Physcis

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Contents

What makes life easier: linearity, static, equilibrium, main term. What makes life hard: nonlinearities, feedback, fluctuation, perturbation, evolution.

Chapter 1

Math

1.1 Definition

Levi-Civita

ϵ_{ijk} is an antisymmetric tensor, which is defined as:

$$\epsilon_{123} = 1, \epsilon_{ijj} = 0, \epsilon_{ijk} = -\epsilon_{jik}$$

1.2 Formulas

1.2.1 Fourier transform

$$\delta^3(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}}$$

1.2.2 Integral

$$I = \int_{-\infty}^{\infty} dk \frac{e^{ikr} - e^{-ikr}}{k}$$

Note that the integrand does not blow up as $k \to 0$, so

$$I = \lim_{\delta \to 0} \left[\int_{-\infty}^{\infty} dk \frac{e^{ikr} - e^{-ikr}}{k + i\delta} \right]$$

So we have pole at $k = -i\delta$. For e^{ikr} we must close the contour up to get exponential decay at large k. This misses the pole, so this term gives zero. So

$$I = \int_{-\infty}^{\infty} dk \frac{-e^{-ikr}}{k + i\delta} = -(2\pi i)(-e^{-\delta r}) = 2\pi i e^{-i\delta r}$$

Useful Integrals

$\int_0^\infty e^{-\alpha x} x^n dx = \frac{n!}{\alpha^{n+1}}$

1.3 Functions

1.3.1 Interesting functions

Weierstrass function

$$f(x) = \sum_{n=0}^{\infty} a^n cos(b^n \pi x) \tag{1.1}$$

Continuous but not derivative everywhree.

1.3.2 Frequently used functions

Legendre Polynomials and spherical harmonics

$$Y_l^m(\theta, \phi) = N e^{im\phi} P_l^m(\cos\theta) \tag{1.2}$$

which satisfies

$$r^2 \nabla^2 Y_l^m(\theta, \phi) = -l(l+1) Y_l^m(\theta, \phi)$$

Standard convention :

$$P_l^{-m}(\cos\theta) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(\cos\theta),$$

In QM:

$$Y_l^m(\theta, \phi) = (-1)^m sqrt \frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} P_{lm}(\cos\theta) e^{im\phi}$$

Where P_{lm} are associated Legendre polynomials without the Condon-Shortley phase.

Though of different definition, all of them satisfy

$$Y_l^{m*}(\theta,\phi) = (-1)^m Y_l^{-m}(\theta,\phi)$$

Associate Legendre Polynomials:

1.3.3 Bessel Function

1.3.4 Differentive Equations

First Order Equation

$$\frac{dV}{dt} + \eta(t)V(t) = f(t)$$

so:

$$\frac{1}{I}\frac{d(I(t)V(t))}{dt} = \frac{dV(t)}{dt} + \frac{I'(t)}{I(t)}V(t)$$

if:

$$\frac{I'(t)}{I(t)} = \eta(t) \rightarrow I(t) = exp(\int^t dt'\eta(t'))$$

so:

$$\frac{d(I(t)V(t))}{Idt} = f(t)$$

Floquet theory

$$\ddot{x} + \omega_0^2(1+\mu cos(\nu t))x = 0 \tag{1.3}$$

Analogy to QM(Schrodinger Equation):

$$-\frac{\hbar^2}{2m}\psi'' + U(x)\psi = E\psi$$

where $U(x) = -U_0 cos\frac{2\pi x}{a}$. So let

$$k^2 = \frac{2mE}{\hbar^2}, u = \frac{U_0}{E}, \nu = \frac{2\pi}{a}$$

u is small, we will get

$$\psi'' + k^2\psi = -k^2 u cos(\nu x)\psi$$

1.3.5 Integrated functions

Γ function

$$\Gamma(z) \equiv \int_0^{+\infty} dte^{-t}t^{z-1} = 2\int_0^{+\infty} dxe^{-x^2}x^{2z-1}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin\pi z}$$

1.4 Group

1.4.1 Representation of Group

A **representation** of a group G on a vector space V over a field K is a group homomorphism from G to GL(V), the

1.5 Principles

1.5.1 Symmetry

For a theory, study the symmetry of its EOM, and then specify how physical quantities transform in order to preserve the symmetry.

Chapter 2

Classical Mechanics

Chapter 3

Electramegnetic Mechanics

3.1 Formulas

$$\vec{A}_{tr} = \vec{n} \times (\vec{n} \times \vec{A}) = \vec{A} - (\hat{n}\vec{A})\hat{n} = (\infty - n \cdot n^T)\vec{A}$$

let n being the column vector $n = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

3.2 Tensor

the stress tensor T^{ij} is the force per area. The force per volumn f^j i: the minus divergence of the stress tensor

$$f^j = -\partial_i T^{ij}$$

Conservation law:

$$\partial_t g^j + \partial_i T^{ij} = 0$$

where g^j is the momentum per volume. So $f^j = \partial_t g^j$, the force is the time derivative of the momentum. In electrostatic:

$$f^j = \rho E^j$$

So:

$$\rho E^j = \partial_i T_E^{ij}$$

To get

$$T_E^{ij} \equiv -E^i E^j + \frac{1}{2} E^2 \delta^{ij}$$

The force:

$$
\begin{aligned}
F^j &= \int_v d^3\vec{r}\rho(\vec{r})E^j = \int_v d^3\vec{\nabla}\cdot\vec{E}E^j \\
&= \int_v d^3(\nabla_i E^i)E^j \\
&= \int_v d^3[\nabla_i(E^iE^j) - E^i\nabla_i E^j] \\
&= \int_v d^3[\nabla_i(E^iE^j) - E^i\nabla_j E^i] \\
&= \int_v d^3[\nabla_i(E^iE^j) - \frac{1}{2}\nabla_j E^2] \\
&= \int_v d^3[\nabla_i(E^iE^j) - \frac{1}{2}\nabla_i E^2\delta^{ij}] \\
&= \int_v d^3\nabla_i[(E^iE^j) - \frac{1}{2}E^2\delta^{ij}] \\
&= \int_S dSn_iT_E^{ij} \\
&= -\int_S dSn_iT_E^{ij}
\end{aligned}
$$

For magnetism:

$$
T_B^{ij} = -\frac{1}{\mu}B^iB^j + \frac{1}{2\mu}B^2\delta^{ij}
$$

So

$$
f_{em}^j = -\partial_i T_E^{ij} - \partial_i T_B^{ij} - \frac{1}{c}(\partial_t D\times B)^j \quad = (\nabla\cdot D)E^j + ((\nabla\times\vec{H})\times\vec{B})^j - al_tD\times B)^j
$$

3.3 Maxwell Equation

$$
\begin{cases}
\nabla\cdot E = \dfrac{\rho}{\epsilon} \\
\nabla\times B = \dfrac{1}{c}\left(J + J_D + J_{ind}\right) \\
\nabla\cdot B = 0 \\
\nabla\times E = -\dfrac{1}{c}\partial_t B
\end{cases}
\longrightarrow
\begin{cases}
\nabla\cdot D = \rho \\
\nabla\times H = \dfrac{1}{c}J + \dfrac{1}{c}\partial_t D \\
\nabla\cdot B = 0 \\
\nabla\times E = -\dfrac{1}{c}\partial_t B
\end{cases}
\tag{3.1}
$$

Symmetry of Maxwell Eqn:

$$\begin{cases} \nabla \cdot \vec{E} = \dfrac{\rho}{\epsilon} \\ \nabla \times B - \dfrac{1}{c}\partial_t \vec{E} J = \dfrac{\vec{J}}{c} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} - \dfrac{1}{c}\partial_t \vec{B} = 0 \end{cases} \tag{3.2}$$

For $\rho = \vec{j} = 0$, then there is a symmetry between $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$.

$$\begin{matrix} \nabla \cdot \vec{E} = 0 & & \nabla \cdot \vec{B} = 0 \\ -\dfrac{1}{c}\partial_t \vec{E} + \nabla \times \vec{B} = 0 & \Longleftrightarrow & -\dfrac{1}{c}\partial_t \vec{B} - \nabla \times \vec{E} = 0 \end{matrix}$$

Though we derive the $\vec{E}$,$\vec{B}$ symmetry from free conditions ($\rho = \vec{j} = 0$), it **looks like ???** we can apply them in any conditions.

where

$$J_D = \epsilon \partial_t E; \quad J_{induced} = c\chi_m^B (\nabla \times B)$$

Note that the Maxwell equation is relationship about **E** and **B**, the $\vec{D}$ and $\vec{H}$ is introduced to help reduce the equations to a more concised form.

