $LI_{\mathrm{g},k}\times LI_{\mathrm{d},k}$ matrix used to truncate a set of L impulse responses of length $I_{\mathrm{d},k}$ to a length $I_{\mathrm{g},k}$, i.e.,

$$\mathbf{T}_k = \begin{bmatrix} \mathbf{I}_{LI_{\mathrm{g},k}} & \mathbf{0}_{LI_{\mathrm{g},k}\times L(I_{\mathrm{d},k}-I_{\mathrm{g},k})} \end{bmatrix}. \tag{22}$$

Finally, let us define $\mathbf{\Psi}_k$ as a $LI_{\mathrm{d},k}\times LI_{\mathrm{d},k}$ matrix used to set to 0 the first $I_{\mathrm{g},k}$ samples of a set of L impulse responses of length $I_{\mathrm{d},k}$, i.e.,

$$\mathbf{\Psi}_k = \begin{bmatrix} \mathbf{0}_{LI_{\mathrm{g},k}\times LI_{\mathrm{g},k}} & \mathbf{0}_{LI_{\mathrm{g},k}\times L(I_{\mathrm{d},k}-I_{\mathrm{g},k})} \\ \mathbf{0}_{L(I_{\mathrm{d},k}-I_{\mathrm{g},k})\times LI_{\mathrm{g},k}} & \mathbf{I}_{L(I_{\mathrm{d},k}-I_{\mathrm{g},k})} \end{bmatrix}. \tag{23}$$

The authors in [10] showed that the P_k-order approximation of the optimal subband filters can be computed with the superfast solver as

$$\mathbf{g}_{\mathrm{ap},k} = \mathbf{g}_{\mathrm{fd},k} + \mathbf{T}_k \mathbf{F}^H_k \bar{\mathbf{H}}^{\dagger}_k \bar{\mathbf{H}}_k \left(\sum_{p=0}^{P_k} \left[\mathbf{F}_k \mathbf{\Psi}_k \, \mathbf{F}^H_k \bar{\mathbf{H}}^{\dagger}_k \bar{\mathbf{H}}_k \right]^p \right) \mathbf{F}_k \mathbf{\Psi}_k \, \mathbf{F}^H_k \bar{\mathbf{H}}^{\dagger}_k \bar{\mathbf{d}}_k, \tag{24}$$

where $\mathbf{g}_{\text{fd},k}$ is a $LI_{\text{g},k}\times 1$ vector defined as

$$\mathbf{g}_{\text{fd},k} = \mathbf{T}_k \mathbf{F}_k^H \bar{\mathbf{H}}_k^\dagger \bar{\mathbf{d}}_k \ . \tag{25}$$

The expression (24) can be alternatively expressed as

$$\mathbf{g}_{\text{ap},k} = \mathbf{g}_{\text{fd},k} + \sum_{p=0}^{P_k} \mathbf{T}_k \mathbf{F}_k^H \bar{\mathbf{\Delta}}_k \mathbf{F}_k \mathbf{r}_{p,k}, \tag{26}$$

where

$$\mathbf{r}_{p,k} = \begin{cases} \mathbf{\Psi}_k \mathbf{F}_k^H \bar{\mathbf{H}}_k^\dagger \bar{\mathbf{d}}_k & \text{for } p = 0 \\ \mathbf{\Psi}_k \mathbf{F}_k^H \bar{\mathbf{\Delta}}_k \mathbf{F}_k \mathbf{r}_{p-1,k} & \text{otherwise} \end{cases}, \tag{27}$$

which is the p-th corretion term that can be computed recursively.

We summarize in Table 2 the steps required to compute the P_k-th order approximation of the subband filters for wPM-S using the superfast solver. The DFTs/IDFTs in Table 1 are of size $N = I_{\text{d},k}$.

