

# Phyiscis

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## 0.1 Principles

**What makes life easier:** periodicity, symmetry, linearity (constant), causality, conservation, static, equilibrium, main term, limitation, independent, binary system.

**What makes life hard:** nonlinearities, feedback, coupling (correlation), fluctuation, perturbation, evolution.

### 0.1.1 Symmetry

For a theory, study the symmetry of its EOM, and then specify how physical quantities transform in order to preserve the symmetry.

- Continuous symmetry
  - Gauge symmetry: phase transition
- Discrete symmetry

### 0.1.2 others

#### **correspondence principle**

which requires that in the limit of large quantum numbers the complete quantum mechanical description must agree with the classical one.

#### **Uncertainty principle**

Conjugated variables can't be identified simultaneously. ( $x$  &  $p$ ,  $t$  &  $E$ ,  $\theta$  &  $L$ )

#### **Minimum Action principle**

The action of a process being minimum.

#### **Practical principle**

It does not matter that a theory is formulated in terms of infinite quantities as long as observable quantities are finite. Extensive use is made of complex quantities in optics, and there is no objection to that as long as the observables are real.

### 0.1.3 Rules

- Consistency check
- Intuitive physical picture

# Part I

## Introduction



# Chapter 1

## Introduction

EOM: one can derive it from action (the smallest action principle). As for the action, it should obey some symmetric rules, for example, in relativity, the action should be *Lorentz invariant* and *Gauge invariant*.

In terms of  $\dot{q} = \frac{\partial q(t)}{\partial t}$ , one can also write  $q(t) = \int \dot{q} dt$ , so  $\dot{q}$  and  $q$  are equivalent, not one base on another, both can be independent.

The tools of QM: Perturbation theory (P.T. both time-dependent and time-independent)

Two keys objects in QM: operators and wavefunctions, if you keep operators invariant and wavefunction change with time, you get *Schrodinger Picture*; on the other hand, if you keep wavefunction invariant and change operators with time, you get *Heisenburg Picture*. No matter in which picture, both operators and wavefunctions depends on *representations*, usual representations include r rep., k rep., Bloch state rep. etc.

QFT is just QM with an infinite number of Harmonic Oscillator (H.O.)

In QM (QFT), the basic idea behind renormalization – infinities can appear in intermediate calculations, but they must drop out of physical observables.

## 1.1 Doubts

Some doubt about physics

### 1.1.1 Classical Physics

#### Newton's Second Law

Why it is relationship between acceleration and force, rather than velocity and force?

#### Action

$S = \int L dt$ , why L, but not other quantities, e.g. H.

#### Lagrangian

What's the intuitive physical meaning of  $L = E_k - E_p$

### 1.1.2 Statistical Physics

#### Enthalpy, Free energy, Gibbs Free energy

What do H,F,G stand for? Physical meaning?

#### Entropy

How do we know the entropy defined in  $\Delta\Gamma = \frac{d\Gamma}{dE}\Delta E = e^S$  is the same one as we defined in  $dE = TdS - PdV$  ?

## 1.2 Notation

### 1.2.1 Electrodynamics

The reason a conductor puts boundary conditions on the EM field is that the electrons move around to compensate for an applied field. But there is a limit on how fast the electrons can move. The resulting cutoff frequency is called the *plasma frequency*

## Maxwell Eqn

$$\left. \begin{aligned} \vec{\nabla} \cdot \vec{E} &= \rho \\ \vec{\nabla} \times (\vec{\nabla} \varphi) &= 0 \\ \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) &= 0 \end{aligned} \right\} \Rightarrow \left. \begin{aligned} \vec{\nabla} \cdot \vec{E} &= \rho \\ \vec{\nabla} \times \vec{B} &= \frac{1}{c} \vec{j} + \frac{1}{c} \partial_t \vec{E} \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ -\vec{\nabla} \times \vec{E} &= \frac{1}{c} \partial_t \vec{B} \end{aligned} \right\} \quad (1.1)$$

With  $\vec{E}$  and  $\vec{B}$  still connected with each other, there is one **DOF**.

Now that EM wave can only transport translated wave ( $\vec{k} \cdot \vec{E} = 0$ ), then how does the static electric potential  $\varphi$  could produce  $\vec{E}$  in the direction of *vecr*?

## 1.2.2 Statistical Physics

### Partisian func

When count from particle (one particle can occupy how many states), remember the  $\frac{1}{N!}$  factor! When count from state (how many particles in one state), don't need  $\frac{1}{N!}$ .

## 1.2.3 Quantum Mechanics

### Baker-Campbell-Hausdorff

if  $[A, B] = \mathbb{C}$ , then:

$$e^A e^B = e^B e^A e^{[A, B]}, e^A e^B = e^{A+B+\frac{1}{2}[A, B]}, e^{A+B} = e^{B+A}$$

### Angular Momentum

Rep. of J in the Hilbert Space are discrete (finite)  $\Leftarrow$  subspace.

$$Y_l^m(\theta, \phi) = \langle \theta, \phi | l, m \rangle = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}$$

### Harmonic Oscillator

What's the physical meaning of the energy eigenstate  $|\phi\rangle$ , what's its relationship to  $|x\rangle$  state?

# Chapter 2

## Constants

### 2.1 Constants

Table 2.1: Symbols

|  |
|--|
| $\nabla^2 = \frac{1}{r^2} \partial_r (r^2 \partial_r) + \frac{1}{r^2} \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{r^2 \sin^2 \theta} \partial_\phi^2$ |
| $\nabla^2 = \frac{1}{r} \partial_r (r \partial_r) + \frac{1}{r^2} \partial_\theta^2 + \partial_z^2$  |

### 2.2 Scale

#### 2.2.1 optics

#### 2.2.2 Resistivity

#### 2.2.3 Superconductivity

### 2.3 Collections

Table 2.2: Constants

| Constant                       | S.I.   | Gauss   |
|--------------------------------|--|---|
| $h$                            | $6.626 \times 10^{-34} \text{ J} \cdot \text{s}$   | $4.1357 \times 10^{-15} \text{ eV} \cdot \text{s}$    |
| $\hbar = \frac{h}{2\pi}$       |  | $6.5821 \times 10^{-16} \text{ eV} \cdot \text{s}$    |
| $\hbar c$                      |  | $200 \text{ MeV} \cdot \text{fm}$                     |
| $\epsilon_0$                   | $8.854 \times 10^{-12} \text{ F} \cdot \text{m}^{-1}$<br>$8.854 \times 10^{-12} \text{ C}^2 \cdot \text{N}^{-1} \cdot \text{m}^{-2}$   |   |
| $\frac{1}{4\pi\epsilon_0}$     | $8.9876 \times 10^9 \text{ V} \cdot \text{m} \cdot \text{C}^{-1}$  |   |
| $\frac{e}{4\pi\epsilon_0}$     | $1.438 \times 10^{-9} \text{ V} \cdot \text{m}$  |   |
| $\frac{e^2}{4\pi\epsilon_0}$   | $2.301 \times 10^{-28} \text{ C} \cdot \text{V} \cdot \text{m}$  | $1.438 \times 10^{-9} \text{ eV} \cdot \text{m}$      |
| $\mu_0$                        | $4\pi \times 10^{-7} \text{ H} \cdot \text{m}^{-1}$<br>$1.257 \times 10^{-6} \text{ N} \cdot \text{A}^{-2}$<br>$1.257 \times 10^{-6} \text{ T} \cdot \text{m} \cdot \text{A}^{-1}$<br>$1.257 \times 10^{-6} \text{ Wb} \cdot \text{A}^{-1} \cdot \text{m}^{-1}$<br>$1.257 \times 10^{-6} \text{ V} \cdot \text{s} \cdot \text{A}^{-1} \cdot \text{m}^{-1}$ |   |
| Boltzmann's const $k$          | $1.308 \times 10^{-8} \text{ J} \cdot \text{K}^{-1}$   | $8.617 \times 10^{-5} \text{ eV} \cdot \text{K}^{-1}$ |
| Stefan-Boltzmann $\sigma$      | $5.67 \times 10^{-8} \text{ W} \cdot \text{m}^{-2} \cdot \text{K}^{-4}$  |   |
| Larmor radiation ( $v \ll c$ ) | $P = \frac{2}{3} \frac{q^2 a^2}{4\pi\epsilon_0 c^3}$   | $P = \frac{2}{3} \frac{q^2 a^2}{c^3}$                 |

Table 2.3: Frequently used constants

|                    |  |
|--------------------|--|
| Bohr radius        | $R_n = n^2 a_0; a_0 = 4\pi\epsilon_0 \hbar^2 / m e^2 = 0.529 \times 10^{-8} cm$  |
| Bohr Energie       | $E_n = -\frac{1}{2} \frac{e^2}{4\pi\epsilon_0 a_0} \frac{1}{n^2} = -\frac{1}{2} \frac{\hbar^2}{m a_0^2} \frac{1}{n^2}$ |
| Bohr magneton      | $\mu_B = \frac{e}{2m_e}$   |
| Fine structure     | $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$  |
| Compton wavelength | $\lambda = \frac{\hbar}{mc}$   |
| k(photon)          | $\frac{10^4}{\lambda[\mu m]} [cm^{-1}]$  |
| E(photon)          | $\frac{1.242}{\lambda[\mu m]} [eV]$  |

Table 2.4: atom and lattice

| description                      | value   |
|----------------------------------|---|
| Electron frequency in H.O. model | $\omega_0 = \sqrt{\frac{k}{m}} \sim 10^{16} \text{ rad/s}$                                      |
| Atomic excited state lifetime    | 2 ns  |
| lattice constant                 | $a \sim \text{\AA}, \frac{e^2}{a} \sim eV, v \sim \sqrt{\frac{e^2}{am}} \sim 10^8 \text{ cm/s}$ |
| Debye temp.                      | $\sim 300K$   |
| plasmon                          | $\hbar\omega_p = \hbar \left( \frac{4\pi n e^2}{m} \right)^{\frac{1}{2}} \sim 10eV$             |
| minimum band gap                 | C:5.5eV, Si:1.1eV, Ge:0.76eV  |

Table 2.5: optics

| description        | value                    |
|--------------------|--------------------------|
| f of visible light | $\omega \sim 10^{15}$ Hz |

Table 2.6: Classification of solids using resistivity

| class         | typical R ( $\Omega \cdot cm$ ) | example |
|---------------|---------------------------------|---------|
| metal         | $10^{-6}$                       | copper  |
| semimetal     | $10^{-3}$                       | bismuth |
| semiconductor | $10^{-2} - 10^9$                | silicon |
| insulator     | $10^{14} - 10^{22}$             | diamond |

Table 2.7: Superconductivity

| description       | value                  |
|-------------------|------------------------|
| transition energy | $10^{-5} - 10^{-2} eV$ |

Table 2.8: Wavelengths of Common Lasers

| Source         | nm                                    |
|----------------|---------------------------------------|
| ArF            | 93                                    |
| KrF            | 48                                    |
| Nd:YAG(4)      | 66                                    |
| XeCl           | 08                                    |
| HeCd           | 25, 441.6                             |
| N <sub>2</sub> | 37.1, 427                             |
| XeF            | 51                                    |
| Nd:YAG(3)      | 54.7                                  |
| Ar             | 88, 514.5, 351.1, 363.8               |
| Cu             | 10.6, 578.2                           |
| Nd:YAG(2)      | 32                                    |
| HeNe           | 32.8, 543.5, 594.1, 611.9, 1153, 1523 |
| Kr             | 47.1, 676.4                           |
| Ruby           | 94.3                                  |
| Nd:Glass       | 060                                   |
| Nd:YAG         | 064, 1319                             |
| Ho:YAG         | 100                                   |
| Er:YAG         | 15 940                                |



# Chapter 3

## Math

### 3.1 Useful Equations

- Associated Laguerre polynomials:

$$L_n^m(x) = (-1)^m \frac{d^m}{dx^m} L_{n+m}(x) \quad \text{for } x \geq 0$$

- Associated Legendre polynomials:

$$P_l^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \quad \text{for } x \geq 0$$

- Beta function

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$$

- Complete elliptic integral of the first kind:

$$K(k) = F(k, \frac{\pi}{2}) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

- Complete elliptic integral of the second kind:

$$K(k, \frac{\pi}{2}) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta$$

- Complete elliptic integral of the third kind:

$$\Pi(\nu, k, \frac{\pi}{2}) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - \nu \sin^2 \theta) \sqrt{1 - k^2 \sin^2 \theta}}$$

- Confluent hypergeometric function:

$$F(a, c, x) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)x^n}{\Gamma(c+n)n!}$$

- Regular modified cylindrical Bessel function:

$$I_{\nu}(x) = i^{-\nu} J_{\nu}(ix) = \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{k! \Gamma(\nu+k+1)} \quad \text{for } x \geq 0$$

- Cylindrical Bessel function of the first kind:

$$J_{\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k! \Gamma(\nu+k+1)} \quad \text{for } x \geq 0$$

- Irregular modified cylindrical Bessel function:

$$K_{\nu}(x) = \frac{\pi}{2} i^{\nu+1} (J_{\nu}(ix) + i N_{\nu}(ix)) = \begin{cases} \frac{I_{-\nu}(x) - I_{\nu}(x)}{\sin \nu \pi} & \text{for } x \geq 0 \text{ and } \nu \notin Z \\ \frac{\pi}{2} \lim_{\mu \rightarrow \nu} \frac{I_{-\mu}(x) - I_{\mu}(x)}{\sin \mu \pi} & \text{for } x < 0 \text{ and } \nu \in Z \end{cases}$$

- Cylindrical Neumann function of the first kind

$$N_{\nu}(x) = \begin{cases} \frac{J_{\nu}(x) \cos \nu \pi - J_{-\nu}(x)}{\sin \nu \pi} & \text{for } x \geq 0 \text{ and } \nu \notin Z \\ \lim_{\mu \rightarrow \nu} \frac{J_{\mu}(x) \cos \mu \pi - J_{-\mu}(x)}{\sin \mu \pi} & \text{for } x < 0 \text{ and } \nu \in Z \end{cases}$$

- Incomplete elliptic integral of the first kind:

$$F(k, \phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad \text{for } |k| \leq 1$$

- Incomplete elliptic integral of the second kind:

$$E(k, \phi) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \theta} d\theta \quad \text{for } |k| \leq 1$$

- Incomplete elliptic integral of the third kind:

$$\Pi(k, \nu, \phi) = \int_0^{\phi} \frac{d\theta}{(1 - \nu \sin^2 \theta) \sqrt{1 - k^2 \sin^2 \theta}} \quad \text{for } |k| \leq 1$$

- Exponential integral:

$$Ei(x) = - \int_{-x}^{\infty} \frac{e^{-t}}{t} dt$$

- Hermite polynomials:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

- Hypergeometric series:

$$F(a, b, c, x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{x^n}{n!}$$

- Laguerre polynomials:

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) \quad \text{for } x \geq 0$$

- Legendre polynomials:

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l \quad \text{for } |x| \leq 1$$

- Riemann zeta function:

$$Z(x) = \begin{cases} \sum_{k=1}^{\infty} k^{-x} & \text{for } x > 1 \\ 2^x \pi^{x-1} \sin\left(\frac{x\pi}{2}\right) \Gamma(1-x) \zeta(1-x) & \text{for } x < 1 \end{cases}$$

- Spherical Bessel functions of the first kind:

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x) \quad \text{for } x \geq 0$$

- Spherical associated Legendre function:

$$Y_l^m(\theta, \phi) = (-1)^m \left[ \frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{\frac{1}{2}} P_l^m(\cos \theta) e^{im\phi} \quad \text{for } |m| \leq l$$

- Spherical Neumann functions, Spherical Bessel functions of the second kind:

$$n_n(x) = \left( \frac{\pi}{2x} \right)^{\frac{1}{2}} N_{n+\frac{1}{2}}(x) \quad \text{for } x \geq 0$$

Number of independent solution to a derivative equation ???