To solve it, using iterative method (in power of $1/c$)

i) 0th order:

$$\begin{matrix} \nabla \cdot E^{(0)} = \rho & \nabla \times B^{(0)} = 0 \\ \nabla \times E^{(0)} = 0 & \nabla \cdot B^{(0)} = 0 \end{matrix}$$

ii) 1th order:

$$\begin{matrix} \nabla \cdot E^{(1)} = 0 & \nabla \times B^{(1)} = \dfrac{j}{c} + \dfrac{1}{c}\partial_t E^{(0)} \\ \nabla \times E^{(1)} = 0 & \nabla \cdot B^{(1)} = 0 \end{matrix}$$

iii) 2th order:

$$\begin{matrix} \nabla \cdot E^{(2)} = 0 & \nabla \times B^{(2)} = 0 \\ \nabla \times E^{(2)} = -\dfrac{1}{c}\partial_t B^{(1)} & \nabla \cdot B^{(2)} = 0 \end{matrix}$$

iv) 3th order ...

$$\nabla \cdot E^{(3)} = 0 \qquad \nabla \times B^{(3)} = \frac{1}{c}\partial_t E^{(3)}$$
$$\nabla \times E^{(3)} = -\frac{1}{c}\partial_t B^{(1)} \qquad \nabla \cdot B^{(3)} = 0$$

So in the quasi-static approximation, we get

$$E = E^{(0)} + E^{(2)} + \dots$$
$$B = B^{(1)} + B^{(3)} + \dots$$

3.3.1 Boundary conditions

$$\begin{cases} \vec{e}_n \cdot (\vec{D}_2 - \vec{D}_1) = \sigma \\ \vec{e}_n \times (\vec{H}_2 - \vec{H}_1) = \dfrac{\vec{\alpha}}{c} \\ \vec{e}_n \cdot (\vec{B}_2 - \vec{B}_1) = 0 \\ \vec{e}_n \times (\vec{E}_2 - \vec{E}_1) = 0 \end{cases} \tag{3.3}$$

Note that here σ and $\vec{\alpha}$ are **free** charge and **free** current. The induced charges and current make trivial contribution due to infinitesimal integral volume (or area).

3.3.2 Plain wave

If $J = \rho = 0$, then

$$\begin{aligned} \nabla \times (\nabla \times E) &= \nabla(\nabla \cdot E) - \nabla^2 E \\ &= -\frac{1}{c}\partial_t(\nabla \times B) = -\frac{1}{c^2} \cdot \epsilon\mu \cdot \partial_t^2 E \end{aligned}$$

let

$$n^2 = \epsilon\mu$$

So

$$\nabla^2 E - \frac{n^2}{c^2}\partial_t^2 E = 0$$

Because $E = E_0 e^{-i\omega t}$, so we get

$$\nabla^2 E + \frac{n^2}{c^2}\omega^2 E = 0$$

$$E = E_0 e^{i(kx-\omega t)}, \quad k^2 = n^2\omega^2/c^2$$

let $B = B_0 e^{i(kx-\omega t)}$

$$\nabla \times E = -\frac{1}{c}\partial_t B = \frac{i\omega}{c}B$$
$$\vec{B} = \frac{c}{i\omega}\nabla \times E = \frac{c}{\omega}\vec{k} \times \vec{E}$$

3.3.3 Metal

$$\begin{aligned}\nabla \times H &= \frac{1}{c}(J + \partial_t D) \\ &= \frac{1}{c}(\sigma E - i\omega\epsilon E) \\ &= \frac{\sigma - i\omega\epsilon}{c}E\end{aligned}$$

So

$$\nabla \times (\nabla \times E) = -\nabla^2 E = -\frac{1}{c}\nabla \times \partial_t B = -\frac{1}{c}\partial_t(\nabla \times \mu H) = -\mu\frac{\sigma - i\omega\epsilon}{c^2}(-i\omega)E = k^2 E$$

where,

$$k^2 = \frac{\omega^2\mu(\epsilon + i\sigma/\omega)}{c^2} = \frac{\omega^2\mu\hat{\epsilon}}{c^2}$$

Helmholtz eqn:

$$(\nabla^2 + \frac{\omega^2\mu\hat{\epsilon}}{c^2})E = 0$$

3.4 EM in relativity

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{pmatrix} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{3.4}$$

Dual basis $F_{\mu\nu}$, when lowering or raising the 0 index, change the sign; while for other indecies, nothing happen:

$$F^{0i} = -F^{i0} = F^0{}_i = -F_0{}^i = -F_{0i} = F_{i0} = -F_i{}^0 = F^i{}_0$$
$$F^{ij} = -F^{ji} = F^i{}_j = F_i{}^j = F_{ij} = -F_{ji} = -F_j{}^i = -F^j{}^i$$

So:

$$F_{\mu\nu} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{3.5}$$

$$F_{\mu\nu}F^{\mu\nu} = 2(\vec{B}^2 - \vec{E}^2)$$

The Maxwell Eqn:

$$\begin{gathered} \partial_\mu F^{\mu\nu} = \Box A^\nu - \partial_\nu(\partial_m u A^\mu) = \Box A^\mu = 0, \\ \partial_\mu F^{\nu\rho} + \partial_\nu F^{\rho\mu} + \partial_\rho F^{\mu\nu} = 0 \end{gathered} \tag{3.6}$$

Define:

$$\tilde{F}^{\mu\nu} \equiv \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & -E^z & E^y \\ -B^y & E^z & 0 & -E^x \\ -B^z & -E^y & E^x & 0 \end{pmatrix} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{3.7}$$

where:

$$\epsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & even permutation of 0,1,2,3 \\ -1 & odd permutation of 0,1,2,3 \\ 0 & otherwise \end{cases}$$

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = 2\vec{B}\cdot\vec{E}$$

Define:

$$T_{[\mu,\nu]} \equiv \frac{1}{2}(T_\mu\nu - T_\nu\mu)$$

$$T_{[\mu_1,\mu_2,\mu_3]} \equiv \frac{1}{3!}[(T_{\mu_1\mu_2\mu_3} - T_{\mu\nu_1\mu_3\mu_2}) - (T_{\mu_2\mu_1\mu_3} - T_{\mu_2\mu_3\mu_1}) + (T_{\mu_3\mu_1\mu_2} - T_{\mu_3\mu_2\mu_1})] \tag{3.8}$$

Then the complete Maxwell Eqns:

$$\begin{gathered} \partial_\mu F^{\mu\nu} = A^\nu \\ \partial_\nu \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\partial_{[\nu}F_{\alpha\beta]} = 0 \end{gathered} \tag{3.9}$$

Transformation of $\vec{E}$ and $\vec{B}$

in a boost along x:

$$\begin{aligned} E'_{//} &= E_{//} & B'_{//} &= B_{//} \\ E'_\perp &= \gamma E_\perp + \vec{\beta}\times\vec{B} & B'_\perp &= \gamma B_\perp - \vec{\beta}\times\vec{E} \end{aligned} \tag{3.10}$$

Chapter 4

Quantum Mechanics

4.0.1 Harmonic Oscillator

$$\langle n'| x |n\rangle = \sqrt{\hbar/2m\omega}(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1})$$

For $V(r) = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}kr^2$

$$\phi_0(r) = [(\frac{\alpha}{\sqrt{\pi}})^{1/2} e^{-\frac{1}{2}\alpha^2 x^2}][(\frac{\alpha}{\sqrt{\pi}})^{1/2} e^{-\frac{1}{2}\alpha^2 y^2}][(\frac{\alpha}{\sqrt{\pi}})^{1/2} e^{-\frac{1}{2}\alpha^2 z^2}]$$

where $\alpha = (\frac{mk}{\hbar^2})^{1/4} = (\frac{m\omega}{\hbar})^{1/2}$

4.1 Rotation

4.1.1 Vector

In QM, we demand that the expectation value of a vector operator transforms under rotation like a classical vector.

$$|\alpha\rangle \rightarrow |\alpha\rangle^R = D(R)|\alpha\rangle$$
$$^R\langle\alpha| V_i |\alpha\rangle^R = \langle\alpha| D^\dagger V_i D |\alpha\rangle = R_{ij}\langle\alpha| V_j |\alpha\rangle$$
$$D^\dagger V_i D = R_{ij} V_j$$