	Operation	Routine	Calls	FLOPs/call
Comp. $\mathbf{g}_{\text{fd}}$	DFT of $h_{ml,k} \to \bar{\mathbf{H}}_{f,k}$	FFTW/`dft`	ML	$5N\log_2 N$
	DFT of $d_{m,k} \to \bar{\mathbf{d}}_{f,k}$	FFTW/`dft`	M	$5N\log_2 N$
	$\bar{\mathbf{H}}_{f,k}^H \bar{\mathbf{H}}_{f,k} + \beta \mathbf{I}_L \to \bar{\mathbf{R}}_{f,k}$	BLAS/`zherk`	N	$4ML(L+1)$
	Solve with Chol. dec. $\bar{\mathbf{R}}_{f,k} \setminus \bar{\mathbf{H}}_{f,k}^H \to \bar{\mathbf{H}}_{f,k}^\dagger$	LAPACK/`zpotrf`	N	$\frac{4}{3}L^3 + 3L^2 + \frac{5}{3}L$
		LAPACK/`zpotrs`	N	$8ML^2$
	$\bar{\mathbf{H}}_{f,k}^\dagger \bar{\mathbf{H}}_{f,k} \to \bar{\mathbf{\Lambda}}_{f,k}$	BLAS/`zgemm`	N	$8ML^2$
	Form $\bar{\mathbf{\Lambda}}_k$ using $\bar{\mathbf{\Lambda}}_{f,k}$	-	-	-
	$\bar{\mathbf{H}}_{f,k}^\dagger \bar{\mathbf{d}}_{f,k} \to \bar{\mathbf{q}}_{\text{opt},f,k}$	BLAS/`zgemv`	N	$8ML$
	Form $\bar{\mathbf{q}}_{\text{opt},k}$ using $\bar{\mathbf{q}}_{\text{opt},f,k}$	-	-	-
	$\mathbf{F}_k^H \bar{\mathbf{q}}_{\text{opt},k} \to \mathbf{g}_{\text{fd},k}$	FFTW/`dft`	L	$5N\log_2 N$
	$\mathbf{\Psi}_k \mathbf{g}_{\text{fd},k} \to \mathbf{r}_{0,k}$	-	-	-
	$\mathbf{T}_k \mathbf{g}_{\text{fd},k} \to \mathbf{g}_{\text{fd},k}$	-	-	-
$p = 0, \ldots, P_k$	$\mathbf{F}_k \mathbf{r}_{p,k} \to \bar{\mathbf{r}}_{p,k}$	FFTW/`dft`	$(P_k+1)L$	$5N\log_2 N$
	$\bar{\mathbf{\Lambda}}_k \bar{\mathbf{r}}_{p,k} \to \bar{\mathbf{r}}'_{p,k}$	BLAS/`zgemv`	$(P_k+1)N$	$8L^2$
	$\mathbf{F}_k^H \bar{\mathbf{r}}'_{p,k} \to \mathbf{r}_{p+1,k}$	FFTW/`dft`	$(P_k+1)L$	$5N\log_2 N$
	$\mathbf{g}_{\text{fd},k} + \mathbf{T}_k \mathbf{r}_{p+1,k} \to \mathbf{g}_{\text{fd},k}$	C/`+`	$(P_k+1)LI_{\text{g},k}$	2
	$\mathbf{\Psi}_k \mathbf{r}_{p+1,k} \to \mathbf{r}_{p+1,k}$	-	-	-
	$\mathbf{g}_{\text{fd},k} \to \mathbf{g}_{\text{ap},k}$	-	-	-

Table 2: Operations required by the superfast solver [10] to compute the P_k-th order approximation of the filters for wPM-S, considering a system with L loudspeakers, M control points, subband filters of length $I_{\text{g},k}$, and targets of length $I_{\text{d},k} = I_{\text{g},k} + I_{\text{h},k} - 1$. The size of the DFTs/IDFTs is $N = I_{\text{d},k}$. We assume that $h_{ml,k}$ and $d_{m,k}$ are known.