## 3.2 Definition

### Levi-Civita

$\epsilon_{ijk}$  is an antisymmetric tensor, which is defined as:

$$\epsilon_{123} = 1, \epsilon_{ijj} = 0, \epsilon_{ijk} = -\epsilon_{jik} \quad (3.1)$$

### Fourier Transform

$$\begin{aligned} \tilde{f}(k) &= \int_{-\infty}^{\infty} dx e^{ikx} f(x) \\ f(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} \tilde{f}(k) \end{aligned} \quad (3.2)$$

If  $f(x) = f(x + R)$ , then

$$\tilde{f}(k) = \int dx e^{ikx} f(x) = e^{-ikR} \int d(x + R) e^{ik(x+R)} f(x + R) = e^{-ikR} \tilde{f}(k) \quad (3.3)$$

So,  $\tilde{f}(k)$  is non-zero only at point with  $kR = 2n\pi$ , or  $k = \frac{2\pi}{R}n$ .

$$f(x) = \sum_k e^{-ikx} \tilde{f}(k) \quad (3.4)$$

## 3.3 Vector Operation

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \quad (3.5)$$

Two point Fourier transformation:

$$f(\mathbf{r}, \mathbf{r}') = \frac{1}{\Omega} \sum_{\mathbf{q}, \mathbf{q}'} e^{i\mathbf{q} \cdot \mathbf{r}} f(\mathbf{q}, \mathbf{q}') e^{-i\mathbf{q}' \cdot \mathbf{r}'} \quad (3.6)$$

## 3.4 Formulas

### 3.4.1 Fourier transform

$$\delta^3(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}}$$

### 3.4.2 Limitation

For

$$f(\omega) = \frac{\sin(N\omega T)}{\sin(\omega T)} \quad (\omega > 0)$$

when  $N \rightarrow \infty$ , we can find that the dominant value lies in

$$\omega = \frac{m\pi}{T} \quad m = 1, 2, 3 \dots$$

with  $\omega = \frac{m\pi}{T} + \Delta\omega$ , we can get:

$$\lim_{N \rightarrow \infty} f(\omega) \approx \frac{(-1)^m \sin(N\Delta\omega T)}{(-1)^m \Delta\omega T} \quad (3.7)$$

Note:  $N\Delta\omega T$  is not infinity small, so we need to keep  $\sin(\cdot)$ .

We also find that:

$$\int_{-\epsilon}^{\epsilon} d(\Delta\omega) \frac{\sin(N\Delta\omega T)}{\Delta\omega T} = \frac{1}{T} \int_{-N\epsilon T}^{N\epsilon T} dx \frac{x}{x} = \frac{\pi}{T} \quad (3.8)$$

So:

$$\lim_{N \rightarrow \infty} f(\omega) \approx \frac{(-1)^m \sin(N\Delta\omega T)}{(-1)^m \Delta\omega T} = \sum_{m=1} \frac{\pi}{T} \delta(\omega - \omega_m) \quad (3.9)$$

Similarly:

$$\lim_{N \rightarrow \infty} f_1(\omega) = \left( \frac{\sin(N\omega T)}{\omega T} \right)^2 \approx \left( \frac{(-1)^m \sin(N\Delta\omega T)}{(-1)^m \Delta\omega T} \right)^2 \quad (3.10)$$

$$\int_{-\epsilon}^{\epsilon} d(\Delta\omega) \left( \frac{\sin(N\Delta\omega T)}{\Delta\omega T} \right)^2 = \frac{N}{T} \int_{-N\epsilon T}^{N\epsilon T} dx \frac{x}{x} = \frac{N\pi}{T} \quad (3.11)$$

$$f_1(\omega) = \sum_{m=1} \frac{N\pi}{T} \delta(\omega - \omega_m) \quad (3.12)$$

$$\lim_{\alpha \rightarrow 0^+} \frac{1}{x \pm i\alpha} = P \left( \frac{1}{x} \right) \mp i\pi\delta(x) \quad (3.13)$$

### 3.4.3 Integral

$$I = \int_{-\infty}^{\infty} dk \frac{e^{ikr} - e^{-ikr}}{k}$$

Note that the integrand does not blow up as  $k \rightarrow 0$ , so

$$I = \lim_{\delta \rightarrow 0} \left[ \int_{-\infty}^{\infty} dk \frac{e^{ikr} - e^{-ikr}}{k + i\delta} \right]$$

So we have pole at  $k = -i\delta$ . For  $e^{ikr}$  we must close the contour up to get exponential decay at large  $k$ . This misses the pole, so this term gives zero. So

$$I = \int_{-\infty}^{\infty} dk \frac{-e^{-ikr}}{k + i\delta} = -(2\pi i)(-e^{-\delta r}) = 2\pi i e^{-i\delta r}$$

### Useful Integrals

$$\int_0^{\infty} e^{-\alpha x} x^n dx = \frac{n!}{\alpha^{n+1}}$$

## 3.5 Functions

### 3.5.1 Interesting functions

#### Weierstrass function

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x) \quad (3.14)$$

Continuous but not derivative everywhere.

### 3.5.2 Frequently used functions

#### Legendre Polynomials and spherical harmonics

$$Y_l^m(\theta, \phi) = N e^{im\phi} P_l^m(\cos \theta) \quad (3.15)$$

which satisfies

$$r^2 \nabla^2 Y_l^m(\theta, \phi) = -l(l+1) Y_l^m(\theta, \phi)$$

Standard convention :

$$P_l^{-m}(\cos \theta) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(\cos \theta),$$

In QM:

$$Y_l^m(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{lm}(\cos \theta) e^{im\phi}$$

Where  $P_{lm}$  are associated Legendre polynomials without the Condon-Shortley phase.

Though of different definition, all of them satisfy

$$Y_l^{m*}(\theta, \phi) = (-1)^m Y_l^{-m}(\theta, \phi)$$

Associate Legendre Polynomials:

### 3.5.3 Bessel Function

### 3.5.4 Differential Equations

#### First Order Equation

$$\frac{dV}{dt} + \eta(t)V(t) = f(t)$$

so:

$$\frac{1}{I} \frac{d(I(t)V(t))}{dt} = \frac{dV(t)}{dt} + \frac{I'(t)}{I(t)} V(t)$$

if:

$$\frac{I'(t)}{I(t)} = \eta(t) \rightarrow I(t) = \exp\left(\int^t dt' \eta(t')\right)$$

so:

$$\frac{d(I(t)V(t))}{I dt} = f(t)$$

#### inhomogeneous 2nd order differential equation

$$\frac{d^2x}{dt^2} + a(t)\frac{dx}{dt} + b(t)x = f(t) \quad (3.16)$$

Its solution is given by

$$x(t) = c_1\phi_1(t) + c_2\phi_2(t) + \int_{t_0}^t \frac{\phi_1(\xi)\phi_2(t) - \phi_2(\xi)\phi_1(t)}{W(\xi)} f(\xi) d\xi \quad (3.17)$$

Where  $\phi_1$  and  $\phi_2$  are the 2 orthogonal solutions to the homogeneous 2nd differential equation, and  $W(t)$  is the Wronskian.

$$W(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{vmatrix} \quad (3.18)$$

### Pendulum Equation

$$\begin{aligned}\frac{d^2}{dz^2}\psi + u\cos\psi &= 0 \\ \psi(z) &= \psi(0) + \psi'(0)z - u \int_0^z dz_1 \int_0^{z_1} \cos\psi(z_2)dz_2\end{aligned}\tag{3.19}$$

### Floquet theory

$$\ddot{x} + \omega_0^2(1 + \mu\cos(\nu t))x = 0\tag{3.20}$$

Analogy to QM(Schrodinger Equation):

$$-\frac{\hbar^2}{2m}\psi'' + U(x)\psi = E\psi$$

where  $U(x) = -U_0\cos\frac{2\pi x}{a}$ . So let

$$k^2 = \frac{2mE}{\hbar^2}, u = \frac{U_0}{E}, \nu = \frac{2\pi}{a}$$

$u$  is small, we will get

$$\psi'' + k^2\psi = -k^2u\cos(\nu x)\psi$$

### 3.5.5 Integrated functions

$\Gamma$  function

$$\begin{aligned}\Gamma(z) &\equiv \int_0^{+\infty} dt e^{-t} t^{z-1} = 2 \int_0^{+\infty} dx e^{-x^2} x^{2z-1} \\ \Gamma(z)\Gamma(1-z) &= \frac{\pi}{\sin\pi z}\end{aligned}$$

## 3.6 Group

### 3.6.1 Representation of Group

A **representation** of a group  $G$  on a vector space  $V$  over a field  $K$  is a group homomorphism from  $G$  to  $GL(V)$ , the



## 3.7 Probability Distribution Functions

### 3.7.1 Uniform

### 3.7.2 Exponential

### 3.7.3 Gauss

### 3.7.4 BreitWigner

### 3.7.5 Poisson

## 3.8 Dircvative Functions

RC Oscillation

$$\dot{Q} + \frac{Q}{RC} = I(t) = I_0 e^{-t/\tau} \quad (3.21)$$

to get:

$$Q(t) = A(e^{-t/\tau} - e^{-t/\tau_0}) \quad (3.22)$$

where:

$$\tau_0 = RC$$

If

$$\tau = \tau_0$$

, then the solution is different:

$$Q(t) = I_0 t e^{-t/\tau_0} + A e^{-t/\tau_0} \quad (3.23)$$

# Chapter 4

## Model

### 4.1 Model

Boltzmann distribution

#### 4.1.1 Oscillation

#### 4.1.2 Binary system

#### 4.1.3 Scattering

#### 4.1.4 Drude model

$$m\ddot{\mathbf{x}} = -e\mathbf{E} - \frac{m\dot{\mathbf{x}}}{\tau} \quad (4.1)$$

Where  $\tau$  is the relaxation time, Solve this equation, we will get

$$0 = -e\mathbf{E} - \frac{m\dot{\mathbf{x}}}{\tau} \implies \dot{\mathbf{x}} = \frac{-e\mathbf{E}\tau}{m}$$

This means that if electron reach velocity  $\dot{x} = -eE\tau/m$ , it will move at this constant velocity.

Note the difference between the Drude model and the "accelerating model", which will accelerates electron periodically, with period being the relaxation time  $\tau_1$ , so the average velocity will be:

$$\bar{v} = \frac{1}{2} \frac{-eE\tau_1}{m}$$

So

$$\dot{x} = \bar{v} \implies \tau = \frac{1}{2}\tau_1$$

## 4.2 Approximation

The ultimate weapon of simplifying a question (constructing a model) is approximation. Here are some frequently used approximation.

### 4.2.1 RWA: Rotating Wave Approximation

$$\cos(\omega t)e^{-i\omega_0 t} = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})e^{-i\omega_0 t} = \frac{1}{2}(e^{i(\omega-\omega_0)t} + e^{-i(\omega+\omega_0)t}) \approx \frac{1}{2}e^{i(\omega-\omega_0)t}$$

When  $\omega \sim \omega_0$ , we simply discard the second term because  $\omega + \omega_0 \gg \delta = \omega - \omega_0$ . We can think about it as that the second term oscillates so fast that nothing can response to it (response is small), thus the first term is more effective than the second term.

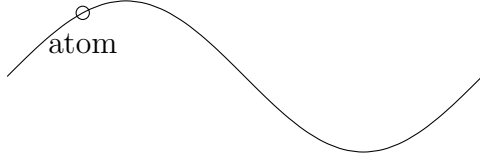
### 4.2.2 Electric Dipole Approximation

For the interaction of light with atom, the wavefunction vanish exponentially on the distance scale of the Bohr radius  $a_0$ .

$$|\vec{k} \cdot \vec{r}| \approx 2\pi a_0 / \lambda \ll 1$$

$$e^{i\vec{k} \cdot \vec{r}} = 1 + i\vec{k} \cdot \vec{r} + \mathcal{O}(\vec{k} \cdot \vec{r})^2$$

Which can be easily seen from the point that an atom is much smaller than the wavelength of light (visible light), so we can think that the electric field applied on the atom is constant at specific time.



### 4.2.3 Slowly Varying Envelope Approximation

The fractional change of the beam's transverse profile over a wavelength distance to be small compared with unity, and similarly for its derivative, which si:

$$\lambda \left| \frac{\partial E_0}{\partial z} \right| \ll |E_0| \text{ and } \lambda \left| \frac{\partial^2 E_0}{\partial z^2} \right| \ll \left| \frac{\partial E_0}{\partial z} \right|$$

#### 4.2.4 Harmonic Approximation

$$U = U_0 + \sum_{n,\alpha} \frac{\partial U}{\partial X_n^\alpha} X_n^\alpha + \frac{1}{2} \sum_{n\alpha;n'\alpha'} \frac{\partial^2 U}{\partial X_n^\alpha \partial X_{n'}^{\alpha'}} X_n^\alpha X_{n'}^{\alpha'} + \dots$$

Truncate the series after second order.

### 4.3 Distribution

#### 4.3.1 Boltzmann Distribution

# Part II

## Definition and Conclusion

# Chapter 5

## Definition

Definition and Conclusion.

### 5.1 General

#### 5.1.1 index

Entries  $(\mathcal{L})_{\mu\nu}$  of a matrix are labelled by rows  $(\mu)$  and columns  $(\nu)$ . You are **free** to move this row and column index up and down at will – what matters is the order of index:

$$(\mathcal{L})_{\mu\nu} = (\mathcal{L}^T)_{\mu\nu} = (\mathcal{L})^\mu{}_\nu = (\mathcal{L}^T)_\nu{}^\mu = L^\mu{}_\nu$$

The much **preferred** placement of the indices surrounding the matrix is just a visual reminder of the individual entries  $L^\mu{}_\nu$  which together form the matrix  $(\mathcal{L})$  and  $(\mathcal{L}^T)$ , and that's all.

The indices labelling  $L^\mu{}_\nu$  can not be raised or lowered randomly, but are raised and lowered with the metric tensor:

$$(g\mathcal{L})_{\mu\nu} = g_{\mu\rho}L^\rho{}_\nu \equiv L_{\mu\nu}$$

and

$$(g\mathcal{L}g)_\mu{}^\nu = g_{\mu\rho}L^\rho{}_\sigma g^{\sigma\nu} \equiv L_\mu{}^\nu$$

## Vector Index

$$\begin{aligned} x^i &= (x, y, z); & x_i &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ x^\mu &= (t, x, y, z); & x_\mu &= \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \end{aligned} \quad (5.1)$$

### 5.1.2 Cross Section

$$\sigma = \frac{\text{scattered power}}{\text{incident intensity}} \quad (5.2)$$

quasi-periodic:

$$\phi(x + a) = e^{ika} \phi(x) \quad (5.3)$$

There is a phase shift, so it is not rigorously periodic.