With infinitesimal transform, we will find:

$$[J_i, V_j] = i\hbar\epsilon_{ijk} V_k \tag{4.1}$$

Let $V_0 = V_Z$ and $V_{\pm1} = \frac{\mp V_x - iV_y}{\sqrt{2}}$, then

$$[J_\pm, V_\mp] = \sqrt{2}V_0, \quad [J_\pm, V_0] = \sqrt{2}V_\pm$$
$$[J_\pm, V_\pm] = 0, \quad [J_0, V_q] = qV_q$$

Where

$$J_\pm \equiv J_x \pm iJ_y \quad J_{\pm1} = \mp\frac{1}{\sqrt{x}}J_\pm, \quad \text{and} \quad J_0 = J_z \tag{4.2}$$

4.1.2 Tensor

We define spherical tensor of rank k with $(2k+1)$ components labelled by q as

$$D^\dagger(R)T_q^{(k)}D(R) = \sum_{q'=-k}^{k} D_{qq'}^{(k)*}T_{q'}^{(k)} \tag{4.3}$$

4.1.3 Spin-$1/2$

$$D(R_{\hat{n}}(\phi)) = e^{-i\frac{\phi}{\hbar}\hat{n}\cdot\vec{S}} = e^{-i\frac{\phi}{2}\hat{n}\cdot\vec{\sigma}} = cos(\frac{\phi}{2}) - i\hat{n}\cdot\vec{\sigma}sin(\frac{\phi}{2}) = \begin{pmatrix} \cos\left(\frac{\phi}{2}\right) - in_z\sin\left(\frac{\phi}{2}\right) & (-in_x - n_y)\sin\left(\frac{\phi}{2}\right) \\ (-in_x + n_y)\sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) + in_z\sin\left(\frac{\phi}{2}\right) \end{pmatrix}$$

4.1.4 Spin-1

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4.4}$$

Wigner-Echart theorem

The matrix elements of tensor operators with respect to angular momentum eigenstates.

$$\langle\alpha', j'm'|\, T_q^{(k)}\, |\alpha, jm\rangle = \frac{\langle jk; mq|jk; j'm'\rangle}{\sqrt{2j+1}}\langle\alpha', j'|\, T^{(k)}\, |\alpha, j\rangle \tag{4.5}$$

Zeeman-effect

This is effect only in **weak magnetism**

Paschen-Back effect

This happen with **strong magnetism**($\frac{\Phi}{\Phi_0} \gg \alpha^2 \sim 10^{-4}$).

4.2 Symmetry

4.2.1 Parity

Parity of Spherical Harmonics:

$$r \to r, \quad \theta \to \pi - \theta, \quad \phi \to \phi + \pi$$

So

$$Y_{ll}(\theta, \phi) \xrightarrow{P} Y_{ll}(\pi - \theta, \phi + \pi) = e^{il\pi} e^{il\phi} \sin^l(\pi - \theta) = (-1)^l Y_{ll})\theta, \phi$$

Because $L_{\text{-}}$ is an *axial vector*, so $P(L_{\text{-}}) = 1$.

$$L_{\text{-}} Y_{ll} \xrightarrow{P} (-1)^l L_{\text{-}} Y_{ll}$$

To get

$$Y_{lm}(\theta, \phi) \xrightarrow{P} Y_{lm}(\pi - \theta, \phi + \pi) = (-1)^l L_{\text{-}} Y_{l,m+1}(\theta, \phi)$$

4.2.2 Time reversal

$$|jm\rangle \xrightarrow{T} (-1)^m |j, -m\rangle$$

4.3 Young Tableau

A way to describe

4.4 Special Relativity

Two conditions of special relativity.

- general physical rules, which means in different flames, we can observe the same EOM.
- constant c

let $X^\mu = (ct, \vec{x})$, so in order to make c constant, we have

$$-(ct')^2 + \vec{x'}62 = (ct)^2 + \vec{x}^2$$

The **simplest** way to preserve the length is to construct a **linear map**:

$$X'^\mu = (\mathcal{L})^\mu{}_\nu X^\nu$$

Where $(\mathcal{L})^\mu{}_\nu = (\mathcal{L}^T)_\nu{}^\mu = \Lambda^\mu{}_\nu$. Note that for a matrix, what important is the order of index, not their position, so

$$\mathcal{L}^\mu{}_\nu = \mathcal{L}_\mu{}^\nu = (\mathcal{L}^T)_\nu{}^\mu = (\mathcal{L}^T)^\nu{}_\mu$$

Let

$$A = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} \quad B = \begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix}$$

and

$$A_\mu \equiv g_{\mu\nu} A^\nu$$

where

$$\begin{aligned} g_{\mu\nu} &= diag(-1, 1, 1, 1) \qquad (4.6) \\ g^{\mu\nu} g_{\nu\sigma} &= g^\mu{}_\sigma = \delta^\mu{}_\sigma \end{aligned}$$

So, we can define:

$$A \cdot B \equiv -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3 = A^\mu B_\mu = A^\mu g_{\mu\nu} B^\nu$$

Inserting $\mathcal{L}\mathcal{L}^T$ into the above equation:

$$\begin{aligned} A_\mu B^\mu = A^\mu g_{\mu\nu} B^\nu &= (\mathcal{L}^{-1} A')^\mu g_{\mu\nu} (\mathcal{L}^{-1})^\nu \\ &= (\mathcal{L}^{-1})^\mu_\nu A'^\nu g_{\mu\nu} (\mathcal{L}^{-1})^\nu_\rho B'^\rho \\ &= A'^\nu (\mathcal{L}^{-1})^\mu_\nu g_{\mu\nu} (\mathcal{L}^{-1})^\nu_\rho B'^\rho \\ &= A'^\nu (\mathcal{L}^{-1^T})^\mu_\nu g_{\mu\nu} (\mathcal{L}^{-1})^\nu_\rho B'^\rho \\ &= A'^\nu g'_{\nu\rho} B'^\rho \end{aligned}$$

In special relativity, we require that:

$$g'_{\nu\rho} = (\mathcal{L}^{-1^T})^{\mu}_{\nu} g_{\mu\nu} (\mathcal{L}^{-1})^{\nu}_{\rho} = g_{\nu\rho}$$

So we get

$$\mathcal{L}^{-1^T} g \mathcal{L}^{-1} = g \to \mathcal{L}^{-1^T} = g\mathcal{L}g \to \mathcal{L}^{-1} = g\mathcal{L}^T g$$

With these conditions, we can easily find out $\mathcal{L}$. For 2D-coordinate (ct, x), we have

$$\mathcal{L}(v) = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$$

Where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. Usually, we would like to use another quantity in special relativity, which is rapidity:

$$\tanh y \equiv \frac{v}{c} \to y = \frac{1}{2} ln(\frac{1+\beta}{1-\beta})$$

And correspondingly

$$\gamma = \cosh y, \quad \beta\gamma \sinh y$$

Chapter 5

QFT

Questions:

- What's a field, classical field is a spatial distribution, quantum field is a analogy to classical one, but make up of creation and destruction operator. The problem is that we **define** a field which collects creator and destructor operators and it works !!! What's the logic to define such a field ???
- How to construct Lagrangian from a field $\leftarrow$ Lorantz invariant.
- EOM
- Noether theorem. Conserved current and charge
- Symmetry. What's each group? corresponding representation. How to embed particles into lorentz group and unitary group?
- Couple of scalar field to A_μ
- Anticommuting spinors:

$$-(\gamma_0)_{\alpha\beta}\psi_\alpha\psi^*_\beta = (\gamma_0)_{\alpha\beta\psi^*_\beta\psi_\alpha}$$

5.1 Motivation

particle number is not conserved. The creation and destruction of particls, which is possible due to the most famous eqn. of special relativity $E = mc^2$. *Lorentz invariance* guides the definition of particle.

Why QFT: $\boxed{\text{quantum mechanics plus Poincaré symmetry}}$.

$\boxed{\text{Quantum field theory is just quantum mechanics with an infinite number of harmonic oscillators}}$

The Quamtum Mechanics can describe a system with a fixed number of particles in terms of a many-body wave function. The Relativistic Quantum Field Theory with creation and annihilation operators was developed in order to include processes in which the number of particles is not conserved, and to describe the conversion of mass into energy and vice versa. A consequence of relativity is that the number of particles isn't fixed, ($E = mc^2$) though the converse is false: particle production can happen without relativity.

Construct $\mathcal{L}$ from field ϕ and its derivative under the rule of Lorentz Invariant. How to incorporate symmetry ?

QFT is the quantum mechanics of *extensive degrees of freedom* $\langle x|\phi\rangle = \phi(x)$ is a function of space, the wavefunction. This looks like a field. It is not what we mean by field in QFT. meaningless phrases like "second quantization" may conspire to try to confuse you.