From Table 2, the number of FLOPs for the exact solver is:

$$N_{\text{flop}} = \sum_{k=0}^{K/2-1} \left(\left[10L\right] P_k I_{\text{d},k} \log_2 I_{\text{d},k} + \left[8L^2\right] P_k I_{\text{d},k} + \left[2L\right] P_k I_{\text{g},k} + \left[2L\right] I_{\text{g},k} + \right.$$
$$\left. \left[5ML+5M+15L\right] I_{\text{d},k} \log_2 I_{\text{d},k} + \left[\frac{4}{3}L^3+11L^2+\frac{5}{3}L+20ML^2+12ML\right] I_{\text{d},k} \right). \tag{28}$$

4 Subband decomposition

To compute the subband filters for wPM-S, we previously need to compute the subband components $h_{ml,k}$ and $d_{m,k}$ of h_{ml} and d_m, respectively. Then, we discuss in this section the computational complexity required to compute the subband decomposition of a generic signal $a(n)$ of length I_{a}. In particular, we focus on the method proposed in [1], which is a computationally efficient version of the method proposed in [8]. We consider a GDFT filter bank with K subbands, resampling factor R, and prototype filter p of length I_{p}. Given a generic signal a of length I_{a}, we assume that $\widetilde{a}_k$ is the $I_{\widetilde{\text{a}},k}$-length signal at the output of the k-th subband of the analysis filter bank when is fed with a, where

$$I_{\widetilde{\text{a}},k} = \left\lceil \left(I_{\text{a}}+I_{\text{p}}-1\right)/R \right\rceil, \tag{29}$$

and that a_k is the $I_{\text{a},k}$-length subband component of a in the k-th subband, where

$$I_{\text{a},k} = I_{\widetilde{\text{a}},k} - \left\lceil I_{\text{p}}/R \right\rceil + 1 = \left\lceil \left(I_{\text{a}}+I_{\text{p}}-1\right)/R \right\rceil - \left\lceil I_{\text{p}}/R \right\rceil + 1. \tag{30}$$

It is shown in [1] that the subband components a_k of a are given by

$$\mathbf{a}_{\text{opt},k} = \mathbf{E}_k^H \left(\mathbf{P}^T\mathbf{P}\right)^{-1} \mathbf{P}^T \mathbf{W}_k^H \widetilde{\mathbf{a}}_k, \tag{31}$$

where

$$\mathbf{W}_k = \text{diag}\left(\left[e^{j\frac{2\pi R}{K}\left(k+\frac{1}{2}\right)0}, \ldots, e^{j\frac{2\pi R}{K}\left(k+\frac{1}{2}\right)\left(I_{\widetilde{\text{a}},k}-1\right)}\right]\right), \tag{32}$$

$$\mathbf{E}_k = \text{diag}\left(\left[e^{-j\frac{2\pi R}{K}\left(k+\frac{1}{2}\right)0}, \ldots, e^{-j\frac{2\pi R}{K}\left(k+\frac{1}{2}\right)\left(I_{\text{a},k}-1\right)}\right]\right). \tag{33}$$

$$\mathbf{P} = \begin{bmatrix} p_{\downarrow R}(0) & \ldots & p_{\downarrow R}(I_{\widetilde{\delta},k}-1) & 0 & \ldots & 0 \\ 0 & & & & & \\ \vdots & & \mathit{Toeplitz} & & & \\ 0 & & & & & \end{bmatrix}^T, \tag{34}$$