Rotation: The algebraic characterization  $R^T R = \mathbb{I}$  is a much more useful definition of the group than the explicit form of the rotation matrices as a function of  $\theta$ . Another way to define the rotation group is as the set of linear transformations on  $\mathcal{R}^n$  preserving the inner product  $x^i x_i = \delta_{ij} x^i x^j$ :

$$R_{ki} R_{lj} \delta_{kl} = (R^T)_{ik} \delta_{kl} R_{lj} = [(R^T) \delta (R)]_{ij} = (R^T R)_{ij} = \delta_{ij} \quad (5.4)$$

For an oscillating field  $E(t) = E_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}$ , the time average is:

$$\langle E \rangle = \frac{1}{T} \int_0^T E(t) dt = \frac{1}{2} E_0 e^{i\vec{k} \cdot \vec{x}} \quad (5.5)$$

A plain wave  $E = E_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}$  (such as the Fourier mode of a  $\delta$  function) has **no** starting or ending point and it fulfills all the space. So understanding the scatter of a plain wave is some tricky thing.

Velocity:

$$\vec{v} = \nabla_{\vec{p}} E(|\vec{p}|) \quad (5.6)$$

**Lagrangian (Hamiltonian)** is the Legendre transform of the Hamiltonian (Lagrangian)

$$L(q, \dot{q}) = p(q, \dot{q}) \dot{q} - H(q, p(q, \dot{q})), \quad H(q, p) = p \dot{q}(q, p) - L(q, \dot{q}(q, p)) \quad (5.7)$$

where  $p(q, \dot{q})$  is defined as  $\frac{\partial H}{\partial p} = \dot{q}$  and  $\dot{q}(q, p)$  is defined as  $\frac{\partial L(q, \dot{q})}{\partial \dot{q}} = p$

### 5.1.3 Energy

Power

$$\frac{dP}{d\Omega} = \frac{dW}{dt \cdot d\Omega} = r^2 \frac{dW}{dt dA} \quad (5.8)$$

### 5.1.4 Potential

Yukawa potential (screened Coulomb potential)

$$V = V_0 \frac{e^{-\mu r}}{r} \quad (5.9)$$

### 5.1.5 Monte Carlo

**Monte Carlo Method** A method to search for solutions to mathematical problem using a statistical sampling with random numbers.

### 5.1.6 Dirac Delta

**Dimension** Note that  $\delta^3(x)$  and  $\delta(x)$  have different dimensions.

$$[\delta^3(x)] = L^{-3}, \quad [\delta(x)] = L^{-1} \quad (5.10)$$

### 5.1.7 Acronyms

h.c.

Hermit conjugate

## 5.2 CM

## 5.3 QM

Heisenberg EOM:

$$i\partial_t \phi(x) = [\phi, H] \quad (5.11)$$

S-matrix

$$\langle f | S | i \rangle_{heisernberg} = \langle f; \infty | i; -\infty \rangle_{Schrodinger} \quad (5.12)$$



## Harmonic basis

$$\hat{\epsilon}_{-1} = \frac{\hat{x} - i\hat{y}}{\sqrt{2}} \quad \hat{\epsilon}_0 = \hat{z} \quad \hat{\epsilon}_1 = -\frac{\hat{x} + i\hat{y}}{\sqrt{2}}$$

$\hat{\epsilon}_0$  refers to light linearly polarized in the z-direction ( $\pi$ -light). The polarizations  $\epsilon_{\pm 1}$  refer to circularly polarized light with  $-1$  for left-handed ( $\sigma^-$ -light) and  $+1$  for right-handed ( $\sigma^+$ -light)

$$\hat{\epsilon}_q \cdot \vec{r} = \sqrt{\frac{4\pi}{3}} r Y_{1q}(\theta, \phi)$$

### 5.3.1 Symmetry

### 5.3.2 Perturbation Theory

### 5.3.3 Scattering

## Lippmann-Schwinger Equation

$$\begin{cases} i\hbar \partial_t \psi^\dagger(t, \vec{x}) = H \psi^\dagger(t, \vec{x}) \\ \psi^\dagger(t = -\inf, x) = \phi(t, x) = \frac{1}{(2\pi)^3/2} e^{-\frac{i}{\hbar} E t + i \vec{k} \cdot \vec{x}} \end{cases} \quad (5.13)$$

## 5.4 Relativity

Postulates:

- All inertial observers have the same equations of motion and the same physical laws.
- The speed of light is constant for all inertial frames.

Assuming internal frame  $K'$  moves with velocity  $\vec{v} = v_0 \hat{x}$  with respected to rest internal frame  $K$ .

### 5.4.1 Definition

4-vector

$$x^\mu \equiv (ct, x^1, x^2, x^3) = (ct, \vec{x}) \quad (5.14)$$

In calculation, it's usually regarded as a column vector:

$$x^\mu = \begin{pmatrix} ct \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad (5.15)$$

4-velocity

$$U^\mu \equiv \frac{dx^\mu}{d\tau} = (\gamma c, \gamma \vec{v}) \quad (5.16)$$

**metric tensor**

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (5.17)$$

$\beta, \gamma$

$$\begin{aligned} \beta &\equiv \frac{v_0}{c} \\ \gamma &\equiv \frac{1}{\sqrt{1-\beta^2}} \Rightarrow \gamma \geq 1 \end{aligned} \quad (5.18)$$

Play of  $\beta$  and  $\gamma$

$$\begin{aligned} \gamma^2 - \beta^2 \gamma^2 &= \gamma^2 (1 - \beta^2) = 1 \\ 1 - \beta^2 &= (1 + \beta)(1 - \beta) \xrightarrow{\beta \sim 1} 2(1 - \beta) \\ \beta \sim 0 &\Rightarrow \gamma \sim 1 \end{aligned} \quad (5.19)$$

**Proper time**

$$d\tau = \frac{1}{\gamma} dt \quad (5.20)$$

## 5.4.2 Transformation

**Spacetime**

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad (5.21)$$

## Velocity

$$\begin{aligned}c &= c \\v'^x &= \frac{v^x - v_0}{1 - v^x v_0 / c^2} \\v'^y &= \frac{v^y}{\gamma(1 - v^x v_0 / c^2)} \\v'^z &= \frac{v^z}{\gamma(1 - v^x v_0 / c^2)}\end{aligned}\tag{5.22}$$

### 5.4.3 Examples

A object moves in velocity  $\vec{v} = v_0 \hat{x}$ , then in the object rest frame  $K'$ , we measure its length as  $l_0$ ; and the proper time for an event as  $d\tau$ ; then in the lab frame, what we measure are  $L$  and  $dt$ :

length contract effect

$$L = \frac{l_0}{\gamma}\tag{5.23}$$

time dilation

$$dt = \gamma d\tau\tag{5.24}$$

## 5.5 QFT

## 5.6 Nucleus

Radiation length

Mean free path

$dE/dx$

Decay rate

$$\Gamma = 1/\tau\tag{5.25}$$

## 5.7 HEP

Fermi's golden rule: the transition rate between two states is proportional to the matrix element squared:

$$\Gamma \approx |\mathcal{M}|^2 \delta(E_f - E_i) \quad (5.26)$$

The matrix element

$$\mathcal{M} = \langle f | H_{int} | i \rangle \quad (5.27)$$

## 5.8 Miscellaneous

**adjoint representation**

**analyticity**

**associativity of addition and multiplication**  $a + (b + c) = (a + b) + c$  , and  $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

**bilinear** In QFT, a bilinear term means it has exactly two fields. Such as:

$$\mathcal{L}_K \supset \frac{1}{2} \phi \square \phi, \frac{1}{4} F_{\mu\nu}^2, \frac{1}{2} m^2 \phi^2, \frac{1}{2} \phi_1 \square \phi_2, \phi_1 \partial_\mu A_\mu, \dots$$

**Boltzmann distribution**  $n_i = N e^{-\beta E_i}$

**causality**

**charge conjugation C** taking particles to antiparticles

$$C : \quad \psi \rightarrow -i\gamma_2 \psi^* \equiv \psi_C$$

In the Weyl basis,  $\gamma_2^* = -\gamma_2$  and  $\gamma_2^T = \gamma_2$ , so

$$C : \quad \psi^* \rightarrow -i\gamma_2 \psi$$

**chirality** The handedness of a spinor is referred to as its chirality. The **left-handed** and **right-handed** refer to the  $(\frac{1}{2}, 0)$  or  $(0, \frac{1}{2})$  representations of the Lorentz group. This concept only exists for spinors, or more precisely for (A,B) representations of the Lorentz group with  $A \neq B$ . Almost always, chirality means that a theory is not symmetric between left-handed Weyl spinors  $\psi_L$  and right-handed spinors  $\psi_R$

**cluster decomposition principle**

**commutativity of addition and multiplication**  $a + b = b + a$  and  $a \cdot b = b \cdot a$

**contravariant vectors** vectors with upper indices

**covariant vectors** vectors with lower indices

**Dirac spinors**

**Dirac Lagrangian**

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi$$

**Dirac equation**

$$(i\not{D} - m)\psi = 0$$

**Dirac Mass**

$$\mathcal{L}_{\text{Dirac mass}} = m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$$

**Distributivity of multiplication over addition**  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

**equipartition theorem** a body in thermal equilibrium should have energy equally distributed among all possible modes, (mode is a separation of phase space), which means all modes have the same energy.

**faithful representation** A representation in which each group element gets its own matrix is called a *faithful representation*.

**Fermi's golden rule**  $\Gamma \sim |\mathcal{M}|^2 \delta(E_f - E_i)$

**first principle**

**Gauge transform**  $\phi \rightarrow e^{-i\alpha} \phi$

**good quantum number** which doesn't change along time.  $[Q, H] = 0$

**Grassmann Numbers** For Majorana masses to be non-trivial, fermion components cannot be regular numbers, they must be anticommuting numbers. Such things are called Grassmann numbers.

**Helicity** spin projected on the direction of motion is called the helicity.  $\hat{h} = \frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}$

**Legendre transformation**

**Levi-Civita**  $\epsilon_{ijk} = \begin{cases} 1 & \text{normalorder}(123, 231, 312) \\ -1 & \text{reverseorder}(132, 213, 321) \\ 0 & \text{otherwise} \end{cases}$

**Lie algebra** the generators of the Lie group form an algebra called its Lie algebra.

**Lie groups** Lie groups are a class of groups, including the Lorentz group, with an infinite number of elements but a finite number of generators.

**lightlike**  $V^\mu V_\mu = 0$

**Lippmann-Schwinger eqn** In P.T.

$$H = H_0 + V(\text{correction})$$

If

$$H_0 |\phi\rangle = E |\phi\rangle, \quad \text{and} \quad H |\psi\rangle = E |\psi\rangle$$

Then

$$|\psi\rangle = |\phi\rangle + \frac{1}{E - H_0} V |\psi\rangle$$

This is called the **Lippmann-Schwinger equation**. And the inverted object is a kind of Green's function known as the **Lippmann-Schwinger kernel**:

$$\prod_{LS} = \frac{1}{E - H_0}$$

Define operator T, so that

$$V |\psi\rangle = T |\phi\rangle$$

So

$$|\psi\rangle = |\phi\rangle + \frac{1}{E - H_0} T |\phi\rangle$$

and

$$T = V + V \frac{1}{E - H_0} T$$

Solve it perturbatively in V:

$$T = V + V \frac{1}{E - H_0} V + V \frac{1}{E - H_0} V \frac{1}{E - H_0} V + \dots = V + V \prod_{LS} V + V \prod_{LS} V \prod_{LS} V + \dots$$

**little group** The representation of the full Poincaré group is induced by a representation of the subgroup of the Poincaré group that holds  $p^\mu$  fixed, called the *little group*. When  $P_\mu$  is massive, the little group is  $SO(3)$ ; when  $p_\mu$  is massless, the little group is  $ISO(2)$ .

**locality**

**Lorentz group** this is the generalization of the rotation group to include both rotations and boosts.

**Majorana spinor** A spinor whose antiparticle is itself.

**Majorana masses**

$$\mathcal{L} = i\psi_L^\dagger \sigma_\mu \partial_\mu \psi_L + i\frac{m}{2}(\psi_L^\dagger \sigma_2 \psi_L^* - \psi_L^T \sigma_2 \psi_L)$$

The mass terms in this Lagrangian are called **Majorana masses**.

**Majorana fermions** in Dirac spinors:

$$\psi = \begin{pmatrix} \psi_L \\ i\sigma_2 \psi_L^* \end{pmatrix}$$

**pseudo scalar** particles with odd **parity**

**quantize** promote  $x$  and  $p$  as operators and impose the canonical commutation relations:  $[x, p] = i$

**quantum process** time evolution of an open quantum system ???

**Rarita-Schwinger field** spin- $\frac{3}{2}$

**representation** A set of objects that mix under a transformation group is called a representation of the group, though technically the matrix embedding is the representation. A representation is a particular embedding of group elements into operators that act on a vector space. For finite-dimensional representations, this means an embedding of the  $g_i$  into matrices.

**second quantization** canonical quantization of relativistic fields,

$$H_0 = \int \frac{d^3p}{(2\pi)^3} \omega_p (a_p^\dagger a_p + \frac{1}{2})$$

First quantization refer to the discrete mode, for example, of a particle in a box. Second quantization refers to the integer numbers of excitations of each of these modes. There are two features in second quantization:

1. We have many quantum mechanical systems - one for each  $\vec{p}$  - all at the same time.
2. We interpret the  $n$ th excitation of the  $\vec{p}$  harmonic oscillator as having  $n$  *particles*.

### **S-matrix**

**spacelike**  $V^\mu V_\mu < 0$

**SO(n)** the group of nD rotations ( $\det(R) = 1$ )

**time dilation** In relativity, time dilation is a difference in the elapsed time measured by two observers.

**timelike**  $V^\mu V_\mu > 0$

**unitary**  $\Lambda^\dagger \Lambda = 1$

**Weyl spinor** Irreducible unitary spin- $\frac{1}{2}$  representations of the Poincaré group.

$\gamma$  **matrix** In the Weyl basis:

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}$$

$\gamma^5$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

In the Weyl representation:

$$\gamma^5 = \begin{pmatrix} -\infty & \\ & \infty \end{pmatrix}$$



Momentum of wave ???