It is not a coincidence that the harmonic oscillator plays an important role. After all, electromagenetic waves oscillate harmonically.

There are two common ways to quantize a field theory:

- canonical quantization
- Feynman path integral
- Ohter alternatives: perturbation theory

Second quantization: first quantization refers to the discrete modes($d\vec{x}d\vec{p} \sim \hbar$), for example, of a particle in a box. Second quantization refers to the integer numbers of excitations of each of these modes. However, this is somewhat misleading-the fact that there are discrete modes is a classical phenomenon. The two steps really are (1) interpret there modes as having energy $E = \hbar\omega$ and (2) quantize each modes as a harmonic oscillator. In that sense we are only quantizing once.

There are two new features in second quantization:

1. We have many quantum mechanical systems one for each $\vec{p}$ all at the same time.
2. We interpret the nth excitation of the $\vec{p}$ harmonic oscillator as having n **particles**.

5.2 Convention

In relativity, the symmetry refer to invariance after the transformation of *coordinate system*. So the rotation is:

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{5.1}$$

or in other way:

$$R^T \mathbb{1} R = \mathbb{1}$$

5.2.1 4D time-space

$dx^\mu \equiv (dt, d\vec{x})^\mu$
$ds^2 = dt^2 - d\vec{x} \cdot d\vec{x} = \eta_{\mu\nu} dx^\mu dx^\nu$ with

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} \tag{5.2}$$

Lorentz transformation acting on 4-vectors are matrices Λ satisfying

$$\Lambda^T \eta \Lambda = \eta = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Rotation and boost in 4D time-space around x axes is:

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta_x & \sin\theta_x \\ & & -\sin\theta_x & \cos\theta_x \end{pmatrix}, \begin{pmatrix} \cosh\beta_x & \sinh\beta_x & & \\ \sinh\beta_x & \cosh\beta_x & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Lorentz transformation

Scalar field

$$\phi(x^\mu) \to \phi((\Lambda^{-1})^\mu_\nu x^\nu)$$

Vector field

$$V^\mu \to \Lambda^\mu_\nu V^\nu$$

Tensor fields

$$T^{\mu\nu} \to \Lambda^\mu_\alpha \Lambda^] \nu_\beta T^{\alpha\beta}$$

5.2.2 quantization

$$[a_k, a_p^\dagger] = (2\pi)^3\delta^3(\vec{p}-\vec{k}), a_p^\dagger|0\rangle = \frac{1}{\sqrt{2\omega_p}}|\vec{p}\rangle$$

$$\mathbb{1} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}|\vec{p}\rangle\langle\vec{p}|$$

Function derivatives $\frac{\delta\phi(x)}{\delta\phi(y)} = \delta(x-y)$,

$$\frac{\partial(\partial_\alpha A_\alpha)^2}{\partial(\partial_\mu A_\nu)} = 2(\partial_\alpha A_\alpha)\frac{\partial(\partial_\beta A_\gamma)}{\partial(\partial_\mu A_\nu)}g_{\beta\gamma} = 2(\partial_\alpha A_\alpha)g_{\beta\mu}g_{\gamma\nu}g_{\beta\gamma}$$

notation ϕ and π for scalar fields, ψ, ξ, χ for fermions, A_μ, J_μ, V_μ for vectors and h_μ, T_μ for tensors.

Kinetic term Anything with just two fields of the same or different type can be called a kinetic term. Kinetic terms tell you about the free (non-interacting) behavior. Though, sometimes it is useful to think of a *mass term* such as $m^2\phi^2$, as an interaction rather than a kinetic term.

Boundary conditions we will always assume that our fields vanish on asymptotic boundaries. so we can integrate by part:

$$A\partial_\mu B = -(\partial_\mu A)B$$

We define quantum fields as integrals over creation and annihilation operators for each momentum: (Why define it this way ???)

$$\phi_0(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{\omega_p}}(a_p e^{i\vec{p}\vec{x}} + a_p^\dagger e^{-i\vec{p}\vec{x}}) \tag{5.3}$$

$$|\vec{x}\rangle = \phi_0(\vec{x})|0\rangle$$

There is no physical content in the above equation. It is just a definition. The physical content is in the algebra of a_p and $a_p^\dagger$ and in the Hamiltonian H_0. Nevertheless, we will see that collections of a_p and $a_p^\dagger$ in the form of Eq.5.3 are very useful in quantum field theory.

Following this defition, we can get:

$$\pi(\vec{x}) \equiv \partial_t\phi(\vec{x})|_{t=0} = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}(a_p e^{i\vec{p}\vec{x}} - a_p^\dagger e^{-i\vec{p}\vec{x}})$$

$\pi[\phi, \dot{\phi}]$ can also be implicityly defined as:

$$\frac{\partial\mathcal{H}[\phi,\pi]}{\partial\pi} = \dot{\phi}$$

Schrodinger eqn

$$\begin{aligned}
i\partial_t\psi x = i\partial_t \langle x|\psi\rangle &= i\partial_t \langle 0|\,\phi(\vec{x},t)\,|\psi\rangle = i\,\langle 0|\,\partial_t\phi(\vec{x},t)\,|\psi\rangle \\
&= \langle 0| \int \frac{d^3p}{(2\pi)^3}\frac{\sqrt{\vec{p}^2+m^2}}{2\omega_p}(a_p e^{-ipx} - a_p^\dagger e^{ipx})\,|\psi\rangle \\
&= \langle 0|\sqrt{m^2-\vec{\nabla}^2}\phi_0(x)\,|\psi\rangle
\end{aligned}$$

So

$$i\partial_t\psi(x) = \sqrt{m^2-\vec{\nabla}^2}\psi(x) = \left(m - \frac{\vec{\nabla}^2}{2m} + \mathcal{O}(\frac{1}{m^2})\right)\psi(x)$$

To get

$$i\partial_t\psi(x) = -\frac{\vec{\nabla}^2}{2m}\psi(x)$$

5.2.3 Hamiltonian & Lagrangian

Why do we restrict to Lagrangians of the form $\mathcal{L}[\phi,\partial_\mu\phi]$? First of all, this is the form that all "classical" Lagrangians had. If only first derivatives are involved, boundary conditions can be specified by initial positions and velocities only, in accordance with Newton's laws. In the quantum theory, if kinetic terms have too many derivatives, for example $\mathcal{L} = \phi\Box^2\phi$, there will generally be disastrous consequences. For example, there may be states with negative energy or negative norm, permitting the vacuum to decay. But interactions with multiple derivatives may occur. Actually, they must occur due to quantum effects in all but the simplest renormalizable field theories; for example, they are generic in all effective field theories.

Hamiltonian and *Lagrangian* density: (How to connect field ϕ with $\mathcal{L}$ or $\mathcal{H}$???).

$$\mathcal{L}[\phi,\dot{\phi}] = \pi[\phi,\dot{\phi}]\dot{\phi} - \mathcal{H}[\phi,\pi[\phi,\dot{\phi}]]$$

Or inversely:

$$\mathcal{H}[\phi,\pi] = \pi\dot{\phi}[\phi,\pi] - \mathcal{L}[\phi,\dot{\phi}[\phi,\pi]], \frac{\partial\mathcal{L}[\phi,\dot{\phi}]}{\partial\dot{\phi}} = \pi$$

The Halmiltonian corresponds to a conserved quantity - the total energy of the system - while the Lagrangian does not. The problem with Halmiltonian is that they are not Lorentz invariant. It is the 0 component of a Lorentz vector: $P^\mu = (H,\vec{P})$. And $\mathcal{H}$ is the 00 compnent of a Lorentz tensor, the energy-momentum tensor $\mathcal{T}_{\mu\nu}$. Halmiltonians are great for non-relativistic systems, but for relativistic systems we will almost exclusively use Lagrangians.

Time evolution is generated by a hamiltonian H. $i\hbar\partial_t\,|\phi\rangle = H\,|\phi\rangle$

5.2.4 Norther's theorem

If there is such a symmetry that depends on some parameter α that can be taken small (continuous), than we find:

$$0 = \frac{\delta \mathcal{L}}{\delta \alpha} = \sum_n \left\{ \left[\frac{\partial \mathcal{L}}{\partial \phi_n} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \right] \frac{\delta \phi_n}{\delta \alpha} + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha} \right] \right\}$$

If the EOM is satisfied, then it reduces to $\partial \mu J_\mu = 0$, where

$$J_\mu = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha} \tag{5.4}$$

A vector field J_μ that satisfies $\partial_\mu J_\mu = 0$ is called *conserved current*. The total charge Q, defined as:

$$Q = \int d^3 x J_0$$

satisfies

$$\partial_t Q = \int d^3 x \partial_t J_0 = \int d^3 x \vec{\nabla} \cdot \vec{J} = 0$$

Noether's theorem: If a Lagrangian has a continuous symmetry then there exists a current associate with that symmetry that is conserved when the equations of motion are satisfied.