In the previous expression, $p_{\downarrow R}(n)$ is the downsampled prototype filter, i.e, $p_{\downarrow R}(n) = p(nR)$, which is a response of length $I_{\widetilde{\delta},k} = \lceil I_{\text{p}}/R \rceil$. Now, let us define

$$\mathbf{a}_{\text{opt},k}^{\phi} = \mathbf{E}_k \mathbf{a}_{\text{opt},k}, \tag{35}$$

$$\widetilde{\mathbf{a}}_k^{\phi} = \mathbf{W}_k^H \widetilde{\mathbf{a}}_k, \tag{36}$$

which are phase-shifted versions of $\mathbf{a}_{\text{opt},k}$ and $\widetilde{\mathbf{a}}_k$, respectively. Then, instead of directly computing $\mathbf{a}_{\text{opt},k}$, we can compute $\mathbf{a}_{\text{opt},k}^{\phi}$ as

$$\mathbf{a}_{\text{opt},k}^{\phi} = \left(\mathbf{P}^T\mathbf{P}\right)^{-1} \mathbf{P}^T \widetilde{\mathbf{a}}_k^{\phi}, \tag{37}$$

Operation		Routine	Calls	FLOPs/call
Comp. $\mathbf{R}_{\mathrm{p}}$	$p \to p_{\downarrow R}$	-	-	-
	DFT of $p_{\downarrow R} \to P_{\downarrow R}(f)$	FFTW/`dft_r2c`	1	$\frac{5}{2}N\log_2 N$
	$P^*_{\downarrow R}(f)P_{\downarrow R}(f) \to R_{\mathrm{p}}(f)$	C/`*`	$\frac{1}{2}N$	6
	IDFT of $R_{\mathrm{p}}(f) \to \mathbf{R}_{\mathrm{e,p}}$	FFTW/`dft_c2r`	1	$\frac{5}{2}N\log_2 N$
	$\left[\mathbf{R}_{\mathrm{e,p}}\right]_{(0:I_{\mathrm{a},k}-1,0:I_{\mathrm{a},k}-1)} \downarrow \mathbf{R}_{\mathrm{p}}$	-	-	-
Comp. $\widetilde{\mathbf{c}}^{\phi}_k$	$a \to \widetilde{a}_k$	`polyphase` [11]	Q	$(I_{\mathrm{a}}/R)\left[2I_{\mathrm{p}}+ 2K+ 5K\log_2 K\right]$
	Phase shift $\widetilde{a}_k \to \widetilde{a}^{\phi}_k$	C/`*`	$\frac{1}{2}KNQ$	6
	DFT of $\widetilde{a}^{\phi}_k \to \widetilde{A}^{\phi}_k(f)$	FFTW/`dft`	$\frac{1}{2}KQ$	$5N\log_2 N$
	$P^*_{\downarrow R}(f)\widetilde{A}^{\phi}_k(f) \to \widetilde{C}^{\phi}_k(f)$	C/`*`	$\frac{1}{2}KNQ$	6
	IDFT of $\widetilde{C}^{\phi}_k(f) \to \widetilde{\mathbf{c}}^{\phi}_{\mathrm{e},k}$	FFTW/`dft`	$\frac{1}{2}KQ$	$5N\log_2 N$
	$\left[\widetilde{\mathbf{c}}^{\phi}_{\mathrm{e},k}\right]_{0:I_{\mathrm{a},k}-1} \to \widetilde{\mathbf{c}}^{\phi}_k$	-	-	-
Chol. dec. $\mathbf{R}_{\mathrm{p}} \to \mathbf{L}_{\mathrm{p}}$		LAPACK/`dpotrf`	1	$\frac{1}{3}I^3_{\mathrm{a},k}+ \frac{1}{2}I^2_{\mathrm{a},k}+\frac{1}{6}I_{\mathrm{a},k}$
Solve with Chol. dec. $\mathbf{R}_{\mathrm{p}} \setminus \Re\left\{\widetilde{\mathbf{c}}^{\phi}_k\right\} \to \Re\left\{\widetilde{\mathbf{a}}^{\phi}_{\mathrm{opt},k}\right\}$		LAPACK/`dpotrs`	$\frac{1}{2}KQ$	$2I^2_{\mathrm{a},k}$
Solve with Chol. dec. $\mathbf{R}_{\mathrm{p}} \setminus \Im\left\{\widetilde{\mathbf{c}}^{\phi}_k\right\} \to \Im\left\{\widetilde{\mathbf{a}}^{\phi}_{\mathrm{opt},k}\right\}$		LAPACK/`dpotrs`	$\frac{1}{2}KQ$	$2I^2_{\mathrm{a},k}$
$\mathbf{E}^H_k\widetilde{\mathbf{a}}^{\phi}_{\mathrm{opt},k} \to \widetilde{\mathbf{a}}_{\mathrm{opt},k}$		C/`*`	$\frac{1}{2}KQI_{\mathrm{a},k}$	6