# Chapter 6

## EM

Table 6.1: Definition

| description     | SI  | cgs   |
|-----------------|---|---|
| c               | $c = \sqrt{\frac{1}{\mu_0 \epsilon_0}}$   | $c = \sqrt{\frac{\mu_0}{\epsilon_0}}$             |
| radiation power | $P = \frac{2}{3} \frac{q^2 a^2}{4\pi \epsilon_0 c^3} = \frac{q^2 a^2}{6\pi \epsilon_0 c^3}$ | $P = \frac{2}{3} \frac{q^2 a^2}{c^3}$             |
| Poynting vector | $\vec{S} = \vec{E} \times \vec{H} = \vec{E} \times \vec{B} / \mu_0$                         | $\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H}$ |
| E & M           | $\vec{B} = \frac{1}{c} \hat{n} \times \vec{E}$  | $\vec{H} = \hat{n} \times \vec{E}$                |

### 6.1 General

degree of polarization:

$$\text{degree of pol} \equiv \frac{|\sigma_{\perp} - \sigma_{//}|}{\sigma_{\perp} + \sigma_{//}} \quad (6.1)$$

### 6.1.1 Gauge

#### Coulomb Gauge

$$\nabla \cdot \vec{A} = 0 \quad (6.2)$$

#### Lorentz Gauge

$$\nabla \cdot \vec{A} + \partial_t \varphi = 0 \quad (6.3)$$

### 6.1.2 Maxwell Equation

Maxwell Equations in SI units:

$$\begin{aligned} \nabla \cdot E &= \frac{\rho}{\epsilon_0} \\ \nabla \times B &= \frac{1}{c^2} \partial_t E + \mu_0 J \\ \nabla \cdot B &= 0 \\ \nabla \times E &= -\partial_t B \end{aligned} \quad (6.4)$$

$$\left\{ \begin{array}{l} \nabla \cdot E = \frac{\rho}{\epsilon} \\ \nabla \times B = \frac{1}{c} (J + J_D + J_{ind}) \\ \nabla \cdot B = 0 \\ \nabla \times E = -\frac{1}{c} \partial_t B \end{array} \right. \longrightarrow \left\{ \begin{array}{l} \nabla \cdot D = \rho \\ \nabla \times H = \frac{1}{c} J + \frac{1}{c} \partial_t D \\ \nabla \cdot B = 0 \\ \nabla \times E = -\frac{1}{c} \partial_t B \end{array} \right. \quad (6.5)$$

In vacuum environment ( $\epsilon_0 = \mu_0 = 1$ ):

$$\left\{ \begin{array}{l} \nabla \cdot \vec{E} = \rho \\ \nabla \times \vec{B} = \frac{1}{c} \vec{J} + \frac{1}{c} \partial_t \vec{E} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\frac{1}{c} \partial_t \vec{B} \end{array} \right. \quad (6.6)$$

$$\begin{aligned} \nabla \cdot \vec{B} = 0 &\Rightarrow \vec{B} = \nabla \times \vec{A} \\ \nabla \times \vec{E} = -\frac{1}{c} \partial_t \vec{B} &\Rightarrow \vec{E} = -\frac{1}{c} \partial_t \vec{A} - \nabla \varphi \end{aligned}$$

The source equation

$$-\frac{1}{c}\partial_t(\nabla \cdot \vec{A}) - \nabla^2 \varphi = \rho$$

$$\nabla(\nabla \cdot \vec{A} + \frac{1}{c}\partial_t \varphi) + \frac{1}{c^2}\partial_t^2 \vec{A} - \nabla^2 \vec{A} = \frac{\vec{j}}{c}$$

**Coulomb Gauge:**  $\nabla \cdot \vec{A} = 0$

$$-\nabla^2 \varphi = \rho$$

$$\nabla(\frac{1}{c}\partial_t \varphi) + \square \vec{A} = \frac{\vec{j}}{c}$$

If  $\vec{j} = 0$ :

$$-\nabla^2 \varphi = \rho$$

$$\partial_t \varphi = 0$$

Green function:

$$\varphi = \frac{q}{4\pi|\vec{r} - \vec{r}_0|}$$

If  $\rho = 0$ :

$$\varphi = 0$$

$$\square \vec{A} = \vec{j}/c$$

**Lorentz Gauge:**  $\nabla \cdot \vec{A} + \frac{1}{c}\partial_t \varphi = 0$

$$\square \varphi = \rho$$

$$\square \vec{A} = \vec{j}/c$$

Far field ( $r \gg r_0$ , retarded time T):

$$\varphi(t, \vec{x}) = \frac{1}{4\pi r} \int d\vec{r}_0 \rho(T, \vec{r}_0)$$

$$\vec{A}(t, \vec{x}) = \frac{1}{4\pi r} \int d\vec{r}_0 \vec{j}(T, \vec{r}_0)/c$$

## 6.2 Radiation

**Retarded Time**

$$T = t - \frac{|\vec{r} - \vec{r}_0|}{c} \stackrel{r \gg r_0}{\approx} t - \frac{r}{c} + \frac{\vec{n} \cdot \vec{r}_0}{c} \quad (6.7)$$

**E&B field (cgs)**

$$\vec{E}(\vec{r}, t) = q \left( \frac{\hat{n} - \vec{\beta}}{\gamma^2(1 - \vec{\beta} \cdot \hat{n})^3 R^2} \right)_{ret} + \frac{q}{c} \left( \frac{\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]}{(1 - \vec{\beta} \cdot \hat{n})^3 R} \right)_{ret} \quad (6.8)$$

$$\vec{B} = \hat{n} \times \vec{E}$$

where  $\vec{\beta} = \frac{\vec{v}}{c}$ , the first term is proportional to  $\frac{1}{R^2}$ , while the second term is proportional to  $\frac{1}{R}$

**Radiation Power**

$$\frac{dP}{dA} = \vec{S} = c \vec{E} \cdot \vec{B} = \frac{e^2}{16\pi^2} \frac{a^2}{c^3} \cos^2 \theta \quad (6.9)$$

$$\vec{E}_{rad} = -\frac{1}{c} \partial_t \vec{A}_{rad} - \nabla \phi_{rad} = \hat{n} \times \hat{n} \times \frac{1}{c} \partial_t \vec{A}_{rad} = \frac{e}{4\pi r c^2} \hat{n} \times \hat{n} \times \vec{a} \quad (6.10)$$

## 6.3 Relativity

**covariant** vector: lower indices;

**contravariant** vector: upper indices.

**d'Alembertian**

$$\square = \partial_\mu^2 = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2. \quad (6.11)$$

**timelike:**  $V^\mu V_\mu > 0$ , for example (t, 0, 0, 0)

**lightlike:**  $V^\mu V_\mu = 0$ , for example (p, 0, 0, p)

**spacelike:**  $V^\mu V_\mu < 0$ , for example (0, x, 0, 0)

### 6.3.1 4-Vector

- $x^\mu \equiv (ct, \vec{x})$
- $\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_t, \partial_x, \partial_y, \partial_z)$
- $U^\mu = (u^0, \vec{u}) \equiv \frac{dx^\mu}{d\tau} = (\gamma c, \gamma \vec{v})$
- $p^\mu = (E/c, p_x, p_y, p_z)$
- $J^\mu \equiv (c\rho, \vec{J})$
- $A^\mu \equiv (\varphi, \vec{A})$
- $K^\mu = (\omega/c, \vec{k})$

### 6.3.2 E and B

$$\begin{aligned} E'_{//} &= E_{//} & B'_{//} &= B_{//} \\ E'_{\perp} &= \gamma(E_{\perp} + \beta B_{\perp}) & B'_{\perp} &= \gamma(B_{\perp} - \beta E_{\perp}) \end{aligned} \quad (6.12)$$

#### Retarded Field

$$\begin{cases} \square\phi = \rho \\ \square\vec{A} = \vec{j} \end{cases} \Rightarrow \begin{cases} \phi(t, \vec{r}) = \int d^3r_0 \frac{1}{4\pi r} \frac{\rho(T, \vec{r}_0)}{1 - \vec{\beta}(T) \cdot \vec{n}} \\ \vec{A}(t, \vec{r}) = \int d^3r_0 \frac{1}{4\pi r} \frac{\vec{j}(T, \vec{r}_0)}{1 - \vec{n} \cdot \vec{\beta}(T)} \end{cases} \quad (6.13)$$

### 6.3.3 Radiation

$$\vec{E}(t) = \hat{n} \times \hat{n} \times \frac{1}{c} \partial_t \vec{A}_{ret} = \frac{q}{4\pi r c^2} \left[ \frac{\hat{n} \times (\hat{n} - \vec{\beta}) \times \vec{a}}{(1 - \hat{n} \cdot \vec{\beta})^3} \right]_{ret} \quad (6.14)$$

# Chapter 7

## SM

### 7.1 General

Equipartition theorem: a body in thermal equilibrium should have energy equally distributed among all possible modes. (Similar to the equal probability principle)

In equilibrium, the number density is determined by Boltzmann dist.

$$n_i = N e^{-\beta E_i} \quad (7.1)$$

#### 7.1.1 Blackbody Radiation

Quantum explanation: In mode  $\omega_n$ , it can be excited an integer number  $j$  times, giving energy  $jE_n = j(\hbar\omega_n)$  in that mode. The probability of find that much energy in that mode is the same as the probability of finding energy in anything, proportional to the Boltzmann weight  $\exp(-\text{energy}/k_B T)$ . Thus, the expectation value of energy in each mode is

$$\langle E_n \rangle = \frac{\sum_{j=0}^{\infty} (jE_n) e^{-jE_n\beta}}{\sum_{j=0}^{\infty} e^{-jE_n\beta}} = \frac{-\frac{d}{d\beta} \frac{1}{1-e^{-\hbar\omega_n\beta}}}{\frac{1}{1-e^{-\hbar\omega_n\beta}}} = \frac{\hbar\omega_n}{e^{\hbar\omega_n\beta} - 1} \quad (7.2)$$

where  $\beta = 1/k_B T$

# Part III

## Proofs



# Chapter 8

## Classical Mechanics

### 8.1 Oscillation

#### 8.1.1 Forced Oscillation

$$x'' + \omega_0^2 x = F \cos \omega t \quad (8.1)$$

# Chapter 9

## Electromagnetic Mechanics

### 9.1 Electrostatics

Energy density (9.1)

### 9.2 Magnetostatics

Energy density (9.2)

### 9.3 Formulas

$$\vec{A}_{tr} = \vec{n} \times (\vec{n} \times \vec{A}) = \vec{A} - (\hat{n}\vec{A})\hat{n} = (\mathbb{1} - n \cdot n^T)\vec{A}$$

let  $n$  being the column vector  $n = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

### 9.4 Tensor

the stress tensor  $T^{ij}$  is the force per area. The force per volume  $f^j$  is the minus divergence of the stress tensor

$$f^j = -\partial_i T^{ij}$$

Conservation law:

$$\partial_t g^j + \partial_i T^{ij} = 0$$

where  $g^j$  is the momentum per volume. So  $f^j = \partial_t g^j$ , the force is the time derivative of the momentum. In electrostatic:

$$f^j = \rho E^j$$

So:

$$\rho E^j = \partial_i T_E^{ij}$$

To get

$$T_E^{ij} \equiv -E^i E^j + \frac{1}{2} E^2 \delta^{ij}$$

The force:

$$\begin{aligned} F^j &= \int_v d^3 \vec{r} \rho(\vec{r}) E^j = \int_v d^3 \vec{\nabla} \cdot \vec{E} E^j \\ &= \int_v d^3 \vec{r} (\nabla_i E^i) E^j \\ &= \int_v d^3 \vec{r} [\nabla_i (E^i E^j) - E^i \nabla_i E^j] \\ &= \int_v d^3 \vec{r} [\nabla_i (E^i E^j) - E^i \nabla_j E^i] \\ &= \int_v d^3 \vec{r} [\nabla_i (E^i E^j) - \frac{1}{2} \nabla_j E^2] \\ &= \int_v d^3 \vec{r} [\nabla_i (E^i E^j) - \frac{1}{2} \nabla_i E^2 \delta^{ij}] \\ &= \int_v d^3 \vec{r} \nabla_i [(E^i E^j) - \frac{1}{2} E^2 \delta^{ij}] \\ &= \int_S dS n_i T_E^{ij} \\ &= - \int_S dS n_i T_E^{ij} \end{aligned}$$

For magnetism:

$$T_B^{ij} = -\frac{1}{\mu} B^i B^j + \frac{1}{2\mu} B^2 \delta^{ij}$$

So

$$f_{em}^j = -\partial_i T_E^{ij} - \partial_i T_B^{ij} - \frac{1}{c} (\partial_t D \times B)^j = (\nabla \cdot D) E^j + ((\nabla \times \vec{H}) \times \vec{B})^j - \partial_t D \times B)^j$$

## 9.5 Maxwell Equation

$$\left\{ \begin{array}{l} \nabla \cdot E = \frac{\rho}{\epsilon} \\ \nabla \times B = \frac{1}{c}(J + J_D + J_{ind}) \\ \nabla \cdot B = 0 \\ \nabla \times E = -\frac{1}{c}\partial_t B \end{array} \right. \longrightarrow \left\{ \begin{array}{l} \nabla \cdot D = \rho \\ \nabla \times H = \frac{1}{c}J + \frac{1}{c}\partial_t D \\ \nabla \cdot B = 0 \\ \nabla \times E = -\frac{1}{c}\partial_t B \end{array} \right. \quad (9.3)$$

Symmetry of Maxwell Eqn:

$$\left\{ \begin{array}{l} \nabla \cdot \vec{E} = \frac{\rho}{\epsilon} \\ \nabla \times B - \frac{1}{c}\partial_t \vec{E} = \frac{\vec{j}}{c} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} + \frac{1}{c}\partial_t \vec{B} = 0 \end{array} \right. \quad (9.4)$$

For  $\rho = \vec{j} = 0$ , then there is a symmetry between  $\vec{E} \rightarrow \vec{B}$  and  $\vec{B} \rightarrow -\vec{E}$ .

$$\begin{array}{ll} \nabla \cdot \vec{E} = 0 & \nabla \cdot \vec{B} = 0 \\ -\frac{1}{c}\partial_t \vec{E} + \nabla \times \vec{B} = 0 & \Longleftrightarrow -\frac{1}{c}\partial_t \vec{B} - \nabla \times \vec{E} = 0 \end{array}$$

Though we derive the  $\vec{E}, \vec{B}$  symmetry from free conditions ( $\rho = \vec{j} = 0$ ), it **looks like** ??? we can apply them in any conditions.

where

$$J_D = \epsilon \partial_t E; \quad J_{induced} = c \chi_m^B (\nabla \times B)$$

Note that the Maxwell equation is relationship about  $\mathbf{E}$  and  $\mathbf{B}$ , the  $\vec{D}$  and  $\vec{H}$  is introduced to help reduce the equations to a more concised form.

