Marshall Gordon

Enabling Students in Mathematics

A Three-Dimensional Perspective for Teaching Mathematics in Grades 6-12

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To my wife, Paddy, and our children, Ian,
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Overview

Teaching is an extraordinary adventure. Looking at the front covers of physics books where there is an illustration of subatomic particles flying in this direction and that as a result of heightened interaction, it is evident how complicated things are beneath the surface. Mathematics teachers know all about that. Exploding into and out of existence in our students' minds are concerns related and not, questions and ideas, and varied emotions, even when they are listening to what we are saying or watching what we are writing on the board.

Yet the complications do not stop there. The poet T. S. Eliot reminds us that "between the idea and the reality falls the shadow." And mathematics teachers are well aware of the varied shapes the shadow can take, including uninterested students, an insensitive curriculum, excessive administrative pressures, and inappropriate parental involvement. Best of luck with your administration and parents. This book will focus on how mathematics teachers can enable students to become more adept mathematical thinkers, more capable in their mathematics collaborations, and more in charge of their own development as successful mathematics students.

* * *

Clearly, there is much to think about and know to teach well. To be a successful mathematics educator requires our being informed not only about the content knowledge associated with mathematics itself, but the pedagogical content knowledge associated with creating a successful classroom mathematics experience and general pedagogical knowledge associated with knowledge of various teaching strategies and how to foster student learning (Shulman 1986, 1987; Borko and Putnam 1996).

Indeed, teaching mathematics is a complex engagement. G. H. Hardy, the twentieth century British mathematician known for his work in number theory and for discovering the extraordinary Indian mathematician Ramanujan, understood that quite well. His statement, "I would rather lecture than teach," made clear he did not want to get caught up with whether the content actually connected to the students, nor their concerns and questions, or the involved effort and continued discussion needed to ensure their understanding. Yet as mathematics educators working to develop students' mathematical intelligence in grades 6–12, we do need to consider all those aspects. And more: There is the classroom environment that can promote valuable student interactions to think about and the significant influence students' personal qualities play in the thinking/learning process. These are all foundational concerns of a mathematics classroom experience committed to developing students' intellectual, social, and personal capacities essential for a vibrant society.

And of course, there is *the* curriculum. The prescribed body of material to be presented and finished by the end of the school year. It can well be an imposing presence and can rightfully leave us feeling considerable pressure. After all, if we think of a favorite author of ours, and then imagine we get a new book by that author, is anyone really confident enough to predict what page they will be on after reading for two hours? Now consider every student in your class has a mathematics text in their hands that they did not choose. What page will they be on the last day of class—9 or 10 months later? If we cannot know what page we would be on reading our favorite author after reading for just 2 h, how could we possibly know by the end of the school year what page we would be on with a group of students whose interests in and expertise with the material are as varied as they are?

So naturally, the imperative to *cover the curriculum* compels mathematics teachers and textbooks to emphasize presentations of mathematics algorithms and problem-solving techniques as this is the most direct approach to transmit all the content. Yet, this approach surely has its problems. Why, for example, with the mathematics curriculum laid out so clearly with explicit rules and problem-solving procedures associated with each content area is "mathematics widely hated among adults" (Boaler 2008, p. 4)? And as regards the young, why does math anxiety actually exist? We need to take seriously the emotional disturbance and difficulty many students experience while engaging a discipline which celebrates reasoned argument.

This is to say, if we are committed to having mathematics classrooms where students are productively involved, able to analyze problems well, reflect on what they and others say, and are open to changing their minds, then we need to promote, develop, and sustain *that* curriculum. Yet, that is no easy matter. Classroom discussion surely has its difficulties. It is even seen as the source of the "*mathematics teacher's dilemma*". "This dilemma arises in classrooms in which the teacher wishes both to ensure learner participation and to teach particular ideas. The dilemma is how to *elicit the knowledge* from learners that she wants to teach. As long as she genuinely allows learners to express their thinking, a teacher cannot be sure that such expression will contribute towards what she is trying to teach. If the teacher maintains her focus on covering the content of the curriculum, then she may be in danger of missing what learners have to say (Brodie 2009, p. 28; italics added)."

To resolve that dilemma we can, of course, present mathematics material that is so clear in its prescription that it limits the need for conversation and questions. But as just discussed, that approach does not appear to ensure a successful mathematics experience. There is another way. We can have those discussions, but they can be more effective and more productive for both the students and the teacher.

This requires providing students access to the language of more productive mathematical thinkers. In that way, the classroom discussions are more informed and so take less time to develop students' mathematical understanding. To make that happen we need to include *as content* mathematical heuristics, those *problem-clarifying* strategies that mathematically able thinkers draw upon to gain insight into solving mathematics problems. These strategies are fundamentally the "tools of the trade." With students increasingly aware of how to make use of them, there is no need to experience classroom discussions that lack coherence or dedicate so much time to teaching algorithms and procedures. With students more able to think mathematically, the mathematics problems that can be considered and the conversations that can be had can be at a much more engaging and rewarding level for both the students and the teacher. The first section of this book is dedicated to that development.

* * *

Albert Einstein, in reflecting on the common experience of not remembering most of what he learned in school, came to think that "your education is what you know when you forgot what they told you." What is left? Focusing on the positive—productive habits of thought, constructive means of relating, and personal capacities so we can better do things. These would surely be outcomes of a valued and valuable education—positive developments in each of the dimensions of our students' intellectual, social, and personal school experience. Together, they can be said to constitute a *socially responsible* mathematics education.

However, if classroom efforts are primarily given to teacher demonstrations of procedures, student practicing, and their testing, we are likely promoting the development of an adult population trained to look for quick answers, not inclined to think things through nor experienced in the exchange of ideas and competing explanations essential for dealing well with complex issues. Such mathematics classroom experience seems geared toward a limited view of human beings and what it means to be a valued participant in a society dedicated to the fullest development of all of its citizens.

Thinking about what behaviors we would want our mathematics students to demonstrate, we can appreciate that habits are "the mainspring of human action" and "are formed for the most part under the influence of the customs of a group" (Dewey 1954, p. 159). In the mathematics classroom, an appreciation of habit development is apparent when students do homework consistently, are on time to class, bring the right books, etc. Yet of course, habits have a broader compass in every facet of our lives. There are habits associated with personal and social behavior in addition to those associated with thinking that are instrumental for shaping our lived experience in better or lesser ways. That is to say, they influence to an essential degree the individual and collective efforts of our students and our mathematics classroom experience. So the questions that naturally follow are which habits should we seek to promote and develop, and which would be good to eliminate?

For example, if you have taught a while, you may have noticed that if students do not develop confidence in dealing with mathematics questions, they are prone to either believe whatever comes first to their minds, or they cannot trust anything that comes into their minds. Naturally, in the absence of that confidence and trust, students will let impulse make their decisions or remain confused and unsure of how to proceed. As a consequence, they likely develop unproductive habits, unhelpful coping behaviors, and a view of mathematics that is not what we would hope.

Research bears that out. For example, "One of the biggest mistakes students make with math problems is that they often rush in and do *something* with the numbers, without really considering what is being asked of them, whereas successful problem solvers spend some time really thinking about the problem" (Boaler 2008, p. 186; italics in original). This is to be expected, as in a stressful situation, we naturally tend to rush to escape it. So developing students' patience, resilience, and flexibility is *paramount* in helping them get to the state of being able to "spend some time really thinking." Not to mention being more successful on standardized exams that test for understanding, as the coming Common Core State Standards (CCSS) tests are said to be.

Helping students develop mathematical habits of mind and become, in general, more aware of productive means for making good decisions will naturally develop their confidence as they become more successful mathematical thinkers. We can also help them develop their resilience by enabling them to engage complex problems and their trust in themselves as they consider as part of *their curriculum* their own personal/professional development toward becoming more capable mathematics students. Toward that end, the third section of this book focuses on how they can take more thoughtful control of their participation in their mathematics experience.

What about the second section? Whatever standards of decorum or social philosophy a school is following, an appreciation of both the classroom environment and the roles students as adults will have in a democratic society requires promoting their being able to collaborate productively. This is surely an essential activity toward securing a robust society where people are open-minded, listen carefully to each other, and work together to solve complex problems. To promote that social development, the second section will consider how mathematics teachers can develop students' reflective thinking so as to help them become more valued group members, and introduce "multiple-centers" investigations involving engaging mathematics problems that call on those practices to be successful.

* * *

This is all to say that *Enabling Students in Mathematics—A Three-Dimensional Perspective for Teaching Mathematics in Grades 6–12* addresses the cognitive, social, and psychological dimensions that shape students' mathematics learning experience. The object is to help ensure they are capable, cooperative, and confident engaging mathematics. In this complete way, all students can have a productive and enjoyable mathematics experience that would promote their being valued participants in society.

To help secure that life-enriching development, assessment will be part of each dimension of students' mathematics experience. It will also be the focus of the fourth section where grading, homework, and the day-to-day mathematics classroom will be revisited through the lens of values that inform our assessments. For as John Dewey noted, "we learn by doing only if we reflect on what we've done."

(And that would include reflecting on what we have not done.) Without mathematics students and teachers having the opportunity to stop to reflect on how things are going and getting feedback from others as well, there is little chance to see beneath the surface, little chance to decide how to better proceed. Hence, the goal of the assessment considerations is to promote conversations—between students and teachers, between students with themselves, and teachers with themselves—dedicated to developing thoughtful, socially aware, and resilient students of mathematics who will bring their capable selves and energy to the future development of society.

That is what the book is about. Hopefully it will reward your time and thinking.

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Part I
Promoting Mathematics Students' Cognitive Development

Education is not an affair of telling and being told, but an active and constructive process.
John Dewey

Introduction

The students who come to mathematics class naturally arrive with different learning experiences, even if they sat next to each other in earlier years. Each of them tried to make sense of the mathematics from their own perspective, based on their level of interest, their ability to focus, emotions at the moment, and the learning environments they found themselves part of. Yet, despite the differences in their experience, and the successes of a number of students in mathematics, there is considerable evidence things have not gone well for many students. So, it is necessary to consider why learning mathematics is challenging for many students, and what can be done to make it a more valued and valuable educational experience for all students.

The problem seems rooted in what it means to know mathematics. For example, one teacher mentioned in a blog that in light of the Common Core principles and practices she would be adopting, she would stop having her elementary school students multiply length by width to find the area of a rectangle. Instead, she would show why that approach actually works. This seems clearly in the right direction. Students having been told how, but not why some mathematics formula or process works helps locate why many students have had such a poor experience learning mathematics. For "When the focus is on skills and procedures the tendency is to lean away from a problem-based approach to rely on show-and-tell, thereby decreasing opportunities for the students to develop ideas that make sense to them".

That is, understanding remains at a distance from many students' mathematics experience, if their personal classroom experience is one of practicing techniques, as understanding requires making sense of things. To practice division problems, cross multiplication, inverting and multiplying, etc., without having an awareness of what the rationales are for each of these actions is in effect promoting a literally dumb response with regard to solving problems.

Yet, with teachers learning these procedures when they were students, there is the expected outcome that they too would teach that way. With the focus on procedures and not thinking, mathematics remains a confusing experience for many students. "Instead of trying to convey, say, the essence of what it means to subtract fractions, teachers tell students to draw butterflies and multiply along the diagonal wings, add the antennas and finally reduce and simplify as needed. The answer-getting strategies may serve them well for a class period of practice problems, but after a week, they forget. And students often can't figure out how to apply the strategy for a particular problem to new problems".

We can appreciate the clarity experienced in being told formulas as if they were definitions, and by following procedures. But it is a narrow view of what doing mathematics offers and what it means to be educated. Students need to be participants in experiencing and learning about the inventive nature of mathematics along with means to think mathematically. In that direction, teachers of students of all ages might well appreciate Madeline Lampert's transforming a common elementary school way of teaching mathematics, "I, We, You", into what has been called "You, Y'all, We", where the focus is not on teacher's demonstration of a mathematics procedure but students sharing their thinking when engaging a mathematics problem. It clearly transforms the educational experience into one of sensemaking, the objective being not providing an "answer-getting" strategy.

Tara Holm, a mathematician, put it this way:

"Calculators have long since overthrown the need to perform addition, subtraction, multiplication, or division by hand. We still teach this basic arithmetic, though, because we want students to grasp the contours of numbers and look for patterns, to have a sense of what the right answer might be. But what happens next in most schools is the road-to-math-Hades: the single-file death march that leads toward calculus.

We are pretty much the only country on the planet that teaches math this way, where students are forced to memorize formulas and procedures. And so kids miss the more organic experience of playing with mathematical puzzles, experimenting and searching for patterns, finding delight in their own discoveries. Most students learn to detest—or at best, endure—math, and this is why our students are falling behind their international peers.

When students memorize the Pythagorean Theorem or the quadratic formula and apply it with slightly different numbers, they actually get worse at the bigger picture. Our brains are slow to recognize information when it is out of context. This is why real-world math problems are so much harder—and more fascinating—than the contrived textbook exercises.

What I have found instead is that a student who has developed the ability to turn a real-world scenario into a mathematical problem, who is alert to false reasoning, and who can manipulate numbers and equations is likely far better prepared for college math than a student who has experienced a year of rote calculus" (www.bostonglobe.com/opinion/2015/02/12/why-failing-behind-math).

Until mathematics textbooks and teachers focus their energies on student engagement and understanding as the essential component of the mathematics curriculum, students will likely continue to be presented with definitions and demonstrations that create more questions than they answer. A number of such considerations inform this section.

Chapter 1
Developing Students' Mathematical Intelligence

The late physicist, Richard Feynman, walking with his father in the woods, saw some birds on a tree and asked his father what the names of the birds were. His father replied, "Don't worry about the names; watch what they do." He wanted Richard to develop his powers of observation as well his imagination and was giving him time to have questions come to mind that went deeper than the surface knowledge of knowing the names.

This is a lesson mathematics texts might learn. It helps us appreciate that just telling students "the way things are" could flatten what could otherwise be a thoughtful engagement. Here are two instances:

While the mathematician and philosopher Bertrand Russell made clear that definitions are value free, that is, free from considerations of truth or falseness, defining can indeed be a valued learning activity. For example, rather than telling students who the members of the family of four-sided polygons are, we can give them the opportunity to see how they would logically partition quadrilaterals themselves. Their investigations would naturally give them a more intimate understanding of the forms. And as a consequence, they would likely come up with conjectures created by the distinctions they noted. Their experience would likely promote interesting conversations with other students, further opportunities for investigations, and the development of formulating reasoned arguments. Namely, a more inviting, heightened, and personally rewarding, engagement of mathematics.

As for the other example, symbolic representation has a creative aspect as well that deserves being acknowledged. For example, mathematicians in the early seventeenth century found themselves having to decide how to represent the power of a number. How to configure the expression created a real debate until Descartes suggested using counting numbers (rather than Roman numerals, for instance) in the upper right-hand corner (vs. other locations) which gained acceptance. This is to say presenting the accepted symbolic representation to mathematics students as if it was obvious diminishes the interesting experience that is otherwise available. For instance, should there not be a classroom conversation why *m* is the symbol to represent the slope of a line in the coordinate plane? What would seem to be going

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through students' minds when being presented with such a representation without a hint of how such a choice was made?

These instances help point to an important pedagogical problem created by some mathematics textbooks—with telling the primary form of communication, it may well lead the teacher to act in the same manner. But such a taken-for-granted approach would logically tend to diminish the excitement of discovery and the awareness of the sheer inventiveness of mathematics. By telling too often, we likely diminish students' mathematics experience in terms of their potential to ask "what if?" and "what if not?"—questions that can transform the mathematics classroom conversation (Brown and Walter 1983) and, as a consequence, their inclination to look for and secure a "deep understanding."

Being told something is the case, wonder can disappear. But would we not expect, and actually hope, that students would express a concerned confusion at being informed, for example, that the slope of a line is represented by the symbol *m* and not more obviously *s*? And that "right angles are 90°"? Can they not point to the left? And why 90—would 100 divisions be not more pleasing? The great eighteenth-century mathematician, Simon de Laplace, thought so.

Part of the problem is that inasmuch as mathematics textbook writers want to minimize the likelihood of being misunderstood or misleading, there is little reason for them to prompt questions. It is best to make definitive statements and demonstrate procedures and leave the conversations to the mathematics teacher. Such an approach is aided by the belief that students' naïve view based on their limited experience suggests their need to be informed. Rather than seeing their naiveté as being an expression of an open-minded curiosity and flexible capacity for thinking, it may be seen as a limitation. That is a problem we, who are educated, can remedy. After all, it is the naïve view that asks questions that help us all see anew.

* * *

It is natural to lose sight of the questions generated by a naïve intuition that gave birth to ideas, inventions, new paths worth following, etc., as they are lost in the turning of history. For example, the spark of "what if" of the Earl of Sandwich, to put meat between two slices of bread so his hands would not get greasy while playing cards, could one day be of such loss.

The youngsters in front of us come with a lively naiveté if we give them the chance to ask, "What's going on here?" Giving it opportunity to express itself, we create the opportunity for refreshing conversations, interesting conjectures, and new learning experiences that really matter to students—directions their mathematics teachers would likely appreciate well. Such engagements not only add to students' understanding by subtracting doubts and confusion but promote their developing intuition in gaining experience in the investigative process itself. This suggests it is our work as mathematics teachers to create the settings to help shape those opportunities.

Have you ever cut a sandwich on the diagonal? If you try, you will see it is more difficult than cutting it parallel to the sides. Students can understand that cut either way the areas are equal. Something else must be the "why" sandwiches are cut on the diagonal as it is more difficult to do so, yet often the case in restaurants. Comparing the lengths around, they can appreciate why cutting one way rather than the other is practiced, especially when paying for a sandwich in a restaurant.

Suppose, in lieu of stating the area of a circle formula as mathematics textbooks tend to do, we begin with a question—does anyone have any idea as how to get a decent approximation to the area of a circle? Students who would draw a square circumscribing the circle would unconsciously be practicing the helpful habit of mind of *make the problem simpler* and in doing so determine that area to be $4r^2$ (Fig. 1.1). From there, there are many possible conversations of course. If students were familiar with the habit of mind to *take things apart,* they could divide the circumscribed square into 4 unit squares, remove one of the two in the top half, and so make even a better approximation by moving the single square on top so that it is symmetric above the 2 unit squares below (Fig. 1.2). Now it is clear that $3r^2$ is really close to the area of the circle. It actually represents an error of less than 5 % and was an approximation well alive in the recesses of history.

Geometry books often state extraordinary mathematical relationships absent of the heightened emotion that must have accompanied the defining investigation. Indeed, to do so might well fill the book with exclamation marks! Yet the absence of acknowledging the inventive engagements dampens what could generate student interest, investigation, and appreciation. For example, stating the Pythagorean Theorem as if it is an obvious observation would likely leave students confused, for it is

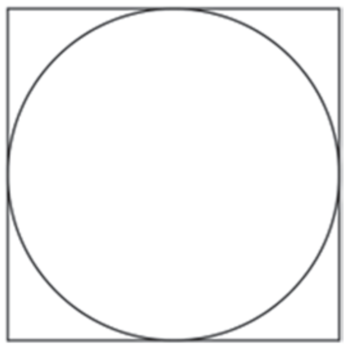

Fig. 1.1 A first approximation to the area of a circle

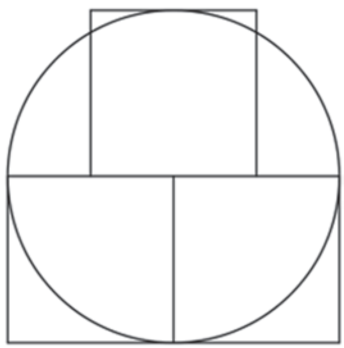

Fig. 1.2 A better approximation

not obvious to see the underlying connection between the lengths of the sides of the right triangle. It was clearly not obvious to the ancient Egyptians, the great builders who used Pythagorean triples but did not know the general relationship. For another example, consider the statement "Prove that two chords that intersect in a circle create similar triangles when their endpoints are connected." Stated that way, it is an assignment—something to be done. But it is extraordinary that it is true! If you want to see why, try drawing two line segments such that their intersection creates segments that share a constant ratio. You could spend a *really* long time at it and still not make that happen—even if you try to have them bisect each other. Now, draw a circle and draw *any* two chords so that they intersect—done!

The great Archimedes' investigation of the area of a circle is presented as a demonstration in clever reasoning. But there would have to be more to the story, of course. He did not have a direct procedure to follow; he had to use an inventive mathematical mind to ask and answer "what's going on here?" And it came into view by his *making the problem simpler!* That essential mathematical habit of mind led him to actually gain better and better area approximations of a circle and finally work with the notion of a limit, which would be the defining element in Newton's and Leibniz's calculus, more than 1800 years later. This is to say that telling youngsters the matter-of-fact information regarding the area of a circle, and so many other formulas and definitions, is a questionable educational practice, especially when the critical mathematical habits of mind that informed the thinking are omitted. In fact, it could actually be seen as promoting a lost educational opportunity.

* * *

To make mathematics "real" need not mean that it is embedded in applications, but rather it connects to the interests of those who engage it. The National Council of Teachers of Mathematics' (NCTM's) thinking that "…the central focus of the classroom environment [is] on sense-making" (1991, p. 57) makes perfectly good sense. To help see why this most significant mathematics experience is often missed, we can try to imagine ourselves as students listening to someone introduce terms and symbols that might be ambiguous to us but are being presented as if there was no confusion other than our own. Would we be motivated to question as a 13-, 15-, or 17-year-old what is seemingly obvious in the face of the teacher's declarative expression? Consider being shown a right angle for the first time. What is "right" about it? Can it not point in any direction, including being upside down? That the statement is visually confusing is compounded by the additional "matter-of-fact" offering that a right angle has 90°. Why 90? That it is half a straight angle begs the question. That it is a quarter-turn around the center of a circle begins to make it interesting, motivating students to ask and desire to understand why, "what's going on here?" Why should a circle have 360°? Would 400, for instance, not be a more reasonable choice?

It would seem that as mathematics educators we would be pleased that students would be perplexed, for as Dewey noted thinking occurs only when we experience a problem. This suggests we ought to consider how we might present some definitions, procedures, and formulas so as to invite student discussion, for in that way they have opportunity to raise conjectures and think more deeply as a consequence.

If in the discussion their intuitions are confirmed, they naturally feel disposed to making other conjectures, which can inspire other valuable conversations. If their intuitions are challenged, here too they are learning, in recognizing that more reflection is often needed. In either case, we create the positive energy essential for promoting a lively mathematics experience, including the development of students' more thoughtful considerations.

With regard to promoting a discussion in the case of the right angle, students could wonder why it would be a focus. Asking them why they would think such an angle would get attention especially in times of early human development, they could come to conjecture that "right" could well be shorthand for "upright" or "correct." Seeing the symbol for the right angle might trigger their imagination. Clearly, in ancient times, buildings being upright could be very challenging, as being just a bit off could well mean gravity would soon become not a supporting but a destructive force. Whether it is "absolutely" true regarding the origin of a *right* angle coming from *upright* (or being the "correct" angle to keep the structure standing) may well be beside the point, as the classroom consensus toward sense making has come up with what appears to be a very reasonable rationale. And as historical research attests, there is often more than one explanation (interpretation/deduction) associated with an event.

That instance was to acknowledge that all mathematical concepts, as all inventions, gain real appreciation when understood as a response to experience. Consider the introduction of numerals. We introduce them to children as if they were things. Yet it took millennia to develop the concept. Prior to the third millennium in both the Mesopotamian and Egyptian civilizations, "The 'four' of 'four sheep' and 'four measures of grain' [were] not written with the same symbol" (Ritter 1989, p. 12). We can appreciate why: Consider four sheep walking around in the meadow and four pieces of grain lying on the floor in a storeroom; how likely are they to suggest the abstract notion of "fourness"?

Hopefully, students can come to appreciate the intellectual leap that had to take place in human thought to create the ethereal objects of numerals. Indeed, the mathematician and philosopher Bertrand Russell claimed that 2 was the first number. These are to suggest that if we want students to "formulate mathematical definitions" (NCTM 1989, p. 40), they will have to be offered opportunities, including vicarious experiences in the form of stories, real or imagined, so that their intuition can be developed and have the opportunity to conjecture and abstract from experience which is so much part of mathematical thinking.

That a right angle has 90° (and not for example the more intuitively appealing 100) needs to be recognized as containing a history as well, or at least some story, that deserves to be shared. (Surely, any unit measure must have a story behind it—like a mile being 5280 ft!) All units of measurement are invented of course, and so it is not enough to use a protractor to give legitimacy to a right angle *being* 90°. That assumes the particular unit as a taken-for-granted measure, as if it had the same concrete reality as the plastic tool in the student's hand. Rather, we could share with our students that in ancient times, going back before ancient Egypt, to Babylon and Mesopotamia—the sky reckoners believed 360 was the number of days in a year.

The Egyptians realized it was 365 as a result of very carefully following the annual return of the star, Sirius, as it appeared when the Nile would again flood, a most propitious time for good or bad. Yet they kept the 360 unit measure as their calendar with 12 months of 30 days and claimed 5 days at the end of the year for holidays! Pretty inventive! Mathematically speaking, it was the better choice. The 365-day unit has only two divisors, while 360 has so many more, allowing for a lot more divisions to distinguish other intervals of time and create relations between different intervals. We will revisit this later, when we consider some implications of the circle being divided into 400 parts, as the great mathematician Laplace argued for.

These instances suggest that to acknowledge the virtue of a naïve perspective and help nurture and develop students' disposition to make mathematical sense, "facts" would not always be presented devoid from their historical or logical roots. Otherwise we are promoting their accepting whatever authority says. But if we want to develop thoughtful, reflective, questioning citizens of a democratic society, acceptance without seeking justification would not be the disposition we would want to promote. Then, with students seeing that it was their wondering that was instrumental in securing a "logical why" and/or a "chronological why" that informs them of their questioning's value. With the mathematics teacher communicating a respect for their questions, their inquisitiveness and thoughtful energies are being recognized. And that would seem to be exactly what is needed to create a classroom environment that promotes the development of their mathematical intelligence.

* * *

In the next chapter, we will consider some standard algorithms and practices that may well be being experienced as more problematic than informative. Especially to an inquiring mind that seeks to make sense of things. With all the energies given to presenting techniques, it is good to remind ourselves that "[a researcher who] studied structural engineers at work for over seventy hours found that although they used mathematics extensively in their work, they rarely used standard methods and procedures" (Boaler 2008, pp. 7–8). This suggests that in sharing with students what is involved in thinking mathematically, how facile one is in applying algorithms would not be the exclusive focus. More completely, mathematics texts often make explicit knowledge that demonstrates "knowing-that," "knowing-how," and sometimes "knowing-why". What will be the focus in the following, the development of students' awareness of problem-clarifying strategies, can be thought of as another knowing: "knowing-to" (Mason and Spence 1999).

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Chapter 2
Presentations into Investigations

As we all recognize, gaining any habit such as learning to read, walk, tie our shoes, etc., begins with being awkward and, as importantly, takes time to secure. But developing productive habits is of course a really good idea despite the complexity of that learning experience. Habits allow us to do things efficiently without giving much of any thought to the behavior, and that allows more time to do other interesting things. Indeed, they would seem essential for becoming a capable and thoughtful mathematics problem-solver. That suggests we need to take seriously what habits we as mathematics teachers should promote, and how we go about doing so. These would seem to be of fundamental concern in our work.