Table 3: Operations required to compute the subband components in all subbands for a set of Q signals a of length I_{a}, considering a GDFT filter bank with K subbands, resampling factor R, and prototype filter of length I_{p}. The size for the DFTs/IDFTs is $N = I_{\widetilde{\mathrm{a}},k}$.

and then, we can compute the optimal solution as $\mathbf{a}_{\mathrm{opt},k} = \mathbf{E}^H_k \mathbf{a}_{\mathrm{opt},k,\phi}$. Now, we can express (37) as

$$\mathbf{a}^{\phi}_{\mathrm{opt},k} = \mathbf{R}^{-1}_{\mathrm{p}}\widetilde{\mathbf{c}}^{\phi}_k, \tag{38}$$

in which $\mathbf{R}_{\mathrm{p}} = \mathbf{P}^T\mathbf{P}$, and $\widetilde{\mathbf{c}}^{\phi}_k = \mathbf{P}^T\widetilde{\mathbf{a}}^{\phi}_k$. Since $\mathbf{R}_{\mathrm{p}}$ is a real matrix, the real and imaginary components of $\mathbf{a}^{\phi}_{\mathrm{opt},k}$ can be independently computed by solving two real-valued systems of linear equations that involve matrix $\mathbf{R}_{\mathrm{p}}$ (which is common for all subbands). Moreover, since $\mathbf{R}_{\mathrm{p}}$ and $\widetilde{\mathbf{c}}^{\phi}_k$ are a correlation matrix and a correlation vector, respectively, they can be computed using FFTs of size $N = I_{\widetilde{\mathrm{a}},k}$.

We summarize in Table 3 the steps required to compute the optimal subband components of a set of Q signals a of equal length I_{a} using the approach previously described. Let us note that $\mathbf{R}_{\mathrm{p}}$ in (38) is common to all subbands and all signals. Then, $\mathbf{R}_{\mathrm{p}}$ and its Cholesky decomposition only need to be computed once for the K subbands and Q signals. Also, it is worth noting that the analysis signals $\widetilde{a}_k$ in Table 3 are computed with the polyphase implementation proposed by [11], which involves DFTs of size K. The other DFTs/IDFTs in Table 3 are of size $N = I_{\widetilde{\mathrm{a}},k}$. From Table 3, the number of FLOPs required to compute the subband components for the Q signals and K subbands is:

$$N_{\mathrm{flop}} = \left[\frac{1}{3}\right] I_{\mathrm{a},k}^{3} + \left[2QK+\frac{1}{2}\right] I_{\mathrm{a},k}^{2} + \left[5\left(QK+1\right)\right] I_{\widetilde{\mathrm{a}},k} \log_2 I_{\widetilde{\mathrm{a}},k} + \left[3QK+\frac{1}{6}\right] I_{\mathrm{a},k} + \left[\frac{Q}{R}\left(2I_{\mathrm{p}}+2K+5K\log_2 K\right)\right] I_{\mathrm{a}} + \left[6QK+3\right] I_{\widetilde{\mathrm{a}},k}. \tag{39}$$