- continuous
- the current is conserved *on-shell*, that is, when the EOM are satisfied. (The field dies out at asymptotic boundary).
- It works for *global symmetries*, parametrized by number α, not only for *local(gauge) symmetries* parametrized by functions $\alpha(x)$.

5.2.5 Energy-Momentum Tensor

There is a very important case of Noether's theorem that applies to a global symmetry of the action, not the Lagrangian. *global* space-time translation $\rightarrow$ energy-momentum tensor.

$$\phi(x) \rightarrow \phi(x + \xi) = \phi(x) + \xi^\nu \partial_\nu \phi(x) + \cdots$$

With infinitesimal change:

$$\frac{\delta\phi}{\delta\xi^\nu} = \partial_\nu\phi, \frac{\delta\mathcal{L}}{\delta\xi^\nu} = \partial_\nu\mathcal{L},$$

So:

$$\delta S = \int d^4x\delta\mathcal{L} = \xi^\nu \int d^4x\partial_\nu\mathcal{L} = 0$$

which leads to:

$$\partial_\nu\mathcal{L} = \frac{\delta\mathcal{L}[\phi_n, \partial_\mu\phi_n]}{\delta\xi^\nu} = \partial_\mu\left(\sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\frac{\delta\phi_n}{\delta\xi^\nu}\right)$$

Or equivalently

$$\partial_\mu\left(\sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\partial_\nu\phi_n - g_{\mu\nu}\mathcal{L}\right) = 0$$

The four symmetries have produced four Noether current, one for each ν:

$$\mathcal{T}_{\mu\nu} = \sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\partial_\nu\phi_n - g_{\mu\nu}\mathcal{L} \tag{5.5}$$

The corresponding conserved current:

$$Q_\nu = \int d^3x\mathcal{T}_{0\nu}$$

Electron has an inherent two-valuedness called spin, while a photon has an inherent two-valuedness called polarization.

5.3 Lagrangian

In QFT, we will use Lagrangian rather than Hamiltonian, because it is Lorentz invariant. To construct Lorentz invariant Lagrangian from field, one just need to add Lorentz invariant terms into it.

A vector field A_μ is just four scalars until we construct it with ∂_μ in the Lagrangian, as in the $(\partial_\mu A_\mu)^2$ part of $F_{\mu\nu}^2$.

5.4 Symmetry

Our universe has a number of apparent symmetries that we would like our quantum field theory to respect. One symmetry is that no place in space-time seems any different from any other place. Thus, our theory should be translation invariant: if we take all our fields $\phi(x)$ and replace them by $\phi(x+a)$ for any 4-vector a^ν, the observable should look the same. Another symmetry is Lorentz invariance: physics should look the same whether we point our measurement apparatus to the left or to the right, or put it on a train. The group of translations and Lorentz translations is called the **Poincaré group**, ISO(1,3) (the isometry group of Minkowski space).

It is possible for two different groups to have the same algebra. For example, the proper orthochronous Lorentz group($det(\Lambda)=+1, \Lambda_0^0>1$) and the full Lorentz group($det(\Lambda)=\pm 1, \Lambda_0^0>1 or \Lambda_0^0<-1$) have the same algebra, but the full Lorentz group has in addtion time reversal and parity. The **proper** Lorentz group is the special orthogonal group SO(1,3), which contains only the elements with determinant 1, so it excludes T and P.

Notice that different representations of the same group G can have different dimensions. **Lie algebra** relations is the properties of G that are inherent in all of its representations.

Assume $D_R(g(\theta))$ as a representation of group G, and

$$D_R(g(\theta\sim 0)) = \mathbb{1} + i\theta_a T_R^a + \mathcal{O}(\theta^2)$$

where $T_R^a \equiv -i\partial_{\theta_a} D_R(g(\theta))|_{\theta=0}$ is the **generator** of G in the representation of R. So

$$D_R(g(\theta)) = e^{i\theta_a T_R^a}$$

Given two such elements $D_R(g(\theta_1)) = e^{i\theta_a^1 T_R^a}$, $D_R(g(\theta_2)) = e^{i\theta_a^2 T_R^a}$, their product must give a third:

$$D_R(g_1)D_R(g_2) = D_R(g_1 g_2) = e^{i\theta_a^3 T_R^a}$$

Expanding the log of both hand sides to second order in the θ(See Maggiore chapter 2.1), we get:

$$\theta_a^3 = \theta_a^1 + \theta_a^2 - \frac{1}{2} f^{bc}{}_a + \mathcal{O}(\theta^3)$$

which implies that:

$$[T^a, T^b] = i f^{ab}{}_c T^c$$

This is the *Lie algebra* of G and the f is the *structure constants* of G. f does not depend on the representation. Note that the normalization of the T^a is ambiguous, and rescaling T rescales f. A common convention is to choose an orthonormal basis:

$$tr(T^a T^b) = \frac{1}{2}\delta^{ab}$$

The Lie algebra is defined in the neighborhood of the identity element, but by conjugating by finite transformations, the tangent space to any point on the group has the same structure, so it determines the local structure. It doesn't know about global, discrete issues, like disconnected components, so different groups can have the same Lie algebra.

A *casimir* of the algebra is an operator made from the generators which commutes with all of them.

The notation of "axis of rotation" is (d=3)-centric. More generally, a rotaion is specified by a (2D) *plane* of rotation. In d=3, we can specify a plane by its normal direction, the one that's left out, $J^i \equiv \epsilon^{ijk} J^{jk}$, in terms of which the so(3) lie algebra is

$$[J^{ij}, J^{kl}] = i(\delta^{jk} J^{il} + \delta^{il} J^{jk} - \delta^{ik} J^{jl} - \delta^{jl} J^{ik})$$

The vector representation is

$$\left(J^{ij}_{(1)}\right)^k{}_l = i(\delta^{ik}\delta^j_l - \delta^{jk}\delta^i_l)$$

The spinor representation is

$$J^{ij}_{(\frac{1}{2})} = \epsilon^{ijk}\frac{1}{2}\sigma^k = \frac{i}{4}[\sigma^i, \sigma^j]$$

For general d, we can make a spinor representation of dimension k (k=2J+1) if we find d $k \times k$ matrices γ^i which satisfy the *Clifford algebra* $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$.

For (faithful) representations of *non-compact* groups, 'unitary' and 'finite-dimensional' are mutually exclusive.

5.4.1 Lorentz group

the symmetry group associated with *special relativity.*

For scalar field, the physical content of Lorentz invariance is that nature has a symmetry under which scalar fields do not transform. Take, for example, the temperature of a fluid, which can vary from point to point. If we change reference frames, the labels for the points change, but the temperature at each point stays the same.

As for vector field, the difference is that the compnents of a vector field at the point x transform into each other as well.

The simplest Lorentz-invariant operator that we can write down involving derivatives is the d'Alembertian:

$$\Box = \partial_\mu^2 = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$$

Objects such as $v^2 = V_\mu V^\mu, \phi, 1, \partial_\mu V^\mu$ are *Lorentz invariant*, meaning they do not depend on the Lorentz frame at all. While objects like: $V_\mu, F_\mu\nu, \partial_\mu, x_\mu$ are *Lorentz covariant*, meaning they do change in different frames, but precisely as the Lorentz transformation dictates.

The Lorentz group is sometimes called O(1,3). This is orghogonal (preserves a metric) group corresponding to a metric with (1,3) signature.

The irreducible representations of the Lorentz group can be constructed from irreducible representations of SU(2).

$$J_i^+ \equiv \frac{1}{2}(J_i + iK_i), J_i^- \equiv \frac{1}{2}(J_i - iK_i)$$

which satisfy

$$\begin{aligned}
[J_i^+, J_j^+] &= i\epsilon_{ijk} J_k^+ \\
[J_i^-, J_j^-] &= i\epsilon_{ijk} J_k^- \\
[J_i^+, J_j^-] &= 0
\end{aligned}$$

Lorentz algebra

Lorentz algebra, so(1,3)

$$\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk} J_k \\
[J_i, K_j] &= i\epsilon_{ijk} K_k \\
[K_i, K_j] &= -i\epsilon_{ijk} J_k
\end{aligned}$$

Note that $[J_i, J_j] = i\epsilon_{ijk} J_k$ is also the algebra for rotations, SO(3), and in fact the J_i generate the 3D rotation subgroup of the Lorentz group.