William James, a psychologist concerned about the educational experience, wrote how preeminent a role habit development should have in schools. As he saw it, "Education, in short, cannot be better described than by calling it *the organization of acquired habits of conduct and tendencies to behavior*" (1899/2008, p. 25, italics in original). And his fellow traveler, John Dewey, also recognized their heightened importance: "We state emphatically that, *upon its intellectual side, education consists in the formation of wide-awake, careful, thorough habits of thinking*" (1933/1936, p. 78; italics in original).

Here, we mathematics educators find ourselves face-to-face with the problem mentioned earlier. Given the press of covering the mathematics curriculum, being able to demonstrate algorithms tend to be the practices that we want our students to develop as habits. Yet, in a number of instances, their efficient demonstration provides little evidence why they work. As a direct consequence when presented without discussion, they can leave students more numb than educated or with a false sense of their mathematical capacity. (Consider, e.g., "to solve a proportion, cross multiply" or "to divide by a fraction, invert and multiply.") They often seem not much different than magic tricks. They work, but of course the question is why, for nothing has otherwise been learned. (Readers interested in seeing a collection of such poor representations see *Nix the Tricks* by Tina Cardone and the MTBoS, NixtheTricks.pdf, updated January 30, 2014.)

Yet, what some students really like about mathematics is having specific techniques that allow them to solve problems. Knowing the division algorithm or how

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to factor a quadratic expression, for example, or in general being able to deal directly with a problem situation by applying a technique demonstrates how efficient one is. But with teachers and texts presenting mathematics where solution models and algorithms are the primary focus to solve sets of problems that students then practice, students will likely have a surface knowledge of mathematics and themselves as mathematics students.

Experienced mathematics educators know well "There is no guarantee in any amount of information, even if skillfully conveyed, that an intelligent attitude of mind will be formed" (Dewey 1937, p. 183). This is made poignantly clear in a National Assessment of Educational Progress question analysis where students were asked to determine the value of $(2/3) \times (2/5)$. The findings were that 70 % of 13-year-olds and 74 % of 17-year-olds could do the multiplication correctly. But when those same students were presented with, "Jane lives 2/5 of a mile from school; when she has walked 2/3 of the way to school, how far has she walked?", students demonstrated very little understanding, with 20 % of the 13-year-olds and 21 % of the 17-year-olds responding correctly.

Surely, efficient means to solving problems should be practiced. In that way, more interesting mathematics problems can be considered. However, what the research says is that "Mathematics learning has often been more a matter of memorizing than understanding" (Kilpatrick et al. 2001, p. 16). And with regard to memorizing algorithms, because of their form being one of technical efficiency, the thinking that gives legitimacy to the procedure is often hidden. So students may memorize a procedure and do well on an exam and yet have no idea why it works. They come to believe "In Math you have to remember, in other subjects you can think about it" (a student quoted in Boaler 2008). In this way, mathematics textbooks and teacher presentations which share the same surface aesthetic can actually be an impediment to student learning and their gaining "deep understanding." The rather exclusive focus on efficient approaches promotes a lack of thoughtful experiences by omitting the otherwise needed time for developing the habit of being able to stay with a problem, that essential quality that life will reward with it happening.

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Fortunately, mathematics algorithms and practices obscuring their rationale can often be re-presented by introducing problem-clarifying strategies, which over time, with dedicated focus, will become mental habits. Such strategies give students a more significant role in shaping the conversation, and so promote their understanding. "A 'habit of mind' means having a disposition toward behaving intelligently when confronted with problems" (Kosta and Kallick 2009). What follow will be companion pieces—standard algorithms and practices lacking explanation presented along with mathematical habits of mind, *problem-clarifying* strategies—a comparison that illustrates the opportunity for students' thoughtful agency rather than their passive acceptance.

The focus will begin with some of the earlier mathematics experiences students are likely to have. For if we want students to "question the teacher and one another; [and] try to convince themselves and one another of the validity of particular representations…, and answers" (NCTM *Professional Teaching Standards*, Standard 3, p. 45), then the place to begin must be with presentations associated with students learning arithmetic procedures.

2.1 The Long-Division Algorithm and *Take Things Apart*

The long-division algorithm is clearly efficient in determining the quotient of a division problem involving multi-digit numbers. Yet, students often have considerable difficulty working with it, so much so that long division tends to be omitted from classroom discussions and is considered by many mathematics educators as not worth the time. This of course does not go uncontested. The authors of "Ten Myths About Math Education And Why You Shouldn't Believe Them" (Budd and Carson et al. 2005), some of whom are mathematicians, are clearly upset with "the snubbing or outright omission of the long division algorithm by NCTM-based curricula…". The conversation continues, and is quite heated. For example, in a recent *Education Week* blog, "Welcome Back, Long Division?", David Ginsburg shares thinking on both sides of this contentious issue and concludes that it should not be invited back (blogs.edweek.org/teachers/coach_gs_teaching_tips/2014/05/welcome_back_long_division).

Yet, there are clearly common instances where long division can be of value. For example: "How many buses will we need to take our 526 students to the museum if each bus holds 36 students?"; "We have seven dozen lollipops. Will we have to buy more so that everyone in our class of 29 students gets the same number?" and "The trip will take 1250 miles. If we can average 55 miles an hour driving, how long will it take us to get there?" These seem to be reasonable considerations where long division would be perfectly good in resolving those questions (even if you do not think a child having four lollipops is reasonable!)

What exactly is the difficulty? One mathematics text made it perfectly clear in presenting the problem of dividing 17 into 231, and asking and answering, "how many 17s are in 23?", to begin the division algorithm. Hopefully, students would be baffled by the question—not the answer, but why *that* would be the question given the problem. If the problem was to determine how many 17s are in 231, why are they being asked to determine how many 17s are in 23? They may well be asking themselves "why is what I am thinking not being discussed?", and in the absence of a response that acknowledges their legitimate confusion, cognitive dissonance becomes the real problem. Here we can see why students could develop an aversion to mathematics, having its beginning with some confounding experiences with arithmetic.

If one did not think about what was going on but just answered the question and continued, the power of the algorithm would be made clear. But blindly following instructions would not be a tenet of a democratic society—quite the contrary. And in this particular case, it apparently has caused more misguided attempts than successful calculations. The algorithm offers an efficient means that makes the problem simpler, but it is not clear why to many students for good reason.

Consider a different way, one where the student's experience shapes the conversation and procedure. Given the problem of how many 17s are in 231, students could be asked "How might you make the problem simpler?" From their earlier experience with division problems, they could well decide to "take away ten seventeens—170," as 170 divided by 17 is clearly 10. (This is what the algorithm is doing, but instead of putting "10" over the 31 in 231, a "1" is put above the 3.) Now that leaves 61 to be divided by 17. From here to determining the final answer of 13 with a remainder of 10, students can subtract multiples of 17 from 61 as a function of the degree of comfort and memorization they have gained, with all arriving at the correct answer. Granted the mathematical actions here to *take things apart* are not as crisp as the explicit procedure of the long-division algorithm. Yet, introducing this *problem-clarifying* heuristic first, as some texts do, would likely make the introduction of the textbook algorithm easier to accommodate after a while. In his blog, Ginsburg quotes Evelyn Hines, a teacher in Georgia, who wrote "… we do teach students to make sense of the problem. For example, 3657 divided by 35—a student looks for a 'friendly number', so he/she might say that 35 times 100 equals 3500 which leaves 657…. By the time they get to 5th grade, they can choose between the strategy or use the standard algorithm after they understand WHY the algorithm works" (emphasis in original). Here the mathematical agency is in the hands of the students, not the procedure.

Unfortunately, the pressure mathematics teachers experience to cover the curriculum in preparation for standardized testing may result in short-circuiting the very conversations students need to develop their number sense, with the time instead being dedicated to practicing the algorithm. With "good" students being those who readily adopt the otherwise puzzling procedure, our work as mathematics educators is made even more perplexing—for, apparently, being insensitive to what appears to be a mathematical action lacking explanation comes to be seen as a desirable quality! While other students for fear of their being seen as ignorant in questioning the evidently successful albeit confounding practice remain silent naturally lose some of the positive energy needed to try to make sense of what's going on. It is not hard to imagine that sometimes the effects would be long-standing.

It is not only this algorithmic procedure of long division that tends to short circuit students' inclination to inquire. It can be also seen in all the basic presentations of addition, subtraction, and multiplication, where the procedures that make sense to a naïve understanding tend to be omitted and in their place are procedures bereft of explanation. (The reader might consider, for example, when adding three-digit numbers the counterintuitive practice that begins with acting on the *least* significant digits. Imagine standing in front of three piles of money—$100, $10, and $1 bills—and being asked how much money was there in total. Which pile suggests it be counted first? Also confusing is the practice of multiplying multi-digit numbers by breaking up the partial products and writing one part of the number in one place and the other part of the number in a different place; surely it is visually and mentally quite perplexing for a number of students.)

Building students' mathematical intuition is essential in their becoming a good problem-solver. In the face of counterintuitive demonstrations, the mathematics experience as a shared journey is lost and the valuable aim of enhancing student–teacher and student–student constructive conversations is diminished. Instead, the interior dialogue students have with themselves becomes, at best, one involving an effort to dismiss their own concerns so as to internalize some perplexing practice.

At such times "… there is no contradiction in their saying, 'I know that such and such is considered to be true, but I do not believe it'" (Confrey 1990, p. 111). This naturally sets up the psychologically discomforting condition of students being supported in their doing what they do not believe in.

2.2 Combining Fractions and *Changing Representation*

It is rather a common classroom experience that when presented with a fraction addition problem such as $\frac{2}{3}+\frac{4}{5}$, students often believe $\frac{6}{8}$ is the sum. In response, some teachers tell students that "you can't add apples and oranges," but this tends to leave many students mystified. After all, they added numerator with numerator and denominator with denominator—in effect, they added apples with apples, and oranges with oranges. (And to compound the confusion, later on when multiplying fractions, students will learn that you can *multiply* "apples" and "oranges"!) Sometimes, students are just told "you can't add that way" without being provided any explanation. But a simple explanation is available. For example, using the model of adding numerators to numerators and denominators to denominators, we would have that $\frac{1}{2}+\frac{1}{2}=\frac{2}{4}$. But that clearly does not make sense since the left-hand side represents a totality of 1, as physical demonstrations would make clear.

Yet, inasmuch as the problem is such a common enduring misunderstanding, there must be something deeper that promotes youngsters adding that way. It could be that the problem beneath the surface is the symbolic representation—or more directly, the absence of any. Namely, from experience, youngsters winning two of three games of checkers on Monday and four of five games on Tuesday know they have won six of eight games. And with that distinction going unmentioned, the confusion naturally persists. Students fall back on their experience outside the classroom, on their very reasonable "combining habit" developed earlier in another context.

When combining numerical fractions, however, the parts are relative to the same whole. This suggests that to help students put the addition of fractions operation into perspective, the teacher, when introducing some *change in representation* such as using pizzas or chocolate cake, make the distinction regarding parts of the same unit being combined a conversation. After a few more similar problems, students will come to appreciate that fractions are associated with some common unit ("two-thirds *of a cup*," etc.). Then it can be made clear in the abstracted, efficient practice of combining fractions where the common unit has been omitted, including using the number line. In this way, students can appreciate the *change of representation* now that they see what was involved. Yet this is not the only time that working with fractions causes considerable confusion, and for good reason.

Later on in their mathematics education, students will be introduced to another algorithm involving fractions—to determine which of two fractions is greater by cross multiplying. This practice also mystifies students, and rightly so. When we compare $\frac{3}{4}$ *and* $\frac{5}{8}$, and determine that the first fraction is greater on cross multiplying, we find that 24 is greater than 20, the "cross-multiplying" algorithm omits the critical understanding—what is going on beneath the surface—that makes clear why it is legitimate to do so. In the absence of a conversation, students naturally come to believe memorizing is what they must do and paradoxically can feel less capable for having done so. They of course deserve to see that the procedure is in effect, creating a comparison of two fractions with the same denominators and looking to see which numerator was greater.

Yet there are other situations where working with fractions can introduce more perplexity to the naïve viewer, as when negative numbers are included. Consider the mathematics statement $\frac{-3}{6} = \frac{4}{-8}$. Simplifying both fractions or cross multiplying can serve to demonstrate that both fractions have the same value. But it would not eliminate students' consternation regarding how the ratio of a smaller number to a larger number could be equal to the ratio of a larger number to a smaller number! That perplexing relationship has a history that stretches back to the seventeenth century, when a number of mathematicians expressed considerable discomfort in accepting such a statement (Kline 1972, p. 252). Surely we would not call upon the field axioms of mathematics to convince youngsters why the ratio of a negative number to a positive equals the ratio of a positive number to a negative. How can we help them appreciate their consternation is indeed legitimate? It is to be appreciated that the discomfort negative numbers have had affected some of the finest mathematicians. (That consideration will come in a while.)

2.3 Invert and Multiply and *Make the Problem Simpler*

Rather than use cross multiplication to determine which of two fractions was greater, we could more reasonably choose to divide one fraction by the other, and if the quotient was greater than one, then the fraction in the numerator would be the greater. But dividing fractions can be quite challenging. In its stead, students are often introduced to the efficient algorithm of "inverting and multiplying." This dual procedure is another mathematics classroom experience which, for many students, is even more challenging in its acceptance.

In being shown that algorithm, students tend to have one of two responses: "Great—an easy way to divide by a fraction!", and "What's going on here?" To help clarify the procedure for all students, there is a conversation that draws upon the habit of mind to *make the problem simpler*. As one of the finest mathematical problem-solvers of the twentieth century, George Polya wrote, "If you cannot solve the proposed problem, could you imagine a more accessible related problem?" (1965, p. 114). (Part III of his *How To Solve It* is a "short dictionary of heuristic" with over 50 brief articles.)

If we apply *make the problem simpler* and determine the result, for example, of dividing 8 by 3/4, we can find a rationale for "invert and multiply," one that students will understand and may well appreciate. (To begin the conversation, it would seem necessary to first make such a problem a legitimate concern. For example, asking how many 3/4 cup servings would there be in an eight-cup recipe, or how many steps would it take to walk across a room 8 yards long if each step was 3/4 yard.)

The reason this problem is chosen is that it is not too easy or too hard to solve by what the students know already. If we asked "what if 8 was divided by 1/2?", some students could yell out "the answer is 16," and then grouse about why they have to sit through another procedure when they were able to solve the problem. So let us see what is involved when 8 is divided by 3/4. Surely, it must be more than 8. (Since 8 divided by 1 is 8, dividing by a smaller number means there must be more left over—as the two physical instances suggested.) It must also be less than 16, since 16 would be the number of halves in 8.

Now some students may conjecture that the answer is 12. This is a common and to be appreciated misconception as students are thinking of additive differences; namely 3/4 is right between 1/2 and 1 on the number line. However, the conjecture is worth acknowledging but not as being wrong. Students are demonstrating that they are thinking about and connecting to the problem, sharing what does seem to be the case to an intuition that is developing. Rather than tell them their conjecture is incorrect, for *conjecturing* is another habit of mind that is worth promoting, we have a wonderful opportunity for them to test their guess, to determine its *plausibility*—another habit of mind that is worth promoting.

In this case, *suppose* 8 divided by 3/4 did equal 12, then $\frac{3}{4}\times 12 = 8$. But the product equals 9. So 12 is too big. The answer must be more toward the middle of 12 and 8. Is it 10? Then $\frac{3}{4}\times 10$ would equal 8. It does not, but it is closer! While the students' conjectures were "wrong," there is no reason to make that the focus. Learning to test answers as being plausible is a mathematical habit of mind to check our thinking; it is a truly valuable strategy. And too, making conjectures is often how we learn, as we learn to test our assumptions.

At this point in time, students see the answer must be close to 10. Further inroads with dividing by a fraction can be made if we *make the problem simpler* by considering what the easiest number to divide by is. Students may initially offer 10 or 2, but after a moment more of reflection they often come upon 1, as with 1 the division is done! So with the focus on the valuable habit of mind to *make the problem simpler*, the problem of determining "how many 3/4s are in 8?" turns into the problem to replace 3/4 with 1 in the denominator, but without changing the value of the fraction.

Two thoughts regarding how the denominator could be made 1 usually come to students' minds: add 1/4 to 3/4, or multiply 3/4 by 4/3. Trying each method is instructive—and to be appreciated is their intuitive judgement that whatever is done to the denominator must be done to the numerator, for otherwise it does not *feel* right. (The power of "what feels right" is surely to be respected, especially the more experience one has. Werner Heisenberg, a winner of the Nobel Prize in Physics, said that if he arrived at an equation that did not feel right, he reconsidered his approach as his developed intuition suggested something was problematic.)

When students try both approaches, they discover that adding the same value to both the numerator and denominator, while intuitively seeming a good solution, is actually a problem. In this instance, adding 1/4 to both creates a new fraction where the numerator is 8 1/4 and the denominator is 1. They know the answer to the original problem has to be closer to 10, as discussed earlier. (At this juncture, the teacher may decide to stop and reinforce that the action of adding the same quantity to the numerator and denominator of a fraction is not generally successful. For example, by pointing out to students that 2/3 would become 3/4 by adding 1 to each the numerator and the denominator.) Now they can try their second idea: multiplying the numerator and denominator by the reciprocal of the denominator. Here the numerator becomes $8 \times \frac{4}{3} = 10\frac{2}{3}$, with the denominator 1.

This answer is indeed plausible, as it is more than $8\frac{1}{4}$ and close to 10. To reenforce that thinking, we can consider another problem to which they likely know the answer: 8 divided by $\frac{1}{2}$. Having students use the method of creating 1 in the denominator by multiplying by the reciprocal of the denominator, students can see that 16 is the answer, which provides supporting recognition. Conversation can then turn to helping students appreciate that when they multiplied the denominator *and* the numerator by $\frac{4}{3}$ in the original problem, they in effect were multiplying the original fraction by $\frac{4}{3}/\frac{4}{3}$—in essence, by 1. This of course ensures the numerical value of the original problem has not been changed, just represented differently. A few more practice problems and students can see that the *change of representation* by "inverting and multiplying" is really efficient—and now makes sense.

In addition, those students who expressed their confusion can deservedly experience the respected nods of their classmates. It was their question of "what's going on here?" that served to promote the worthwhile and needed investigation. Writ large, it is that kind of questioning generated by students' concerned interest which, supported by their mathematics teacher, would be the natural and logical way a mathematics class experience evolves if sense-making was the object. With more such truly educational experiences, as future citizens of a democratic society, our students would become comfortable with offering their concerned responses in the face of a statement made by someone in authority that appeared to them as confusing. (Experience suggests that experience is very much part of their future.)

2.4 Mathematical Slope and *Visualize*

Typically, algebra textbooks define a straight line as "$y = mx + b$." This equation demonstrates clearly how mathematics is so much more dense than spoken language. For instance, students could get a 10–20 page reading assignment, but they would hardly ever get an assignment of that size in a mathematics textbook. In the given linear equation, we find six symbols—the same number as in the word "window." The mathematical expression represents a relationship, actually multiple relationships, with one overriding consideration that could well appear opaque to many students. That is to say a lot is packed into that symbolic representation.

Helping students begin to understand that equality by *taking things apart* enables them to get a hold of its richness, and in that way helps them become more comfortable tinkering with the different quantities in the equation. For one, letting them know that by agreement with the exceptional mathematician Rene Descartes, mathematicians use letters at the front of the alphabet to represent constants, while those at the end are chosen to represent variables. This distinction helps begin to take apart the dense equality statement for students. Also to be appreciated is that the letters in "the middle" are used to represent parameters, quantities that vary given the particular situation. For example, physics equations use the parameter "g" as the constant of gravitational attraction which takes on the value of 32 near Earth, and approximately 5.3 near the moon, as the moon's gravity is about one-sixth that of Earth's.

In the equation "$y = mx + b$," students then see that x and y are variables, and together represent any point in the two-dimensional plane; and that m and b are parameters representing constants, with m the slope of the line, and b the value of the y-intercept as a consequence of $x = 0$. Quite an intricate relationship! What is immediately intriguing to some and confusing to other students is why the symbol m? Surely a fits so much more appropriately with b! (One can share that s was not available as it is taken to represent *distance* in physics formulas, perhaps as a consequence of being the first letter of *stadia*, a unit of distance from earlier times.) And yet, the reason just shared regarding the letters in the middle of the alphabet representing parameters does not really fit the given equation. Inasmuch as a is replaced by m then it would seem that b should also be replaced with some middle-located letter as it also represents a parameter. (Being confused here seems to make perfectly good sense!)

Hopefully, the mathematics teacher can take a naive perspective and appreciate what students are experiencing when presented with the equation of the general line in the x-y plane. There is a lot to deal with. Apparently, it is not known how m became the symbol to represent the slope value. However, with students graphing different lines as a function of their numerical values, they would start to see that the coefficient of the x-term is greater the steeper the lines. Then they would likely come to decide that it would be a good idea to think of that coefficient as the measure of comparative steepness. What else do they associate with steepness? Mountains seem a reasonable response. So why not use m to represent slope?

Descartes was French and wrote in Latin; so using *m* does make sense as it is the first letter of "*mons*," Latin for "mountain," which as a physical presence is surely distinguished one from another in terms of the difficulty of ascent as a function of its slope. Some mathematics historians say this explanation lacks evidence. However, until a better rationale is found for choosing *m,* it seems quite reasonable, given the goal of promoting students' mathematical intelligence to use the students' thinking.

But there is more to discuss here. Part of the usual presentation students receive regarding straight lines is the definition of the slope as "the change in y over the change in x." But why should the slope be defined *that way*? What about "the change in x divided by the change in y?" This may well be a question in the minds of some "naïve" but thoughtful students who are reluctant to ask, and for those students who accept the definition as is, such a consideration helps them to understand that definitions are not chosen without consideration. The resolution will take just a bit of time, but clearly it is worth that as the goal is that students become educated, which requires their making sense of what puzzles them. They can determine the wiser choice. The better slope–ratio representation would be decided by *visualizing* how the contrasting definitions would actually connect or not to graphed lines. In this way, they can appreciate not only their good question, but its resolution. In this way, students develop more sophisticated means of valuing that go beyond impulses of likes and dislikes and the choice made by the authority of others, and promote their own valued and valuable (mathematics) education.

2.5 Arithmetic Series and *Tinkering*

Mathematics textbooks often begin the study and formulation of an arithmetic series by relating the story of the 10-year-old student Carl Gauss, and his teacher who, wanting to promote more student discipline, had his students sum the numbers from 1 to 100. This was extremely tedious and annoying, especially with the "endless" summing on small writing slates! But precocious Carl noted that if the terms of the series $1+2+3+\ldots+98+99+100$ were written again right below those numbers in reverse order, the answer could be immediately determined. Below 1 he wrote 100, below 2 he wrote 99, 98 below 3, etc., and it was clear that at the end of the series 3 would be below 98, 2 below 99, and 1 below 100. With summing those vertical pairings, Carl saw he would have 100 pairs of 101. This being twice the sought-after sum required dividing the product in half, arriving at 5050. Problem solved!

This tends to be where mathematics texts end the story so that students can then practice the procedure, and then proceed to consider other arithmetic series where the common difference is other than 1. The popularity of introducing arithmetic series by this approach can surely be understood and also questioned, especially with regard to its pedagogical value.

We would imagine Carl's teacher was in a state of disbelief, for it would be hours to do the problem in the straightforward manner, not minutes or less! He realized he was dealing with a most precocious 10-year-old mathematical mind. As such, presenting this approach to high school students makes a number of points, none of which seem educationally positive. First, teenagers sitting there could well come to think they are evidently less able than some 10-year-old kid! Also, by showing that procedure, students are robbed of the opportunity to engage the problem and see that they could cleverly uncover the sum themselves. And, in doing so, realize they are pretty good mathematics problem-solvers. It is exactly such experiences that promote positive energies and the mindset needed when dealing with other mathematics problems. (The pedagogical problem is to find problems that are not too hard or too easy, so such productive development can take place.)

Providing students the opportunity to show themselves by their own mathematical thinking that they are more capable than they might otherwise have thought should not be taken lightly. And they surely do not have to compare themselves with a 10-year-old who would become one of the most extraordinary mathematicians of all time.

Consider if the initial arithmetic series presented to students was something like: $1+2+3+4+5+6+7+8$. Students asked to find the sum would of course just add up the numbers. But if they were asked to imagine that the series went on, let us say up to 20, so as to urge them to find means other than manual labor, they would be inclined to see the series as being malleable. Some could consider *taking things apart* by forming the partial even-number series along with the odd-number series. In this case they would see that if they could find the sum of the odd series, the even series would just be 4 more, 1 for each pair, inasmuch as there are 8 terms. Then they could find that the odd series can be summed as 4^2, as by examination they would see that the sums of consecutive odd-number series beginning with 1 appear to be the square of the number of numbers in the series. For example, $1+3=4=2^2$; $1+3+5=9=3^2$, etc. So the sum of the odd series would be 4^2, making 4^2+4 the even-number series sum, resulting in the final sum of 36. And in general, for series with a common difference of 1, if the number of terms n is even, the sum would be $\left(\frac{n}{2}\right)^2+\left(\frac{n}{2}\right)^2+\left(\frac{n}{2}\right)=\frac{n(n+1)}{2}$.

This approach can be adapted to find the sum of an arithmetic series with an odd number of terms. For example, had the series been $1+2+3+\ldots+9$, the sum of the odd numbers would be 5^2, and with there being four even numbers, the sum would be 4^2+4. So if n, the number of terms was odd, the sum would be $\left(\frac{n+1}{2}\right)^2+\left(\frac{n-1}{2}\right)^2+\frac{n-1}{2}$ which also simplifies to $\frac{n(n+1)}{2}$. (The algebraic simplification would be propelled here by student interest, which is exactly the source of energy that is pedagogically desired.)

Other students might *tinker* with the arrangement of the terms of the series and realize that adding the first term with the last, the second with the next to last, etc., ends up creating four sum-pairs of 9; and so the sum is 36. The generalization may well not be immediately obvious, where there are 8 terms and the sum is 36. The search will likely promote further inductive considerations with other series with an even number of consecutive numbers beginning at 1 before coming upon the pattern $\frac{n(n+1)}{2}$. Some students may be drawn to considering series of the same form but with an odd number of terms. For example, with the series of 1–9, the sum would be 4 tens plus one five, or 4.5×10. And, here too, more than one particular instance would be needed before students come to see that in general with n odd, the sum would be $\frac{(n-1)}{2}(n+1)+\frac{(n+1)}{2}$, which as the even number of terms series simplifies to $\frac{n(n+1)}{2}$.

To corroborate their insights, the teacher can then pose that all students determine, regardless of their method, the sum of the first 100 counting numbers. Then the teacher could demonstrate the 10-year-old Gauss's approach that eliminates the need for distinguishing between arithmetic series having an odd or even number of terms, and so students can appreciate their thoughtful efforts along with the more elegant approach of a mathematical prodigy of the highest order.

Now they are (hopefully) psychologically ready to consider arithmetic series that do not begin at 1 and do not increase by 1. They will discover that the approach of separating the initial series into even and odd numbers will not work in general, but that combining opposing pairs will. Naturally, the experience and the memory of the experience and discussion is completely different when students are given the chance to *tinker* or *take things apart* than when presented with "the way" to solve the general series. With their engaging the problem themselves, they get the opportunity to create themselves just as they would want to be—as resilient, thoughtful, capable mathematical thinkers who appreciate that resilience and thoughtfulness are essential qualities when engaging mathematics.