To solve it, using iterative method (in power of  $1/c$ )

i) 0th order:

$$\begin{array}{ll} \nabla \cdot E^{(0)} = \rho & \nabla \times B^{(0)} = 0 \\ \nabla \times E^{(0)} = 0 & \nabla \cdot B^{(0)} = 0 \end{array}$$

ii) 1th order:

$$\begin{array}{ll} \nabla \cdot E^{(1)} = 0 & \nabla \times B^{(1)} = \frac{j}{c} + \frac{1}{c}\partial_t E^{(0)} \\ \nabla \times E^{(1)} = 0 & \nabla \cdot B^{(1)} = 0 \end{array}$$

iii) 2th order:

$$\begin{aligned}\nabla \cdot E^{(2)} &= 0 & \nabla \times B^{(2)} &= 0 \\ \nabla \times E^{(2)} &= -\frac{1}{c}\partial_t B^{(1)} & \nabla \cdot B^{(2)} &= 0\end{aligned}$$

iv) 3th order ...

$$\begin{aligned}\nabla \cdot E^{(3)} &= 0 & \nabla \times B^{(3)} &= \frac{1}{c}\partial_t E^{(3)} \\ \nabla \times E^{(3)} &= -\frac{1}{c}\partial_t B^{(1)} & \nabla \cdot B^{(3)} &= 0\end{aligned}$$

So in the quasi-static approximation, we get

$$\begin{aligned}E &= E^{(0)} + E^{(2)} + \dots \\ B &= B^{(1)} + B^{(3)} + \dots\end{aligned}$$

### 9.5.1 Boundary conditions

$$\left\{ \begin{array}{l} \vec{e}_n \cdot (\vec{D}_2 - \vec{D}_1) = \sigma \\ \vec{e}_n \times (\vec{H}_2 - \vec{H}_1) = \frac{\vec{\alpha}}{c} \\ \vec{e}_n \cdot (\vec{B}_2 - \vec{B}_1) = 0 \\ \vec{e}_n \times (\vec{E}_2 - \vec{E}_1) = 0 \end{array} \right. \quad (9.5)$$

Note that here  $\sigma$  and  $\vec{\alpha}$  are **free** charge and **free** current. The induced charges and current make trivial contribution due to infinitesimal integral volume (or area).

### 9.5.2 Plain wave

If  $J = \rho = 0$ , then

$$\begin{aligned}\nabla \times (\nabla \times E) &= \nabla(\nabla \cdot E) - \nabla^2 E \\ &= -\frac{1}{c}\partial_t(\nabla \times B) = -\frac{1}{c^2} \cdot \epsilon\mu \cdot \partial_t^2 E\end{aligned}$$

let

$$n^2 = \epsilon\mu$$

So

$$\nabla^2 E - \frac{n^2}{c^2}\partial_t^2 E = 0$$

Because  $E = E_0 e^{-i\omega t}$ , so we get

$$\nabla^2 E + \frac{n^2}{c^2} \omega^2 E = 0$$

$$E = E_0 e^{i(kx - \omega t)}, \quad k^2 = n^2 \omega^2 / c^2$$

let  $B = B_0 e^{i(kx - \omega t)}$

$$\nabla \times E = -\frac{1}{c} \partial_t B = \frac{i\omega}{c} B$$

$$\vec{B} = \frac{c}{i\omega} \nabla \times E = \frac{c}{\omega} \vec{k} \times \vec{E}$$

### 9.5.3 Metal

$$\begin{aligned} \nabla \times H &= \frac{1}{c} (J + \partial_t D) \\ &= \frac{1}{c} (\sigma E - i\omega \epsilon E) \\ &= \frac{\sigma - i\omega \epsilon}{c} E \end{aligned}$$

So

$$\nabla \times (\nabla \times E) = -\nabla^2 E = -\frac{1}{c} \nabla \times \partial_t B = -\frac{1}{c} \partial_t (\nabla \times \mu H) = -\mu \frac{\sigma - i\omega \epsilon}{c^2} (-i\omega) E = k^2 E$$

where,

$$k^2 = \frac{\omega^2 \mu (\epsilon + i\sigma/\omega)}{c^2} = \frac{\omega^2 \mu \hat{\epsilon}}{c^2}$$

Helmholtz eqn:

$$(\nabla^2 + \frac{\omega^2 \mu \hat{\epsilon}}{c^2}) E = 0$$

## 9.6 Radiation

Here is an explanation which can help understanding radiation.

This approach is based on the finite speed of light. A charge moving with constant velocity has a radial electric field  $E_r$  (at distance R from the charge), always emerging from the future position of the charge, and there is no tangential component of the electric field ( $E_t = 0$ ). This future position is completely deterministic as long as the velocity is constant. When the velocity of the charge changes, (say it bounces back

during a short time) the future position "jumps", so from this moment and on, the radial electric field  $E_r$  emerges from a new position. Given the fact that the electric field must be continuous, a non-zero tangential component of the electric field  $E_t$  appears, which decreases like  $1/R$  (unlike the radial component which decreases like  $1/R^2$ ).

Hence, at large distances from the charge, the radial component is negligible relative to the tangential component, and in addition to that, fields which behave like  $1/R^2$  cannot radiate, because the Poynting vector associated with them will behave like  $1/R^4$ .

The tangential component comes out (SI units):

$$E_t = \frac{ea \sin(\theta)}{4\pi\epsilon_0 c^2 R}$$

## 9.7 EM in relativity

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{pmatrix} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad (9.6)$$

Dual basis  $F_{\mu\nu}$ , when lowering or raising the 0 index, change the sign; while for other indices, nothing happens:

$$\begin{aligned} F^{0i} &= -F^{i0} = F^0{}_i = -F_0{}^i = -F_{0i} = F_{i0} = -F_i{}^0 = F^i{}_0 \\ F^{ij} &= -F^{ji} = F^i{}_j = F_i{}^j = F_{ij} = -F_{ji} = -F_j{}^i = -F^j{}_i \end{aligned}$$

So:

$$\begin{aligned} F_{\mu\nu} &\equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \\ F_{\mu\nu} F^{\mu\nu} &= 2(\vec{B}^2 - \vec{E}^2) \end{aligned} \quad (9.7)$$

The Maxwell Eqn:

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= \square A^\nu - \partial_\nu(\partial_\mu A^\mu) = \square A^\nu = 0, \\ \partial_\mu F^{\nu\rho} + \partial_\nu F^{\rho\mu} + \partial_\rho F^{\mu\nu} &= 0 \end{aligned}$$

Define:

$$\tilde{F}^{\mu\nu} \equiv \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & -E^z & E^y \\ -B^y & E^z & 0 & -E^x \\ -B^z & -E^y & E^x & 0 \end{pmatrix} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \quad (9.8)$$

where:

$$\epsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{even permutation of } 0, 1, 2, 3 \\ -1 & \text{odd permutation of } 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = 2\vec{B} \cdot \vec{E}$$

Define:

$$T_{[\mu,\nu]} \equiv \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$$

$$T_{[\mu_1,\mu_2,\mu_3]} \equiv \frac{1}{3!}[(T_{\mu_1\mu_2\mu_3} - T_{\mu\nu_1\mu_3\mu_2}) - (T_{\mu_2\mu_1\mu_3} - T_{\mu_2\mu_3\mu_1}) + (T_{\mu_3\mu_1\mu_2} - T_{\mu_3\mu_2\mu_1})] \quad (9.9)$$

Then the complete Maxwell Eqns:

$$\partial_\mu F^{\mu\nu} = J^\nu$$

$$\partial_\nu \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\partial_{[\nu}F_{\alpha\beta]} = 0 \quad (9.10)$$

### Transformation of $\vec{E}$ and $\vec{B}$

in a boost along x:

$$E'_{//} = E_{//} \quad B'_{//} = B_{//}$$

$$E'_{\perp} = \gamma E_{\perp} + \vec{\beta} \times \vec{B} \quad B'_{\perp} = \gamma B_{\perp} - \vec{\beta} \times \vec{E} \quad (9.11)$$



# Chapter 10

## Quantum Mechanics

### 10.1 Questions

#### Operator and Eigen states

In terms of operator and corresponding eigen states, for example:

$$\begin{aligned}\mathbf{r} |r\rangle &= \vec{r} |r\rangle \\ \mathbf{p} |p\rangle &= \vec{p} |p\rangle\end{aligned}\tag{10.1}$$

Is the operator define the eigen states, or the eigen states define the operator? In other word, for any operator, how do we know if it has eigen stated or not? and for any state, can we know that there exist some operator whose eigen state is exactly the given one?

$$\overline{\vec{p} = m \frac{d\vec{r}}{dt} = \frac{im}{\hbar} [\mathcal{H}_0, \vec{r}]}$$

#### Commutation relationship

$$[\mathbf{k}, \mathbf{R}] = -i \implies \begin{cases} \mathbf{R} = i\partial_{\mathbf{k}} \\ \mathbf{k} = -i\partial_{\mathbf{R}} \end{cases}\tag{10.2}$$

Planck formula for BlackBody radiation:

$$U(\omega) = \frac{\hbar\omega^3/\pi^2c^3}{e^{\hbar\omega/k_BT} - 1}$$

### 10.1.1 Harmonic Oscillator

$$\langle n' | x | n \rangle = \sqrt{\hbar/2m\omega}(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1})$$

For  $V(r) = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}kr^2$

$$\phi_0(r) = [(\frac{\alpha}{\sqrt{\pi}})^{1/2} e^{-\frac{1}{2}\alpha^2 x^2}] [(\frac{\alpha}{\sqrt{\pi}})^{1/2} e^{-\frac{1}{2}\alpha^2 y^2}] [(\frac{\alpha}{\sqrt{\pi}})^{1/2} e^{-\frac{1}{2}\alpha^2 z^2}]$$

where  $\alpha = (\frac{mk}{\hbar^2})^{1/4} = (\frac{m\omega}{\hbar})^{1/2}$

## 10.2 Rotation

Rotation of **Hilbert** space:

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k \quad (10.3)$$

$$J_{\pm} = J_x \pm iJ_y \quad (10.4)$$

$$J_{\pm} |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)\hbar} |j, m \pm 1\rangle \quad (10.5)$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (10.6)$$

General rotation for spin:

$$\begin{aligned} D(R_i(\varphi)) &= e^{-\frac{i\varphi}{\hbar}\hat{n}\cdot\frac{\hbar}{2}\vec{\sigma}} = e^{-\frac{i\varphi}{2}\hat{n}\cdot\vec{\sigma}} \\ &= \sum_{k=0}^{\infty} \frac{(-i\varphi/2)^k}{k!} (\sigma_n)^k \quad \sigma_n^2 = 1 \\ &= \sum_{[even]} \frac{(-i\varphi/2)^k}{k!} \cdot \mathbb{1} + \sum_{[odd]} \frac{(-i\varphi/2)^k}{k!} \cdot \sigma_n \\ &= \cos \frac{\varphi}{2} - i\sigma_n \sin \frac{\varphi}{2} \end{aligned} \quad (10.7)$$

Euler Representation:

$$\begin{aligned} R(\alpha\beta\gamma) &= R_{z'}(\gamma) \cdot R_{y'}(\beta) \cdot R_z(\alpha) \\ &= R_{y'}(\beta) R_z(\beta) R_{y'}^{-1}(\beta) \cdot R_{y'}(\beta) \cdot R_z(\alpha) \\ &= [R_z(\alpha) R_y(\beta) R_z(\gamma)] \cdot R_z(\gamma) \cdot R_z(\alpha) \\ &= R_z(\alpha) R_y(\beta) R_z(\gamma) \end{aligned} \quad (10.8)$$

$$|\varphi\rangle_R = |R\varphi\rangle = D(R) |\varphi\rangle \quad (10.9)$$

$$D_{mm'}^j(R) \equiv j\bar{m}D(R) |jm'\rangle \quad (10.10)$$

### 10.2.1 Vector

In QM, we demand that the expectation value of a vector operator transforms under rotation like a classical vector.

$$\begin{aligned} |\alpha\rangle &\rightarrow |\alpha\rangle^R = D(R) |\alpha\rangle \\ {}^R\langle\alpha| V_i |\alpha\rangle^R &= \langle\alpha| D^\dagger V_i D |\alpha\rangle = R_{ij} \langle\alpha| V_j |\alpha\rangle \\ D^\dagger V_i D &= R_{ij} V_j \end{aligned}$$

With infinitesimal transform, we will find:

$$[J_i, V_j] = i\hbar\epsilon_{ijk}V_k \quad (10.11)$$

Let  $V_0 = V_z$  and  $V_{\pm 1} = \frac{\mp V_x - iV_y}{\sqrt{2}}$ , then

$$\begin{aligned} [J_\pm, V_\mp] &= \sqrt{2}V_0, \quad [J_\pm, V_0] = \sqrt{2}V_\pm \\ [J_\pm, V_\pm] &= 0, \quad [J_0, V_q] = qV_q \end{aligned}$$

Where

$$J_\pm \equiv J_x \pm iJ_y \quad J_{\pm 1} = \mp \frac{1}{\sqrt{2}}J_\pm, \quad \text{and} \quad J_0 = J_z \quad (10.12)$$

$$\vec{J} \cdot \vec{V} = J_x V_x + J_y V_y + J_z V_z = J_+ V_+ + J_0 V_0 + J_- V_- = J_0 V_0 - J_{+1} V_{-1} - J_{-1} V_{+1} \quad (10.13)$$

The Projection Theorem:

$$\langle\alpha', jm'| V_q |\alpha, jm\rangle = \frac{\langle\alpha', jm| J \cdot V |\alpha, jm\rangle}{\hbar^2 j(j+1)} \langle jm'| J_q |jm\rangle \quad (10.14)$$

### 10.2.2 Tensor

We define spherical tensor of rank  $k$  with  $(2k+1)$  components labelled by  $q$  as

$$(T_q^k)^R \equiv D^\dagger(R) T_q^{(k)} D(R) = \sum_{q'=-k}^k D_{qq'}^{(k)*} T_{q'}^{(k)} \quad (10.15)$$

$$\begin{aligned} [J_\pm, T_q^k] &= \hbar\sqrt{k(k+1) - q(q \pm 1)} T_{q\pm 1}^k \\ [J_z, T_q^k] &= \hbar q T_q^k \end{aligned} \quad (10.16)$$

### 10.2.3 Spin-1/2

$$D(R_{\hat{n}}(\phi)) = e^{-i\frac{\phi}{\hbar}\hat{n}\cdot\vec{S}} = e^{-i\frac{\phi}{2}\hat{n}\cdot\vec{\sigma}} = \cos\left(\frac{\phi}{2}\right) - i\hat{n}\cdot\vec{\sigma}\sin\left(\frac{\phi}{2}\right) = \begin{pmatrix} \cos\left(\frac{\phi}{2}\right) - in_z\sin\left(\frac{\phi}{2}\right) & (-in_x - n_y)\sin\left(\frac{\phi}{2}\right) \\ (-in_x + n_y)\sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) + in_z\sin\left(\frac{\phi}{2}\right) \end{pmatrix}$$

### 10.2.4 Spin-1

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (10.17)$$

### Wigner-Echart theorem

The matrix elements of tensor operators with respect to angular momentum eigenstates.

$$\langle \alpha', j'm' | T_q^{(k)} | \alpha, jm \rangle = \frac{\langle jk; mq | jk; j'm' \rangle}{\sqrt{2j+1}} \langle \alpha', j' | T^{(k)} | \alpha, j \rangle \quad (10.18)$$

### Zeeman-effect

This is effect only in **weak magnetism**

### Paschen-Back effect

This happen with **strong magnetism** ( $\frac{\Phi}{\Phi_0} \gg \alpha^2 \sim 10^{-4}$ ).