Similar to rotation group, we can generalize to other SO(1,d) by collecting the generators into an antisymmetric matrix J^μ with components $J^{ij} = \epsilon^{ijk} J^k, J^{0i} = K^I = -J^{i0}$(exactly as $\vec{E}, \vec{B}$ are collected into F^μ). This object satisfies:

$$J^{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix}$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} + \eta^{\mu\sigma}J^{\nu\rho} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho})$$

The fundamental (d+1 Dimentional vector) representation matrices solving this equation are

$$(J^{\mu\nu})^{\rho}{}_{\sigma} = i(\eta^{\nu\rho}\delta^{\mu}_{\sigma} - \eta^{\mu\rho}\delta^{\nu}_{\sigma})$$

$$[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho})$$

The Lie algebra is independent of any concrete representation, though we derive them from 4-vector representation. They holds for any representation.

represention

The Dirac representation of the Lorentz group is reducible; it is the direct sum of a left-handed and a right-hadned spinor representation.

γ^5

- $(\gamma^5)^2 = \infty$
- $\{\gamma^5, \gamma^\mu\} = 0$
- Extended Clifford algebra: $\left\{\gamma^M, \gamma^N\right\} = 2g^{MN}$, with $\gamma^M = \gamma^0, \gamma^1, \gamma^2, \gamma^3, i\gamma^5$, and $g^{MN} = diag(1, -1, -1, -1, -1)$

5.4.2 Unitary group

the probability should add up to *1*. $\mathbf{U(N)}$ is defined by its N dimensional representation as $N \times N$ complex unitary matrices $\mathbb{1} = M^\dagger M = MM^\dagger$. This one doesn't arise as a spacetime symmetry, but is crucial in the study of gauge theory.

Unitary representation, which means under the representation, the group element $\mathcal{P}$ has $\mathcal{P}^\dagger\mathcal{P} = 1$. But why the unitarity depends on representation? shouldn't it be representation independent.

Majorana represention

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix} \quad (5.6)$$

The Majorana is another $(\frac{1}{2}, 0)\bigoplus(0, \frac{1}{2})$ representation of the Lorentz group that is phhysically equivalent to the Weyl representation.

5.4.3 Poincaré Group

The group of translations and Lorentz transformation is called the **Poincaré group**, ISO(1,3) (the isometry group of Minkowski space).

Particle can be defined as a set of states that mix only among themselves under Poincaré transformations.

Particles transform under irreducible unitary representations of the Poincaré group.

There are *no finite-dimensional unitary representations of the Poincaré group*

5.4.4 Gauge symmetry

Under a gauge transformation, ϕ can transform as

$$\phi \to e^{-i\alpha(x)}\phi$$

But the derivatives $|\partial_\mu \phi|^2$ is not invariant. we can make the kinetic term gauge invariant using something called a covariant derivative. Let

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$$

So

$$(\partial_\mu + ieA_\mu)\phi \to (\partial_\mu + ieA_\mu + i\partial_\mu \alpha)e^{-i\alpha(x)}\phi = e^{-i\alpha(x)}(\partial_\mu + ieA_\mu)\phi$$

This leads to the definition of *covariant derivative*

$$D_\mu \phi \equiv (\partial_\mu + ieA_\mu)\phi \to e^{-i\alpha(x)} D_\mu \phi$$

5.5 Field

5.5.1 Scalar Field

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\vec{\nabla}\phi \cdot \vec{\nabla}\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2\phi^2) = -\frac{1}{2}(\phi \Box \phi - m^2\phi^2)$$

EOM:

$$-\partial_t^2 \phi + \nabla^2 \phi - m^2 \phi = (\partial_\mu \partial^\mu + m^2)\phi = 0$$

$$\boldsymbol{\phi}(\vec{x}) = \int \frac{d^d k}{(2\pi)^d} \sqrt{\frac{\hbar}{2\omega_k}} (e^{i\vec{k}\cdot\vec{x}} a_k + e^{-i\vec{k}\vec{x}} a_k^\dagger)$$

$$\boldsymbol{\pi}(\vec{x}) = \frac{\partial \mathcal{L}}{\partial_\mu \phi} = \frac{1}{i} \int \frac{d^d k}{(2\pi)^d} \sqrt{\frac{\hbar \omega_k}{2}} (e^{i\vec{k}\cdot\vec{x}} a_k - e^{-i\vec{k}\vec{x}} a_k^\dagger)$$

$$[\phi(\vec{x}), \pi(\vec{x'})] = i\hbar \delta^d(\vec{x} - \vec{x'})$$

$$H = \sum_n (p_n \dot{q}_n) = \int dx (\pi(x)\dot{q}(x) - \mathcal{L})$$

5.5.2 Complex Scalar Field

$$\mathcal{L} = -\phi^* \Box \phi - m^2 \phi^* \phi$$

5.5.3 Klein-Gordon Field

For a massive scalar (spin 0) and neutral (charge 0) field:

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \phi)(\partial^\mu \phi) - m^2 \phi^2]$$

The Euler-Lagrange formula:

$$(\Box + m^2)\phi = 0$$

This is the Klein-Gordon equation. It was quantized by Pauli and Weisskopf in 1934. The Klein-Gordon equation was historically rejected as a fundamental quantum equation because it predicted negative probability density.

5.5.4 Dirac Field

Dirac was looking for an equation linear in E or in ∂_t. For a massive spinor (spin 1/2) field the Lagrangian density is:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi; \quad \bar{\psi} = \psi^* \gamma^0 \quad \text{Dirac adjoint}$$

The 4×4 Dirac matrices $\gamma^\mu (\mu = 0, 1, 2, 3)$ satisfy the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

Define:

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

Any set of four matrices satisfying this equation can be combined to form a 4d representation of so(1,3) in the form

$$J^{\mu\nu}_{Dirac} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

The corresponding EOM is the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0; \quad i(\partial_\mu \bar{\psi})\gamma^\mu + m\bar{\psi} = 0$$

To be explicit, this is shorthand for

$$(i\gamma^\mu_{\alpha\beta}\partial_\mu - m\delta_{\alpha\beta})\psi_\beta = 0$$

The Dirac (γ) matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

where

$$\sigma^\mu \equiv (\mathbb{1}, \vec{\sigma}), \bar{\sigma}^\mu \equiv (\mathbb{1}, -\vec{\sigma}),$$

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Where σ is the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.7}$$

$$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

$$\vec{\sigma}^\dagger = \vec{\sigma}$$

Commutation relation is

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}, [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$$

In the Weyl basis, the Dirac equation is:

$$\begin{pmatrix} -m & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

Quantization of the Dirac field is achieved by replacing the spinors by field operators and using the Jordand and Wigner quantization rules. Heisenberg's EOM for the field operator $\hat{\psi}(\vec{x}, t)$ reads:

$$i\partial_t \hat{\psi}(\vec{x}, t) = [\hat{\psi}(\vec{x}, t), \hat{H}]$$

There are both positive and negative eigenvalues in the energy spectrum. The later are problematic in view of Einsteins energy of a particle at rest $E = mc^2$. Diracs way out of the negative energy catastrophe was to postulate a Fermi sea of antiparticles. This genial assumption was not taken seriously until the positron was discovered in 1932 by Anderson.

5.5.5 Weyl spinor

$$\sigma^\mu \partial_\mu \psi = \mathbb{1}\partial_t \psi - \partial_i \sigma_i \psi = 0$$

Dirac equation for a Weyl spinor:

$$\sigma^\mu \partial_\mu \psi = 0$$

5.5.6 Maxwell Field

$$\tag{5.8}$$

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}^2 - j_\mu A^\mu = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - j_\mu A^\mu \\
&= -\frac{1}{2}(\partial_\mu A_\nu)^2 + \frac{1}{2}(\partial_\mu A_\mu)^2 - j_\mu A^\mu \\
&= (E^2 - B^2)/2 - \phi V + \vec{j}\vec{A}
\end{aligned}$$

Here we use

$$(\partial_\mu A_\nu)^2 = (\partial_\nu A_\mu)^2$$

where J_μ is the external current:

$$J_\mu(x) = \begin{cases} J_0(x) = \rho(x) \\ J_i(x) = v_i(x) \end{cases}$$

Field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

with components

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = -E^i, \quad F^{ij} = \partial^i A^j - \partial^j A^i = -\varepsilon^{ijk} B^k$$

EOM $\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0$

$$-J_\nu - \partial_\mu(-\partial_\mu A_\nu) - \partial_\nu(\partial_\mu A_\mu) = 0$$

which gives

$$\partial_\mu F_{\mu\nu} = J_\nu$$

Lorentz gauge: $\partial_\mu A_\mu = 0$

$$J_\nu = \partial_\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = \Box A_\nu - \partial_\nu(\partial_\mu A_\mu) = \Box A_\nu$$

so

$$A_\nu(x) = \frac{1}{\Box} J_\nu(x)$$

propagator

$$\Pi_A = \frac{1}{\Box}$$

Note that the propagator has nothing to do with the source. In fact it is entirely determined by the kinetic terms for a field.