5 Total computational complexity of wPM-S

The total computational complexity of wPM-S is the combination of the computational complexities required to compute the subband decomposition of h_{ml} and d_m, and the computational complexity required to compute the subband filters. The number of FLOPs required to compute the subband components of h_{ml} are obtained by setting $Q{=}ML$, $I_{\mathrm{a}}{=}I_{\mathrm{h}}$, $I_{\widetilde{\mathrm{a}},k}{=}I_{\widetilde{\mathrm{h}},k}$, and $I_{\mathrm{a},k}{=}I_{\mathrm{h},k}$ in (39), where $I_{\widetilde{\mathrm{h}},k}{=}\left\lceil\left(I_{\mathrm{h}}{+}I_{\mathrm{p}}{-}1\right)/R\right\rceil$ and $I_{\mathrm{h},k}{=}I_{\widetilde{\mathrm{h}},k}{-}\left\lceil I_{\mathrm{p}}/R\right\rceil{+}1$. Similarly, the number of FLOPs required to compute the subband components of d_m are obtained by setting $Q{=}M$, $I_{\mathrm{a}}{=}I_{\mathrm{d}}$, $I_{\widetilde{\mathrm{a}},k}{=}I_{\widetilde{\mathrm{d}},k}$, and $I_{\mathrm{a},k}{=}I_{\mathrm{d},k}$ in (39), where $I_{\widetilde{\mathrm{d}},k}{=}\left\lceil\left(I_{\mathrm{d}}{+}I_{\mathrm{p}}{-}1\right)/R\right\rceil$ and $I_{\mathrm{d},k}{=}I_{\widetilde{\mathrm{d}},k}{-}\left\lceil I_{\mathrm{p}}/R\right\rceil{+}1$. Finally, the number of FLOPs required to compute the subband filters is obtained with (15) for the exact solver and with (28) for the superfast solver.

Bibliography

[1] V. Molés-Cases, G. Piñero, M. de Diego, and A. Gonzalez, "Personal sound zones by subband filtering and time domain optimization," *IEEE/ACM Transactions on Audio, Speech, and Language Processing*, vol. 28, pp. 2684–2696, 2020.

[2] "BLAS (Basic Linear Algebra Subprograms)." [Online] Available: http://www.netlib.org/blas/. Accessed: 2022-04-07

[3] "Intel® oneAPI Math Kernel Library (MKL)." [Online] Available: https://www.intel.com/content/www/us/en/develop/documentation/get-started-with-mkl-for-dpcpp/top.html. Accessed: 2022-04-07

[4] "LAPACK (Linear Algebra PACKage)." [Online] Available: http://www.netlib.org/lapack/. Accessed: 2022-04-07

[5] "FFTW (Fastest Fourier Transform in the West)." [Online] Available: https://www.fftw.org/. Accessed: 2022-04-07

[6] M. Frigo and S. G. Johnson, "FFTW manual," Technical Report, 2020.

[7] E. Anderson, J. Dongarra, and S. Ostrouchov, "LAPACK Working Note 41 Installation Guide for LAPACK," Technical Report, 1994.

[8] J. P. Reilly, M. Wilbur, M. Seibert, and N. Ahmadvand, "The Complex Subband Decomposition and its Application to the Decimation of Large Adaptive Filtering Problems," *IEEE Transactions on Signal Processing*, vol. 50, no. 11, pp. 2730–2743, 2002.

[9] G. H. Golub and C. F. Van Loan, *Matrix Computations*. Baltimore: The Johns Hopkins University Press, 1996.

[10] M. A. Poletti and P. Teal, "A Superfast Toeplitz Matrix Inversion Method for Single- and Multi-Channel Inverse Filters and its Application to Room Equalization," *IEEE/ACM Transactions on Audio, Speech, and Language Processing*, vol. 29, pp. 3144–3157, 2021.

[11] S. Weiss, M. Harteneck, and R. W. Stewart, "On Implementation and Design of Filter Banks for Subband Adaptive Systems," in *Proceedings of the IEEE Workshop on Signal Processing Systems*, 1998.