This is not to say that students have to invent everything they learn in mathematics, but rather that mathematics teachers can help students appreciate what they can come to know by their own mental and emotional energies, their own insights, developing intuition, dedication, and experimentation. *Doing* mathematics can be seen to distinguish what is involved in becoming educated as versus being schooled in mathematics.

The arithmetic series discussion illustrates the tension between presenting material with regard to the aesthetic of efficiency of the discipline of mathematics and promoting the pedagogical aesthetic where students' energetic engagement is the sought-after quality which may or may not converge to uncover, if at all, the elegant mathematical form. That dual consideration would naturally weigh in our decision-making on a regular if not daily basis as teachers of mathematics. Yet of course, students would not be expected to come up with all the mathematical formulas and equations and problem solutions, so there would be times when the teacher could well make a formal presentation for a good purpose.

For example, the mathematical derivation of the extraordinary equation involving all five of the most significant constants in high school mathematics, and only those where $e^{\pi i}+1=0$, can surely be appreciated by many high school students.

This is to say that students can well appreciate a demonstration without having the feeling that they missed the opportunity or would ever come up with such a finding themselves, or feeling less for not being able to. Such a lecture demonstration would seem fine to include if the elements of the argument, including those of imagination, are discussed to student satisfaction and the final expression is realized as the logical conclusion.

2.6 Quadratic Equations and *Make the Problem Simpler*

The reader might be noticing that the heuristic of *make the problem simpler* has been drawn upon a number of times. That should not be surprising, as it has many variations. So much so that the mathematician Keith Devlin, who writes the monthly column "Devlin's Angle" in the Mathematical Association of America's monthly magazine, the *American Mathematical Monthly*, made the emphatic point that the heuristic of making the problem simpler is "the way we do mathematics!" And physicist Steven Carlip would seem to agree. He writes "ask a physicist too hard a question, and a common reply will be, 'Ask me something easier'. Physics moves forward by looking at simple models that capture pieces of a complex reality" (*Scientific American*, April 2012, p. 42).

However, mathematics texts do not tend to point out that most valuable *problem-clarifying* strategy and how essential it often is. Consider the problem: Find x, such that $x^2 - 3.5x = 11$. This is a hard problem as stated—unless one knows what to do, and then it is not a problem. Without having a procedure clearly in mind, what we can determine from the problem as stated is that $x > 3.5$, since the difference is positive. But after that it seems it could be any number, and while "guess and check" is a time-honored habit of mind that could be attempted of course, there is little reason to believe in its efficacy here, especially if the values could be fractions or irrational numbers. This would seem to explain why mathematics textbooks presenting quadratic equation problems usually begin with stating: "set one side equal to 0," regardless of the particular numerical values. Doing so, the problem has been made much simpler.

But again, that strategy of *make the problem simpler* tends not to be mentioned. Instead, the textbook demonstration turns to factoring and solving many such problems, while little has been made of the fact that the original problem was really difficult. So the first and most important question that would have been best to ask was, "How can we make the problem simpler?" But with texts not tending to be written with the object of engaging the reader in a conversation, the valuable thinking behind the action is lost as the factoring algorithm gains the focus. Having *made the problem simpler* via *changing representation* should be celebrated as a wonderful idea (tool), one that has enormous application. Yet, it is factoring quadratic polynomials that is made the focus and as a relatively heavily practiced activity, which is questionable as the coefficients have to be very carefully chosen so that the factoring algorithm can be readily put into practice. Henry Pollak, who had been a leading mathematician at AT&T remarked, "there are two types of numbers: real numbers and numbers in mathematics textbooks," given the extraordinarily messy coefficients he experienced working with real situations. How much time to give to factoring polynomials deserves a conversation. That technique while valuable in the theory of equations, as a procedure for all high school students needs to be weighed in light of the technology that allows students to see quadratic equations and find excellent approximations if not exact roots graphically, and all the other mathematics that could be included were there more time. What would seem quite valuable to include in the consideration of quadratic equations is another instance of *making the problem simpler*, with completing the square—a lovely technique for simplifying complexity. Indeed, working with the quadratic equation and the mental action of *making the problem simpler* ought to be being appreciated together.

* * *

In this chapter, mathematical *problem-clarifying* strategies were presented in comparison with the prevailing model of teaching mathematics procedures. The concern is that the latter emphasis tends not to promote "deep conceptual understanding," or "deep learning," but often deep confusion. However the problem is deep-seated. Apparently, mathematics teachers tend to believe they are focusing on problem-solving in their classrooms when they actually may not be. Gill and Boote (2012) point out that "In a key cross-cultural study of mathematics education, although 70% of US teachers said that their videotaped lessons aligned with the NCTM standards to at least a fair degree, most of the observed lessons were inconsistent with the intent of the standards. For example, 96% of US students' time during seatwork was spent practicing procedures…. Further, 78% of US teachers were about as likely to simply state concepts as develop them" (www.tcrecord. Org/PrintContent.asp?ContentID=16718). That would seem to be a real problem that needs to be resolved as soon as possible.

This bifurcated view is apparently also shared by students and employers. "For example, while 59% of students said they were well prepared to analyze and solve complex problems, just 24% of employers said they had found that to be true of recent college graduates…. The gap between how prepared students feel and employers' assessment of them has been established. The question now is what students, employers, and colleges are going to do about it?" (Fabris 2015).

It would seem that question needs to be addressed by the educational community at large if there is to be a truly systemic response, not hit-or-miss. Procedures that make doing a class of problems easier make life easier in school and out. So students may well conclude they are capable mathematical thinkers based on a limited view of what solving complex problems entails. However, for most students—the vast majority, the times in their lives that mathematical algorithmic practices will be called upon would seem to be very limited. What is of fundamental and rather universal value is the learning experience they could have regarding the development of their creative and dedicated thinking, individually and collaboratively. To make that more possible, students need time to experiment, create, and draw upon *heuristics*—*problem-clarifying* strategies, mental actions that can reshape an amorphous problem situation into one that can be worked with. In this way, we mathematics educators are seeding society so that it could well be more productive for all of its participants, including mathematics students' employers.

The formal textbook demonstration-practice format is problematic. Demonstrating procedures or presenting equations or explanations as if they were obvious and forgetting about "the invention, ingenuity, observation, exercised" (Dewey 1936, p. 212) that was most likely at the root of the uncovering is a questionable pedagogical practice. With such an approach, the inquiry experience has been flattened out of recognition. It is important to share with students that "the logical formulations [as textbook presentations usually are] are not the outcome of any process of thinking that is personally undertaken and carried out; the formulation has been made by another mind and is presented in a finished form, apart from the processes by which it was arrived at" (1936, p. 79–80). In the absence of that recognition, we can understand Dewey going on to say that "the adoption by teachers of this misconception of logical method has probably done more than anything else to bring pedagogy into disrepute" (1936, p. 81).

* * *

When we make students' engagement the focus—not the text or teacher presentation, we can truly appreciate that "'Meaningful' [mathematics] of course means: meaningful to the learners" (Freudenthal 1981, p. 144). Ultimately, with repeated opportunity for student inquiry, it would seem reasonable to believe that *all* students can come to develop greater patience and resilience when faced with the complexity of problematic situations, and so become more capable mathematics students. Rather than their telling themselves "I forgot—I don't see why," their growing intuition and capacity to draw upon mathematical habits of mind would greatly promote their seeing why. With such experiences rather than telling themselves "I don't know what to do," they can find themselves telling themselves "I don't know what to do *yet*—let's see what *I can do* to gain some insight, some understanding…, make some progress."

There is considerable authority (including Poincare and Polya) to support the statement that "authentic mathematical activity" is to "get a sense of mathematics as human invention, as certain habits of mind, that is more engaging and meaningful than learning a procession of given facts, methods and question-types" (Watson 2008, p. 3). With that perspective in practice, there would likely be reports that differ considerably from the finding that less than half (46 %) the students met the American College Testing (ACT) benchmark in mathematics as measured on the Condition of College and Career Readiness exam (Adams 2012). To promote more thoughtful engagement seems essential given our increasingly technologically driven society.

With students using their own intuitive approaches to initiate investigations and solve problems, its pedagogical value is apparent as students come to see if and how what they thought connects to the time-honored mathematical expression. The interested reader can find, for example, child-invented valid means of combining numbers (cf. Yackel et al. 1990). These are to suggest that we give serious consideration to whether *any* accepted formal presentation be made prior to mathematics students having the opportunity to discuss and make introductory sense of the relationship themselves. We need to appreciate that while we have a goal in mind, if it is only that they "get it," where the "it" is the means to solve a class of mathematics problems or master a technique or procedure, then we are likely to make less of the educational *process* of the mathematics experience than the *product*; and that seems a narrow educational goal.

Perhaps those of us teaching mathematics for a number of years need to remind ourselves "that what is an old story to [the teacher] may arouse emotion and thought in the child" (Dewey 1936, p. 212). This suggests that in our role as initiators of students' mathematics experience, we share our earlier wonder and our interest, and acknowledge theirs—for that is the essential spark needed to generate thinking and to promote dedicated effort. Such engagements will not be as time-wise efficient as presenting procedures and having students practice them. But in a collaborative classroom environment, the mathematics students could realistically feel they were part of a "community of scholars," participants in a shared educational journey, as they have opportunity to use problem-clarifying strategies to help shed light where there was none. Toward their successful engagement, the next chapter is given to promoting all students being capable mathematical thinkers.

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Chapter 3
Habits of Mind—The Heart of the Mathematics Curriculum: Some Instances

Learning how to do anything well, including thinking, can be associated with two sets of behaviors—those that are valuable with regard to any learning situation and those that are task-specific. As regards the former, without being able to stick with a problem and without being able to attempt different ways of engaging it, there is little reason to believe that a problem of any significance will be handled well. And with regard to task-specific behaviors, some action is called for, and applying a particular technique or practice is required.

Yet, as the earlier chapters have discussed, there are those behaviors that provide valuable, actually essential, means for making headway toward gaining clarity when none is evident. In the amorphous “middle ground,” where we are not at the point of resolving the difficulty with an explicit move but rather at an uncertain distance, we need to draw upon some mental action, some heuristic(s) that can help us move through the confusion to clearer ground. As mathematics educators, it is here we have opportunity to share the fund of mathematical agency, the collection of thoughtful actions that have served the mathematics community in its creative and dedicated past, to enrich our mathematics students’ thinking and lives.

That problem-solving is thought of as the ”heart of mathematics” (Halmos 1980) surely makes sense. And with that understanding, mathematical habits of mind would be the heart of the mathematics curriculum. Made available as content would make doing mathematics—creating and solving mathematics problems—all the more possible for every student. In that direction, some practices may well come naturally to mind, such as *looking for patterns,* or *guessing,* perhaps even *arguing by counterexample*. However, others that shed light on complex mathematical situations may need to be uncovered with the mathematics teacher helping take part in the question-asking. And as students come to see how *tinkering, taking a problem apart, visualizing,* and other heuristics are critically valuable in making headway, they naturally become disposed to including those problem-clarifying strategies as part of how they think. As a consequence, all students require less and less direction in engaging mathematics problems.

Having a set of problem-clarifying tools, students gain a sophisticated way of seeing. So, we can appreciate Aristotle’s concern that “It is a matter of real im-

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portance whether our early education confirms us in one set of habits or another. It would be nearer the truth to say that it makes a very great difference indeed. In fact all the difference in the world" (Aristotle 1971, *Ethics,* Book 2, Chap. 1, 1103b1–25). For this development to occur, it would seem we need to dedicate classroom conversations to making the mathematical thinking process as legitimate as the mathematics procedures that tend to fill textbooks. This perspective, which focuses on the development of heuristics as content, has a long history acknowledged by major mathematicians through the ages, including Archimedes, Descartes, Leibniz, Poincaré, and Polya.

More recently, if we as mathematics educators agree with the National Council of Teachers of Mathematics (NCTM) that "problem solving must be the focus of school mathematics" (1980, p. 1), then "seeking solutions, not just memorizing procedures; exploring patterns, not just memorizing formulas; [and] formulating conjectures, not just doing exercises" (National Research Council 1989, p. 84) would be the picture of every mathematics classroom. Yet, seeking solutions by exploring patterns and formulating conjectures is only possible if mathematics teachers promote that focus.

There are indeed instances of that focus becoming part of the mathematics classroom conversation. Michal Yerushalmy (1997) writes that "The ability to generalize, especially when the generalization requires a major breakthrough in habits of mind, is one indication of algebraic reasoning." Al Cuoco (1998) shared his wish list regarding what he would have wanted to know when he began his teaching career, and mathematical habits of mind was in the top three. Also, Sharon Friesen, cofounder of the Galileo Educational Network, works with mathematicians and mathematics educators to "create mathematical investigations and problems that enable students to look for connections, identify patterns and relationships, [and] make conjectures" (September 2006).

Yet, Lianghuo Fan and Yan Zhu (2007), in comparing mathematics series at the lower secondary level used in China, the USA, and Singapore, highlight the Singapore series as it devotes a chapter to specific heuristics, including "draw a diagram," "change your point of view," and "use an equation" (p. 71). In contrast, they noted "the majority of problems in the US books…were routine, traditional, and moreover of single-step" (p. 69) and as a consequence limited the necessity for heuristic approaches.

However, efforts are being made in the USA to include habits of mind as part of a mathematics curriculum with more than traditional problems. For one, the Park School of Baltimore's mathematics department, inspired by Paul Goldenberg's 1996 article, wrote its first iteration of a mathematics program for grades 9–11 that treats habits of mind as content in 2006–2008. (Interested readers can access the material at https://www.dropbox.com/sh/ltbj5o5nrg228ws/AACmjY9lulUHKlCsus-8CV7wxa.) In addition, the *CME Project* high school mathematics textbook series (2008), led by Al Cuoco, a colleague of Goldenberg's at Education Development Center (EDC), also provide habits-of-mind activities and associated discussions as part of their content. There may well be a number of other mathematics texts that include habits of mind, especially for the earlier grades, as such an approach is deservedly gaining recognition.

How best to make heuristics available to students is an open question. However, it is clear that if problem investigations were involved enough so as to require problem-clarifying strategies, mathematics students would see their necessity and value. And with these strategies included from time to time, we could expect that even the most capable learners would prosper as well. But it is hard to imagine that such practices would become part of all students' mathematical thinking in the absence of some focused conversation, such as including a variety of mathematics problems calling for the same habit(s). Research is needed in this regard. But what is eminently clear is that the absence of problem-clarifying strategies precludes the opportunity for engaging more complex problems for a vast population of mathematics students and continues to keep most students limited to learning algorithms and solving one- or two-step problems. Thinking mathematically, thinking inventively, would seem to require a change of emphasis and focus in mathematics classrooms.

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The cognitive psychologist Jerome Bruner relates that "The young child approaching a new subject or an unfamiliar problem either has recourse to the less than rigorous techniques of intuition or is left motionless and discouraged. So too the scientist operating at a far reach of his capacities in his chosen field" (1971, p. 96). However, he points out "there are ways for using the mind in a fashion designed to save work, to make seemingly difficult problems easier, to bring a complicated matter into the range of one's attention. One learns to make little diagrams or to use a matrix…. Or one asks, encountering a new problem, whether there is anything like it one has encountered and solved before" (op cit., 95). And though, in 1971, "one rarely speaks of them, and surely there are no courses for teaching them," Bruner believes "as a start that any new curriculum contain a syllabus designed to teach the economical tricks of the trade as early and as effectively as possible" (op cit., 96). In that direction, the Conference Board of the Mathematical Sciences stated that "mathematics courses for teachers should develop the habits of mind of a mathematical thinker…." (2001, p. 2). For those habits of mind ("economical heuristics") make for an informed mathematical intuition and so provide the resources toward becoming a better mathematical thinker.

This clearly makes sense. As teachers of mathematics, we naturally want to help students think better mathematically. But as mentioned earlier, when they are stuck on a problem and we urge them to "Think!" with their not being familiar with problem-clarifying strategies, we can expect they would more likely be thinking about the discomfort they feel. Consider the problem: "Find y if $80 = 5(y - 79)$." If the student is not clear on the procedure of distributing, they may well be stumped, and emotional disturbance may well fill their thoughts. Had they been introduced earlier to the problem-clarifying strategy "create an easier similar problem," they could have a valuable idea how to begin. For example, they could come up with: "Find x if $80 = 5x$." And *that* problem they can readily solve. Now the *change of representation*, where $x = y - 79 = 16$, makes the final step for solution clear. It could be said that this approach lacks the elegance of directly applying the distributive property. Yet, the problem has been transformed from a frog or toad of a problem to a prince or princess of a problem by *making the problem simpler* with a change of representation. And such *problem-clarifying* strategies serve to develop and support students' intuition so that it becomes more alert to when such strategies can be applied to other similar contexts. And the teacher could in that moment revisit the distributive property so as to make that approach more viable for that student.

Here's a more extended application. Graph theory has established itself as a valuable area of study in contemporary society and in high school as there is considerable material that interests students. Consideration of complete graphs offers students the opportunity to see if they can determine the general relationship between the number of nodes and the number of edges of such a graph. To begin the investigation, students draw three dots representing nodes and connect all the pairs of nodes establishing edges (connecting segments). They determine that in a complete graph with three nodes, there are three edges. Using the same procedure for a four-node complete graph, as well as for a five-, six-, seven-, and eight-node complete graph, they generate data determining the associated number of edges. The information of carefully counting (including more than one pair of eyes) yields can be expressed as a set of ordered pairs, $\{(n, \mathrm{e})$: (3, 3), (4, 6), (5, 10), (6, 15), (7, 21), (8, 28)$\}$. Students see that as the number of nodes increase by 1, the number of edges increase arithmetically by 3, 4, 5, 6, and 7. The question, what is the general relationship (expressed in mathematical notation), suggests itself.

How to proceed? It seems a perfect time for the mathematics teacher to introduce the method of finite differences or a sophisticated counting procedure. Either would do the job. But that would mean the students in effect would solve the problem by just following the bouncing ball provided by the given method. They would be passive responders with little agency. How to have them provide the generating idea required not saying anything.

A group of 10th graders couldn't see how to express the general relationship connecting *all* the points. What they did see was that for the pairs with an odd number of nodes, there was a relationship they recognized. *Taking things apart,* they wrote the odd-node pairs as: $(3, 3\times 1)$, $(5, 5\times 2)$, $(7, 7\times 3)$. To test their conjecture, they drew a nine-sided complete graph and found just what they were hoping for: that figure had $9\times 4=36$ edges. After a while, they found a pattern for the case when n is odd: the number of edges would be $n\times\frac{n-1}{2}$ or $\frac{n(n-1)}{2}$.

What about the sequence for the number of nodes being even? That is, given the points (4, 6), (6, 15), and (8, 28), is there a pattern that can be uncovered? Quiet time for examination and small-group conversations led to the pattern discovery that $6=4\times 1+2$; $15=6\times 2+3$; and $28=8\times 3+4$. The prediction was that for $n=10$, there would be $10\times 4+5=45$ edges. Careful drawing and a number of counters confirmed exactly that. So for n even, the number of edges would be $n\times ?+\frac{n}{2}$. How to represent "?" algebraically? It was clear that it is always 1 less than the added on term. Then for $n=12$, the number of edges would be $12\times 5+6$. With some discussion, the number of edges for an even number of nodes in a complete graph was written as $n\times(\frac{n}{2}-1)+\frac{n}{2}$. It seemed that two equations were needed to discuss the relationship. Then, the question was raised on how the complicated expression might be simplified. Doing so, with dedicated energy, it became clear that both formulas were really the same: $\frac{n(n-1)}{2}$!

Was it worth the time and effort? It all depends on what is valued. The habits of mind *finding patterns* and *taking things apart* are clearly instrumental in doing mathematics. And not just mathematics: "The natural way for any scientist to think about a problem was to break it into parts" (Bolles 1997, p. 277) in addition to seeking patterns. In the present case, the effort took a class period, but it did not require giving students a problem-solving technique for them to apply; they had the chance to think and apply two problem-clarifying strategies that led to conjectures and their testing. And it was very clear that the students appreciated their dedicated effort. After all, they were seeing themselves uncovering the beauty of mathematics and in the process, learning about how capable they actually are. Here they found they were more capable than they thought, and the excitement and recognition was well deserved. (There would be positive energy to draw upon later.)

The opportunity to introduce such strategies can arise often, and there is always a value judgement to be made, of course. For example, when students are presented with a conversion statement such as "one inch is approximately equal to 2.54 centimeters" and asked to write it in mathematical symbols, it is not uncommon for them to create a literal translation, and write "$i = 2.54c$." Even asked to "check their answer," they return to the equation and see that it literally maps onto the English expression. So, it checks. Then, they find out they should have written it as "$c = 2.54i$," or "$i = 2.54/c$," which they may or may not understand why that is the case, as it seems so counterintuitive. Had the students developed the practice of *testing for plausibility*, they would have been inclined to check their equation's legitimacy, and this would have provided them the needed insight without having to learn they could not do it.

They likely know 1 m = 100 cm, which is a bit more than a yard, 36 in. Thus, their equation $i = 2.54c$ would determine that 254 in. was equivalent to 100 cm; namely, a length more than 20 ft was equal to the length of a meter stick. A developing intuition suggested a literal translation; a plausibility test made clear they needed to do some rethinking. Doing so, their intuition develops more productively. And they have a happier mathematics experience realizing their growing capability.

This is to suggest that with mathematical habits of mind at the center of the mathematics curriculum, *all* students can come to have tools that promote productive inquiry. Given the increasing success they could well experience, they would naturally be inclined to persist and try alternative means to gain insight—that is, develop further useful attitudes, most especially be more patient, resilient, and flexible in their thinking. Exactly those qualities that promise good conversations to inform future society's decision-making.

* * *

An educated populace, in the sense that they are thoughtful and capable of learning from setbacks and dealing well with complexity, is an essential goal of a democratic society. To promote that development in mathematics classrooms requires students learning to navigate relatively complicated mathematical situations and developing clarifying means that will serve them (and us) well as they become adults. Surely, introducing and working with problem-clarifying strategies take time. But it is the only game in town that is in all our best interests.

One of the finest mathematicians of the twentieth century greatly concerned about the teaching of mathematics, George Polya, wrote 70 years ago that mathematics should "first and foremost teach young people to THINK" (1945/1957, p. 100; capitals in original). And he went on to say, "'Teaching to think' means that the mathematics teacher…should stress know-how, useful attitudes, and desirable habits of mind" (op cit.) And seven decades later, that focus continues. The NCT M has in recent years produced *Professional Standards for Teaching Mathematics* and *Principles and Standards for School Mathematics* (1991; 2000), where habits of mind can be seen to be a foundational concern. For example, they write "Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts and from the earliest grades."

In discussing problem-solving for grades 6–8, they mention specific mathematical heuristics that would serve to develop a more mathematically able problem-solver. For example, "Habits of persistence and curiosity" are pointed to as ways of thinking that would "serve [students] well…in everyday life and in the workplace" (Chap. 3, *Principles and Standards for School Mathematics*). How to promote that occurrence is clear: Focus on interesting problems that require problem-clarifying strategies and promote mathematical habits of mind. However, with straightforward problems and algorithms, the primary classroom focus, they would be superfluous. As a direct consequence, there is no reason to expect that persistence and curiosity would become more the way those mathematics students come to think.

Were the discussion of useful attitudes and habits of mind an integral part of their mathematics experience—enough to be thought of as literally *content*—students would be inclined to develop a more sophisticated and resilient inner voice that would shape and focus their internal conversations toward constructive ends, rather than the dead ends and feelings of inadequacy that are often the consequence of their being faced with a problem that does not fit the model they practiced. It would seem that consideration deserves very careful attention. More thoughtful development, of course, means the development of more valuable and valued human beings in the role of society's decision-makers.

* * *

As we all recognize, most habits take determination and continued commitment, but once we gain that ability, it is part of who we are and it can make life so much more engaging. But before a habit language becomes common to students' mathematical thinking, the opportunities for their application must be many, and the participants need to appreciate that new and valuable practice requires time and thought. Miles Davis, the fine musician, captured the dedicated evolving nature of the learning experience when he said, "Sometimes it takes a long time to sound like yourself." For *all* students to gain a realistic self-confidence in their developing capacities, there would have to be considerable opportunity for them to engage in relatively complex problems. This would likely require the omission of extended practice activities associated with solving one- or two-step mathematics problems. The emphasis on the new standards and concern for students understanding mathematics suggests such a rewriting is in order. The following sections are given to how a habits-of-mind orientation can become more the way things are in the mathematics classroom.

I. Tools of the Trade

The mathematical habits-of-mind list that follows was chosen based on the subject matter knowledge and pedagogical content knowledge of the Upper School Mathematics faculty at the Park School of Baltimore during the years 2006–2008. Financially supported by the school and private foundations, the mathematics faculty's goal was to select a *workable* collection of heuristics—not too many and not too few, which would hopefully become integral to the mathematics classroom conversation and to the mathematical thinking of all students over time. It was understood that such development required a mathematics curriculum that would promote their consideration and application. The reader will note that the habits-of-mind practices have value beyond the mathematics classroom, which is a quality that was also appreciated. Namely, such problem-clarifying strategies could actually help solve problems in all facets of one's life.

Different mathematics educators take different habits of mind as their focus, as mentioned earlier. Indeed, the list of practices that have served the mathematics community well over time is quite extensive as you would imagine (just Google "Mathematical Habits of Mind"). The ones discussed here have continued to be of value over a number of years for students in grades 6–12. How to promote their becoming more common in the classroom mathematics conversation will be discussed in the section following their description.

The Heuristic Set

1. *Look for patterns:* look for patterns amongst a set of numbers or figures
2. *Tinker:* to play around with numbers, figures, or other mathematical expressions in order to learn something more about them or the situation; experiment
3. *Describe:* to describe clearly a problem, a process, a series of steps to a solution; modulate the language (its complexity or formalness) depending on the audience
4. *Visualize:* to draw, or represent in some fashion, a diagram in order to help understand a problem; to interpret or vary a given diagram
5. *Represent symbolically:* to use algebra to solve problems efficiently and to have more confidence in one's answer; and also so as to communicate solutions more persuasively, to acquire deeper understanding of problems, and to investigate the possibility of multiple solutions
6. *Prove:* to desire that a statement be proved to you or by you; to engage in dialogue aimed at clarifying an argument; to establish a deductive proof; to use indirect reasoning or a counterexample as a way of constructing an argument
7. *Check for plausibility:* to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, Internet, or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations

8. *Take things apart:* to break a large or complex problem into smaller chunks or cases, achieve some understanding of these parts or cases, and rebuild the original problem; to focus on one part of a problem (or definition or concept) in order to understand the larger problem
9. *Conjecture:* to generalize from specific examples; to extend or combine ideas in order to form new ones
10. *Change or simplify the problem:* to change some variables or unknowns to numbers; to change the value of a constant to make the problem easier; to change one of the conditions of the problem; to reduce or increase the number of conditions; to specialize the problem; to make the problem more general
11. *Work backwards:* to reverse a process as a way of trying to understand it or as a way of learning something new; to work a problem backwards as a way of solving
12. *Reexamine the problem:* to look at a problem slowly and carefully, closely examining it and thinking about the meaning and implications of each term, phrase, number, and piece of information given before trying to answer the question posed
13. *Change representations:* to look at a problem from a different perspective by representing it using mathematical concepts that are not directly suggested by the problem; to invent an equivalent problem, about a seemingly different situation, to which the present problem can be reduced; to use a different field (mathematics or other) from the present problem's field in order to learn more about its structure
14. *Create:* to invent mathematics both for utilitarian purposes (such as in constructing an algorithm) and for fun (such as in a mathematical game); to posit a series of premises (axioms) and see what can be logically derived from them

For these practices to become internalized would require, as has been said, creating mathematics situations where their value can be appreciated on a number of occasions. Indeed, the only reason any of us put up with the awkwardness and difficulties involved in learning most of any new behavior is it makes life easier. And until such practice becomes natural, part of who we want to be, it takes our effort and reflection and our appreciative understanding that development takes time.