5.5.7 Proca (Massive Vector Boson) Field

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - j_\mu A^\mu$$

EOM:

$$\Box A^\nu - \partial^\mu(\partial_\mu A^\mu) + m^2 A^\nu = j^\nu$$

spinors

Dirac spinors have both left- and right-handed compnents. Theny can be massive or massless.

Weyl spinors are always massless and can be left- or right-handed. When embedded in Dirac spinors they satisfy the constraint $\gamma_5 \psi = \pm\psi$.

Majorana spinors are left- or right handed. When embedded in Dirac spinors they satisfy the constraint $\psi = \psi_C = -i\gamma_2\psi^*$

5.6 Interaction

time ordered amplitude:

$$D_F(x,y) \equiv \langle 0| T\{\phi(x)\phi(y)\} |0\rangle = \langle 0| \phi(x)\phi(y) |0\rangle \theta(x^0 - y^0) + \langle 0| \phi(y)\phi(x) |0\rangle \theta(y^0 - x^0)$$
$$(\Box + m^2) D_F(x,y) = -i\hbar\delta^{(4)}(x-y)$$

Let $|\Omega\rangle$ being the vacuum state of interacting QFT, ($|\Omega\rangle \neq |0\rangle$), then

$$\begin{aligned}
&\partial_t \langle\Omega| T\{\phi(x)\phi(x')\} |\Omega\rangle \\
=&\partial_t[\langle\Omega| \phi(x)\phi(x') |\Omega\rangle \theta(t-t') + \langle\Omega| \phi(x')\phi(x) |\Omega\rangle \theta(t'-t)] \\
=&\langle\Omega| (\partial_t\phi(x))\phi(x') |\Omega\rangle \theta(t-t') + \langle\Omega| \phi(x')\partial_t\phi(x) |\Omega\rangle \theta(t'-t)] \\
&+ \langle\Omega| \phi(x)\phi(x') |\Omega\rangle \delta(t-t') + \langle\Omega| \phi(x')\phi(x) |\Omega\rangle \delta(t'-t)] \\
=&\langle\Omega| T\{(\partial_t\phi(x))\phi(x')\} |\Omega\rangle + \langle\Omega| [\phi(x), \phi(x')] |\Omega\rangle \delta(t-t') \\
=&\langle\Omega| T\{(\partial_t\phi(x))\phi(x')\} |\Omega\rangle
\end{aligned}$$

So, the second derivative:

$$\begin{aligned}
&\partial_t^2 \langle\Omega| T\{\phi(x)\phi(x')\} |\Omega\rangle \\
=&\partial_t \langle\Omega| T\{(\partial_t\phi(x))\phi(x')\} |\Omega\rangle \\
=&\langle\Omega| T\{(\partial_t^2\phi(x))\phi(x')\} |\Omega\rangle + \langle\Omega| [\partial_t\phi(x), \phi(x')] |\Omega\rangle \delta(t-t') \\
=&\langle\Omega| T\{(\partial_t^2\phi(x))\phi(x')\} |\Omega\rangle - i\hbar\delta^{(4)}(t-t')
\end{aligned}$$

So for free field ($(\Box + m^2)\phi(x) = 0$):

$$\begin{aligned}
(\Box + m^2) \langle\Omega| T\{\phi(x)\phi(x')\} |\Omega\rangle = \langle\Omega| T\{(\Box + m^2)\phi(x) \quad \phi(x')\} |\Omega\rangle = -i\hbar\delta^{(4)}(x-x') \\
= - i\hbar\delta^{(4)}(x-x')
\end{aligned}$$

Define:

$$\langle\Omega| T\{\phi(x)\phi(x_2)\cdots\phi(x_n)\} |\Omega\rangle = \langle\phi(x)\phi(x_2)\cdots\phi(x_n)\rangle \tag{5.9}$$

So:

$$\Box\langle\phi(x)\phi(x_2)\cdots\phi(x_n)\rangle = \langle(\Box\phi(x))\phi(x_2)\cdots\phi(x_n)\rangle - i\hbar\sum_{j=1}^{n}\delta^{(4)}(x-x_j)\langle\phi(x)\phi(x_2)\cdots\phi(x_{j-1})\phi(x_{j+1})\cdots\phi$$

One more definition:

$$\phi_x \equiv \phi(x), \qquad \phi_j \equiv \phi(x_j) \tag{5.10}$$

So, in general:

$$\Box\langle\phi_x\phi_2\cdots\phi_n\rangle = \langle(\Box\phi_x)\phi_2\cdots\phi_n\rangle + ... \quad = \langle(\frac{\partial\mathcal{L}_{int}}{\partial\phi_x}phi_2\cdots\phi_n\rangle - i\hbar\sum_{j=1}^{n}\delta^{(4)}(x-x_j)\langle\phi_x\phi_2\cdots\phi_{j-1}\phi_{j+1}\cdots$$

For interacting QFT: $((\Box + m^2)\phi = \frac{\partial\mathcal{L}_int}{\partial\phi})$
Define shorthands:

$$\delta_{xi} \equiv \delta(x-x_i), \quad D_{xi} \equiv \langle\phi_x\phi_i\rangle \tag{5.11}$$

So for free massless scalar:

$$\Box_x D_{x1} = -i\delta_{x1}$$

So:

$$\begin{aligned} D_{12} = \langle\phi_1\phi_2\rangle &= \int d^4x\delta_{x1}\langle\phi_x\phi_x\rangle \\ &= i\int d^4x\Box_x D_{x1}\langle\phi_x\phi_2\rangle \\ IBPtwice &= i\int d^4x D_{x1}\Box_x\langle\phi_x\phi_2\rangle \\ &= \int d^4x D_{x1}\delta_{x2} = D_{21} \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} \langle\phi_1\phi_2\phi_3\phi_4 &= \int d^4x\delta_{x1}\langle\phi_x\phi_2\phi_3\phi_4\rangle = i\int d^4x D_{x1}\Box_x\langle\phi_x\phi_2\phi_3\phi_4\rangle \\ &= i\int d^4x D_{x1}(-i)(\delta_{x2}D_{34} + \delta_{x3}D_{24} + \delta_{x4}D_{23}) \\ &= D_{21}D_{34} + D_{31}D_{24} + D_{41}D_{23} \\ &= D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23} \end{aligned}$$

5.7 Terminology

free spinors

Dirac spinors

Weyl representation

spinor representation

$(\frac{1}{2}, \frac{1}{2})$ representation

projective representation

Lorentz generators

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

Lorentz group O(1,3); tensor representations $\phi, A_\mu, h_{\mu\nu}$, spinor represontations: Weyl spinors ϕ_L, ϕ_R. A Dirac spinor ψ transforms in the reducible $(\frac{1}{x}, 0) \oplus (0, \frac{1}{2})$ representaion. The next step towards quantizing a thory with spinors is to use these Lorentz group representaions to generate irreducible unitray representaions of the Poincaré group.

Chapter 6

Examples

6.1 Dimensional analysis

6.1.1 Black body radiation

$$I(\omega) \equiv \frac{1}{V}\frac{d}{d\omega}E(\omega)$$

This has units of $[Energy] \times [time] \times [distance]^{-3}$ that can be constructed out of $\omega, k_B T$ and c. So easily:

$$I(\omega) = const \times c^{-3} w^2 k_B T$$

6.2 First Principle

6.2.1 Special relativity field

$$\text{special relativity scalar field}\phi \xrightarrow{simplest possible\, Lorentz invariant} \Box\phi = 0$$

One solution is:

$$\phi(x) = a_p(t)e^{i\vec{p}\cdot\vec{x}}$$

So we get:

$$(\partial_t^2 + \vec{p}\cdot\vec{p})a_p(t) = 0$$

This is exactly the equation of motion of harmonic oscillator. General solution:

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3}[a_p(t)e^{i\vec{p}\cdot\vec{x}} + a_p^*(t)e^{-i\vec{p}\cdot\vec{x}}]$$

Chapter 7

doubts

7.1 Doubts

Some doubt about physics

7.1.1 Classical Physics

Newton's Second Law

Why it is relationship between acceleration and force, rather than velocity and force?