## 10.3 Symmetry

### 10.3.1 Parity

Parity of Spherical Harmonics:

$$r \rightarrow r, \quad \theta \rightarrow \pi - \theta, \quad \phi \rightarrow \phi + \pi$$

So

$$Y_l(\theta, \phi) \xrightarrow{P} Y_l(\pi - \theta, \phi + \pi) = e^{il\pi} e^{il\phi} \sin^l(\pi - \theta) = (-1)^l Y_l(\theta, \phi)$$

Because  $L_-$  is an *axial vector*, so  $P(L_-) = 1$ .

$$L_- Y_l \xrightarrow{P} (-1)^l L_- Y_l$$

To get

$$Y_{lm}(\theta, \phi) \xrightarrow{P} Y_{lm}(\pi - \theta, \phi + \pi) = (-1)^l L_- Y_{l,m+1}(\theta, \phi)$$

### 10.3.2 Time reversal

$$|jm\rangle \xrightarrow{T} (-1)^m |j, -m\rangle$$

## 10.4 Young Tableau

A way to describe

## 10.5 Scattering

In the case of an electron of mass  $m_e$  scatters off a proton of mass  $m_p$ . From non-relativistic QM, the xsection should be given by the Born approximation:

$$\left(\frac{d\sigma}{d\Omega}\right)_{Born} = \frac{m_e^2}{4\pi^2} |\tilde{V}(\vec{k})|^2 \quad (10.19)$$

Where the Fourier transform of the potential is given by:

$$\tilde{V}(\vec{k}) = \int d^3x e^{-i\vec{k}\cdot\vec{x}} V(\vec{x})$$

$\vec{k}$  is the difference in the electron momentum before and after scattering, sometimes called the *momentum transfer*. For Coulomb potential,  $V(x) = \frac{e^2}{4\pi|\vec{x}|}$ ,  $\tilde{V}(\vec{k}) = \frac{e^2}{k^2}$ , So

$$\left(\frac{d\sigma}{d\Omega}\right)_{Born} = \frac{m_e^2}{4\pi^2} \left(\frac{e^2}{k^2}\right)^2$$

# Chapter 11

## Special Relativity

### 11.1 Relativistic generalization from Classical results

The power of radiation is (cgs units):

$$P = \frac{2}{3} \frac{q^2 a^2}{c^3}$$

Generalize it to **covariant** form:

$$P = \frac{2}{3} \frac{q^2}{m^2 c^3} |\dot{\vec{P}}|^2$$

The power  $P$  should be *Lorentz invariant*, so  $|\dot{\vec{P}}|^2$  should include the Lorentz scalar by taking the inner product of the four-acceleration  $a^\mu = dp^\mu/d\tau$  with itself, so:

$$P = -\frac{2}{3} \frac{q^2}{m^2 c^3} \frac{dp_\mu}{d\tau} \frac{dp^\mu}{d\tau}$$

where:

$$\frac{dp_\mu}{d\tau} \frac{dp^\mu}{d\tau} = \beta^2 \left( \frac{dp_0}{d\tau} \right)^2 - \left( \frac{d\vec{p}}{d\tau} \right)^2$$

The above inner product can also be written in terms of  $\vec{\beta}$  and its time derivative:

$$P = \frac{2q^2\gamma^6}{3c} \left[ (\dot{\vec{\beta}})^2 - (\vec{\beta} \times \dot{\vec{\beta}})^2 \right]$$

## 11.2 Relativity

Two conditions of special relativity.

- general physical rules, which means in different frames, we can observe the same EOM.
- constant  $c$

let  $X^\mu = (ct, \vec{x})$ , so in order to make  $c$  constant, we have

$$-(ct')^2 + \vec{x}'^2 = (ct)^2 + \vec{x}^2$$

The **simplest** way to preserve the length is to construct a **linear map**:

$$X'^\mu = (\mathcal{L})^\mu{}_\nu X^\nu$$

Where  $(\mathcal{L})^\mu{}_\nu = (\mathcal{L}^T)_\nu{}^\mu = \Lambda^\mu{}_\nu$ . Note that for a matrix, what important is the order of index, not their position, so

$$\mathcal{L}^\mu{}_\nu = \mathcal{L}_\mu{}^\nu = (\mathcal{L}^T)_\nu{}^\mu = (\mathcal{L}^T)^\nu{}_\mu$$

Let

$$A = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} \quad B = \begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix}$$

and

$$A_\mu \equiv g_{\mu\nu} A^\nu$$

where

$$\begin{aligned} g_{\mu\nu} &= \text{diag}(-1, 1, 1, 1) \\ g^{\mu\nu} g_{\nu\sigma} &= g^\mu{}_\sigma = \delta^\mu{}_\sigma \end{aligned} \tag{11.1}$$

So, we can define:

$$A \cdot B \equiv -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3 = A^\mu B_\mu = A^\mu g_{\mu\nu} B^\nu$$

Inserting  $\mathcal{L}\mathcal{L}^T$  into the above equation:

$$\begin{aligned} A_\mu B^\mu &= A^\mu g_{\mu\nu} B^\nu = (\mathcal{L}^{-1} A')^\mu g_{\mu\nu} (\mathcal{L}^{-1})^\nu{}_\rho B'^\rho \\ &= (\mathcal{L}^{-1})^\mu{}_\nu A'^\nu g_{\mu\nu} (\mathcal{L}^{-1})^\nu{}_\rho B'^\rho \\ &= A'^\nu (\mathcal{L}^{-1})^\mu{}_\nu g_{\mu\nu} (\mathcal{L}^{-1})^\nu{}_\rho B'^\rho \\ &= A'^\nu (\mathcal{L}^{-1T})^\mu{}_\nu g_{\mu\nu} (\mathcal{L}^{-1})^\nu{}_\rho B'^\rho \\ &= A'^\nu g'_{\nu\rho} B'^\rho \end{aligned}$$

In special relativity, we require that:

$$g'_{\nu\rho} = (\mathcal{L}^{-1T})^\mu_\nu g_{\mu\nu} (\mathcal{L}^{-1})^\nu_\rho = g_{\nu\rho}$$

So we get

$$\mathcal{L}^{-1T} g \mathcal{L}^{-1} = g \rightarrow \mathcal{L}^{-1T} = g \mathcal{L} g \rightarrow \mathcal{L}^{-1} = g \mathcal{L}^T g$$

With these conditions, we can easily find out  $\mathcal{L}$ . For 2D-coordinate  $(ct, x)$ , we have

$$\mathcal{L}(v) = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$$

Where  $\beta = \frac{v}{c}$  and  $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ . Usually, we would like to use another quantity in special relativity, which is rapidity:

$$\tanh y \equiv \frac{v}{c} \rightarrow y = \frac{1}{2} \ln\left(\frac{1+\beta}{1-\beta}\right)$$

And correspondingly

$$\gamma = \cosh y, \quad \beta\gamma \sinh y$$



# Chapter 12

## QFT

Questions:

- What's a field, classical field is a spatial distribution, quantum field is a analogy to classical one, but make up of creation and destruction operator. The problem is that we **define** a field which collects creator and destructor operators and it works !!! What's the logic to define such a field ???
- How to construct Lagrangian from a field  $\leftarrow$  Lorentz invariant.
- EOM
- Noether theorem. Conserved current and charge
- Symmetry. What's each group? corresponding representation. How to embed particles into lorentz group and unitary group?
- Couple of scalar field to  $A_\mu$
- Anticommuting spinors:

$$-(\gamma_0)_{\alpha\beta}\psi_\alpha\psi_\beta^* = (\gamma_0)_{\alpha\beta}\psi_\beta^*\psi_\alpha$$

### 12.1 Motivation

*particle* number is not conserved. The creation and destruction of particles, which is possible due to the most famous eqn. of special relativity  $E = mc^2$ . *Lorentz invariance* guides the definition of particle.

Why QFT: quantum mechanics plus Poincaré symmetry.

Quantum field theory is just quantum mechanics with an infinite number of harmonic oscillators

The Quantum Mechanics can describe a system with a fixed number of particles in terms of a many-body wave function. The Relativistic Quantum Field Theory with creation and annihilation operators was developed in order to include processes in which the number of particles is not conserved, and to describe the conversion of mass into energy and vice versa. A consequence of relativity is that the number of particles isn't fixed, ( $E = mc^2$ ) though the converse is false: particle production can happen without relativity.

Construct  $\mathcal{L}$  from field  $\phi$  and its derivative under the rule of Lorentz Invariant. How to incorporate symmetry ?

QFT is the quantum mechanics of *extensive degrees of freedom*  $\langle x|\phi\rangle = \phi(x)$  is a function of space, the wavefunction. This looks like a field. It is not what we mean by field in QFT. meaningless phrases like "second quantization" may conspire to try to confuse you.

It is not a coincidence that the harmonic oscillator plays an important role. After all, electromagnetic waves oscillate harmonically.

There are two common ways to quantize a field theory:

- canonical quantization
- Feynman path integral
- Other alternatives: perturbation theory

**Second quantization:** first quantization refers to the discrete modes ( $d\vec{x}d\vec{p} \sim \hbar$ ), for example, of a particle in a box. Second quantization refers to the integer numbers of excitations of each of these modes. However, this is somewhat misleading-the fact that there are discrete modes is a classical phenomenon. The two steps really are (1) interpret these modes as having energy  $E = \hbar\omega$  and (2) quantize each mode as a harmonic oscillator. In that sense we are only quantizing once.

There are two new features in second quantization:

1. We have many quantum mechanical systems – one for each  $\vec{p}$  – all at the same time.
2. We interpret the  $n$ th excitation of the  $\vec{p}$  harmonic oscillator as having  $n$  **particles**.

## 12.2 Convention

In relativity, the symmetry refer to invariance after the transformation of *coordinate system*. So the rotation is:

$$R = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (12.1)$$

or in other way:

$$R^T \mathbb{1} R = \mathbb{1}$$

### 12.2.1 4D time-space

$$dx^\mu \equiv (dt, d\vec{x})^\mu$$

$$ds^2 = dt^2 - d\vec{x} \cdot d\vec{x} = \eta_{\mu\nu} dx^\mu dx^\nu \text{ with}$$

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} \quad (12.2)$$

Lorentz transformation acting on 4-vectors are matrices  $\Lambda$  satisfying

$$\Lambda^T \eta \Lambda = \eta = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Rotation and boost in 4D time-space around x axes is:

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos \theta_x & \sin \theta_x \\ & & -\sin \theta_x & \cos \theta_x \end{pmatrix}, \begin{pmatrix} \cosh \beta_x & \sinh \beta_x & & \\ \sinh \beta_x & \cosh \beta_x & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

### Lorentz transformation

Scalar field

$$\phi(x^\mu) \rightarrow \phi((\Lambda^{-1})^\mu_\nu x^\nu)$$

Vector field

$$V^\mu \rightarrow \Lambda^\mu_\nu V^\nu$$

Tensor fields

$$T^{\mu\nu} \rightarrow \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta}$$

### 12.2.2 quantization

$$[a_k, a_p^\dagger] = (2\pi)^3 \delta^3(\vec{p} - \vec{k}), \quad a_p^\dagger |0\rangle = \frac{1}{\sqrt{2\omega_p}} |\vec{p}\rangle$$

$$1 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} |\vec{p}\rangle \langle \vec{p}|$$

**Function derivatives**  $\frac{\delta\phi(x)}{\delta\phi(y)} = \delta(x - y)$ ,

$$\frac{\partial(\partial_\alpha A_\alpha)^2}{\partial(\partial_\mu A_\nu)} = 2(\partial_\alpha A_\alpha) \frac{\partial(\partial_\beta A_\gamma)}{\partial(\partial_\mu A_\nu)} g_{\beta\gamma} = 2(\partial_\alpha A_\alpha) g_{\beta\mu} g_{\gamma\nu} g_{\beta\gamma}$$

**notation**  $\phi$  and  $\pi$  for scalar fields,  $\psi, \xi, \chi$  for fermions,  $A_\mu, J_\mu, V_\mu$  for vectors and  $h_\mu, T_\mu$  for tensors.

**Kinetic term** Anything with just two fields of the same or different type can be called a kinetic term. Kinetic terms tell you about the free (non-interacting) behavior. Though, sometimes it is useful to think of a *mass term* such as  $m^2\phi^2$ , as an interaction rather than a kinetic term.

**Boundary conditions** we will always assume that our fields vanish on asymptotic boundaries. so we can integrate by part:

$$A\partial_\mu B = -(\partial_\mu A)B$$

We define quantum fields as integrals over creation and annihilation operators for each momentum: (Why define it this way ???)

$$\phi_0(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\omega_p}} (a_p e^{i\vec{p}\vec{x}} + a_p^\dagger e^{-i\vec{p}\vec{x}}) \quad (12.3)$$

$$|\vec{x}\rangle = \phi_0(\vec{x}) |0\rangle$$

There is no physical content in the above equation. It is just a definition. The physical content is in the algebra of  $a_p$  and  $a_p^\dagger$  and in the Hamiltonian  $H_0$ . Nevertheless, we will see that collections of  $a_p$  and  $a_p^\dagger$  in the form of Eq.12.3 are very useful in quantum field theory.

Following this defition, we can get:

$$\pi(\vec{x}) \equiv \partial_t \phi(\vec{x})|_{t=0} = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_p}{2}} (a_p e^{i\vec{p}\vec{x}} - a_p^\dagger e^{-i\vec{p}\vec{x}})$$

$\pi[\phi, \dot{\phi}]$  can also be implicitly defined as:

$$\frac{\partial \mathcal{H}[\phi, \pi]}{\partial \pi} = \dot{\phi}$$

## Schrodinger eqn

$$\begin{aligned}
i\partial_t \psi &= i\partial_t \langle x | \psi \rangle = i\partial_t \langle 0 | \phi(\vec{x}, t) | \psi \rangle = i \langle 0 | \partial_t \phi(\vec{x}, t) | \psi \rangle \\
&= \langle 0 | \int \frac{d^3 p}{(2\pi)^3} \frac{\sqrt{\vec{p}^2 + m^2}}{2\omega_p} (a_p e^{-ipx} - a_p^\dagger e^{ipx}) | \psi \rangle \\
&= \langle 0 | \sqrt{m^2 - \vec{\nabla}^2} \phi_0(x) | \psi \rangle
\end{aligned}$$

So

$$i\partial_t \psi(x) = \sqrt{m^2 - \vec{\nabla}^2} \psi(x) = \left( m - \frac{\vec{\nabla}^2}{2m} + \mathcal{O}\left(\frac{1}{m^2}\right) \right) \psi(x)$$

To get

$$i\partial_t \psi(x) = -\frac{\vec{\nabla}^2}{2m} \psi(x)$$

## 12.2.3 Hamiltonian & Lagrangian

Why do we restrict to Lagrangians of the form  $\mathcal{L}[\phi, \partial_\mu \phi]$ ? First of all, this is the form that all "classical" Lagrangians had. If only first derivatives are involved, boundary conditions can be specified by initial positions and velocities only, in accordance with Newton's laws. In the quantum theory, if kinetic terms have too many derivatives, for example  $\mathcal{L} = \phi \square^2 \phi$ , there will generally be disastrous consequences. For example, there may be states with negative energy or negative norm, permitting the vacuum to decay. But interactions with multiple derivatives may occur. Actually, they must occur due to quantum effects in all but the simplest renormalizable field theories; for example, they are generic in all effective field theories.