To help move that evolution along, the heuristic set could appear on the classroom wall, for example above the front board, in a print large enough to be read from anywhere in the classroom. That way, when students were stuck working on a mathematics problem, they could look up and be provided with a gentle reminder as how they might proceed. Also, in addition to the answers section in the back of the text, which in their explicit expression are often of little, if any, value, habits-of-mind suggestions as how to continue engaging particular problems could be included as well. In this way, students could have a hint of how they might reengage any mathematics problems they were stuck on.

In terms of classroom practice, three approaches are offered.

II. Introducing Their Practice

The first approach is as mentioned earlier: Make problem-clarifying strategies an essential part of the classroom conversation by providing problems that cannot be solved readily. In this way, all students see that solutions are not always found by directly applying an explicit mathematical technique but require a way of engaging the problem that provides valuable means for proceeding. Then, with a problem solved, taking a moment to appreciate what exactly was done, enough to have students write down the heuristic and so in effect "take it in." That is, appreciate the process as much as the product.

A second way for making some heuristic more available to students is by providing a set of problems that focus on the application of that particular strategy after students have had opportunity to see its value in engaging a problem together. Then, students could practice applying that problem-clarifying strategy in a number of problem settings. As a consequence, they would become more familiar with the heuristic and contexts in which it has proven valuable. In this way, the strategy would likely become more readily habituated and students' intuitions correspondingly more developed; of course, taking notes here as well makes good sense.

A third means is to present the strategy in a graduated format, in the form of a three-stage model. Namely, students are presented with sets of problems that can be productively engaged by a specific heuristic in a three-stage sequence, where:

Stage 1—*presents the heuristic* in response to a problem

Stage 2—provides a problem that *suggests the heuristic*

Stage 3—includes a problem that *tacitly requires the heuristic*

A few three-stage applications follow for each of three problem-clarifying strategies: visualizing, looking for patterns, and tinkering.

3.1 Visualizing

Making the Heuristic Explicit The teacher could share, "At times it's valuable to think of numbers as represented by geometric shapes as they provide another way to think about numbers and uncover relationships. For example, see if you can create a visual argument to determine whether or not $3^2+5^2=8^2$." As the students' pictures make clear, when a 3×3 square is put next to a 5×5 square, the resulting figure falls short of creating an 8×8 square as the form is 8 units on only one side.

To solidify that understanding, they can be asked "What does this suggest about whether squaring is or is not distributive? —that is, would you argue that $(3+5)^2=3^2+5^2$?" This consideration can help prevent the common algebraic error many students intuitively believe is true: namely that $(a+b)^2=a^2+b^2$, as a consequence of thinking it is analogous to $2(a+b)=2a+2b$. With this application, hopefully students would be more inclined to bring geometric/visual considerations to algebraic problems and vice versa. And that would be more likely, of course, if the particular heuristic would be introduced in other ways.

Suggesting the Heuristic The teacher could ask, “Since $8^2 = (3+5)^2 = (3+5)(3+5)$, how could the right side of the equality represent a square 8 by 8?” They know the result of multiplying on the left side of the equal sign must be 64, so they are led to see that four products must be involved on the left and can be represented geometrically. With their drawing the 3-square contiguous to the 5-square, students can see that the base is now 8 units long. Now, with adding a 5×3 (width × height) rectangle above the 5-square, and a 3×5 rectangle above the 3-square, they can create the desired 8-square. In this way, the experience can lead to the general understanding that $(a+b)^2 = a^2 + 2ab + b^2$, helping to make the algebraic symbolism more concrete.

Tacitly Requiring The teacher could ask, “If PQ and QR are diagonals of two faces of a cube, what would be the measure of angle PQR?”

3.2 Looking for Patterns

Making the Heuristic Explicit “Create a pattern for a numerical sequence by introducing the next three terms to the sequence whose first three terms are 1, 2, 3. However, ‘4, 5, 6’ has been taken.” The object here is for students to appreciate what Leibniz proved regarding sequences: that the next terms can be any numbers as long as one can create a rule defining the sequence.

So, for example, students might create a rule such as the second set of three numbers as being twice (or ten times) the values of the first set, and the third set as being three times (or 100 times) the value of the first set, etc. Now there is the opportunity to algebraically represent the generalized expressions. For the sequences just provided, students could come to create the expressions *ka, kb,* and *kc,* where *a, b,* and *c* are the initial triplet, and *k* is the constant multiple. While the other sequence could be written as: $a10^n, b10^n, c10^n$, where $n = 0, 1, 2, \ldots$, it is clearly an opportunity for students to write many sequence generators in the language of algebra. The teacher might offer as the next triplet, 10, 29, and 66, as the consequence of the sequence generator, $S(n) = n + (n-1)(n-2)(n-3)$, where n represents the counting numbers, 1, 2, 3,…. Sharing this with students allows them to create literally infinite sets of triplets building on this sequence generator that Poincaré used to help Binet understand in his design of intelligence quotient (IQ) tests that marking students wrong or right if they get the next number of a sequence depending on Binet’s answer is not a good idea. Working with Poincaré’s generator, students can experience a liberating moment by joining their imagination with one of the very finest of mathematicians and come to see that polynomials could be applied here as in other situations. (Such an investigation creating various sequences based on an initial triplet also serves as a metaphor that wherever one is in life, the next step is open.)

Suggesting the Habit “The ancient Greeks appreciated expressing numbers in geometric form. For example, they created triangular numbers, including a single point to represent the first ‘triangle’ so as to complete a pleasing pattern. Creating the next three, the sequence would be 1, 3, 6, 10 as each new triangular number can be represented by a row of points beneath the prior row determined by adding 1 more point to the prior row. For example, the second triangular number is 3 as it adds 2 points to the prior row of 1 point, creating a total of 3 points, etc. The ancient Greeks also created square numbers, also beginning with a point for the first square number (the 1×1 square) followed by a 2×2 square, 3×3 square, etc. Can you find any relationship between the sequences of triangular numbers and square numbers? If you can, can you represent your finding without using words?” Careful observation will help students uncover that the kth square number is equal to the sum of the k-1st and kth triangular number. To demonstrate that relationship, students can draw a line just above the main diagonal of any square number, and the two triangular numbers will appear above and below the diagonal. (This is surely a nice visualizing problem as well.)

Tacitly Requiring “What is the unit’s digit of 3^{101}? What is the unit’s digit of 2573^{101}?” The student will appreciate that listing powers of 3 and their corresponding numerical values might not provide all the structural insight being sought; the power needs a change of representation.

3.3 Tinkering

Making the Heuristic Explicit “We’ve seen how the quadratic equation $h(t) = -16t^2 + 40t + 3$ has served as a model for relating how high an object that was initially 3 feet off the ground and thrown up with an initial velocity of 40 ft/sec would be at a given time t. Now consider the general quadratic equation, $f(x) = ax^2 + bx + c$. Describe carefully what the effects of each of the constants a, b, and c are on the shape of the graph of f(x).” This tinkering activity, best done on a graphics calculator, or a graphing program such as the comparatively more efficient Desmos, provides students with the opportunity to appreciate the value of varying one parameter at a time so that each of the constant’s unique roles can be studied. A particular insight hopefully students come upon is that of comparing what happens when the lead coefficient changes sign, as well as what happens when any of the coefficients take on the value of 0. Parenthetically, it is important to note that the particular case was introduced before the general. Providing a particular instance first gives students an opportunity to immediately see how the construct of a quadratic equation can be applied. Then, tinkering with the general expression by considering the effect on the shape of the curve of changing each of the coefficients makes for an informative understanding of the parabolic form.

Suggest the Heuristic “What is an easy way to multiply any number, n, by 99? The product would always have what form?”

Tacitly Requiring "John draws a triangle and two of its altitudes. He then states, 'See…any two altitudes of a triangle will intersect inside the triangle'. Is John's conjecture correct? If so, explain why. If not, give a counterexample." Tinkering will provide the opportunity to draw different triangles, not all of which are acute or right triangles.

* * *

As mentioned earlier, there are other venues for presenting a habits-of-mind orientation. The Problem-Solving Strategy Inventory created by the British Columbia Ministry of Education (Parliament Buildings, Victoria, BC V8V 2 M4, Canada) asks students to make explicit whether they considered any strategies when they were working on a problem. The list of strategies include: make a table, look for a pattern, work backward, etc., and space for "other." This is, of course, a valuable self-assessment practice as it has students reflect on what they have or have not tried and provides specific means for them to choose from. The list can naturally be added to as new practices are seen to have value. Having students not only make a list of the particular strategies but also the contexts that prompted their application helps them see where they may be applied in the future. Collecting such information (in a "toolkit") promotes those practices naturally coming into existence.

In this way, the learning experience provides the opportunity to *re-view*— not just review but revisit with fresh eyes. Namely, considering "What did we do" to make progress, that is "What were the *heuristic means* used to make progress?" helps make the big point. In this way, students can acknowledge the worth of the critical mental actions that moved things forward—the *problem-clarifying* strategies they used that enabled them to secure the mathematics solution. With reflection, they realize they are more capable than they were; and that would seem a most essential element of an education.

It is worth repeating that there is more to take from the experience: the student's recording of the effort. Most often students, if they take notes, write down the mathematical argument that concludes some investigation or the algorithm that makes the solution explicit. Namely, they tend to record the *product* of the experience, the "logical bones" of the framework. And this is what the teacher would hope and expect would become a habit, especially with using textbooks that provide a model. However, considering problems that go beyond applying a particular *problem-solving* technique should compel students to also make a habit of making note of *the process,* the organic development. Without their including the *problem-clarifying* strategy(ies) that enabled the problem to be engaged productively, all that the "good" notetaker will have will be the skeleton of the engagement: the problem and the formal solution. The essential middle ground will be nowhere to be found and that suggests their note-taking, and likely their mathematics education, will unnecessarily be of limited value. However, with the solution approaches on the board being annotated, the mathematics teacher can help students see the inquiry process and how the particular problem-clarifying strategies helped to generate progress. In this way, after a while, the valuable practice of note-taking of *the thinking process* becomes a new habit—the earlier inconvenience of its practice is forgotten, and the gaining of a more successful way of connecting what was known with what was not is gained, just like any other habit worth having.

III. Assessing a Habits-of-Mind Classroom

A habits-of-mind language is likely not what we as mathematics teachers have been accustomed to in working with much of the mathematics curriculum. Nor has it likely been part of the conversation we as students experienced in mathematics classes. Indeed, it could be said that mathematical habits of mind is a hidden curriculum, a middle ground one cannot see for standing on it.

However, with a mathematics curriculum that includes more mathematics problems that are not directly solvable by a set procedure but calls for additional mathematical thinking, there could be *informal* assessments. For example, mathematics teachers could observe how successful students are on their homework where a heuristic was needed as well be alert to the frequency of problem-clarifying strategies language that is being used during classroom problem-solving conversations. Such feedback provides valuable understanding regarding which practices have become or are becoming part of students' mathematical thinking. Also, students self-reporting from time to time with regard to which heuristics are becoming more a part of their thinking would be a valuable means to learn how they are progressing. Especially, if students understood that the object of their self-reporting is not to impress the mathematics teacher with what the teacher wants to hear.

In this way, there is in effect an ongoing habits-of-mind assessment conversation, which the students can see is important to the teacher and valuable to their mathematical thinking. Additionally, formal assessments also provide opportunity for such feedback. In the early going, it may be reasonable to let students use a problem-clarifying strategies list on exams as it makes opportunity for their growing appreciation, and as a direct consequence, those practices become increasingly more in mind. After a while, students will hopefully see these strategies are literally the door to light and happiness in solving mathematics problems of any complexity. That is, familiarity could here generate contentment as the positive emotional expression of a developing realistic confidence.

* * *

The commitment by the teacher to making *problem-clarifying* strategies a primary focus, explicitly embedded in mathematics classroom conversations, requires as said more than once a commitment to ensuring the curriculum calls for that thinking and language. Namely, mathematics problems that promote and support students asking questions of the sort: "Any patterns here?", "What tinkering might be good?", "How could I represent the situation more simply?", "Is my answer plausible?", etc. need to be made available consistently, perhaps by the teacher's including such during a problem-solving engagement. Such questions give shape and clarity to the inquiry process, to the thinking that can carry one through. Naturally, it would take time to develop, but it would seem the thing to do: A habits-of-mind orientation promotes reflection, experimentation, and the opportunity for testing one's thinking toward constructing convincing demonstrations. Such an approach, while initially taking more time, deserves the attention of all mathematics educators as those means are essential for students doing mathematics well, for thinking productively.

To promote that happening, it would seem the object is to make thinking a more conscious activity in the classroom rather than leave it to the subconscious (where most of our thinking seems to occur) as in the serendipitous moment of the student who somehow sees what to do and is called on. For in helping all students become more aware of the thinking process, they can all become more capable mathematics students and thoughtful individuals. Naturally, a mathematics classroom experience committed to such thinking would mean that the content of *problem-clarifying* strategies would appear often—for introduction, be fostered, and acknowledged on a regular basis—so those practices in effect come to be integral to the classroom conversation. As John Dewey noted, it is our habits that give us character; and by implication, as mathematics educators, we have an obligation to locate and work with those habits that give the discipline of mathematics its character.

Indeed, it could not be some preconceived quantity of what is to be learned that should be our focus, for surely no one knows how much anyone, including themselves, no less a collection of individuals, could learn in some period of time. Rather, it is the desired quality of that experience that should be made explicit and sought after. It is also worth recognizing that whatever the educational times and the accompanying claims of what particular mathematics content ought to be being taught, or whether it should emphasize the basics, or applications, or mathematical structure, what would always be good and right and true to include are problem-clarifying practices that make engaging mathematics the exciting and rewarding enterprise that it is. Inasmuch as heuristics actually promote productive thinking in general, their focus could well enrich the personal and societal decision-making throughout our students' lives, enabling them to deal better with the complicated problems that are sure to be made "available" to them as they get older.

All in all, with the K-12 goal to create a community of life-long learners who are thoughtful, capable, and caring citizens in a global society, the fundamental concern of the mathematics community would seem to be that of fostering a classroom culture, wherein the participants have opportunity to develop and share the "tools of the trade" that help unify and liberate their individual and collaborative mathematical thinking.

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Part II
Promoting Mathematics Students' Social Development

A society is a number of people held together because they are working along common lines, in a common spirit, and with reference to common aims. The common needs and aims demand a growing interchange of thought and growing unity of sympathetic feelings.
John Dewey

Introduction

The classroom experience can be seen along a spectrum from each student investigating whatever is found interesting to those where all students are presented with the same material. Each is a society in microcosm. So John Dewey makes the case that before deciding what the educational experience is to be, the decision must be made regarding what type of society is desired. With that decision made, the classroom experience would naturally follow promoting those behaviors and relationships reflective of the society to be furthered. If we mathematics educators take this seriously, then we need to be clear what practices and interactions we value as expressions of the citizens of the society we want to promulgate in our classrooms.

Toward making those objectives more evident, the first chapter in this section, "Lessons from a Third-Grade Mathematics Classroom," looks at the values that shaped the practice in that mathematics classroom, where students learn they can depend on themselves and each other in the engaging process of becoming educated. This is not to say "everybody is equal," for the teacher is the one in charge, of course. What is evident is that the process of coming to make sense of things is very much in practice, with students sharing their thinking and questioning, listening to one another so as to gain a richer understanding, and everyone appreciating that it does take work, individually and collectively, to solve problems. While the classroom experience is with respect to a third-grade mathematics class, its relevance and value to grades 6–12 and the society being sought is quite apparent.

The chapter that follows, "Sharing a Language for Productive Inquiry," focuses on the questions mathematics teachers can ask to enrich students' reflective thinking. Inasmuch as the questions we ask determine what we think about and serve as the connective tissue when students are working together, helping students come to see what questions tend to be productive when engaging problematic situations would seem a valuable part of their mathematics education. With the inquiry process seen as content worth studying and drawing upon, every student could develop further their reflective capacity so as to support their own and each other's investigative/creative efforts in the mathematics classroom. In this way, all students have the opportunity to be valued participants in a society open to hearing the thinking of all of its citizens.

The last chapter in this section considers "Collaborative Mathematics Investigations." Here, students truly have the opportunity "to see the importance of building community in their mathematical work" (Su 2010, p. 168). These student investigations begin with a focus problem established with the teacher that can be pursued in more than one direction, making for "multiple centers" of engagement. The opportunities for exploration, changing one's mind, listening carefully, working with someone else's ideas, being able to explain oneself clearly, compromising, and leading and following make for everyone's benefit. Then, with resolution of the focus problem, multiple investigations again become available as students pursue questions or aspects they found interesting during the initial investigation but did not get a chance to act on at the time. In this way, students have opportunity to give shape to and inform their education and enrich the collective mathematics experience and the mathematics curriculum for all involved. In this way, students come to experience that their interests and collaborative efforts matter, and together they create a community that is fun and rewarding being part of.

Chapter 4
Lessons from a Third-Grade Mathematics Classroom

In what follows, we have the opportunity to "listen in" on the thinking of teachers and administrators reflecting on a mathematics class they observed, along with the thinking of some readers who reacted to the article written about that class experience. How their values and concerns connect or not to those of ours as mathematics teachers give us an opportunity to reflect further on what values shape the mathematics experience we provide and hope to provide.

A third-grade mathematics class engaged in "collaborative inquiry learning" was the subject of a newspaper article, "Elementary math focuses on real-life problem solving" that appeared in The Windsor Star (NCTM *Smart Brief,* May 21, 2013). The writer observed "kids are learning math slowly, intensively and collaboratively. Instead of mowing through work sheets, they're breaking off into groups, discussing the different ways to solve the problem, drawing diagrams, writing out answers in sentence form and presenting their solutions to their classmates, who respond with raised hands and their own ideas."

What makes it of special interest here includes the negative assessments provided by some of the article's readership. One person wrote the hour-long investigation to determine "'if a grasshopper covers 25 centimetres each jump and jumps eight times, how far does it go?' was quite literally a waste of time." Yet not only was there criticism directed at the problem-consideration time frame, for "with a strong knowledge of the basics it shouldn't take more than a minute to solve the problem," but also with students working in groups, as it is "very easy to sit back and let others do the thinking in a group setting." Another reviewer concluded, "In a time-strapped world we should teach the kids to solve the simple problems quickly and move on to bigger questions." These observations would seem hard to argue with.

Surely, mathematics educators can imagine the grasshopper problem being presented to a class and after a short while asking for hands and along with student responses would be explanations how multiplication as repeated addition works in this situation. This could well have been done in a few minute's time. That approach would clearly be not uncommon. Indeed, it would seem to be the standard way to engage the problem in contrast to the class at St. Angela Catholic Elementary School in Windsor, Ontario.

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What led the folks there to design the mathematics experience as they did were the objectives being sought as a consequence of the values they held. They believed it would be best for students to work alone and collaboratively in an investigative effort to make mathematical sense of that question. Apparently, the educational consequences were appreciable. A 3-year evaluation process done by the Ministry of Education found: "improved student performance," "improved teacher's belief in their ability to teach math," and "increased interest and engagement by students." These are clearly not minor outcomes. But let us consider the value framework that set the class in motion.

Considerably more than a solution to the mathematics problem was being sought. Underlying collaborative inquiry learning is the commitment to and understanding that *the process of engagement* is an essential aspect of *the learning product*. In contrast to coming up with a procedure and an answer as efficiently as possible as directed by the mathematics teacher, a common spirit of engagement is promoted so that a number of successful approaches may be uncovered. The school's principal shared that "we are preparing kids who are patient problem solvers." And the youngster who was reported to push his thinking further so as to determine the answer in meters (not just centimeters) demonstrated how the experience promoted problem posing/creating as well. A teacher who observed that class in effect responded to the reader who saw the hour activity as a waste of time in "a time-strapped world." Her comment was that "By working through and struggling through one problem, we slow everything down in this hectic, hectic world and say there's a lot of richness in this task." Both agree that life is complicated and knowing how to solve mathematics problems deserve priority. But the school is seeing the greater gains made with an eye toward an unknown but guaranteed complex future. For the "real-life" problem considered was not just that of determining the distance the grasshopper hopped.

The real-life educational problem apparently had to do with the teacher promoting students' interests so they would engage the mathematics question with dedicated energy, and in the investigative process develop and share their perspectives and ideas, and work together to make sense of things. Their thoughtful considerations apparently led to developing various representations, along with raising and testing ideas, in an effort to gain mathematical solutions. The shared interest meant students would have to listen carefully to each other and try to express themselves clearly. Such a collaborative environment would suggest the opportunity for students becoming more open-minded, more adept at recognizing essential information, and more successful in securing a satisfying understanding.

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It is not being claimed here that the classroom format is *the* model for every mathematics class. But to highlight that whatever mathematics classroom experiences we offer be looked at through the lens of the personal, social, and intellectual values we seek to promote. If, for example, we wanted to promote self-reflective, socially responsive, confident individuals, then we would design our classroom environment so that those qualities would have the best or at least a good chance of being present. The third-grade lesson was to point out that regardless of the classroom approach, whether a lecture, guided discovery, flipped, blended, constructivist or whatever, the question remains, what exactly are the objectives that we are working towards. After all, student's intellectual, social, and psychological development would seem the foremost consideration from the perspective of a society enlivened by its citizens.

Learning mathematics offers a wonderful context in which to promote all three dimensions of that development. Its commitment to imagination, experimentation, and reflection are at the foundation of thinking (and doing). And its social value can be found not only in its essential role in the sciences, the physical as well as the social, but in the collaborative efforts that have generated so much interesting and valuable mathematics, including those experienced in our classrooms where students learn to work together, compromise, and respect each other's efforts. In addition, with the opportunity to learn how to deal well with challenges, requiring being more flexible in thinking, and how to draw upon emotional reserves so to secure the resilience needed to proceed, such practices promote students' psychological development.

Essentially, the consideration of that third-grade mathematics classroom was to raise the questions underlying this chapter and book: what are the intellectual, social, and psychological capacities that we believe ought to being developed and promoted and what practices would enable their being secured in the best interests of society. Knowing that, we can shape our students' mathematics experience and have a realistic confidence in their gaining success.

What seems a good rule of thumb is as Dewey noted: We need to think how we can prepare students for a future we do not know by giving them command of themselves. That would seem to mean that in their best interests and those of a democratic society, we help them develop as reflective, resilient, and responsive human beings. With that as our aim, we can better decide how to structure the mathematics classroom experience. Then assessment naturally follows, as it would be looking to see whether or not those valued dispositions and practices are in fact developing.

Such a perspective helps ensure we do not take too narrow a view of what is a productive environment for learning mathematics. It implies that we would question, for example, the value of having students spend their time on work sheets as a common mathematics classroom activity, even if it seemed some students enjoyed doing so. For *"It is possible for the mind to develop interest in a routine or mechanical procedure, if conditions are continually supplied which demand that mode of operation and preclude any other sort"* (Dewey in Dworkin, italics in original, pp. 108–109). Spending classroom time practicing procedures that have little viability in the world would not seem socially responsible from the school's perspective or personally responsible from the student's. As Dewey noted: "were all instructors to realize that the quality of mental process, not the production of correct answers, is the measure of educative growth something hardly less than a revolution in teaching would be worked" (1916/1944, p. 176).

In that direction, it seems fundamentally important to "psychologize" our mathematics lessons so that they connect to where the students are and create a learning environment that promotes those values essential for developing a more thoughtful, more open-minded, and more socially responsive citizenry. How to help promote that development is the aim of the next two chapters.

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Chapter 5
Sharing a Language for Productive Inquiry

A critically important but hidden expression of students' successful mathematics experience is the questions they ask themselves. For it is the questions that we ask that gives shape to our thinking, our effort at making sense of things. Were students increasingly able to consider and share thoughtful questions, they would be more able to build on each other's thinking, and in essence establish more consistently productive collaborative inquiry. Societal implications are evident.

That suggests an essential part of our work as mathematics educators would be to share what problem-solving experience has made clear are effective questions to ask. For trying harder to reach a goal only has benefit if there is an appreciative understanding of valuable means. For that exact reason, weaker students tend to appreciate a highly structured learning environment given to explicit instruction, what is referred to as "fully-guided instruction" (Clark et al. 2012). It not only lessens the stress of not knowing how to solve a problem, but the stress of not knowing what to ask to move things forward. Such directed instruction is a qualitatively different mathematics experience than that shaped by students' productive questions. And those qualities matter when considered from the perspective of being a valued citizen in a dynamic society.

A major obstacle to students developing their heuristic capacity, in addition to it being muted by a mathematics experience that rather exclusively focuses on presenting step-by-step procedures, is that essential thinking is usually invisible in the presentation of final mathematical arguments. This is especially frustrating for diligent students who work really hard taking notes. For the valuable questions asked and mathematical problem-clarifying strategies drawn upon that actually fostered the choices of the particular solution approaches are generally omitted from the formal demonstration. That is why it was suggested prior that we make more visible the critical questions and the reflections and heuristic actions that led to the technical moves to secure the desired conclusions. In that way, we promote the chance for transfer of learning by providing "thick description" of the thinking that underlies the formal presentation. In this way, the questions pursued and problem-clarifying approaches that were called upon in shaping the argument are made as evident as well, not just the polished product.

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What gives the questions teachers ask their heightened import is not only that they shape the classroom conversation but importantly inform students how they themselves should go about inquiring. This suggests that the nature and quality of the questions students would be asked would be very much part of the mathematics teacher's planning and curriculum. Professional development and evaluation experts can be found who agree with the importance of that focus. For example, Charlotte Danielson, in her fourth edition of *The Framework for Teaching—Evaluation Instrument* writes: "Questioning and discussion are the only instructional strategies specifically referred to in the Framework for Teaching, a decision that reflects their central importance to teacher's practice" (2013, p. 63). And while some of us are more inquisitive than others and some of us are really good at presenting lectures, the questions we ask are pretty much decisive regarding the understanding that will be pursued. So in terms of promoting an educated populace, the work of the mathematics teacher would seem to be in good part to model good questions so as to determine where needed energies are best placed to promote productive decision-making. With students learning to raise questions that enable progress to be made in disentangling complicated mathematics problems, they become more thoughtful and interesting to themselves and others.

Having the emotional resilience to handle the state of doubt and confusion that often accompanies perplexing considerations is of course essential, as our emotions create the immediate environment in which those questions can be asked and pursued. So it is important for students to know that "It is primarily through the recognition of error that we are enabled to achieve major insights and proceed with learning" (Raths et al. 1986, p. 166). Many mathematicians discuss how mathematics literally evolves from the errors made (cf., Lakatos 1976; Davis and Hersh 1981), and scientists as well, of course. The inventor and billionaire, James Dyson, regarding the mental environment he creates at his workplace—"I'm very keen on wrong thinking. That's the creative bit: watching the failures" (TIME, Aug 19, 2013, p. 60).