Action

$S = \int Ldt$, why L, but not other quantities, e.g. H.

Lagrangian

What's the intuitive physical meaning of $L = E_k - E_p$

7.1.2 Statistical Physics

Enthalpy, Free energy, Gibbs Free energy

What do H,F,G stand for? Physical meaning?

Entropy

How do we know the entropy defined in $\Delta\Gamma = \frac{d\Gamma}{dE}\Delta E = e^S$ is the same one as we defined in $dE = TdS - PdV$?

7.2 Notation

7.2.1 Electrodynamics

The reason a conductor puts boundary conditions on the EM field is that the electrons move around to compensate for an applied field. But there is a limit on hwo fast the electrons can move. The resulting cutoff frequency is called the *plasma frequency*

Maxwell Eqn

$$\left.\begin{aligned} \overrightarrow{\nabla}\times(\overrightarrow{\nabla}\varphi)=0 \\ \overrightarrow{\nabla}\cdot(\overrightarrow{\nabla}\times\overrightarrow{A})=0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \overrightarrow{\nabla}\cdot\overrightarrow{E}=\rho \\ \overrightarrow{\nabla}\times\overrightarrow{B}=\frac{1}{c}\overrightarrow{j}+\frac{1}{c}\partial_t\overrightarrow{E} \\ \overrightarrow{\nabla}\cdot\overrightarrow{B}=0 \\ -\overrightarrow{\nabla}\times\overrightarrow{E}=\frac{1}{c}\partial_t\overrightarrow{B} \end{aligned}\right\} \tag{7.1}$$

With $\overrightarrow{E}$ and $\overrightarrow{B}$ still connected with each other, there is one **DOF**.

7.2.2 Statistical Physics

Partisian func

When count from particle (one particle can occupy how many states), remember the $\frac{1}{N!}$ factor! When count from state (how many particles in one state), don't need $\frac{1}{N!}$.

7.2.3 Quantum Mechanics

Baker-Campbell-Hausdorff

if $[A,B]=\mathbb{C}$, then:

$$e^Ae^B=e^Be^Ae^{[A,B]},e^Ae^B=e^{A+B+\frac{1}{2}[A,B]},e^{A+B}=e^{B+A}$$

Angular Momentum

Rep. of J in the Hilbert Space are discrete (finite) $\Leftarrow$ subspace.

$$Y_l^m(\theta,\phi)=\langle\theta,\phi|l,m\rangle=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(cos\theta)e^{im\phi}$$

Harmonic Oscillator

What's the physical meaning of the energy eigenstate $|\phi\rangle$, what's its relationship to $|x\rangle$ state?

Chapter 8

Terminology

adjoint representation

analyticity

associativity of addition and multiplication $a + (b + c) = (a + b) + c$, and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

bilinear In QFT, a bilinear term means it has exactly two fields. Such as:

$$\mathcal{L}_K \supset \frac{1}{2}\phi\Box\phi, \frac{1}{4}F_{\mu\nu}^2, \frac{1}{2}m^2\phi^2, \frac{1}{2}\phi_1\Box\phi_2, \phi_1\partial_\mu A_\mu, \ldots$$

Boltzmann distribution $n_i = Ne^{-\beta E_i}$

causality

charge conjugation C taking particles to antiparticles

$$C: \quad \psi \to -i\gamma_2\psi^* \equiv \psi_C$$

In the Weyl basis, $\gamma_2^* = -\gamma_2$ and $\gamma_2^T = \gamma_2$, so

$$C: \quad \psi^* \to -i\gamma_2\psi$$

chirality The handedness of a spinor is referred to as its chirality. The **left-handed** and **right-handed** refer to the $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ representations of the Lorentz group. This concept only exists for spinors, or more precisely for (A,B) representations of the Lorentz group with $A \neq B$. Almost always, chirality means that a theory is not symmetric between left-handed Weyl spinors ψ_L and right-handed spinors ψ_R

cluster decomposition principle

commutativity of addition and multiplication $a + b = b + a$ and $a \cdot b = b \cdot a$

contravariant vectors vectors with upper indices

covariant vectors vectors with lower indices

Dirac spinors

Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

Dirac equation

$$(iD\!\!\!\!/ - m)\psi = 0$$

Dirac Mass

$$\mathcal{L}_{\text{Dirac mass}} = m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$$

Distributivity of multiplication over addition $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

equipartion theorem a body in thermal equilibrium should have energy equally distributed among all possible modes, (mode is a seperation of phase space), which means all modes have the same energy.

faithful representation A representation in which each group element gets its own matrix is called a *faithful representation*.

Fermi's golden rule $\Gamma \sim |\mathcal{M}|^2 \delta(E_f - E_i)$

first principle

Gauge transform $\phi \to e^{-i\alpha}\phi$

good quantum number which doesn't change along time. $[Q, H] = 0$

Grassmann Numbers For Majorana masses to be non-trivial, fermion compnents cannot be regular numbers, they must be anticommuting numbers. Such things are called Grassmann numbers.

Helicity spin projected on the direction of motion is called the helicity. $\hat{h} = \frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}$

Legendre transformation

Levi-Civita $\epsilon_{ijk} = \begin{cases} 1 & normalorder(123, 231, 312) \\ -1 & reverseorder(132, 213, 321) \\ 0 & otherwise \end{cases}$

Lie algebra the generators of the Lie group form an algebra called its Lie algebra.

Lie groups Lie groups are a class of groups, including the Lorentz group, with an infinite number of elements but a finite number of generators.

lightlike $V^\mu V_\mu = 0$

little group The representation of the full Poincaré group is induced by a representation of the subgorup of the Poincaré group that holds p^μ fixed, called the *little group.* When P_μ is massive, the little group is SO(3); when p_μ is massless, the little group is ISO(2).

locality

Lorentz group this is the generalization of the rotation group to include both rotations and boosts.

Majorana spinor A spinor whose antiparticle is itself.

Majorana masses

$$\mathcal{L} = i\psi_L^\dagger \sigma_\mu \partial_\mu \psi_L + i\frac{m}{2}(\psi_L^\dagger \sigma_2 \psi_L^* - \psi_L^T \sigma_2 \psi_L)$$

The mass terms in this Lagrangian are called **Majorana masses**.

Majorana fermions in Dirac spinors:

$$\psi = \begin{pmatrix} \psi_L \\ i\sigma_2 \psi_L^* \end{pmatrix}$$

pseudo scalar particles with odd **parity**

quantize promote x and p as operators and impose the canonical commutation relations: $[x, p] = i$

quantum process time evolution of an open quantum system ???

Rarita-Schwinger field spin-$\frac{3}{2}$

representation A set of objects that mix under a transformation group is called a representation of the group, though technically the matrix embedding is the representation. A representation is a particular embedding of group elements into operators that act on a vector space. For finite-dimensional representations, this means an embedding of the g_i into matrices.

second quantization canonical quantization of relativistic fields,

$$H_0 = \int \frac{d^3p}{(2\pi)^3}\omega_p(a_p^\dagger a_p + \frac{1}{2})$$

First quantization refer to the discrete mode, for example, of a particle in a box. Second quantization refers to the integer numbers of excitations of each of these modes. There are two features in second quantization:

1. We have many quantum mechanical systems - one for each $\vec{p}$ - all at the same time.
2. We interpret the nth excitation of the $\vec{p}$ harmonic oscillator as having n *particles.*

S-matrix

spacelike $V^\mu V_\mu < 0$

SO(n) the group of nD rotations ($det(R) = 1$)

timelike $V^\mu V_\mu > 0$

unitary $\Lambda^\dagger \Lambda = 1$

Weyl spinor Irreducible unitary spin-$\frac{1}{2}$ representations of the Poincaré group.

γ **matrix** In the Weyl basis:

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}$$

γ^5

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

In the Weyl representation:

$$\gamma^5 = \begin{pmatrix} -\infty & \\ & \infty \end{pmatrix}$$

Table 8.1: Symbols

$\nabla^2 = \frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2}\frac{1}{\sin\theta}\partial_\theta \sin\theta \partial_\theta + \frac{1}{r^2 \sin\theta^2}\partial_\phi^2$
$\nabla^2 = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\theta^2 + \partial_Z^2$

Table 8.2: Frequently used constants

Constant	S.I.	Gauss
h	6.626×10^{-34}	
Bohr magneton	$\mu_B = \frac{e}{2m_e}$	
Fine structure	$\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$	