*Hamiltonian and Lagrangian density:* ( How to connect field  $\phi$  with  $\mathcal{L}$  or  $\mathcal{H}$  ??? ).

$$\mathcal{L}[\phi, \dot{\phi}] = \pi[\phi, \dot{\phi}] \dot{\phi} - \mathcal{H}[\phi, \pi[\phi, \dot{\phi}]]$$

Or inversely:

$$\mathcal{H}[\phi, \pi] = \pi \dot{\phi}[\phi, \pi] - \mathcal{L}[\phi, \dot{\phi}[\phi, \pi]], \quad \frac{\partial \mathcal{L}[\phi, \dot{\phi}]}{\partial \dot{\phi}} = \pi$$

The Halmiltonian corresponds to a conserved quantity - the total energy of the system - while the Lagrangian does not. The problem with Halmiltonian is that they are not Lorentz invariant. It is the 0 component of a Lorentz vector:  $P^\mu = (H, \vec{P})$ . And  $\mathcal{H}$  is the 00 component of a Lorentz tensor, the energy-momentum tensor  $\mathcal{T}_{\mu\nu}$ . Halmiltonians are great for non-relativistic systems, but for relativistic systems we will almost exclusively use Lagrangians.

Time evolution is generated by a hamiltonian H.  $i\hbar \partial_t |\phi\rangle = H |\phi\rangle$

### 12.2.4 Noether's theorem

If there is such a symmetry that depends on some parameter  $\alpha$  that can be taken small (continuous), then we find:

$$0 = \frac{\delta \mathcal{L}}{\delta \alpha} = \sum_n \left\{ \left[ \frac{\partial \mathcal{L}}{\partial \phi_n} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_n)} \right] \frac{\delta \phi_n}{\delta \alpha} + \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha} \right] \right\}$$

If the EOM is satisfied, then it reduces to  $\partial_\mu J_\mu = 0$ , where

$$J_\mu = \sum_n \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha} \quad (12.4)$$

A vector field  $J_\mu$  that satisfies  $\partial_\mu J_\mu = 0$  is called *conserved current*. The total charge  $Q$ , defined as:

$$Q = \int d^3x J_0$$

satisfies

$$\partial_t Q = \int d^3x \partial_t J_0 = \int d^3x \vec{\nabla} \cdot \vec{J} = 0$$

**Noether's theorem:** If a Lagrangian has a continuous symmetry then there exists a current associate with that symmetry that is conserved when the equations of motion are satisfied.

- continuous
- the current is conserved *on-shell*, that is, when the EOM are satisfied. (The field dies out at asymptotic boundary).
- It works for *global symmetries*, parametrized by number  $\alpha$ , not only for *local(gauge) symmetries* parametrized by functions  $\alpha(x)$ .

### 12.2.5 Energy-Momentum Tensor

There is a very important case of Noether's theorem that applies to a global symmetry of the action, not the Lagrangian. *global* space-time translation  $\rightarrow$  energy-momentum tensor.

$$\phi(x) \rightarrow \phi(x + \xi) = \phi(x) + \xi^\nu \partial_\nu \phi(x) + \dots$$

With infinitesimal change:

$$\frac{\delta\phi}{\delta\xi^\nu} = \partial_\nu\phi, \frac{\delta\mathcal{L}}{\delta\xi^\nu} = \partial_\nu\mathcal{L},$$

So:

$$\delta S = \int d^4x \delta\mathcal{L} = \xi^\nu \int d^4x \partial_\nu\mathcal{L} = 0$$

which leads to:

$$\partial_\nu\mathcal{L} = \frac{\delta\mathcal{L}[\phi_n, \partial_\mu\phi_n]}{\delta\xi^\nu} = \partial_\mu \left( \sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)} \frac{\delta\phi_n}{\delta\xi^\nu} \right)$$

Or equivalently

$$\partial_\mu \left( \sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)} \partial_\nu\phi_n - g_{\mu\nu}\mathcal{L} \right) = 0$$

The four symmetries have produced four Noether current, one for each  $\nu$ :

$$\mathcal{T}_{\mu\nu} = \sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)} \partial_\nu\phi_n - g_{\mu\nu}\mathcal{L} \quad (12.5)$$

The corresponding conserved current:

$$Q_\nu = \int d^3x \mathcal{T}_{0\nu}$$

Electron has an inherent two-valuedness called spin, while a photon has an inherent two-valuedness called polarization.

## 12.3 Concepts

Schwinger-Dyson equation:

$$(\square_x + m^2)\langle\phi_x\phi_1\cdots\phi_n\rangle = \langle\mathcal{L}'_{int}[\phi_x]\phi_1\cdots\phi_n\rangle - i\hbar \sum_j \delta^4(x-x_j)\langle\phi_1\cdots\phi_{j-1}\phi_{j+1}\cdots\phi_n\rangle \quad (12.6)$$

Where  $\phi_x \equiv \phi(x)$  and  $\phi_j \equiv \phi(x_j)$ ,  $\angle\phi_x\phi_1\cdots\phi_n\rangle \equiv \langle\Omega| T\{\phi_x\phi_1\cdots\phi_n\} |\Omega\rangle$ ,  $\mathcal{L}'_{int}[\phi] = \frac{d}{d\phi}\mathcal{L}_{int}[\phi]$

## 12.4 Lagrangian

In QFT, we will use Lagrangian rather than Hamiltonian, because it is Lorentz invariant. To construct Lorentz invariant Lagrangian from field, one just need to add Lorentz invariant terms into it.

A vector field  $A_\mu$  is just four scalars until we construct it with  $\partial_\mu$  in the Lagrangian, as in the  $(\partial_\mu A_\mu)^2$  part of  $F_{\mu\nu}^2$ .

## 12.5 Symmetry

Our universe has a number of apparent symmetries that we would like our quantum field theory to respect. One symmetry is that no place in space-time seems any different from any other place. Thus, our theory should be translation invariant: if we take all our fields  $\phi(x)$  and replace them by  $\phi(x + a)$  for any 4-vector  $a^\nu$ , the observable should look the same. Another symmetry is Lorentz invariance: physics should look the same whether we point our measurement apparatus to the left or to the right, or put it on a train. The group of translations and Lorentz translations is called the **Poincaré group**, ISO(1,3) (the isometry group of Minkowski space).

It is possible for two different groups to have the same algebra. For example, the proper orthochronous Lorentz group ( $\det(\Lambda) = +1, \Lambda_0^0 > 1$ ) and the full Lorentz group ( $\det(\Lambda) = \pm 1, \Lambda_0^0 > 1 \text{ or } \Lambda_0^0 < -1$ ) have the same algebra, but the full Lorentz group has in addition time reversal and parity. The **proper** Lorentz group is the special orthogonal group SO(1,3), which contains only the elements with determinant 1, so it excludes T and P.

Notice that different representations of the same group G can have different dimensions. **Lie algebra** relations is the properties of G that are inherent in all of its representations.

Assume  $D_R(g(\theta))$  as a representation of group G, and

$$D_R(g(\theta \sim 0)) = 1 + i\theta_a T_R^a + \mathcal{O}(\theta^2)$$

where  $T_R^a \equiv -i\partial_{\theta_a} D_R(g(\theta))|_{\theta=0}$  is the **generator** of G in the representation of R. So

$$D_R(g(\theta)) = e^{i\theta_a T_R^a}$$

Given two such elements  $D_R(g(\theta_1)) = e^{i\theta_1^a T_R^a}$ ,  $D_R(g(\theta_2)) = e^{i\theta_2^a T_R^a}$ , their product must give a third:

$$D_R(g_1)D_R(g_2) = D_R(g_1g_2) = e^{i\theta^3 T_R^a}$$



Expanding the log of both hand sides to second order in the  $\theta$  (See Maggiore chapter 2.1), we get:

$$\theta_a^3 = \theta_a^1 + \theta_a^2 - \frac{1}{2} f^{bc}{}_a + \mathcal{O}(\theta^3)$$

which implies that:

$$[T^a, T^b] = i f^{ab}{}_c T^c$$

This is the *Lie algebra* of G and the  $f$  is the *structure constants* of G.  $f$  does not depend on the representation. Note that the normalization of the  $T^a$  is ambiguous, and rescaling T rescales  $f$ . A common convention is to choose an orthonormal basis:

$$\text{tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$$

The Lie algebra is defined in the neighborhood of the identity element, but by conjugating by finite transformations, the tangent space to any point on the group has the same structure, so it determines the local structure. It doesn't know about global, discrete issues, like disconnected components, so different groups can have the same Lie algebra.

A *casimir* of the algebra is an operator made from the generators which commutes with all of them.

The notation of "axis of rotation" is (d=3)-centric. More generally, a rotation is specified by a (2D) *plane* of rotation. In d=3, we can specify a plane by its normal direction, the one that's left out,  $J^i \equiv \epsilon^{ijk} J^{jk}$ , in terms of which the so(3) lie algebra is

$$[J^{ij}, J^{kl}] = i(\delta^{jk} J^{il} + \delta^{il} J^{jk} - \delta^{ik} J^{jl} - \delta^{jl} J^{ik})$$

The vector representation is

$$\left( J_{(1)}^{ij} \right)_l = i(\delta^{ik} \delta_l^j - \delta^{jk} \delta_l^i)$$

The spinor representation is

$$J_{(\frac{1}{2})}^{ij} = \epsilon^{ijk} \frac{1}{2} \sigma^k = \frac{i}{4} [\sigma^i, \sigma^j]$$

For general  $d$ , we can make a spinor representation of dimension  $k$  ( $k=2J+1$ ) if we find  $d$   $k \times k$  matrices  $\gamma^i$  which satisfy the *Clifford algebra*  $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$ .

For (faithful) representations of *non-compact* groups, 'unitary' and 'finite-dimensional' are mutually exclusive.

### 12.5.1 Lorentz group

the symmetry group associated with *special relativity*.

For scalar field, the physical content of Lorentz invariance is that nature has a symmetry under which scalar fields do not transform. Take, for example, the temperature of a fluid, which can vary from point to point. If we change reference frames, the labels for the points change, but the temperature at each point stays the same.

As for vector field, the difference is that the components of a vector field at the point  $x$  transform into each other as well.

The simplest Lorentz-invariant operator that we can write down involving derivatives is the d'Alembertian:

$$\square = \partial_\mu^2 = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$$

Objects such as  $v^2 = V_\mu V^\mu, \phi, 1, \partial_\mu V^\mu$  are *Lorentz invariant*, meaning they do not depend on the Lorentz frame at all. While objects like:  $V_\mu, F_{\mu\nu}, \partial_\mu, x_\mu$  are *Lorentz covariant*, meaning they do change in different frames, but precisely as the Lorentz transformation dictates.

The Lorentz group is sometimes called  $O(1,3)$ . This is orthogonal (preserves a metric) group corresponding to a metric with  $(1,3)$  signature.

The irreducible representations of the Lorentz group can be constructed from irreducible representations of  $SU(2)$ .

$$J_i^+ \equiv \frac{1}{2}(J_i + iK_i), J_i^- \equiv \frac{1}{2}(J_i - iK_i)$$

which satisfy

$$\begin{aligned} [J_i^+, J_j^+] &= i\epsilon_{ijk} J_k^+ \\ [J_i^-, J_j^-] &= i\epsilon_{ijk} J_k^- \\ [J_i^+, J_j^-] &= 0 \end{aligned}$$

#### Lorentz algebra

Lorentz algebra,  $\mathfrak{so}(1,3)$

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k \\ [J_i, K_j] &= i\epsilon_{ijk} K_k \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k \end{aligned}$$

Note that  $[J_i, J_j] = i\epsilon_{ijk}J_k$  is also the algebra for rotations,  $SO(3)$ , and in fact the  $J_i$  generate the 3D rotation subgroup of the Lorentz group.

Similar to rotation group, we can generalize to other  $SO(1,d)$  by collecting the generators into an antisymmetric matrix  $J^\mu$  with components  $J^{ij} = \epsilon^{ijk}J^k$ ,  $J^{0i} = K^i = -J^{i0}$  (exactly as  $\vec{E}, \vec{B}$  are collected into  $F^\mu$ ). This object satisfies:

$$J^{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix}$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} + \eta^{\mu\sigma}J^{\nu\rho} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho})$$

The fundamental (d+1 Dimensional vector) representation matrices solving this equation are

$$(J^{\mu\nu})^\rho{}_\sigma = i(\eta^{\nu\rho}\delta^\mu_\sigma - \eta^{\mu\rho}\delta^\nu_\sigma)$$

$$[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho})$$

The Lie algebra is independent of any concrete representation, though we derive them from 4-vector representation. They holds for any representation.

## representation

The Dirac representation of the Lorentz group is reducible; it is the direct sum of a left-handed and a right-handed spinor representation.

$\gamma^5$

- $(\gamma^5)^2 = \mathbb{1}$
- $\{\gamma^5, \gamma^\mu\} = 0$
- Extended Clifford algebra:  $\{\gamma^M, \gamma^N\} = 2g^{MN}$ , with  $\gamma^M = \gamma^0, \gamma^1, \gamma^2, \gamma^3, i\gamma^5$ , and  $g^{MN} = \text{diag}(1, -1, -1, -1, -1)$

## 12.5.2 Unitary group

the probability should add up to 1.  $U(N)$  is defined by its N dimensional representation as  $N \times N$  complex unitary matrices  $\mathbb{1} = M^\dagger M = M M^\dagger$ . This one doesn't arise as a spacetime symmetry, but is crucial in the study of gauge theory.

Unitary representation, which means under the representation, the group element  $\mathcal{P}$  has  $\mathcal{P}^\dagger \mathcal{P} = 1$ . But why the unitarity depends on representation? shouldn't it be representation independent.

### Majorana representation

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix} \quad (12.7)$$

The Majorana is another  $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$  representation of the Lorentz group that is physically equivalent to the Weyl representation.

### 12.5.3 Poincaré Group

The group of translations and Lorentz transformation is called the **Poincaré group**, ISO(1,3) (the isometry group of Minkowski space).

**Particle** can be defined as a set of states that mix only among themselves under Poincaré transformations.

Particles transform under irreducible unitary representations of the Poincaré group.