This is to suggest that learning to appreciate the positive side of making errors needs to be part of the classroom conversation inasmuch as "To many persons both suspense of judgment and intellectual search are disagreeable; they want to get them ended as soon as possible" (Dewey 1933/1936, p. 16). To help change that scenario, when students engaged with a problem do not know what to do they need to know what to ask. To make that more possible would seem an essential consideration in shaping the mathematics classroom conversation.

How we can help all students of mathematics appreciate the process of thinking through the confusion and for a good number dealing with the attending discomfort they experience is an essential part of our work. What we need to help them understand and develop the capacity for is that "To be genuinely thoughtful we must be willing to *sustain and protract* that state of doubt which is the stimulus to thorough inquiry…" (Dewey 1933/1936, p. 16; italics added). And our students will be more inclined to do that not only as a consequence of our modeling questions that promote productive thinking, but by there being the class time and outside time that give them the opportunity to spend thinking, posing questions, and engaging them.

That space creates the needed room for their thinking productively, and an environment where they feel safe sharing their thinking—namely an environment where the spirit of inquiry provides the intellectual, social, and emotional energy to engage mathematics problems for extended periods of time.

While questioning would seem to be of the very fabric of the mathematics classroom experience, not only what to ask but when can be a complex affair. For instance, if we as the mathematics teacher ask for an answer to a question that is not time-wise appropriate, this could well promote impulsive thinking and what students offer could well be off the mark. This complication strongly suggests that silence must be appreciated in the mathematics classroom experience. It creates a needed space in which the participants can take stock of things. Research suggests that increasing the "wait time" between our asking a question and calling on a student tends to elicit a response more associated with the student's true belief (Rowe 1974). And in doing so that is surely time-wise efficient.

(An interesting class experience is to have students do nothing for 1 min. With students not watching the clock but consciously living in the minute, they usually find it to be longer than they think. A lot longer, especially if students have hardly had a silent minute to think in a mathematics classroom where covering the curriculum is the driving force. Hopefully, students have silent times to think alone, even if they are in groups.)

To make the content of inquiry an integral part of the classroom conversation clearly takes time and focus. A framework valuable in this regard is adapted from *Teaching for Thinking* (Raths et al. 1986). To use it well, they suggest the teacher tape themselves for 10 min a day, and use the checkoff list to evaluate the nature of the questions asked. Their work with teachers led them to conclude that "Should you do this, the likelihood is great that you will begin to notice a marked change in the quality of your interaction with students" (pp. 174–175).

As all frameworks, this framework can be worked with, and it has been in what follows. It was extended to include teacher's asking questions that promote students summarizing their and other student's findings. This of course is a helpful means to promote students listening carefully to themselves and others, and provides needed opportunity to reflect on shaping what a well-rounded yet concise evaluation sounds like. Added also are questions associated with the earlier focus, namely those drawing upon *problem-clarifying* strategies. For example, asking students how they might *make the problem simpler* or *take things apart* or *check for plausibility* helps students become more comfortable using the "tools of the trade."

Clearly, productive questioning lays at the very heart of the intellectual framework that makes doing mathematics well or most anything else involving thoughtful decision-making. Using this assessment lens as a means for promoting more thoughtful student questioning/thinking can become an essential and explicit part of the classroom mathematics curriculum. That is, if we think of questions as the foremost determinant in shaping the classroom discussion, which seems to make sense. Introducing this framework into our mathematics teaching might well mean we have to alter our own practice, which is not an easy matter. But that of course is what is involved in doing anything better—namely, making changes so as to gain more productive habits. Here it is:

Coding sheet for tallying teaching for thinking responses (Raths et al. 1986, p. 175; adapted).

Consider each of the following being presented as questions—"Can anyone…?"

- Provide a paraphrase reflecting main idea
- Share conjectures they have
- Make clear the information that is available
- Provide an example to illustrate idea
- Make clear the assumptions being held
- Share how they came to hold a particular view/idea
- Evaluate a shared idea
- Consider alternatives given an apparent dead end
- Share supporting evidence
- Share any hypotheses that seem plausible
- Share exactly what needs to become clear
- Test a particular hypothesis
- Provide a summary
  And
- Make the problem simpler
- Take things apart
- Test for plausibility
- Change representation (e.g., from particular to general or vice versa, algebraic to geometric or vice versa)
- Tinker (e.g., make a variable a constant, introduce numbers)
- Make a diagram (e.g., of what is known and what is not, a web, a table)
- Organize a proof (reasonable argument)
- Apply the principle to new situations

This list divides questions into those that focus on what the situation is, and having made that clear, what questions would move things forward. Students are asked to reflect on what has been said to ensure a secure hold of what is known (by asking them to paraphrase, provide an example, make clear what the assumptions are, share supporting evidence, explain how they came to an idea, and make explicit what remains confusing); and what students could think about or are thinking about to move the conversation forward (by sharing conjectures, considering alternatives, sharing a hypothesis, testing a particular hypothesis). The partition helps students appreciate that thinking/problem solving is as much about taking stock of where they are in the problem-solving process as well what might be some promising direction(s). It is made clear that not only is it a good idea to stop and consider what really can be held as solid as versus what remains in need of evidence, but also that dead ends are to be expected to be found in the problem-solving landscape. With students hearing this language on a regular basis in class, it would seem those mental actions would in time become "furniture of the mind" as they work their thinking into a coherent framework.

Yet of course, this is not to suggest that students would not come up with these or their own questions with regard to both clarifying where the thinking is at and what could be promising directions to pursue. Problematic situations promote their own questions, and with students having the time to reflect on those situations there would surely be opportunities for thoughtful conversations toward their resolution. Becoming educated means to expand our understanding, and productive questions naturally provide the setting.

Ultimately, with mathematics teachers promoting and supporting conversations that help students formulate clarifying questions and potentially valuable directions to take, there is good reason to believe such conversations would have a significant impact on students' thinking and their mathematics learning. With thoughtful question-asking a common mathematics experience across the grades, societal implications are suggestive.

When we ask ourselves a question, we are requesting a response from ourselves, and hence feel obligated to respond. We have, in effect, baited the hook and as the direct consequence are all the more likely to "get a bite" if we keep at it. Whether the catch is worth keeping or should be tossed back, that is part of the tension and excitement of thinking. But the question set offered here could well be expected to promote more rewarding ideational moments. The investigations presented in the next chapter consider an inquiry environment.

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Chapter 6
Collaborative Mathematics Investigations

Group-based learning activities are quite a common phenomenon in school for good reason. Research supports the practice as a productive means for student learning, and grouping represents in microcosm what will extend to students' lives as adults working together. With sufficient opportunities for group investigations, mathematics educators can promote students' personal, social, and intellectual development, including the asking of good questions that help ensure productive interactions essential for a vibrant society.

Academically speaking, student collaboration is very promising. For instance, "When students worked in small groups, taking significant responsibility for planning, undertaking, and reporting on research into subject matter, most scored significantly higher on content-area tests of math, history, literature, science, geography, and reading comprehension" (Harvey and Daniels 2009). And "When students engage socially in talk and activity about shared problems or tasks, their questions can stimulate not only themselves but, also, another group member to use the relevant thinking strategies and processes (e.g., hypothesizing, predicting, explaining) in their search for an answer" (Chin and Osborne 2008, p. 3). In group investigations where mathematics students have to draw upon problem-clarifying strategies and questions that promote reflective thinking, there is good reason to think those activities would be truly enriching educational experiences. Yet, grouping students may not be as simple as dividing by a single-digit group-size divisor.

As any experienced mathematics educator knows, teaching the entire class at the same pace has the potential to make the classroom experience very trying—in particular, for those who learn more readily and those who need more time, not to mention the teacher. Hence, the pedagogical adage of "aim for the middle." Some educators, apparently not sold on the aesthetic of "the greatest good for the greatest number," have in effect used the heuristic of *take things apart* by having small groups of students work on different facets of the curriculum simultaneously. This is obviously quite a challenging experience for the teacher trying to keep all the ideational balls in the air at the same time. Differentiating instruction is a complex undertaking, yet it offers a valuable educational opportunity in responding to each student (cf. Tomlinson 2001).

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A variation is to have students of similar academic abilities grouped together, with the hope and expectation that everyone can participate. (This seems to be particularly true with mathematics, science, and language classes.) There are potential gains as well as losses with this hierarchical approach. Clustering classes this way means there is little if any opportunity for some students to interact with some others, and those students who struggle would only have each other to count on. And this could be problematic for a democratic society where people of varying abilities and backgrounds would need to share perspectives and try to work together to reach an acceptable resolution.

A response to dealing with the range of student abilities has been to group together students so that the more able and/or more inclined help those who have greater difficulties learning/connecting to the material. The problem here is that those who are naturally more capable and/or more dedicated do not have the opportunity to explore matters as deeply as they might hope due to their obligation to others in the group; as importantly, the less able or less inclined tend to find themselves on the receiving end of the educational experience, too rarely feeling they have added to the group's understanding. Such a practice, while well intentioned, could fail to foster productive interactions and positive feelings.

Grouping students for working on projects is gaining following in classrooms. The most prominent seems to be project-based learning (PBL) where "students go through an extended process of inquiry in response to a complex question, problem, or challenge" (cf. "What is PBL?" www.bie.org/about/what_is_pbl and "8 Essentials for Project-Based Learning" in *Educational Leadership,* September 2010, Vol. 68, No. 1). This approach provides opportunity for students' collective efforts to be shared with others in the community. While some students may well "have difficulty with open-ended situations and with ill-defined problems" (Thomas 2000, p. 29), the intervention of "providing a structural set of inquiry steps for students to follow" (Thomas 2000) would seem a questionable solution. That surely precludes the opportunity for promoting flexible thinking and students learning to respond well to novel situations. So it clearly requires a careful balancing act regarding what to discuss without being too prescriptive. It would suggest both reminding students of what questions would make for furthering the inquiry as well as having them reflect on possible problem-clarifying strategies or questions they think/feel might be of assistance in helping shape a productive inquiry.

Whatever the grouping format, it would surely be a goal that every student has the opportunity to have their voice be valued in promoting the collective understanding. For "While what we call intelligence may be distributed in unequal amounts, it is the democratic faith that it is sufficiently general so that each individual has something to contribute, and the value of each contribution can be assessed only as it enters into the final pooled intelligence constituted by the contribution of all" (Dewey in Ratner 1939, p. 403). To increase the likelihood that every student would be appreciated in any group investigation would in part be to give time prior to forming groups so that everyone has time to think about the problem to be investigated. This individual reflection-time increases the chance that more students will have something to bring to the group to begin the investigation. It could well be a finding gained by applying an insightful question or a productive heuristic, including coming up with what seems a promising conjecture, or just a realization that some direction is closed, given an instance of some quantitative analysis. Seeding the conversation seems essential, and so everyone needs time to formulate their thinking.

However, as mathematics educators know well, getting all students involved is not an easy matter. With students at a distance from the focus of the conversation, the teacher's experience and observation can well be that "I taught it, but they didn't learn it." That statement has the same logic as the salesperson who claims they sold something but no one bought it, as Dewey noted. In life-enriching contrast, with students of "one mind" working on a shared concern from varied points of view, our work as mathematics educators does not require creating/struggling with student discipline, for their very engagement in the learning experience does that.

To help ensure that happening "it is part of the educator's responsibility to see equally to two things: First, that the problem grows out of the conditions of the experience being had in the present, and that it is within the range of the capacity of students; and, secondly, that it is such that it arouses in the learner an active quest for information and for production of new ideas" (Dewey 1938, p. 79). It is in such settings shaped by students' focused collaborative energies, including the questions that serve to clarify where things presently stand in an effort to move the conversation forward, we can appreciate the difference between keeping students busy and seeing them involved in something they find engaging. (This is not to say of course that for a classroom experience to be valuable all the participants must be working in groups. For an intellectually stimulating whole-class lecture or a "good" problem to ponder are each alone quite sufficient to engage students individually as well.)

Yet working in groups can provide a setting for students to ask questions and play out ideas they are not particularly sure of or comfortable with "announcing" to the whole class. Additionally, having the opportunity to sit facing each other promotes interactions that require listening carefully to one another and the need to reflect on questions and statements shared. It is not unrealistic to expect in such a setting everyone respecting the group's effort for what is uncovered as a consequence of the give-and-take is often what none of the parties knew would be the resolution. Also, such a setting provides for continuous feedback—i.e., an ongoing "ungraded assessment," from which all can learn, and each can become a more productive member of the group. If we as mathematics educators are concerned with students' social development and their capacity to engage challenging questions collaboratively, then grouping would be a common experience in our classrooms.

* * *

A "multiple-centers" investigation, though not being argued as the everyday classroom-grouping procedure, divides the class by student interest in response to a *focus problem*. It need not be a "real-world" problem, but it does need to be devised so as to ensure more than one approach is available to shed light on the inviting but challenging investigation. As a direct consequence, a broader mathematical understanding can develop than when working toward finding a single path to a solution. The varied entry points naturally increase the chance for a greater number of students to participate and for uncovering a variety of interesting avenues to pursue.

When enough time has resulted in individuals and small groups having come to findings including questions they believe are significant, the teacher aware of each group's state of understanding either helps them focus their efforts further or, finding the time sufficient, brings everyone together. It could be to celebrate the victory of the conceptual hunt or to bring questions for everyone's consideration to see if some further headway can collectively be made. In either case, discussion is given to how ideas have come together and what heuristics and critical questions were made good use of.

With the eventual solution of the focus problem, students then have the opportunity to self-select extensions of the initial investigation. Multiple centers are again formed, but now with students working alone or in new groups as a function of their newfound interests as a consequence of the initial investigation that the teacher agrees has the potential to add to the collective educational experience. After a second interval of time, students are brought back together, and this time their efforts are presented to the class in a quasi-cohesive ordering determined by the teacher based on how those multiple-centers investigations fit together, including incomplete findings and unanswered questions. In this way, the learning experience established by students' individual and collaborative inquiries and initiatives is acknowledged, and every student can be recognized for having added to the general understanding.

Depending on the teacher's thinking, extensions that enrich the collaborative effort can come from other disciplines, not necessarily from mathematics. For example, with regard to the mathematics of fountain arcs which is one of the multiple-centers investigations to follow, student findings include the result of researching the history of fountains as that focus was considered a valuable addition to their learning experience with connections made between the evolution of fountains and humanity's development being established.

That some questions may go unanswered needs to be appreciated as well—only television mystery shows find the solution in the allotted time however complicated the problem. In the reality of the mathematics classroom, students learn that problems may well take an extended period of time to make headway, which of course means they may not be resolved in the present, if at all. (The physicist J. Robert Oppenheimer wanted to share with 14-year olds the most difficult problems in physics so that they would have years to play with them. With their not yet having bought into the accepted way of seeing of the educated group, he was acknowledging the virtue of having a naïve perspective and the human imagination, given time.)

When engaging the focus problem, because of the varied perspectives that arise, students realize it really is important to listen carefully to one another, most especially to those who hold competing hypotheses and contrary viewpoints. Different perspectives may promote at times a friendly competition as well as a more comprehensive understanding and calls for compromise in promoting further avenues for consideration. Such an interactive engagement makes it eminently clear that "…education is a process of living and not a preparation for future living" (Dewey 1897, p. 79).

* * *

Four multiple-centers investigations follow. The mathematical content ranges from a sixth grade investigation to one at the precalculus level.

The first, "Four-of-a-Kind," promotes students creating/uncovering patterns by tinkering with four of the same number and any combination of the basic four arithmetic operations. Along the way, the investigation provides multiple opportunities for students to practice arithmetic and to uncover mathematical relationships that promote their inventing symbolic representations to express the general findings. In so doing, a connection is made between working with numbers and the value of algebra, which is a foundational concern in mathematics education.

The second, "Would 400 be Better than 360?" provides students with the opportunity to use their imagination in coming to decide whether a circle ought to be divided into 360 or 400 parts. The latter suggestion, as will be discussed, was the suggestion-cum-claim of Simon de Laplace, one of the finest mathematicians of the eighteenth century. The investigation allows for varied explorations with number and geometry, and introduces the significant role aesthetics can play in decision making in mathematics.

The third, "Mathematics of a Fountain Arc," engages mathematics students in trying to create a mathematical model that represents the path of a water fountain arc rising from the surface of the fountain at a particular angle and returning to the fountain at a given distance away. Here students have the opportunity to transform the basic quadratic equation associated with the height a projectile reaches at a particular time with another quadratic equation that requires additional parameters.

The last multiple-centers investigation, "Approximating the Area of a Simple Closed Curve," requires students to create a mathematics formula and test its accuracy. The opportunity to satisfy their mathematical intuition and develop convincing arguments leads them to explore a variety of geometric forms, come face-to-face with a dead end, and finally determine the best "approximating" rectangle to represent the area of a simple closed curve.

6.1 Four-of-a-Kind

A pervasive educational concern, from state education departments to work-force study groups, focuses on students having success in algebra. The general belief is that "Algebra, in particular [has] become increasingly essential to educational advancement and career opportunities" (Ball 2003, p. 3). How to create that bridge begins of course with enabling students to make the transition from working with numbers successfully to working with abstract symbols so they not only feel comfortable but actually enjoy engaging and representing general mathematical relationships symbolically.

This multiple-centers investigation is directed at fostering that development. It provides a number of opportunities for students of all abilities to collaborate in uncovering number patterns, determine the need for an order of operations, and come naturally and logically to incorporate symbolic representations as the result of their generalized findings.

It begins with students being asked to represent the number 1 using four of each of the numerals from 1 to 10 drawing upon the basic operations of addition, subtraction, multiplication, and division. For example, $1\times1\times1\times1$, $\frac{2+2}{2+2}$, to 4 tens, as in $\frac{10+10-10}{10}$. With students working alone and together in uncovering many equivalent expressions, this investigation provides multiple opportunities for students to become more comfortable and facile with the basic arithmetic operations. In addition, it creates many opportunities for raising questions that require justification and promote generalization, which are the other two practices along with representation that are considered "central to the learning and use of mathematics" (Ball 2003, p. 36). Because students uncover many instances where the representations have the same form, the investigation promotes students inventing symbols to express those general relationships. Some used an asterisk, others a wagon wheel with four spokes, others a large X, and others a square form to represent that any of the numerals 1–10 could be used all four times to create the numerical value of 1. In effect, they were recognizing the value of using symbols, and in so doing developing an algebraic awareness and appreciation.

To represent some of their findings here, the commonly accepted symbol, *n,* where *n* is any number from 1 to 10 is used. For example, students readily found that for any *n,* $\frac{n+n}{n+n}=1$, and many other ways to represent one, including: $(n+n)\div(n+n)$, $\frac{(n+n)-n}{n}$, $\frac{n+(n-n)}{n}$, $(n\div n)\times(n\div n)$, $(n\div n)+(n-n)$, and $(n-n)+(n\div n)$. Some students claimed the initial expression and the first in the list here were different, as the first was a fraction representation of 1 with the second representing the division process. They also saw the need for parentheses upon realizing that number operations are binary and came to see the commutative property of addition and in pretty sophisticated terms. As students reflected on these patterns, they drew upon another key habit of mind, that of *proving,* and with the essential assistance of algebra they appreciated that in many of their findings all the counting numbers from 1 to 10, and beyond, would be included!

With the focus problem solved, a number of activities were then established by students' questions. They were excited to see if they could express other numerical values from 2 to 10 using four of each kind. (Some youngsters decided to name their group after the name of the number they were investigating.) Valuable assistance was provided by the teacher who put up large sheets of paper around the room and into the hall with a different numeral at the top of each sheet. When students found a new way to express a particular number using four of any number from 2 to 10, they wrote it on the appropriate sheet. After a good while for investigation and small-group discussions, students walked around looking at the sheets to look at all the findings and see if they could uncover other generalizations and represent them symbolically. (The excitement of the conceptual hunt was much in evidence.)

This multiple-centers investigation naturally led students to draw upon the problem-clarifying strategies of *tinker* along with *consider similar problems,* and they found themselves considering how grouping quantities work, including working with parentheses. They invented commutative and associative rules, determining where they worked and where they did not. They shared hypotheses that seemed plausible and provided examples to illustrate their ideas, and asked a number of questions on the list offered earlier. All in all, the investigation provided a variety of opportunities for instructive student exploration including final considerations taking the form of algebraic representations.

When all students came back together after their multiple inquiries, there was opportunity for the teacher to help them verbalize what they had written and share their supporting evidence. For example, some of the "cross-number" discoveries students made by observing different sheets in the hallway included their noting that in addition to $6-(6+6)/6=4$ and $7-(7+7)/7=5$, the other digits also fit that form, being "two less," except for 4 twos as the form generated 0. Some students, expressing the mathematician's desire for completeness, wanted to expand the number set to include 0, but others argued for leaving that to another time. The ensuing conversation uncovered that in general $n-\frac{(n+n)}{n}$ could be written as $n-2$, which they really appreciated for its simplicity. Similarly, $6+(6+6)/6=8$ and $7+(7+7)/7=9$ could be represented as $n+\frac{(n+n)}{n}=n+2$.

Other generalizations promoted more conversations about algebra. For example, students found instances such as $(7\times 7-7)/7=6$ that led to generalizing to any number n, with $\frac{(n\times n-n)}{n}=n-1$. This was quite perplexing to a number of students, so returning to numerical examples and changing representation made a big difference. Namely, with the expression $(7\times 7-7)/7$ rewritten as $(7\times 7)/7-7/7$, students saw that they were really finding $7-1=6$. And with the generalization, they saw that the final expression did make sense. Another form they enjoyed discovering was prompted by their representing 5 using 4 tens in the form $\frac{10\times 10}{10+10}$. They generalized it as $\frac{n\times n}{2n}=\frac{n}{2}$. This helped reinforce for some students that when there is no symbol written between a number and a letter it is "saying multiplication."

Finally, putting together a page of all the generalizations they had uncovered provided a summary of students' efforts and made clear the big ideas they had captured in algebraic expressions and the associated questions; and each expression attested to its efficiency and elegance. Here a real appreciation for engaging mathematics, including the helpful heuristics of tinkering, looking for patterns, and changing representations, was experienced. Yet, despite good efforts, limitations were also found, as 10 could only be created using 4 twos, fives, eights, nines, and tens; and 5 could be expressed using 4 twos to 4 tens except for 4 eights. But even with the empty spaces in their charts, they knew they had made a good effort and wondered if they would be filled in later or perhaps never because it could not be done. That too helped them appreciate the excitement that mathematics provides. Overall, they were very pleased with their investigative efforts. And, as often is the case with rich inquiry, they received the gift to wonder further.

Several features of this unit are worth highlighting in addition to its success at making the entire enterprise of algebra both intelligible and exciting:

- There is a game-like quality to this challenge, but the competition is with the material and not between students. It was not "Who can do this first?" but rather, "Can we figure this out?"
- Key here was the way the unit was introduced, using an introductory *focus problem* to get things going and making sure everyone could see what the real issues were.
- Then student interest extended their investigations to all the numbers from 2 through 10. Interest groups were formed and student's questions promoted multiple avenues and strategies, and they posted their findings and considered the work of others. This meant that all individuals and groups could be valued participants in the conversation which produced the kind of synergy that can make group work exciting. It would include the more adventurous working alone and the socially inclined playing out considerations with others.
- Further, because of the multiple approaches, the richness of the problem came to the surface and students fed off of its complexity, becoming excited about creating their own grammar and proofs.
- Finally, the various occasions for collective reflection gave the teacher many of those "golden" moments to underline themes, particular questions, strategies, and habits of mind that had been employed. And of course, the value of symbolic representation and its expression as algebra.

6.2 Would 400 Be Better Than 360?

This multiple-centers investigation helps students understand how a different way of thinking can promote exciting mathematical considerations and uncover from history some really interesting decisions.

Students in grades 6–8 are told, if not earlier in their education, that a circle has 360° and a triangle 180°. That the former has no apparent angles, while the latter is constituted by them and is half the measure of the former, hopefully piques students' imaginations as two obviously disparate shapes seem to share an intimate quantitative relationship. A further distinction that was needed to be made was that the statement "a circle has 360°" was an expression of agreement, not the consequence of a formal argument as was "a triangle has 180°." Each would require discussion but of a different kind.

The *focus problem* was prompted by the extraordinary claim of the exceptional eighteenth century mathematician and scientist Simon de Laplace. In exhilarated response to the reality of the French Revolution and its promise of a new beginning, he declared he would create a new beginning for standard measures. He would divide the day into 10 h, the hour into 100 min, the minute into 100 s, temperature into 100° from the boiling to the freezing point of water, and divide the circle into

400 parts! (The interested reader may wish to read about Laplace in Stephen Jay Gould's delightful essay "The Celestial Mechanic and the Earthly Naturalist" in his *Dinosaur in a Haystack—Reflections in Natural History,* 1995.)

Laplace's celebratory impulse to re-quantify standard measures naturally suggested to some students to wonder about other common measures they had learned (Why is a foot 12 in.? A mile 5280 ft? In addition, the length of a meter, which was also created in recognition of the French Revolution). These instances provide a valuable lesson in students' coming to understand that "the way things are" is often the consequence of an individual's or group's efforts. As such, it may deserve to be greatly appreciated or be reconsidered, especially if a new way of thinking creates a better situation, such as having greater application or broader or deeper explanation or value. From such considerations students can appreciate how humanity's understanding grows. (The interested reader can see how metric time actually plays out at http://mathwithbaddrawings.com/2015/04/16/metric-time/.)

From Laplace's inspiration, the focus problem was "Is there any reason to think that a 400 unit circle would be better than a 360 unit circle?" What "better" meant and what it means when making assumptions or definitions in mathematics created ongoing conversations in the investigation.

Some students reasoned that a circle would still look like a circle even if it had 400 and not 360 divisions. And the area and its circumference would also remain as they are, disassociated as they are from any angle measure. A general conversation arose regarding how 400° could fit where 360° fit! This impossibility prompted the realization that the size of a degree would have to change. The new degree would have to be nine-tenths of the original degree! Would it still be called a "degree?" While there was some interest in inventing a new name, the students were informed one already existed, a "gradian." To express a measure in gradians, the superscript circle was replaced with a superscript g. That agreement allowed students to put their energies to consider whether dividing a circle by 360 or 400 parts would be the better choice. The focus on dividing prompted some students to consider factorizations, others divisors, while others wanted to explore some geometric consequences of using 400.

Those students considering factors found the prime factorization of 360 to yield 2, 2, 2, 3, 3, 5, while 400 had 2, 2, 2, 2, 5, 5. That 360 had three distinct prime factors while 400 had only two was counterintuitive to what some students thought would be the more "complex" number, as 400 was greater than 360 and both were even. That observation provided an opportunity for their gaining a more informed intuition. The factorization helped them appreciate that looking at things from different angles could yield deeper understanding (a truth they appreciated from studying history, though using the word "angle" in this circumstance caused a not insignificant amount of laughter).

Those students considering the divisors found for 360: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 18, and a bunch more; for 400: 1, 2, 4, 5, 8, 10, 16, 20, 25, 40, and more. Their investigations led them to the finding that 360 having more divisors was the more "complex" number in this context as well. Sharing their findings with the other group led them to wonder whether there was a connection—was there actually a quantitative relationship between the number of prime factors and the number of divisors? The teacher suggested that that investigation would be a good one to pursue after the focus problem was resolved. Such a suggestion was not an uncommon occurrence where the initial investigation prompted interest in more than one direction, but with collaboration a key concern, it seemed best to keep everyone focused on the same problem. (The tension between “the one and the many” often cast a shadow over the educational road to be taken.) Multiple investigations can make for an appreciated resolution, when teachers think appropriate.