There are *no finite-dimensional unitary representations of the Poincaré group*

### 12.5.4 Gauge symmetry

Under a gauge transformation,  $\phi$  can transform as

$$\phi \rightarrow e^{-i\alpha(x)} \phi$$

But the derivatives  $|\partial_\mu \phi|^2$  is not invariant. we can make the kinetic term gauge invariant using something called a covariant derivative. Let

$$A_\mu \rightarrow A_\mu + \frac{1}{e} \partial_\mu \alpha$$

So

$$(\partial_\mu + ieA_\mu)\phi \rightarrow (\partial_\mu + ieA_\mu + i\partial_\mu \alpha)e^{-i\alpha(x)}\phi = e^{-i\alpha(x)}(\partial_\mu + ieA_\mu)\phi$$

This leads to the definition of *covariant derivative*

$$D_\mu \phi \equiv (\partial_\mu + ieA_\mu)\phi \rightarrow e^{-i\alpha(x)} D_\mu \phi$$

## 12.6 Field

### 12.6.1 Scalar Field

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\vec{\nabla}\phi \cdot \vec{\nabla}\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) = -\frac{1}{2}(\phi\Box\phi - m^2\phi^2)$$

EOM:

$$-\partial_t^2\phi + \nabla^2\phi - m^2\phi = (\partial_\mu\partial^\mu + m^2)\phi = 0$$

$$\phi(\vec{x}) = \int \frac{d^dk}{(2\pi)^d} \sqrt{\frac{\hbar}{2\omega_k}} (e^{i\vec{k}\cdot\vec{x}} a_k + e^{-i\vec{k}\cdot\vec{x}} a_k^\dagger)$$

$$\pi(\vec{x}) = \frac{\partial\mathcal{L}}{\partial_\mu\phi} = \frac{1}{i} \int \frac{d^dk}{(2\pi)^d} \sqrt{\frac{\hbar\omega_k}{2}} (e^{i\vec{k}\cdot\vec{x}} a_k - e^{-i\vec{k}\cdot\vec{x}} a_k^\dagger)$$

$$[\phi(\vec{x}), \pi(\vec{x}')] = i\hbar\delta^d(\vec{x} - \vec{x}')$$

$$H = \sum_n (p_n \dot{q}_n) = \int dx (\pi(x) \dot{q}(x) - \mathcal{L})$$

### 12.6.2 Complex Scalar Field

$$\mathcal{L} = -\phi^*\Box\phi - m^2\phi^*\phi$$

### 12.6.3 Klein-Gordon Field

For a massive scalar (spin 0) and neutral (charge 0) field:

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\phi)(\partial^\mu\phi) - m^2\phi^2]$$

The Euler-Lagrange formula:

$$(\Box + m^2)\phi = 0$$

This is the Klein-Gordon equation. It was quantized by Pauli and Weisskopf in 1934. The Klein-Gordon equation was historically rejected as a fundamental quantum equation because it predicted negative probability density.

### 12.6.4 Dirac Field

Dirac was looking for an equation linear in  $E$  or in  $\partial_t$ . For a massive spinor (spin 1/2) field the Lagrangian density is:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi; \quad \bar{\psi} = \psi^*\gamma^0 \quad \text{Dirac adjoint}$$

The  $4 \times 4$  Dirac matrices  $\gamma^\mu (\mu = 0, 1, 2, 3)$  satisfy the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

Define:

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

Any set of four matrices satisfying this equation can be combined to form a 4d representation of  $so(1,3)$  in the form

$$J_{Dirac}^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

The corresponding EOM is the Dirac equation:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0; \quad i(\partial_\mu\bar{\psi})\gamma^\mu + m\bar{\psi} = 0$$

To be explicit, this is shorthand for

$$(i\gamma_{\alpha\beta}^\mu\partial_\mu - m\delta_{\alpha\beta})\psi_\beta = 0$$

The Dirac ( $\gamma$ ) matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

where

$$\sigma^\mu \equiv (\mathbb{1}, \vec{\sigma}), \quad \bar{\sigma}^\mu \equiv (\mathbb{1}, -\vec{\sigma}),$$

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Where  $\sigma$  is the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (12.8)$$

$$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

$$\vec{\sigma}^\dagger = \vec{\sigma}$$

Commutation relation is

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}, [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$$

In the Weyl basis, the Dirac equation is:

$$\begin{pmatrix} -m & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

Quantization of the Dirac field is achieved by replacing the spinors by field operators and using the Jordand and Wigner quantization rules. Heisenberg's EOM for the field operator  $\hat{\psi}(\vec{x}, t)$  reads:

$$i\partial_t\hat{\psi}(\vec{x}, t) = [\hat{\psi}(\vec{x}, t), \hat{H}]$$

There are both positive and negative eigenvalues in the energy spectrum. The later are problematic in view of Einstein's energy of a particle at rest  $E = mc^2$ . Dirac's way out of the negative energy catastrophe was to postulate a Fermi sea of antiparticles. This genial assumption was not taken seriously until the positron was discovered in 1932 by Anderson.

### 12.6.5 Weyl spinor

$$\sigma^\mu\partial_\mu\psi = \mathbb{1}\partial_t\psi - \partial_i\sigma_i\psi = 0$$

Dirac equation for a Weyl spinor:

$$\sigma^\mu\partial_\mu\psi = 0$$

### 12.6.6 Maxwell Field

(12.9)

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}^2 - j_\mu A^\mu = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - j_\mu A^\mu \\ &= -\frac{1}{2}(\partial_\mu A_\nu)^2 + \frac{1}{2}(\partial_\mu A_\mu)^2 - j_\mu A^\mu \\ &= (E^2 - B^2)/2 - \phi V + \vec{j}\vec{A} \end{aligned}$$

Here we use

$$(\partial_\mu A_\nu)^2 = (\partial_\nu A_\mu)^2$$

where  $J_\mu$  is the external current:

$$J_\mu(x) = \begin{cases} J_0(x) = \rho(x) \\ J_i(x) = v_i(x) \end{cases}$$

Field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

with components

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = -E^i, \quad F^{ij} = \partial^i A^j - \partial^j A^i = -\varepsilon^{ijk} B^k$$

$$\text{EOM } \frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} = 0$$

$$-J_\nu - \partial_\mu (-\partial_\mu A_\nu) - \partial_\nu (\partial_\mu A_\mu) = 0$$

which gives

$$\partial_\mu F_{\mu\nu} = J_\nu$$

Lorentz gauge:  $\partial_\mu A_\mu = 0$

$$J_\nu = \partial_\mu (\partial_\mu A_\nu - \partial_\nu A_\mu) = \square A_\nu - \partial_\nu (\partial_\mu A_\mu) = \square A_\nu$$

so

$$A_\nu(x) = \frac{1}{\square} J_\nu(x)$$

**propagator**

$$\Pi_A = \frac{1}{\square}$$

Note that the propagator has nothing to do with the source. In fact it is entirely determined by the kinetic terms for a field.

### 12.6.7 Proca (Massive Vector Boson) Field

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - j_\mu A^\mu$$

EOM:

$$\square A^\nu - \partial^\mu (\partial_\mu A^\mu) + m^2 A^\nu = j^\nu$$



**spinors**

**Dirac spinors** have both left- and right-handed components. They can be massive or massless.

**Weyl spinors** are always massless and can be left- or right-handed. When embedded in Dirac spinors they satisfy the constraint  $\gamma_5\psi = \pm\psi$ .

**Majorana spinors** are left- or right handed. When embedded in Dirac spinors they satisfy the constraint  $\psi = \psi_C = -i\gamma_2\psi^*$

## 12.7 Interaction

time ordered amplitude:

$$D_F(x, y) \equiv \langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle = \langle 0 | \phi(x) \phi(y) | 0 \rangle \theta(x^0 - y^0) + \langle 0 | \phi(y) \phi(x) | 0 \rangle \theta(y^0 - x^0) \\ (\square + m^2) D_F(x, y) = -i\hbar \delta^{(4)}(x - y)$$

Let  $|\Omega\rangle$  being the vacuum state of interacting QFT, ( $|\Omega\rangle \neq |0\rangle$ ), then

$$\begin{aligned} & \partial_t \langle \Omega | T \{ \phi(x) \phi(x') \} | \Omega \rangle \\ &= \partial_t [ \langle \Omega | \phi(x) \phi(x') | \Omega \rangle \theta(t - t') + \langle \Omega | \phi(x') \phi(x) | \Omega \rangle \theta(t' - t) ] \\ &= \langle \Omega | (\partial_t \phi(x)) \phi(x') | \Omega \rangle \theta(t - t') + \langle \Omega | \phi(x') \partial_t \phi(x) | \Omega \rangle \theta(t' - t) \\ & \quad + \langle \Omega | \phi(x) \phi(x') | \Omega \rangle \delta(t - t') + \langle \Omega | \phi(x') \phi(x) | \Omega \rangle \delta(t' - t) \\ &= \langle \Omega | T \{ (\partial_t \phi(x)) \phi(x') \} | \Omega \rangle + \langle \Omega | [\phi(x), \phi(x')] | \Omega \rangle \delta(t - t') \\ &= \langle \Omega | T \{ (\partial_t \phi(x)) \phi(x') \} | \Omega \rangle \end{aligned}$$

So, the second derivative:

$$\begin{aligned} & \partial_t^2 \langle \Omega | T \{ \phi(x) \phi(x') \} | \Omega \rangle \\ &= \partial_t \langle \Omega | T \{ (\partial_t \phi(x)) \phi(x') \} | \Omega \rangle \\ &= \langle \Omega | T \{ (\partial_t^2 \phi(x)) \phi(x') \} | \Omega \rangle + \langle \Omega | [\partial_t \phi(x), \phi(x')] | \Omega \rangle \delta(t - t') \\ &= \langle \Omega | T \{ (\partial_t^2 \phi(x)) \phi(x') \} | \Omega \rangle - i\hbar \delta^{(4)}(t - t') \end{aligned}$$

So for free field ( $(\square + m^2)\phi(x) = 0$ ):

$$\begin{aligned} (\square + m^2) \langle \Omega | T \{ \phi(x) \phi(x') \} | \Omega \rangle &= \langle \Omega | T \{ (\square + m^2)\phi(x) \phi(x') \} | \Omega \rangle = -i\hbar \delta^{(4)}(x - x') \\ &= -i\hbar \delta^{(4)}(x - x') \end{aligned}$$

Define:

$$\langle \Omega | T \{ \phi(x) \phi(x_2) \cdots \phi(x_n) \} | \Omega \rangle = \langle \phi(x) \phi(x_2) \cdots \phi(x_n) \rangle \quad (12.10)$$

So:

$$\square \langle \phi(x) \phi(x_2) \cdots \phi(x_n) \rangle = \langle (\square \phi(x)) \phi(x_2) \cdots \phi(x_n) \rangle - i\hbar \sum_{j=1}^n \delta^{(4)}(x-x_j) \langle \phi(x) \phi(x_2) \cdots \phi(x_{j-1}) \phi(x_{j+1}) \cdots \phi(x_n) \rangle$$

One more definition:

$$\phi_x \equiv \phi(x), \quad \phi_j \equiv \phi(x_j) \quad (12.11)$$

So, in general:

$$\square \langle \phi_x \phi_2 \cdots \phi_n \rangle = \langle (\square \phi_x) \phi_2 \cdots \phi_n \rangle + \dots = \langle (\frac{\partial \mathcal{L}_{int}}{\partial \phi_x} \phi_2 \cdots \phi_n) \rangle - i\hbar \sum_{j=1}^n \delta^{(4)}(x-x_j) \langle \phi_x \phi_2 \cdots \phi_{j-1} \phi_{j+1} \cdots \phi_n \rangle$$

For interacting QFT:  $((\square + m^2)\phi = \frac{\partial \mathcal{L}_{int}}{\partial \phi})$

Define shorthands:

$$\delta_{xi} \equiv \delta(x-x_i), \quad D_{xi} \equiv \langle \phi_x \phi_i \rangle \quad (12.12)$$

So for free massless scalar:

$$\square_x D_{x1} = -i\delta_{x1}$$

So:

$$\begin{aligned} D_{12} &= \langle \phi_1 \phi_2 \rangle = \int d^4x \delta_{x1} \langle \phi_x \phi_2 \rangle \\ &= i \int d^4x \square_x D_{x1} \langle \phi_x \phi_2 \rangle \\ \text{IBP twice} &= i \int d^4x D_{x1} \square_x \langle \phi_x \phi_2 \rangle \\ &= \int d^4x D_{x1} \delta_{x2} = D_{21} \end{aligned} \quad (12.13)$$

and

$$\begin{aligned} \langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle &= \int d^4x \delta_{x1} \langle \phi_x \phi_2 \phi_3 \phi_4 \rangle = i \int d^4x D_{x1} \square_x \langle \phi_x \phi_2 \phi_3 \phi_4 \rangle \\ &= i \int d^4x D_{x1} (-i) (\delta_{x2} D_{34} + \delta_{x3} D_{24} + \delta_{x4} D_{23}) \\ &= D_{21} D_{34} + D_{31} D_{24} + D_{41} D_{23} \\ &= D_{12} D_{34} + D_{13} D_{24} + D_{14} D_{23} \end{aligned}$$

Perturbation Theory in coupling constant  $g \ll 1$ : truncate at  $O(g^2)$   
 Feymann Rules:

1. For all external points, draw a fixed point with a line going out from it.
2. (a) contract lines together – Feymann Propagator  
 (b) line-split: an interaction-vertex  $\prod i\mathcal{L}'_{int}(\phi)$
3. for given order in P.T. in coupling constant, sum of all diagrams with all lines contracted and intergragte over position of all vertices.

## 12.8 Terminology

free spinors

Dirac spinors

Weyl representation

spinor representation

$(\frac{1}{2}, \frac{1}{2})$  representation

projective representation

Lorentz generators

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

Lorentz group  $O(1,3)$ ; tensor representations  $\phi, A_\mu, h_{\mu\nu}$ , spinor representations: Weyl spinors  $\phi_L, \phi_R$ . A Dirac spinor  $\psi$  transforms in the reducible  $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$  representaion. The next step towards quantizing a thory with spinors is to use these Lorentz group representaions to generate irreducible unitray representaions of the Poincaré group.

# Chapter 13

## Examples

### 13.1 Dimensional analysis

#### 13.1.1 Black body radiation

$$I(\omega) \equiv \frac{1}{V} \frac{d}{d\omega} E(\omega)$$

This has units of  $[Energy] \times [time] \times [distance]^{-3}$  that can be constructed out of  $\omega, k_B T$  and  $c$ . So easily:

$$I(\omega) = \text{const} \times c^{-3} \omega^2 k_B T$$

### 13.2 First Principle

#### 13.2.1 Special relativity field

special relativity scalar field  $\phi \xrightarrow{\text{simplest possible Lorentz invariant}} \square \phi = 0$

One solution is:

$$\phi(x) = a_p(t) e^{i\vec{p} \cdot \vec{x}}$$

So we get:

$$(\partial_t^2 + \vec{p} \cdot \vec{p}) a_p(t) = 0$$

This is exactly the equation of motion of harmonic oscillator. General solution:

$$\phi(x, t) = \int \frac{d^3 p}{(2\pi)^3} [a_p(t) e^{i\vec{p} \cdot \vec{x}} + a_p^*(t) e^{-i\vec{p} \cdot \vec{x}}]$$

### **13.2.2 Random walker**

The RMS distance of a random walker grows as a square root of the time

# Part IV

## Application

## 13.3 Application

### 13.3.1 time

Definition of time:

- sun
- pendulum – 1929
- quartz – 1967
- atomic beam resonance – 1967
- atomic fountain – 1989
- optical clock – 2020 ?

## Part V

# Timeline of Important Physics