After writing out the complete divisor listings, some students realized that it was not necessary to divide beyond a certain point. They saw that the factors of 360 included 18 and 20, and writing the divisors as pairs of factors made the process shorter. That is, there was no reason to find any divisors after 20 as they would be part of an earlier pair—for example, 25 would have been found when dividing by 16. The question of when they could stop looking for new divisors was answered when they examined carefully the list of divisors with 400. It was clear that once they had the divisor 20, the division resulted in 20 being the other factor, so then further factors would just match an earlier pair. For 360, which did not have a whole number square root, they came to see the last pair they would need to find would be the whole number closest to the square root of 360. They were very eager to share their finding with the entire class.

Meanwhile other students, apparently more visually inclined, saw that a 400^g circle meant a quarter-turn resulted in a 100^g rotation. That is, a “right” angle would be 100 not 90 were the circle to be partitioned into 400 parts. Some students saw that as a real improvement over the initial “odd choice” (created of course from the assumption of a 360° circle). So, if a right angle now has 100^g, what were some implications? For example, what would be the sum of the angles of a triangle? Some students conjectured that since a 360° circle is associated with a 180° triangle, then, since the forms are not changing, the “parallel” would be that a 400^g circle would be associated with a 200^g triangle. The teacher shared that thinking was an instance of argument by analogy, and that was an interesting topic for future investigation as well.

They tried to push further to try to prove that the sum of the angles of a triangle must be 200^g. Not having seen the 180° triangle proof that begins and ends with the masterly stroke of a line drawn through a vertex of the triangle parallel to an opposite side meant they were only able to consider special cases. For example, they saw that with a right angle now 100^g, a square would have 400^g. And as a diagonal divides the area of the square into two equal halves, that would include the angles as well. So, for isosceles right triangles, the sum of the angles would be 200^g. They agreed that would be the case no matter how small or large the square was since the angles would be the same. Whether they generalize to other triangles led some students to work alone to see what they could discover/create.

Some realized that if a right angle was 100^g, half would be 50^g, a quarter angle 25^g, and then get quite messy. But being able to create those angle measures by folding a right angle carefully in half and then in half again, they were able to construct a 75^g angle as well. Then they drew other triangles where one of the angles was 50^g and another was 75^g. The question was if the other angle was 75^g so that the triangle would have a sum of 200^g. They drew triangles with an angle of 125^g and an angle of 25^g as well, and then determined if the remaining angle was 50^g. Those collaborations led to the conjecture that *all* triangles have a sum of 200^g. It took a moment to recall that it all depended on if there would be 400^g in a circle. And a conversation led to the appreciation that a handful of demonstrations do not make for a *completely* convincing argument.

Other students made other connections. Some saw that *if* the sum of the angles of a triangle was 200^g, *then* an equilateral triangle would have really numerically messy angle measures ($200^g/3 = 66 \times 2/3^g$ each). Here the question of whether 400 was *really* a better choice for a circle was raised again. Given that the triangle is the building block of all figures with straight sides (as a convincing consequence of drawing figures and generalizing), with the "perfect" triangle having such arithmetically unpleasant angle measures, the role of aesthetics naturally arose. Comparison with the 180° equilateral triangle which has 60° angles naturally had much more appeal.

The conjecture was also made that inasmuch as a triangle has 200^g and a square 400^g, then a 5-sided figure would have 600^g, and "it keeps going." Those groups were asked to describe "it keeps going" in the language of mathematics. After a while they were convinced that the sum of the angles would be "200^g times the number of sides - 400^g." They were asked to see if they could make the statement in mathematical symbols and simplify it, and they did. They enjoyed their success, with their dedicated effort providing that emotionally satisfying moment.

Bringing individuals and groups back together made for a dynamic learning experience with a number of students and groups sharing what they found. Here the teacher had students revisit all the high points, including the interesting relationships, supporting evidence, helpful questions, and valuable habits of mind associated with the different journeys so that everyone gained a more informed education. The realization that being able to prove something in general is not an easy matter was seen as something that was true. (This suggested that a viable next topic for the class would be "proving"—considering that what mathematics statements they knew or could easily conjecture would serve as appropriate curriculum.)

With the further opportunity for pursuing their own investigations, some students chose to look into the gradian as a unit of measure while others went to find out more about analogies, and others pursued some of Laplace's other claims. For instance, they considered what Laplace's 10-h day and 100-min h would do to our time and daily schedules (make things "slower *and* faster!" and more complicated and wearing!). Interestingly the conversation turned to uncover the question why 60 min had come to constitute an hour, and a Google investigation led to find base 60 established by the Babylonians and Mesopotamians. Compare and contrast as valuable means for promoting learning was clearly in practice with students appreciating that 60 was a basic unit of angular measure while a basic unit of linear measure was 12. Some students appreciated learning about the *invention* of the

Table 6.1 In search of a relationship

n	Prime factorization	Divisors
12	2, 2, 3	1, 2, 3, 4, 6, 12
10	2, 5	1, 2, 5, 10
30	2, 3, 5	1, 2, 3, 5, 6, 10, 15, 30

meter—that 10,000,000 of them laid end to end was the distance from the North Pole along a meridian to the Equator through Paris! "Why do *that*?" (Time would tell.) Further researching provided some wonderful stories from history, including some competing explanations.

Others went back to seeing if there was a relationship between the prime factorization of a number and its number of divisors. *Make a table* being a time-honored problem-clarifying strategy led students to create the following (Table 6.1).

The first two cases led some students to immediately "see the rule!," but the third case meant they had to put their initial conjecture aside. (This gave the teacher the opportunity to mention the well-practiced "hasty generalization" that needs to be kept an eye on.) After further observation and reflection they came to the thinking that they did not see any pattern. They were then asked if they could make the problem simpler. What would that mean became the focus. They agreed they would choose numbers whose prime factorization could be written as a power of a single number (see Table 6.2).

These cases suggested the number of divisors was "always one more than the number of prime factors!" How could they know that relationship worked *all* the time? Time for thinking and imaginative observation. Someone saw why and excitedly explained it to the group who, after thinking about it, agreed. Now their attention turned to working with numbers with "mixed prime factors."

All involved appreciated the investigation—their own and each other's efforts and findings. They liked learning that the math facts from history included some measures that were the result of social agreement, while other measures came about from logical arguments that also required some basic agreements. Also of interest was how aesthetics plays a role in mathematics, including what is involved in proving. Those students who had worked on trying to uncover the relationship between the number of prime factors and the number of divisors did so, after a while of dedicated effort, and their efforts were roundly appreciated. A report on analogies came a few days later and also received wide appreciation.

Table 6.2 A more informative set

n	Prime factorization	Divisors
9	3, 3	1, 3, 9
27	3, 3, 3	1, 3, 9, 27
8	2, 2, 2	1, 2, 4, 8
16	2, 2, 2, 2	1, 2, 4, 8, 16

6.3 Mathematics of a Fountain Arc

Looking at something in more than one way, as exemplified by analogies, metaphors, competing paradigms in science, and antagonists in literature and history, naturally adds richness to our experience and in general serves to heighten one's thinking. The Fountain Arc investigation creates such an opportunity. The initial discussion is provided by considering quadratic equations representing parabolic curves and the heights reached as a function of time. The focus then turns to the parabolic curve that the common water fountain arc demonstrates, and the uncommon considerations needed to describe it mathematically. If the investigation is given full exploration it provides an engaging science, technology, engineering, and math (STEM) activity.

To help students understand the quadratic equation in the context of space and time, *taking things apart* is really valuable. For example, consider $y = -16t^2 + 40t + 3$. Considering the terms separately uncovers that there is the upward velocity of 40 ft/s for the projectile, which is initially at a height of 3 ft. off the ground that interacts with the downward accelerating force of gravity near the Earth's surface. If there was no gravity the object would travel 40 ft for every second in the air and after 10 s be 403 ft. above the ground. But it would never reach that height, of course, as gravity's downward force results in the object's time in the air to be less than 3 s since $16t^2 > 40t + 3$, for $t = 3$.

A side conversation that seemed worth including early on was to ensure that all students retained their familiarity with how to convert feet per second into miles per hour. That would give them a renewed opportunity to practice solving proportions, and since some of the high school audience already drove cars, and others were soon to, it would also serve to help them better appreciate the very concrete but abstract measure of "60 miles per hour" that many of them drove at. Finding that 60 miles per hour was actually equivalent to 88 ft/s not only allowed them to determine what 40 ft/s was in miles per hour, it also helped them realize that 60 miles per hour is moving really fast—pretty close to the height of a 9-story building in 1 s, or the length of a football field in little more than 3 s. Hopefully, they would gain greater respect for driving at such speeds.

Moving the subject back to the quadratic equation by considering different gravitational forces and other velocities as initial starting points (assisted by their graphics calculators) helped students become more aware of their impact on the shape of the parabolic arc. After students had gained a comfortable understanding, another parabolic arc was presented, this time one made by the path of a water arc in a fountain. Closer consideration made clear a different way of seeing the quadratic equation was required. The *focus problem* came into view.

Now the angle at which the water left the surface must somehow be accounted for, and the horizontal distance the water traveled so as to ensure that the water came down inside the fountain. Here was the same visual representation of a parabolic arc, only this time with more to consider, a lot more. How to somehow include these other parameters in the equation became the driving question. There seemed to be too much to put into one equation!

There are many sites on the Internet where students can see fountain arcs, including viewing them in motion. The variations are considerable and often dazzling. They can find water jets that shoot straight up and rise over 300 m (more than the height of a 90-story building), along with their initial velocities. Looking at the spectacle, students found themselves wondering how much water is in the air at any time; how long did the water that leaves the surface remain in the air, and a number of other interesting mathematical questions to be pursed in time. These connections promoted students seeing the problem as “real”—in their own terms—and so worth their time and effort.

They broke up into interest groups all sharing the same concern: How to *make the problem simpler*. They agreed the initial height could be set to 0 as the water arc begins its journey from the surface of the water in the fountain. (If the water was to come out of the mouth of a stone dolphin for example, that can be dealt with later.) How to make it simpler? Asked to *describe* clearly what they were trying to do led some to focus on the exit angle from the surface of the water and sketches that left out the effects of gravity. Visualizing, they saw that if the water left the surface at an angle A, then $(\tan A)x$ would be the height of the water x horizontal units from where it began, were gravity not a factor. But of course it was a factor, so some students established the equation $y = -16t^2 + (\tan A)x$. They realized they had to be able to express one of the variables in terms of the other. Time to think, *make a diagram, provide an example to make an idea concrete….*

Making a drawing really helped. In this way, some students could see that the right triangle used to represent $\tan A$ as y/x could have a hypotenuse represented by vt (see Fig. 6.1) as that would be the distance traveled free of the effect of gravity. That meant $\cos A = \frac{x}{vt}$, and solving for t in terms of x resulted in $t = \frac{x}{v(\cos A)}$. This *change of representation* was very much appreciated.

Their general equation could then be represented as a function of one variable with $y = -16(\frac{x}{v\cos A})^2 + (\tan A)x$! Students picked different values for the angle and the initial velocity, and graphed them on their calculators. Now there was the opportunity to design fountain arcs with systems of equations and entertain and be entertained by all sorts of interesting questions.

Meanwhile, other students also taken back by “all the variables” made *the problem simpler* by considering separately the horizontal component x in terms of t, and the vertical component y also in terms of t. They did this by seeing that if the

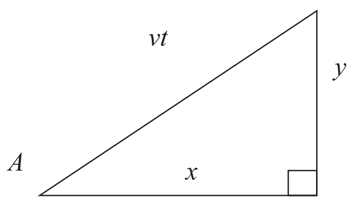


Fig. 6.1 In search of an understanding

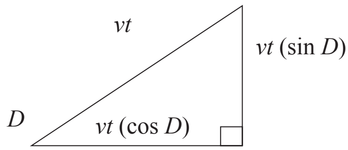


Fig. 6.2 Simplifying the complexity

hypotenuse is vt, then the horizontal component, x, as a function of time, could be represented by $x = vt(\cos D)$, and the vertical component $y = vt(\sin D)$ (see Fig. 6.2). But gravity naturally affected the height, so the equations they came up with were $x = (v\cos D)t$, and $y = -16t^2 + (v\sin D)t$. (The situation naturally prompted a short discussion of "parametric" equations.)

Now these students could pick initial values for the angle of the water as well the velocity at which it would leave the surface of the water's edge and determine the height and horizontal distance traveled. At this time, there was enough understanding to create fountain designs containing a number of water arcs. Some students considered how to make two arcs with different velocities reach the same height as a function of different exit angles. Others got involved in determining an equation where the horizontal distance traveled equaled the maximum height. Twice the maximum height. (They were playing with the equations in lieu of playing in the water and having a really good time.)

Students who came up with the height as a function of horizontal distance realized they could not use that formula to create water jets as the 90° angle was required and tan (90°) is undefined. Others who had found that $y = -16t^2 + (v\sin D)t$ could. They saw that when the measure of angle $D = 90°$, then, $y = -16t^2 + vt$. Reflecting on that equation, students appreciated what it said: distance up $= vt$ affected by gravity down, $-16t^2$. *Plausible* indeed. This gave them greater confidence in their equations.

In addition to creating a variety of quadratic equations to represent water arcs and water jets (the latter are known as "degenerate parabolas"), lots of other questions were played out. Students created their own designs with water arcs and jets at a variety of angles and velocities reaching different heights. Some of the mathematically ambitious students accounted for a horizontal and vertical shift in their equations, corresponding to where they wanted their fountain arcs to begin in the fountains they designed.

With the focus problem resolved, because of the heightened (sic) interest there were multiple centers for further investigation. Some students interested in engineering read up about pipe size and power requirements (e.g., at www.tryengineering.org/lessons/waterfountain.pdf); others made laminar nozzles after seeing them on YouTube.com, where water turbulence is so reduced the water leaving the nozzle appears as made of glass! Some students examining water jets determined how long the water would be in the air as a function of height $(t = 0.5\sqrt{h})$. Others determined the maximum horizontal distance for any angle A as $\dfrac{2v^2(\sin A)(\cos A)}{32}$. Then they found the maximum distance for any velocity would occur when $A = 45°$ (this finding came about from a number of directions, including experimenting, intuiting, and using the double angle formula for the sine).

More opportunities were explored and shared as well. Determining the equation for the water arc at one of the school fountains required careful measurements, of course, given the scale. And finding the angle of the water from the faucet was another challenge, resulting in the angle formula $A = \tan^{-1}(\frac{2h}{x})$. Other students with special capacities in drawing helped other students design fountains; while others interested in history researched fountain designs and uses. Because of the multiple considerations, when presentations were made, everyone had something to offer that was appreciated. (What is also to be appreciated is that a good number of the considerations listed on the coding sheet for promoting students' thinking were alive and well applied in this investigation.) With their mathematical understanding, they were even more attentive to fountain designs they found on the Internet.

6.4 Approximating the Area of a Simple Closed Curve

The last multiple-centers investigation involves students engaging the physical world, but more as a thought experiment. It provides another opportunity for students to appreciate coming at a mathematics problem from a number of different perspectives and to engage questions and propose solutions based on what is thought to be convincing evidence and ultimately make a decision if what is proposed is plausible. Going at this problem from a relatively naïve perspective (without recourse to calculus) made for a rich investigation.

The problem is raised by presenting data provided in geographic tables, such as with lakes or states. There we usually find the given dimensions of maximum width, maximum length, and area. A number of students expressed that they always wondered how an irregular form, like a lake or a state, had its area of square miles determined. Drawing a simple closed irregular curve on the board and embedding it in a rectangle (Fig. 6.3) made clear that the data given was not self-explanatory. That there should be a way to approximate the area of a simple closed curve, such as the shape of a lake or a town, with a better fitting rectangle created the opportunity for another multiple-centers investigation.

To gain a better approximation some students created right triangles in the corners of the rectangle whose areas they subtracted from the area of the rectangle. The

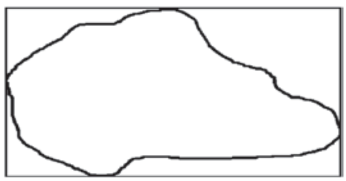


Fig. 6.3 Bounding a simple closed curve

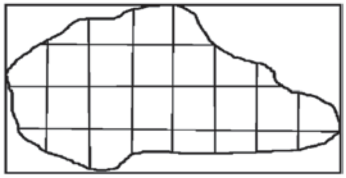


Fig. 6.4 Partitioning the simple closed curve

question arose how successful would that approach be with "really different" corner regions of the closed curve. Another group divided the region by drawing vertical and horizontal segments within the figure with the underlying goal of counting easily determined areas of square units (see Fig. 6.4). Naturally, there were irregular regions that remained to be accounted for here as well. Working carefully, these students came up with approximations they thought were good. Having both groups revisit the focus problem led to the realization that their approaches were at somewhat of a distance from the search for an "approximating" rectangle. However, their effort was not wasted—their estimate of the area would serve as a test of their final approximation formula. (It seemed fair to assume they better appreciated the notion of listening carefully.)

The search to *make the problem simpler* triggered the thought for some students to find an average, a representative value, amidst the otherwise random collection of lengths and widths. The decision of how many vertical and horizontal segments to draw was the result of their agreeing on what "looked right." Now they took a *guess:* The rectangle with average height and average length would provide a good approximation. A number of students thought that was a promising idea. Some students went off to work with scale drawings, others to consider a form whose area they could determine and use to test whether their conjecture of average height × average width yielded a good approximation. A trapezoid came to the minds of some students, while a circle did to others as the latter more closely represented the curvilinear shape of the region in question. All group efforts were directed at determining if the conjecture was *plausible,* and if they could establish a *proof.*

* * *

Students who had decided to use a circle realized that as it has axial symmetry there was no need to make a distinction between length and width, which meant determining an average length or width would serve as the average value in both dimensions. (Some students were concerned about using a form where both values were equal. They would need to decide whether to seek another approach or see what happens with their initial "special case.")

A radius of 8 cm and vertical chords 2 cm apart were chosen. Then, after drawing seven vertical segments, represented by the six chords and the diameter (Fig. 6.5), they agreed that measuring by hand was too inaccurate.

Others in the group realized they could use the Pythagorean relationship to determine the lengths of the chords exactly! A series of right triangles was created in a semicircle with the diameter, the hypotenuse, and the right angle formed on the circumference, as an angle on a semicircle would be a right angle if the chords creating the angle went to the opposite edges of the diameter. The perpendiculars

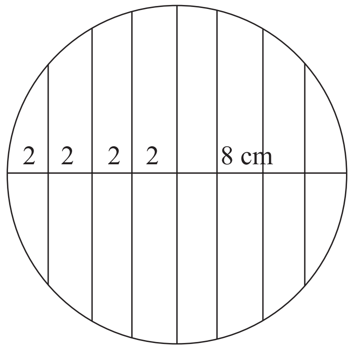


Fig. 6.5 Partitioning the circular region

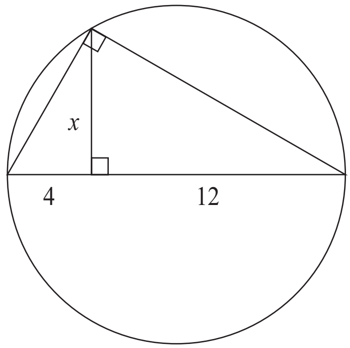


Fig. 6.6 A means for measuring chord lengths

drawn to the hypotenuse created similar triangles (Fig. 6.6). With the hypotenuse length 16 cm and the horizontal segments of lengths 6, 4, and 2 cm from the center vertical segment toward the edge of the circle, they determined the vertical lengths of the altitudes drawn to the hypotenuse of the three right triangles to be exactly $\sqrt{60}, \sqrt{48},$ and $\sqrt{28}$ cm. To find the total length they saw that the chords represented by the vertical segments were twice as long plus the length of the diameter of 16 cm.

The average of the sum of these seven segments $(2\sqrt{60}+2\sqrt{48}+2\sqrt{28}+16+2\sqrt{60}+2\sqrt{48}+2\sqrt{28})/7 = 13.69$ cm served as both the average length and average width of an "approximating" rectangle. This finding yielded an area of 187.49 cm^2. The question was how good an approximation it was in comparison with the area of the circle, which was $\pi(8)^2 \approx 201.06$ cm^2. Was 187.49 cm^2 close enough to 201.06 so that the conjecture would be kept? The relative error was $(201.06-187.49)/201.06=0.07$. Here students appreciated that to determine "how close is close enough?" drew upon their intuition—an informed judgement that included their feelings. More segments to create a better approximation seemed called for. They wanted to feel more convinced.

With the radius still 8 cm, instead of seven chords, they considered 15 chords by dividing the horizontal axis into 1 cm segments instead of two. Now the sum of the vertical segments including the diameter could be determined, accounting for the entire segments on both sides of the diameter as:

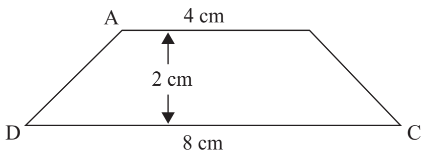


Fig. 6.7 Another variation on making the problem simpler

$4(\sqrt{63}+\sqrt{60}+\sqrt{55}+\sqrt{48}+\sqrt{39}+\sqrt{28}+\sqrt{15})+16=197.75$ cm. Since there were 15 measurements, the average height was approximately 13.18 cm, and so the area of the "approximating" rectangle yielded 173.80 cm^2. The relative error here was $(201.06-173.80)/201.06=0.14$! This value was twice the earlier error measure. It seemed the more data the more reason to claim the choice of the averaging rectangle was not a good fit. The conjecture was rejected. Time to regroup.

Those students who chose to use a trapezoid considered an isosceles trapezoid, *making the problem even simpler* for it made for fewer calculations. Some wondered if it was so specialized that it might not generate a finding that could be applied to irregular closed curves; others argued that it would, based on their airtight argument that "an area was an area." But all agreed if the approximation formula was not good in the simpler case, there would not be any reason to think it would work with a more irregular form.

Considering the isosceles trapezoid, ABCD (Fig. 6.7), where the top base is 4 cm, the bottom base is 8 cm, and the height is 2 cm, yielded an area of $1/2(2)(4+8)=12$ cm^2. Now they knew what the area was exactly.

To get an average height, they drew vertical segments 1 cm apart such that the sum of the seven vertical segments $(1+2+2+2+2+2+1)$ was found as 12 cm; the average height $=12/7$ cm. Some students intuitively thought that the average width would be 6 cm, the average of the two bases of 4 and 8 cm. Assuming the average width was 6 cm, the average width times the average height yielded the area of the "approximating" rectangle as $\frac{12}{7}(6)=10\,\frac{2}{7}\,\text{cm}^2$, which yielded a relative error of 0.14. Was that enough to reject the approximation formula? They agreed it was worth creating more vertical segments to get a better approximation. However, some students focused on how well the average height represented the set of vertical segments drawn to the bases. They saw that the average height could not be 2 cm no matter how many segments were chosen—all the vertical segments outside the inner rectangle of the trapezoid would be less than 2 cm! So they now believed more deeply that the initial conjecture could not be true. The students had come up with what they thought was definitely a *counterexample*. Now their problem was to try and come up with some alternative given the apparent dead end.

Students who had decided to work with scale drawings, unknowingly following the advice of Archimedes that careful drawings are very informative, also saw that the average height × average width generated an approximation that was too small.

All agreed: A different direction was needed.

* * *

Was all the work for naught? Clearly not. Much had been learned. Should the teacher have told the students earlier that the conjecture would not hold up? Hopefully not. Part of becoming a thoughtful and capable thinker is being able to withstand setbacks and have the resilience to bounce back from disappointments of misdirection. (Apply the discussion regarding the value of learning from failures here.) Inasmuch as life tends to present all of us with considerable "opportunities" for problem-solving, if students do not develop their capacity for resilience and persistence from their school mathematics problem-solving experiences, but actually the opposite, a great disservice will have been done to them and to society. For these students could well "enter" the larger world having developed an aversion to working on problems that are complicated and all that implies. Clearly, that is not in their or our best interests.

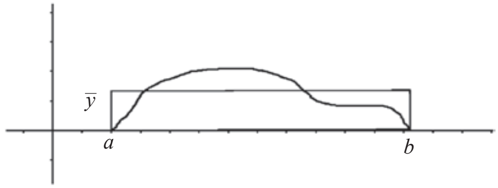
Besides, it is to be appreciated that there is learning when conjectures are found to not work. As noted earlier, students have added to their knowledge by eliminating some approach that seemed reasonable but was not. Here students can appreciate that an idea is at bottom a tool for decision-making—and when dealing with complex situations, we often discover that we need a different tool for the job than initially anticipated. That is to say, the reality of problem-solving experiences of coming up with models that do not work is to naturally be expected when engaging complex situations. If students are going to develop the mental flexibility to rethink their mathematics effort, which is essential when working on complicated problems, they need opportunities to have such experiences in mathematics class, and it needs to be appreciated rather than just taking from the experience that they were wrong. That is, that time is not "wasted." It is a critical part of their real education, which is informed by learning from one's mistakes in judgment as well as practice.

* * *

How to regroup and create further collaborative energy and focus needed to be thought through. New hypotheses were needed. Perhaps the best rectangle approximation was one found by finding three-quarters of the average length and average width. That did not "feel right." Was there anything they could learn from the formula for the trapezoid became the focus for some students. The standard formula of $\frac{h}{2}(b_1 + b_2)$—was saying $\frac{1}{2}$ the maximum height times the sum of the bases was the area. How did that relate to the area formula being sought? It did not. But with a *change of representation* the area formula could be re-presented as $h\frac{(b_1 + b_2)}{2}$, which "spoke" to them: The maximum height × average width would be the "approximating" rectangle formula! That this should be the best approximation to the curved form became their hypothesis. But, only *if* the average of the two bases actually did represent the average of all the widths. They became involved in trying to determine whether that was the case. That consideration was surely at the heart of their investigation. (The notion of a "lemma" was introduced.)

Those students who had come to think that the successful approximating rectangle was found by finding the average height and the maximum width of the closed shape wanted to share their findings. Some student looked at a triangle and realized that the formula 1/2 $b \times h$ was really a statement of average width (of the 0 length

Fig. 6.8 Gaining a visual understanding



at the top and the base length, b) and the maximum height, h. Though here too they realized that the assumption that the average of the "bases" needed more fleshing out why it "had to be."

Students who had been working with the circle had a curved region to test the hypothesis on. They multiplied the maximum width—the diameter's length of 16 cm—by the average length, 13.69 cm (with seven chord partitions), and 13.18 cm (with 15 chords), and found increasingly better area approximations of 219.04 and 210.88 cm^2 (with decreasing relative errors of 0.09 and 0.05)! They were eager to share their supporting evidence. Those who were using the trapezoid were also excited, as their "approximation" was exactly the area of their trapezoid. But the assumption of how to determine the average width needed more corroboration.

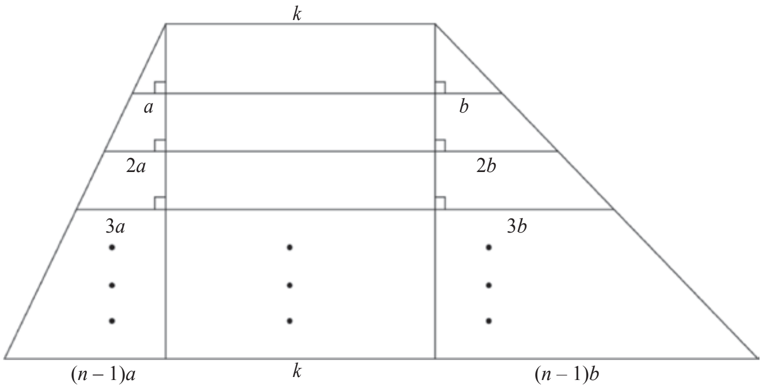
In calculus, students may learn that the average value of a function, $\overline{y}$, is equal to the area under the curve divided by the maximum width (Fig. 6.8). With the subject of the formula being the area under the curve, it is seen to equal the product of the average height, $\overline{y}$, and the maximum width—what the students had determined. Sharing this conclusion with the students was well appreciated given their dedicated efforts to uncovering it!

* * *

What deserved recognition was that the students did not just dismiss the initial conjecture after the first set of calculations, but sought further evidence for a stronger argument before being willing to accept or reject their conjecture. That is a mark—a high mark—of thoughtful engagement. That demonstration represents not only facts and techniques they tried and learned, but the commitment they made—ideas they tried, and the persistence and flexibility they brought to working with the complex situation. The solution finally arrived at came from a number of directions including careful drawing. It was the result of an extended dedicated, thoughtful, and collaborative effort that would seem to be what an educational experience is at its best.

Here are the investigations that followed:

1. Coming up with an argument that for any trapezoid (or triangle) the average of the bases is equal to the average of any number of horizontal segments drawn parallel to the given bases and at equal distances from each other had many students' attention. Drawing a set of parallel segments equidistant from each other and the bases along with dropping perpendiculars from the end points of the shorter base to the larger base provides the means to argue there is indeed an arithmetic sequence. With this argument, the claim that the average of all the segment lengths is equal to the average of the first and last segment (the bases) can be made. See Fig. 6.9.



$$S = k + k + (a + b) + k + 2(a + b) + \cdots + k + (n - 1)(a + b)$$

$$S = \frac{n\,(k + k + (n - 1)(a + b))}{2}$$

$$\frac{S}{n} = \frac{k + k + (n - 1)(a + b)}{2}$$

Fig. 6.9 Solving the needed argument

2. Some students who enjoyed geography used their approximating rectangle and the data of maximum dimensions of length and width and total area provided by Google to determine what the associated average widths and lengths would have to be.
3. A Google search to find out how areas of irregular closed forms are determined led some students to reading about a planimeter, though with no firm conclusions secured.
4. Students who worked on the circle approximations wanted to use the new "approximating" rectangle of maximum width × average length for any circle with radius *r*. The maximum width of a chord was of course the diameter. Since the area of the unit circle was π, it naturally followed that the average length of a chord was $\frac{\pi}{2}$, which was aesthetically pleasing. However, research on the Internet found the average length of a chord in a unit circle was found to be both $\frac{\pi}{2}$ and $\frac{4}{\pi}$! What is going on here? Is the second response really a typo that was to represent the length of a chord in a semicircle? As it turns out, the two answers are arrived at depending on how the chords are chosen. If the perpendicular chord is drawn through a point on the diameter, the average value formula yields the familiar value of $\frac{\pi}{2}$; but if the point is chosen on the circumference and the chord is drawn perpendicular to the diameter, the average value of the chord is the unfamiliar measure (cf. http://ocw.mit.edu). That there was a distinction was

truly counter-intuitive to say the least! An understanding would come with their formal study of calculus.

5. Some students wanted to use a variation of the new formula to determine the density of the human body by dividing their weight by their volume since they could use average width × maximum height—until they realized three dimensions were needed! A new question surfaced: Would the approximation be best if the third dimension was taken as an average or a maximum value, or something in between?
6. To test the new formula in another context the teacher offered that some students consider the enclosed region created by the intersection of $y=x$ and $y=x^2$, from $x=0$ to 1. How could they apply what they learned? Careful observation provided the maximum height—the maximum difference between the curves. This was corroborated by using calculators and finding the greatest value of $y=x-x^2$ by letting x take on values from 0 to 1 by increments of 0.10. Now the question of average width needed to be considered. A direct way of finding that would be to take the end points of the region created by the lines $y=0$ to 1, with increments of 0.10. For example, the left bound at $y=0.3$ falls on the line $y=x$, and so the left-most point is (0.3, 0.3). The right-most point when $y=0.3$ falls on the curve $y=x^2$ at (0.55, 0.3). This allows the width to be determined as 0.25 when $y=0.3$. With this procedure repeated for $y=0.1$ up to 0.9, there are now a set of widths from which the average width could be determined. Students were interested in seeing that as the number of widths increased, the closer the rectangle's area approached 1/6. They were delighted to be told that was exactly the area between the curves determined by calculus.
   Others realized that the area of the region under the line $y=x$ from 0 to 1 to the x-axis could be determined as that of a right triangle. Then they could determine what the area under the curve $y=x^2$ from 0 to 1 would have to be.

* * *

Multiple-centers investigations surely take more time than textbook presentations. However, if we believe society would prosper from students' intellectual, social, and psychological development, it would seem the experiential gains would be well worth the cost in time. In looking for potential focus problems, history and art can be of service especially for younger students. For example, history provides stories involving the creation of units of measure, the rationales for their invention, and the opportunity for comparative judgements. Why did the metric system not catch on in the USA given it is used all over the world? Why there would be such a unit of measure of a mile takes us to Rome, and their army, where a mile is associated with a *mille,* from the Latin, representing 1000. Art and design and architecture also draw (!) upon measures, such as ratios, with the golden rectangle, the rise to run of a step or the pitch of a roof, Le Corbusier's unit of measure, a figure on a screen seeming to walk into the distance, etc. Their connection to students' aesthetics, including the most visually pleasing ratio for a rectangle determined by a random sample of students, choice of the most functional step ratio requiring building materials and the question of cost affecting such decisions, etc.

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Part III
Promoting Mathematics Students' Psychological Development

The students are alive, and the purpose of education is to promote and guide their self-development.
Alfred North Whitehead

Introduction

Alfred North Whitehead was a mathematician and philosopher who had a deep interest in education. The quotation that introduces this section is from his *Aims of Education*. Students having the chance to share their thinking and ask questions in valuable interchange that promotes their development as thoughtful capable individuals Whitehead understood as what is at bottom of a real education.

To see a mathematics classroom where students are expressing their desire to understand by the questions they ask, offering conjectures, listening to other students' ideas, testing their arguments, stepping back to rethink the situation when something important is not clear, and then working alone and in groups and discussing their thinking…. What an exciting educational experience to see, especially if you are the mathematics teacher!

What if it is otherwise? What if the students are not comfortable with not knowing? What if they do not think carefully? What if they are fearful of offering their perspective? What if they just blurt out answers without thinking? What if they have a hard time focusing? What if their feelings are hurt when their ideas do not hold up? What if they continue to demonstrate counter-productive behaviors? What then?

Creating a learning environment that supports the cognitive, social, *and* psychological development of all the students in the class is a challenge indeed. What makes it especially complex is that they are not distinct domains. While students' insightful thinking makes breakthroughs, it is their emotional state that either promotes or limits the energy they bring to their learning experience. Their thinking, as our own, is surely impacted by whether they can be patient when dealing with perplexing issues or are inclined to make an impulsive decision to end the stress they feel. Being confident, they tend to find themselves able to take things on, including comments that challenge their ideas, and stay with the confusion longer; being insecure, they could be overwhelmed by almost anything they are asked to do, even a question that asks them to explain their thinking.

That is why the second part of Whitehead's sentence, "the purpose of education is to promote and guide their self-development," deserves attention. If students do not have a *realistic* confidence to draw upon in their mathematics classroom experience, there is little reason to believe they have the resilience, patience, and flexibility to focus so as to be effective mathematical thinkers. That is why teaching "struggling" mathematics students is a challenge. They lack much of any belief in their own capacities, and so have little energy to give to trying to affect the situation for the better. But not to make a sustained effort to help them transform that lack of agency into a positive force means their mathematics education could well be one they would want not to remember. Not only was it a waste of their time (spelled l-i-f-e) but the time spent only served to inform them of how incapable they were. Clearly, no one wants such an "education."

The authors of both the revised and the "new" taxonomy of educational objectives (Anderson et al. 2001; Marzano and Kendall 2007) well understand how intimately emotions are linked to thinking. This is corroborated by research (cf. de Sousa 1987) which finds that our reasoning is actually constructed to support and justify our feelings. Indeed, "Knowledge is a small cup of water floating in a sea of emotions" (Dewey quoted in Fishman and McCarthy 1998, p. 21).

This seems so much the case: it seems so much easier to listen to someone saying what we believe than what we do not; easier to see what we believe than not; easier to accept a statement said by a friend than a stranger, etc. Experience, regardless of whether it is outside of school or in the classroom, makes evident how truly important it is at times to deal with one's emotional state. Lack of comfortable control can well make impossible the opportunity to focus and think productively, and as a consequence, the outcome tends to be one of making poor decisions. Unfortunately, for many students in mathematics classes, this is exactly their ongoing experience.

The first chapter in this section takes as its focus how we can provide a psychologically supportive mathematics experience for all students, and shares some research findings in that regard. The companion chapter considers ways in which students can develop, with their mathematics teacher's assistance, their more capable selves as a consequence of acting on their personal curriculum. Inasmuch as the decisions mathematics teachers make establish the emotional climate, that is where we will start.

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Chapter 7
Providing a Supportive Mathematics Classroom

If all students are to come to the “belief in the value of diligence and in one’s own efficacy” (Ball 2003, p. 9), they need a classroom environment that helps them secure positive control over their actions and their emotions. Otherwise, there is little, if any, reason to think the mathematics classroom would be a satisfying place for them to be. Students who often have a hard time could well blame their mathematics teacher or the textbook as the source of the problem, or they may justify their not knowing by claiming/believing they are “one of those who can’t get math.” Leaving those students to always return to that disturbing emotional state and self-defeating belief when faced with the next mathematics problem would seem to ensure they will continue to have a poor mathematics experience and naturally think poorly of whatever and whoever was directly involved, perhaps including themselves. How we can help them negotiate the tension in a constructive manner is essential, for the confusion is indeed a veritable storm for some students in mathematics, making their gaining any sure footing impossible.

For those students, it does not seem that an emotional pep talk is going to change that reality. It is not enough to make their expectations positive in the absence of any real assistance that literally serves to deal with the difficulties. In the absence of having a personal sense of agency as gained from having a language for productive inquiry and a problem-clarifying strategy focus, struggling students are often left to develop confidence based on “procedure-based understanding rather than deep conceptual understanding,” which results in advances limited to further low-level mathematics courses (Buckley 2010, p. 1). Namely, such practice is “a bridge to nowhere.”

Students who have had to struggle with mathematics tend to lack the capacity to focus on mathematics as their emotional selves tend to focus on their lacking that very capacity. Mostly, any new problem can promote their having discomforting feelings once again as “What to do?” is often met with frozen silence and/or frantic considerations. Without additional commitment to enable all students to develop a realistic sense of themselves as being capable, mathematics educators and textbooks will likely continue to present and have students practice procedures to solve classes of problems in a misguided effort to claim those students learned mathemat-

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ics and had a comfortable experience. As discussed, this approach is of limited value and application, a short-term solution with a questionable consequence.

Yet, some students may well appreciate that approach. For the teacher, demonstrating procedures has in effect turned a problematic conceptual problem into a simple mechanical one, which is naturally seen by some students as good as it eliminates the need for thinking and, most significantly, the accompanying discomforting emotional experience. However, there does not seem to be any reason to think such a surface mathematics education will have any long-term positive effects other than perhaps some warm feelings held by some students toward some teachers. As the new Scholastic Aptitude Test (SAT) and Common Core State Standards (CCSS) report, the coming exams are given to evaluate students' mathematical understanding, not the capacity to demonstrate standard procedures.

Without emotional resilience, little is possible. Yet, the disturbance some students experience when dealing with mathematics begins much earlier than high school or when preparing for the SAT or American College Testing (ACT). We need to appreciate that students' negative feelings associated with school mathematics have been found with 5-year-olds and extends to perhaps half the students who study mathematics (Boaler 2012). Sian Beilock, a University of Chicago researcher, found as Jo Boaler that anxiety begins as early as first- and second-grade mathematics—and she found it affecting negatively about *half of the high-achieving* first and second graders ("Study: Math anxiety starts young", September 12, 2012, www.upi.com/Science_News, italics added).

These are disturbing findings both because of the emotional discomfort those students experience and because mathematics is an area of study acknowledged to be essential for functioning effectively in a technological society. What also needs to be appreciated is that "math anxiety is *only weakly related* to overall intelligence. Moreover, the small correlation between math anxiety and intelligence is probably inflated because IQ tests include quantitative items on which individuals with math anxiety perform more poorly than those without math anxiety" (Ashcraft 2002, p. 182; italics added).

Efforts to shed light on students' mathematics anxiety have been and continue to be very much the subject of educational research, for it is so pervasive. Indeed, it is the only school subject associated with a phobic reaction. Recent findings point to a genetic link, to a general anxiety that goes beyond a particular aversion to mathematics, that accounts for a portion of it. Yet, that research concludes that environmental factors play the greater role ("Math Anxiety is Linked to Genetics, Study Finds"; Curriculum Matters, blogs.edweek/curriculum/2014/04/study_math_anxiety_linked_to_g.html). In that direction, it has been found "that the more anxious their female teachers were about math, the more likely girls—but not boys—were to endorse gender-related stereotypes about math ability. In turn, the girls who echoed those stereotypical beliefs were performing less well than other students in math by year's end" (Sparks 2011).

We can compare that experience of not knowing and promoting tension with observing a realistically confident student engaging a challenging mathematics problem. Having a sense of "one's own efficacy," it is a positive stress that prompts further dedicated effort. But when a struggling student engages a mathematics problem, there is rarely any positive energy to draw upon, and the student soon abandons the effort having provided themselves some reason(s) why doing so was the right thing to do. Unfortunately, the felt disturbance of stressed students tends to promote a self-fulfilling resolution because of the physiological consequences. The growing body of research regarding how math stress literally affects the participant's brain in dealing with the disturbing situation makes evident why the behavior is what it is (Young et al. 2012; Beilock et al. 2010).

Researchers at the Numerical Cognition Laboratory in Ontario observed that "it's very much as though individuals with math anxiety use up the brainpower they need for the problem" (Sparks 2011). In effect, the working memory that would have otherwise been given to the problem is short-circuited by worrying about not being able to be successful with the problem! More completely, "neural circuits responsible for conscious self-control are highly vulnerable to even mild stress. When they shut down, primal impulses go unchecked and mental paralysis sets in" (Arnsten et al. 2012, p. 48).

Communication between people can be a complicated affair. In the mathematics classroom, it is likely even denser. There, "What a participant says…contains information about his understanding of the topic, his interpretation of the situation, his expectations of what the others might know, as well as his present emotional concerns" (Bauersfeld 1980, p. 34). However, students who are not comfortable with their own ability are not inclined to share their thinking or lack thereof. More fundamentally, to help them participate constructively in the classroom conversation—just to the point of trying to keep their focus—requires enabling them to have positive conversations with themselves. This is essential. Otherwise, their thoughts will likely continue to be given to habitual feelings of insecurity, mental escape, and failure.

We need to make classroom space secure enough and classroom conversations open enough to promote students' internal conversations regarding what they can do to think more effectively. In effect, help them develop different habits, productive habits with regard to how they deal with mathematics and how they deal with their emotions. Surely, it makes sense that students' "academic self-concept" is clearly established in the research as an essential element with respect to learning (Felson 1984; Marsh 1991).

In an earlier chapter, productive habits were partitioned into those that are valuable with regard to specific behaviors and those that are valuable more generally. Both are essential to gaining a disciplined freedom to engage problem situations in the mathematics classroom and in life. Practices such as listening carefully, having patience when working on challenging issues or problems, having resilience when an approach fails, being flexible, not wearing ourselves out with negative thinking, having the capacity to organize our thoughts, etc., are practices that we as mathematics educators and adults appreciate as actually essential habits and dispositions we all would want. So, it seems such considerations should be part of the mathematics classroom conversation if we are to affect change for the better, for it is exactly those practices that would be absent from the behaviors of mathematics students who tend to be stressed.

Students who feel insecure when presented with a mathematics problem that does not fit a mold they have practiced can find themselves immediately stating they cannot do it after just looking at the problem. That allows them to immediately exit their disturbing state. Or they could claim the answer is whatever comes into their frantic minds as a ready means to end the disturbance as soon as possible by finding momentary security in some haphazard claim. Or they could act out and express their disturbance so as to disrupt the class. Should we hope they grow out of such problematic, self-defeating practices? Should we penalize them?

Unfortunately, those students discomforted by their lack of a sense of efficacy could well feel compelled to various problematic short-term coping strategies that will alleviate their disturbance. That includes disrupting the class, which is an overt statement of disturbance that serves for the moment to relieve their personal stress by providing a way for "getting back" at the system. For another life-diminishing behavior, students who continually forget to bring pencils or notebooks to mathematics class have realized that a problem-solving strategy arrived at consciously or unconsciously eliminates the stress that using a pencil and book tend to promote. It also seems to be the case that the most stressed students tend to finish exams quickly, so they can leave as soon as possible. In this way, problematic behaviors become habituated as these students' internal conversations focus on the discomfort and disturbance their mathematics experience promotes. Some students develop phobic reactions and others may "solve the problem" by quitting school. This emotionally complicated and disturbing journey needs mathematics educators' and the school's attention. In the absence of their changing the self-defeating conversations they have with themselves, little good can be expected. Otherwise, there is no reason things will get better for a good number of students.

This is to say, the line between helping students' personal development and cognitive development cannot be seen as a border that cannot be crossed. Our emotions inform our thinking, and our thinking informs our emotions, as we know. It is an intricate engagement, and our assistance as mathematics educators could well make the difference between a student coming to think of themselves realistically as a more capable mathematics student and person or not. Having that perspective, hopefully, the productive habits of mind associated with mathematics and those that are generic to every learning situation would be discussed as content. Their presence as their absence plays a fundamental role in shaping the external and internal learning environments for the better or worse. Most importantly, students need to understand and appreciate that they cannot help what they think—stuff comes into their minds. What they can do is think about what they thought, and decide what they think about it, with the goal of acting to make a situation better. With that being the case, they and society naturally progress. But, of course, this is really difficult for students whose mathematics experience usually promotes stress and discomfort.

For little can be expected in the way of effective thinking in the absence of students having a "productive disposition." It underlies what is recognized as fundamental to "competent learning" as it "depend[s] on the way in which people approach, think about, and work with mathematical tools and ideas" (Ball 2003, p. 32). More explicitly, "…this focus…calls attention to aspects of mathematical proficiency that are often left implicit in instruction, going beyond knowledge and skills to include the habits, tools, dispositions, and routines that support competent mathematical activity" (Ball 2003, p. 11). Clearly, these aspects need to become more explicit, made subject for discussion. For a problem-solving approach (including avoidance) is intimately shaped (including distorted) by the individual's psychological disposition and their emotional sense of the challenging experience. It is the disquieting foundation upon which their behavior follows.

* * *

Surely, experienced mathematics educators do not need research findings to know that not intervening to help students with their learning difficulties will likely result in more problematic moments. It is especially clear how powerful our emotions are in affecting our thought and action and how difficult it is to change any habits, including those not in our own best interests. Yet, Sian Beilock's research has led her to think that "essentially, overcoming math anxiety appears to be less about what you know and more about convincing yourself to just buckle down and get to it" ("Ways to ease math anxiety studied", October 20, 2011, www.upi.com/Science_News).

What would help them "buckle down"? It would likely require considerable discipline and positive internal conversations. Perhaps developing "mindful practices" would help (cf. "Mindful Exercises Improve Kids' Math Scores", Mandy Oaklander, January 26, 2015, Time.com/3682311/mindfulness-math/). Many anxious students do try to "buckle down," but when they sit for an exam, stress often floods their thinking. What experience has proven helpful in that tension-filled "moment" is for them to talk to themselves constructively—to remind themselves that they have been doing the homework and classwork, taking notes, and asking questions in class. Indeed, they would best have that talk before the exam, and perhaps more than once. Focusing on what they have done that are the right things to do tends to lessen their anxiety. Their consciously reflecting on the concrete evidence of their positive effort in effect provides emotional support that can respond directly to the voice in their heads telling them "I can't do it." So, it is essential they are doing what they tell themselves as it is exactly the evidence needed to overcome that depressing thinking they experience when taking exams.

There is also research to support the positive effects of students reflecting at the end of class on what has been accomplished. Post-lesson reflection has been found to lead to "a stronger feeling of self-efficacy, which in turn leads to improved performance" ("Post-Lesson Reflection Boosts Learning", Ellen Wexler, June 6 2014, *Education Week* blog). This suggests that with the last few minutes of class given to such an activity, students who have little confidence in what they have come to know can actually see how substantial their learning has been. Making that practice a habit could have an enduring positive effect. It could well be valuable during that time to have students check off the big ideas in their notes and take a moment to ask themselves if they really know what those big ideas are about. It is especially helpful that they have an example or two that represent the big idea in action. Such reflective practices would likely make for valuable learning habits. And, of course, there are other environmental effects that can be promoted to help students be more mathematically able.

While there is research that points to girls being less successful in mathematics, research also suggests that in a "learning environment where students learned math through collaboration, working together to solve complex, multi-dimensional, open-ended problems, boys and girls performed equally well. Both boys and girls scored at higher levels than the students who had learned math traditionally" ("Expert: Change math classroom, not girls", April 9, 2012, www.upi.com/Health_News). That would suggest in mathematics classrooms where there are primarily whole-group discussions, it is important to be alert to whether there is a tacit, if not explicit, competition for who answers first. The emotional strain that puts on some students, especially females, makes their participation suffer, whereas working in groups tends to eliminate that heightened problematic competition.

Another classroom practice that has promoted more productive thinking is "wait time," a teaching technique associated with Mary Budd Rowe's 1974 study. She found that when teachers allowed students *3 s* rather than the usual *1 s* to answer a question, their thinking changed profoundly. So, it is easy to appreciate Dewey's observation that "The holding metaphorically, of a stop watch over students in a test or in class, exacting prompt and speedy response from them, is not conducive to building up a reflective habit of mind" (1916/1944). And there is physiological evidence to support that view: "research conducted at Stanford (regarding timed tests) reveal[ed] that math anxiety changes the structure and workings of the brain" (Boaler, "Timed Tests and the Development of Math Anxiety", *Education Week* blog, July 3, 2012). Given extra time, students are apparently able to relax more and, with feeling less strain, can naturally think more effectively.

This research in effect asks the mathematics teacher who is planning to give an exam to decide whether the object of giving the exam is to determine if students can demonstrate their mathematical understanding or if they can demonstrate it within some predetermined (institutional) unit of time chosen without regard to the students. Of course, how much time is enough time is a good question that seems best determined by the situation, including what is best for the student. Yet Beilock and Willingham's research (2014) led them to conclude that to help reduce mathematics students' anxiety, mathematics teachers should stop giving timed tests.

Also, they recommend teachers be careful when "consoling students who are struggling" so as not to validate "a student's opinion that he's not good at math" (Beilock and Willingham, "Reducing Math Anxiety: What Can Teachers Do?", *Education Week* blog, June 10 2014). Such consolation would naturally feed into their negative thinking—making not doing well an expected outcome. Carol Dweck's work on mindsets led her to understand that students who view intelligence as a fixed trait "do not view it as something that can be improved with effort" (2014). That is why it is really important for students to see intelligence as malleable so they understand that they can develop a more educated mathematical intelligence. (Most likely, they can recognize that their intelligence has grown with respect to their ability engaging electronic devices!)

One way to help students feel more in charge of their situation is that instead of presenting mathematics problems to students to solve, they get to choose the problem(s). This suggestion will be made again with regard to student homework, but in an effort for students to express their interests and their developing sense of self, providing a variety of settings for problem selection allows for problem posing as well. And rather than it just be from the set that accompanies the text or chapter, it could be more open for decision-making. Steve Brown (1994, p. 178) makes the following suggestions with regard to student selection:

- Anticipated difficulty of solution
- Relationship to problems already understood or solved
- Potential for the problem to open new territory
- Embeddedness within a particular branch of mathematics
- Potential for the problem to clarify what is not well understood
- Similarity of problem(s) to those already defined in the field

Such a list of problems could be in effect a standing offer as students develop confidence and capacity to feel increasingly comfortable choosing more challenging or diverse problems. In this way, they become very aware of their own agency, which, of course, is the most significant dimension of their psychological development.

There is also evidence that we can help students become more capable in mathematics by deciding what not to do. Eugene Geist, the author of *Children Are Born Mathematicians* (2001) "works with math teachers to create 'anxiety-free classrooms…' [He] advises teachers to have students focus on learning mathematical processes, rather than relying on the answer keys in the textbooks, which can undermine both their own and the teacher's confidence in their math skills" (Sparks 2011). This same understanding was shared with regard to girls working on challenging mathematics problems. Penelope Peterson reporting on some of her research writes, "focusing on the mathematical strategy rather than the answer was particularly important following a girl's high-level mathematics response and for encouraging girls' high-level achievement" (Peterson 1988, p. 11).

To be a successful thinker, as we all know, requires a "disciplined mind," one that in its best expression "…takes delight in the problematic, and cherishes it until a way out is found that approves itself upon examination" (Dewey 1929, p. 182). How we as mathematics educators can help promote that psychological experience for our students is clearly an aspiration worth our efforts. One path appears with the mathematics teacher modeling what resilience and flexible thinking look like by taking the time to demonstrate what can be done when faced with a dead end working on a mathematics problem. Students observing how one can step back and bring new energy to the effort by taking the problem apart or otherwise making the problem simpler so as to gain new direction gives them insight into the constructive emotional dimension of the problem-solving process. It is such modeling efforts that help students learn how to persevere—how to find alternative ways of making inroads and the resilience that are needed to promote productive efforts.

One way students can gain positive energy when they are working on a problem that has them stuck is to get up and take a walk. This will send more oxygen to their brains and so provide more energy for thinking. In addition, during the walk, their subconscious can work on the problem, and by taking their conscious mind off the problem, they are free to relax while their thinking goes on at a different level of awareness. The great mathematician and philosopher of science, Henri Poincaré, remaining stumped by a mathematics problem after a considerable period of deliberation decided to go off to the beach. He reported that as soon as he put his foot up on the bus step, he realized how to solve the problem. Of course, he brought a very positive energy to his holiday. This is to suggest that during exams, some students may well need to take a break, including a walk.

Before having students hand their exams in—and most especially those students who tend to want to get done quickly—suggest they take a walk so as to clear their head. Then, upon returning, they hopefully can look with fresh eyes at their work and, having gained some emotional distance, can be critically productive of their earlier effort. A caveat: the walk has to be considerable; experience has made clear if they return quickly, they just look "over" their work, not actually reading it critically, but in effect just nodding their way through their reviewing, in effect having a quick eye exam. What is sought is a "re-view," a rethinking, not a glossing over of the details. Having them determine how they know their answers are plausible often helps construct a rationale for rethinking.

Another way to help students secure some positive energy when stuck with a mathematics problem can be found in engaging a mental exercise: They can go to the past or the future to get some positive energy and bring it back to the present. After they stop laughing and thinking how strange their mathematics teacher is who shared the idea with them, they are ready to listen. Here is how it works: Have students close their eyes and think of some really positive experience they have had or hope to have. In focusing on that desired past or imagined future, they will literally feel positive energy. And that positive energy can be used in that very moment to regain resilience to engage a problem once again. Of course, as with learning any practice, it requires discipline to be done well. For some students may well want to stay in the pleasant mental place they just created and, as a consequence, be mentally absent from the classroom considerations!

Another action we as their mathematics teachers can take toward setting a positive psychological climate is to take seriously what day to give a quiz, test, or exam—for why should we give it when our students are not ready? We want them to demonstrate to both themselves and to us their mathematics teachers that they have come to a decent understanding, not that they have not. They can tell us the latter just by our asking, without our spending valuable time creating a formal albeit complicated means to have them communicate they have not, which only produces a poor score and likely poor feelings on both sides of the classroom desk. Hopefully, students would look at taking an exam as a "Question-and-Answer Opportunity" to demonstrate they have come to make good sense of what has been being discussed and worked with in mathematics class. For that to happen, students naturally need to be involved with the decision-making regarding when that time should be.

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This is all to suggest that students can well use more than a few comforting words, though these are naturally appreciated, to have a more productive mathematics experience. Being open to discussing how to be more emotionally productive as mathematics students is not an easy matter. Yet, their concerns shape the conversations they have together and with themselves and so actually constitute the *lived* curriculum. So, to promote a more efficient and life-enriching mathematics classroom, students need to be given time and assistance to develop themselves so that their emotions do not push them to tell themselves and others the results of ineffectual thinking. That is, they need the opportunity to develop a positive state of mind associated with productive intellectual effort. And that very significant concern deserves a place in the classroom curriculum.

If, for example, initially discomforted mathematics students can come to see and appreciate (and even laugh at) the statement that "confusion is the foothills before the mountain of understanding," then they may be able to relax a bit more, understanding that the confusion is not *their* problem but a sign that there are things needing clarification—that it is not that they are not capable but that *situations are confusing*. Promoting that belief, and saying it on a number of occasions, and in a number of ways, could help them focus more productively. After all, in the best interests of a pluralistic society, we would want them to be "self-directed" learners so that "when confronted with complex and sometimes ambiguous and intellectually challenging tasks, [they would] exhibit the dispositions and habits of mind required to be self-managing, self-monitoring, and self-modifying" (Costa and Kallick 2004, p. 51).

In support of that developing awareness, students may well appreciate learning the difference between *assimilating* and *accommodating* new knowledge. As the cognitive psychologist Jean Piaget (1952) made clear, there is a categorical distinction that is worth being made regarding our receptivity to new experiences. Namely, when it agrees with our basic understanding or does not challenge our thinking, it is relatively easy to *assimilate* some information or technique, etc. For example, when students come to see how matrices can be used to solve a system of two equations in two unknowns, they "get" (assimilate) quite readily how the procedure can be extended to n equations in n unknowns. However, to *accommodate* something new means we have to make adjustments to our thinking, which is not an experience as fluid as assimilating something new. So, for example, after mentally *accommodating* the logic that underlies the procedure to bisect a line segment with a compass, students can more likely *assimilate* the practice of bisecting an angle with a compass.

This is to say when students report that something they are learning is "hard," what they mean is they are having a difficult time accommodating the material. To give students intellectual and emotional support can require another way of explaining the at present confusing concept/practice as what is apparently abstract needs to be made more concrete (as whatever is abstract becomes concrete with understanding) . To provide emotional support in their developing the capacity to hang in there so as to accommodate challenging mathematical ideas, we can share with them that doing so can be quite a challenge. Some truly great mathematicians (Descartes, Pascal, and Euler) found it really difficult to accommodate negative numbers into their mathematical thinking. Descartes worked to eliminate them if possible from any equations; "Pascal regarded the subtraction of 4 from 0 as utter nonsense" (Kline 1972, p. 252); and Euler put negative numbers after positive infinity—so otherworldly did he find them.

To secure a new way of understanding is not an easy transition to make when what is required is that we change the way we have thought about things or are faced with something quite different from anything we have considered prior. And students who are being asked in effect to change their own beliefs regarding the difficult learning experiences in mathematics they have been having is really difficult for them to accommodate. That is why we need to help them develop their heuristic and reflective thinking as both are invaluable in becoming more flexible and productive in varied settings, where they can actually see some positive development. As said earlier, for as every mathematics educator knows there are things worth repeating, we need to make room for conversations regarding the role emotions play in thinking so all mathematics students can develop practices that help them cope more productively.

The assimilation and accommodation of productive habits that yield constructive means to deal with perplexity requires work and time but of course is worth the effort. For securing a valuable "habit means the formation of intellectual and emotional disposition as well as an increase in ease, economy, and efficiency of actions" (Dewey 1916/1944, p. 48). That formation process ultimately requires being resilient in the face of setbacks inasmuch as "the acquiring of habits is due to… our ability to vary responses till we find an appropriate and efficient way of acting" (Dewey 1916/1944, p. 49). And this goal lies at the heart of what the National Council of Teachers of Mathematics (NCTM) sees as student academic goals: "adaptive reasoning, strategic competence, and productive disposition."

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The philosopher of education Maxine Greene urged those of us who are educators to have a conception of education "…as a process of releasing persons to become different, of provoking persons to repair lacks and to take action to create themselves" (1988, p. 22). This seems both eminently reasonable and practically essential for the collaborative effort that is a productive democratic society. How we go about doing that is clearly *not* a minor element of our work as mathematics educators. The process of inquiry with its emotional dimension must be given its just due; after all, it shapes our mathematics classroom experience. Enabling students to mentally step outside their experience to evaluate it, including their own thinking and feeling, gives students what could be the needed distance to look at themselves in a more productive light. The next chapter considers how students can focus on developing their more capable mathematical selves by pursuing a personal/professional goal of their choosing and having the support and commitment of their mathematics teacher in their working to secure it.

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Chapter 8
Including Students' Goals

The premise of this book is that the aim of education being the development of thoughtful, socially responsive human beings who have an appreciative understanding of what is required to do things well should inform and shape students' mathematics experience. Namely, the way mathematics is made available should enable students to grow as self-reflective, cooperative, and resilient individuals.

How to promote the third dimension further is explored in this chapter. It continues the conversation with the last where the focus was on how mathematics teachers can help students develop their more capable selves by providing a psychologically supportive environment. Now that thinking is expanded to consider students' personal/professional goals as part of the curriculum in their mathematics experience. Enabling students to reflect on how things are going with a constructive perspective is of course essential. To give that effort direction students need to uncover what goal(s) they have. With those in mind, there is the opportunity for discussions regarding what would be effective means for their becoming more the person/student they want to be by being more able to deal with challenging situations, and most especially their own behavior.

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What is becoming increasingly clear is that "The best predictor of kids' academic success is … how willing they are to persist at challenging tasks and how well they plan ahead, pay attention, remember and follow instructions, and control their impulses and emotions. These so-called executive functions, also known collectively as self-regulation or self-control, have long been considered a key life skill" (http://connect.oregonlive.com/staff/awang/index.html). And there is no reason to think that such a perspective does not extend to students of adolescent age and the mathematics students in our classes.

How seriously students take their personal/professional development would seem a function of how seriously we, their teachers, do. In that direction, they would need to appreciate that they, as all of us, are malleable material, something that can be worked with. That it is true can be made clear to them when they consider the fundamental role habits have played and do play in their lives. Not only did habits

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make possible their being able to read, write, and walk, they shape how we think, do things, and feel (Dewey 1922, p. 25). With that appreciative realization, there is the opportunity for students to ask themselves what new habit(s) they want to secure, or what old habit(s) they want to discard, so as to make their mathematics experience more productive—toward their becoming more capable individuals. In the absence of that awareness, and the opportunity to work at securing/omitting specific practices, it would seem there is little chance for them to become more productive mathematics students.

It is quite clear to any experienced mathematics educator that helping students think and act more intelligently when engaging mathematics extends beyond the practice and application of mathematical techniques. For example, determining what the problem is actually asking (Polya's "understanding the problem") and sustaining disbelief (i.e., being comfortable playing out ideas) are essential considerations. In addition, being patient analyzing a situation and not letting errors in judgement or execution trigger problematic behavior, etc. are also quite essential for being successful mathematics students. Helping students develop these intellectual/emotional capacities would seem to be part of our work as mathematics educators, inasmuch we establish the classroom environment. We determine "whether social conditions obstruct the development of judgment and insight or effectively promote it" (Dewey in Bernstein 1966, p. 285).

Discussing these practices and other productive actions in class help students get a practical sense of what they could do toward having more rewarding present and future experiences. To make that valuable development actually happen, students need time and opportunity on a regular basis to think about their behavior, including how they speak to themselves when engaging a mathematics problem. For making changes in one's practices including eliminating some problematic mental or emotional practice takes considerable mental and emotional effort to accommodate, of course. This suggests the initial presentation by mathematics educators regarding the opportunity for such development needs to acknowledge how challenging such an endeavor could be in the present along with how very worthwhile it is for the future of their lives.

In that direction, students also need to appreciate that a goal without an awareness of what would be signposts of progress would likely have as much consequence as making a wish. Closing one's eyes and having it come true is a rare event usually associated with special days of the year. Wishing precludes the volition of the individual, and so it is not generally a good problem-solving strategy. For example, telling students to "be patient" may work, but likely not if it is not clear to them what signs of their being patient explicitly look like.

Signs of progress are most important as they are concrete expressions that thought and action are coming together. Until those signpost behaviors can be made clear and integrated into their practice, it is difficult to imagine gains would be made. So, with regard to the student wanting to become more patient when working on mathematics problems, suggesting they read the problem more than once, make clear to themselves what is given and what is to be determined, and perhaps underline the critically important information, along with taking time to play out ideas and test their conclusion for plausibility, would help them be more aware of what would be concrete evidence of their becoming more patient. And those explicit practices can well serve as a guide and gauge for them and their mathematics teacher as to how things are developing.

Yet, even if students are open to change, the effort to develop a new habit or discard an old one is of course quite challenging. Accommodating new behaviors into one's life naturally makes life more complicated at the time; so, the effort can be experienced as discomforting enough to convince them to stop. Students need to appreciate that it takes time and effort to learn any new habit, such as tying one's shoes, driving a car, learning how to successfully *make a problem simpler,* or deciding exactly how to *tinker* for best results. Also, it takes time and effort to rid oneself of a bad habit. So they need to understand the discomfort, and seeming unnaturalness will start to fade as new productive actions begin to appear, as mind and body once again move to unify. As well recognized, with commitment, effort, and time, the dance and dancer become one.

To help support them in sustaining their effort, they need to be reminded most likely more than once that once we get a good habit, it is ours forever, and once we replace a bad habit, we have made ourselves better forever. Indeed, it will likely take hours of efforts to accommodate a valuable change, but then it is theirs for decades to come. At bottom, such considerations are essential for our students living more productive lives, being more valued members of society, and their having a more successful mathematics experience.

It is here we find the opportunity for assessment for the public good. With students developing productive behaviors and valuable attitudes and dispositions as a consequence of their learning mathematics, the public is clearly "getting their money's worth." It represents students' personal/professional developing awareness of what they could do to make a situation more productive by their thinking and acting more constructively. And the mathematics classroom, where imagination, experimentation, and reflection are hallmarks of a successful effort, is a perfect setting for that development.

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Relatively younger students learn how to be "change agents" with changing things outside and around themselves. They have naturally done so many times just growing up. In mathematics class, it could be with regard to deciding where to stack graph paper, whether to work in groups or not that day, to go over a homework question or not, or what to do to test some hypothesis, etc. With those actions, a consequence of their imagination, experimentation, and reflection, not impulse as a consequence of their disturbance, they are likely to make valuable decisions and result in the participants feeling good about themselves, and rightfully so.

Developing productive behaviors, such as being relatively more resilient when dealing with perplexing situations, or being more open to other's thinking, are not minor attributes. These are practices we would associate with capable, responsible individuals, and the possessor of those behaviors would be in a relatively ideal position to gain a valuable education and be successful in general. Making those qualities part of the mathematics classroom conversation helps students reflect on what practices could significantly affect their experience for the better. In that direction, conversations along with writing activities could help uncover what each student thinks their personal/professional concerns (curriculum) are about. Enabling them to get in touch with what really concerns them, what they really want to be able to do as mathematics students and developing young people, deserves our attention. Not to incorporate these concerns into their mathematics learning experience, to not see their personal/professional development as an essential part of their mathematics experience, would seem to be a major omission in school practice, at least from the perspective of the student and society.

After all, if we want to have a populace capable of thoughtfully evaluating how things are going, and being imaginative in an effort to find comprehensive solutions, and supportive enough to make things better, surely school experience ought to promote that development. Being aware of hasty decision-making or adversarial thinking or lacking a sense of self enough to come to a thoughtful decision ought to be considered with regard to each student's mathematics education. Also, being open to listening carefully including to contrary points of view and being comfortable with changing one's mind are behaviors worth having and discussing. To help students develop those personal and social capacities is not an easy matter (as a fortune cookie attests, "it is more difficult to judge oneself than to judge others"). Promoting consideration of self-evaluation and self-development seems to be the best way to ensure that students become more reflective and more in control of their own thought and behavior, and so more capable of creating desired habits—that is, the mathematics student and person they want to be.

What mathematics educators can do to promote that happening would seem to require a supportive environment, of course. Enabling students to focus on their development by helping them create a self-assessment tool that they could use to evaluate and reflect upon how things are going could provide instrumental means. Otherwise, we can likely expect that whatever poor habits students have when they enter our mathematics class regarding their thinking and acting and emotional responding will most likely be what we will see when they graduate, and so on as adults. As such, that developmental absence represents a great loss of opportunity and productive decision-making for them and for all of us.

Just as an artist steps back and decides what would be needed to make their effort more satisfying, students can take seriously their developing artistry in shaping their mathematics experience. To support and codify that effort beyond providing emotional support, the teacher can have the student write down three behaviors that would serve as signposts, demonstrations that would be signs they are making progress toward a new practice they wish to gain or an old practice they want to rid themselves of. This gives them a personal/professional checklist to keep in mind, something concrete to hold onto that provides both support and perspective. It is especially important for those students who are trying to change some counterproductive behavior they have developed in response to their stressful experiences in mathematics classrooms. For example, students who bring to their experience old feelings of discomfort when engaging new mathematics problems can hear themselves saying quite quickly "I can't do it," as that gives them license to stop making an effort. The reward is ending the stressful experience. But that is surely not what they or we as their mathematics teachers would hope would actually be the desired outcome. So having those signpost behaviors close–by, e. g., where they are studying can serve as gentle reminders to keep their eyes on the prize.

Becoming more of who they want to be requires of course both the mathematics teacher giving airtime to such discussions and students being in the right mindset. This suggests that part of our work as mathematics educators would be to meet with each of our students from time to time to hear how their personal/professional development is going. Doing so tells them that the school values and supports their effort. In this way, students have a respected authority on their side in their pursuit of a habit they want to have or change. And the more they can see or be helped to see signs of their development, the better. Of course, not as a grade on a report, but as a quality that can be worked with and the effort appreciated.

In the latter direction, it makes good sense that their efforts are included on evaluations. In this way, those efforts are acknowledged as being important, which of course they are—essential, actually. That suggests collecting student evaluations of how they see their development progressing. This provides teachers with additional information and helps students not lose hold of their own concerns amidst the obligations of all their school and other responsibilities.

Of course, the longer we wait as part of students' mathematics education to discuss how valuable it is for them to look at how things are going and to decide what they would want to happen, the more difficult naturally it is for valuable change to occur. The thinking would likely be "Since I made it this far doing what I do, why do I need to introduce or stop anything now?" (Even we mathematics teachers could feel that way.) However, with a thoughtful, aware, and sensitive mathematics teacher, and a supportive environment that extends across years, the educational journey can be significantly rewarding for everyone involved, including the students and their mathematics teachers as well as students' families, school administrators, and society. Helping students develop those mental, emotional, and physical habits would seem to be what Einstein had in mind in reflecting on what education could provide at its best.

One classroom practice that has proven valuable in this direction is to begin with a handout (adjusted of course for the developmental age of the students) in the first days of the school year. Here is one that is shared with high school mathematics students.

Welcome!

I'd appreciate your letting me know what I can do to make our mathematics experience valuable to you. Feel free to share in writing what you've experienced in mathematics classes that you appreciated and what you didn't. All those thoughts can help shape what you want to happen this year.

Also, reflecting on what you want to happen for yourself this year can be really important. After all, we can think of ourselves as artists of our experience. This allows us to step back and ask ourselves what our experience needs to be more fulfilling, and then work at making it happen. For example, do we want to become more patient engaging mathematics problems so we have more resilience to keep going? Do we want to become less stressed on exams so we can think more successfully? Or do we want to experiment more or become more playful in our thinking when we don't know how to progress with a mathematics problem? Each and all of these developments can make our mathematics experience and life a lot more rewarding. And we can surely expect our efforts at change to be awkward early on; but like learning to walk, once we have it, once we develop a new and better practice, it is ours forever. Making any such positive change makes life, including school, easier, more productive, and more enjoyable. And mathematics class is a perfect place to make that happen.

The list that follows contains practices that are valuable to have and others worth replacing so as to be more successful in mathematics, school in general, and beyond. To consider how you might have a more personally valuable educational year, put a check mark next to any that seem worth your pursuing seriously. After you take some time to think about the ones you've checked, go back and *circle the one* you feel most strongly about or most eager to begin with. You'll know objectively: Your emotions will tell you which is the most important. Even if you find yourself checking off more than one (some students check them all!), realistically we can work with one at a time for best effect.

As the Artist of Your Educational Experience, What Do You Want to Make Happen?

____ read more effectively	____ think more productively
____ relax more during exams	____ get more out of homework
____ make better reasoned arguments	____deal with confusion better
____ prepare better for exams	____ be more open to others' ideas
____ take better notes in class	____ participate more in class
____ eliminate "careless" errors	____ be more patient
____ develop more persistence	____ observe more carefully
____ be less critical of oneself and/or others	____ ____________________ (other)

Now that you picked one, it needs to be given the attention it deserves. We can all appreciate it takes real effort to change, as anything worthwhile tends to take work and dedication. Here are three observations that you might find really helpful in your effort. The first reminds us that "The self is not something ready-made, but something in continuous formation through choice of action." So while we may think that the way we do things is "written in stone," it's not; it's up to us to look at how we do things and decide if there is something we should do that would help us make things better. The second, as how best to proceed, you would do well to consider: "The nearest way to glory—a shortcut as it were—is to strive to be what you wish to be thought to be." That's an interesting notion: First decide how you would want others and yourself to think about you as a person, and then do those things so that it actually happens. And, even if we're not musicians, we can appreciate that "Sometimes you have to play a long time before you sound like yourself." That's to say, we are developing creatures, and that development toward doing things better takes time and effort, of course.

At our next class, we'll take this opportunity further, toward making your mathematics experience more personally/professionally rewarding. So bring your reflections on your earlier mathematics experience, and read the list carefully and see where your interests really lie.

After all, this isn't just another school year coming up, it's a year of your life, and time is not a renewable resource. Hopefully together we can make learning mathematics really valuable and special this year.

Such considerations of course take dedicated focus, energy, and time, and such reflections are not easy. Looking at what we do not like is not comforting. But as the great architect Frank Lloyd Wright noted, doing so we get more in touch with what we do appreciate. So to help our students take seriously what they are asking of themselves requires assistance, which includes providing time for reflection, discussion, and support. Expressing concern and interest individually and collectively regarding how things are going demonstrates to students that their mathematics teacher takes their personal/professional development seriously. Without that concerted focus, after a while students get caught up being students, namely, doing what is being asked of them by others. In this taken-for-granted way, they lose sight of what they were asking of themselves, and how very important it was.

To help students focus on their goal, to have it become an explicit part of their personal/professional development project, they could write two copies of that goal with three associated behaviors they believe represent progress (signposts) in that direction. Using 5×8 index cards are durable and provide a sufficient amount of room. If they give one copy to the mathematics teacher, their effort can continue to be supported. They can put their copy somewhere at home or in their locker where they will see it often, so it can serve as a gentle reminder to help sustain their effort. What is critically important is making sure that the behaviors they view as being signs of progress are literal enough to be recognized.

The *Assessment Standards for School Mathematics* (NCTM 1995) makes clear that student assessment of their own mathematical efforts is really helpful to their development. Hopefully students, especially those who struggle with mathematics, take it seriously. If they do not make an effort at some concrete change, it is pretty certain what their experience will be. To continue to be of help, mathematics teachers need to be in touch to ask their students if they wish to discuss any aspects of the challenging opportunity, or just to share how things are going. Having them write about their efforts from time to time may help as well as gentle reminders; keeping a journal would likely promote reflection and evaluation.

During the first few weeks of the school year it is especially valuable to provide quasi-regular "advertisements" expressing the hope that they are taking *their* personal/professional development as mathematics students seriously. Giving it—that is, themselves—the attention they deserve is a really good idea. Reminding students that it takes a lot of energy to get a rocket ship off the ground, but once it does, it really moves and that we as their mathematics teachers are available to discuss ways in which we can help makes clear that additional energy may be required to get things going and that it is available.

As we can appreciate, this is not an easy activity to keep alive in the mathematics classroom as "real" learning usually means focusing all their attention on the subject matter content. Over the course of the year, it is to be expected that some students will give up on their development effort, while others will give it sporadic attention, while some will continue to take their curriculum seriously. Hopefully, they all appreciate that the ultimate value in such an effort means they can make their lives, in school and out, more productive and enjoyable. The social and political ramifications suggest themselves: A more thoughtful populace open to and capable of dealing with difficult issues would be a more realistic goal. Toward promoting that occurrence, the degree of effort is the essential assessment. And it is surely no easy matter; it takes continuing attention. Hopefully they will come to believe "The capability and willingness to assess their own progress and learning is one of the greatest gifts students can develop" (Stenmark 1989). It needs to be truly appreciated by them and us, their mathematics teachers.

The student assessment that contains the prior quotation includes a self-assessment questionnaire that is used as a follow-up to a mathematics activity or project. Here, students have the opportunity to reflect on "how well the group functioned and how well the student participated." The student is presented with fill-ins; for example, "What could you have done to make your group work better?", and extends to promoting questions for journal writing, such as "What I still don't understand is ________." In this way, students naturally become more reflective about their learning mathematics, their participation in groups, and how things are going with regard to their personal/professional development as a successful student.

To get feedback from students regarding their experiences, goals, and concerns associated with learning mathematics seems to make good sense. After all, it is their education that we are involved in, helping create with them. While their view can be considered naïve, there is much to learn from such a perspective. As noted earlier, it is the asking of such questions that often get to the heart of matters, including questioning statements, arguments, and decisions (including those mathematical) that seemed quite certain but upon further reflection lose some of their firmness. Asking students what they enjoyed learning the prior year and why, and what was less well received, for example, could well provide valuable insight for the mathematics considerations to follow. Not only does asking them about their prior mathematics learning experience make clear to them that you are interested in what they think and have experienced, but also helps them uncover what they believe is important in their learning experience.

Such opportunity for reflection would help them become more aware of their growing personal, social, and intellectual development and could give further clarity to specific behaviors associated with their goals. There would be much to talk and think about. That information, as real or imagined or one-sided as it may be, informs you and them of their expectations and concerns for the coming school year, and provides valuable information to help shape the collective opportunity. After all, it is a year of everyone's life. And that deserves careful consideration and spirited dedication to be given its just due.

Yet, of course, more can be looked at to gain a more complete sense of how students and the mathematics classroom experience could be more successful. Considerations of grading, homework, and classroom observations in the following chapters will fill out the picture.

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Part IV
Assessing Students' Mathematics Experience

Assessment for learning is any assessment for which the first priority in its design and practice is to serve the purpose of promoting pupils' learning.
Paul Black

Introduction

The assessments we make are natural consequences of being alive. They are expressions of what we value—our concerns and interests. If they are important enough, they point to where we should put our energies, and what we think of the consequences of whether we did or did not. What follows is not "the latest" on assessment of classroom practices; that apparently is being written pretty much nonstop. It is not the first time that assessment for the public good is being promoted. Surely, it goes back to the earliest of times in the development of our species, where the education of the young was directed at learning what was needed to enable the continuance and success of the group. And today, it is not hard to imagine the elders naturally want the young to be thoughtful and act with the best of intentions and practices that would promote the further well-being of the community. So, with a pluralistic society's center for the general transmission of its culture and values being the school, it would be expected the formal education experience would reflect those concerns. After all, making explicit what is valued creates the foundation of our efforts, and coming to understand how things can be improved makes securing what we value all the more possible.

* * *

That mathematics teachers would give their energies to reflecting on pretty much every aspect of the classroom experience they are overseeing, including their own practice, and make additions and subtractions, and yes, multiplications and divisions, based on their assessments comes with the territory for every mathematics educator dedicated to establishing a learning environment that impacts students' lives for the better.

In addition to the mathematics teacher's personal/professional reflections of their effort, there are many student-assessment frameworks, including standards-based, performance-based, and "authentic" assessment. What will be considered here is referred to as "socially responsible" assessment. While the latter description may well be thought to contain all those described prior, the more generic assessment includes the eye tests we give youngsters, so we do not confuse their capacity to read with their capacity to see. They are the homework given to help students gain further capacity and clarity, along with the mathematics exams they take when they say they are ready. It also includes the write-ups that acknowledge gains students have made in their personal/professional development as students, and where it would be good for them to put their energies further. All in all, "socially responsible" assessments as discussed here are judgments regarding what makes sense for schools and both mathematics students and teachers to do to further the student's psychological/ emotional, social, and cognitive development.

In that direction, earlier chapters included discussion of an assessment inventory of teacher classroom questions that promote students coming to think more ably, including the extent to which mathematical habits of mind are part of the conversation. Also, a student self-assessment inventory regarding concerns they have with regard to their own development was given focus in the prior chapter, for doing so, we acknowledge their concerns and help them take their development into their own hands. How we could more completely assess the educational experience with the goal being our mathematics students' further development as thoughtful, cooperative, and confident human beings would seem we consider the role that grading, homework, and classroom interaction play in that effort.

All in all, if our assessments, in whatever expression, do not support the development of mathematics students' thought and behavior from the perspective of securing a more capable, cohesive, responsive society toward the betterment of all its citizens, then regardless of the pep rallies, the words in the school philosophy, or who speaks at graduation, how we are affecting the future is surely in question.

Chapter 9
Grades and Tests

As noted earlier, we consciously and unconsciously assess what we value—it actually seems involuntary at times as we get a visceral read of a situation that concerns us. In school, as mathematics teachers, the "read" is in general more complicated. Interacting with and observing students over extended periods of time provide a variety of information regarding a number of significant aspects of their learning experience. This includes the extent to which they are engaging the material, developing their thinking, where their interests lie, the difficulties if any that endure, their wherewithal to stay with a challenging problem, how they engage with others toward securing a common understanding, and what would be valuable considerations for each toward their further development. It is a complex affair, as mathematics educators well know.

Working with young people with the focus on their cognitive, social, and personal/professional developments as mathematics students is clearly a challenging endeavor. That we can claim that we capture months and a year of such a rich and varied engagement with a number or letter grade does not add up. Indeed, to collapse their educational experience into the "efficient" expression of a symbol representing an obscure measure would seem a travesty of the experience. It would seem such grading is best left to labeling meat. Though here too the evaluation is suspect (anyone want to write a short paragraph on meat that is a B-?).

An assessment framework as represented by a letter or number is not only ambiguous, it provides little or no useful information other than for creating a dumb stratification system. Does it help students understand how they are progressing or what they have come to make sense of, have facility with, and where we hope to see them progress? Does it help next year's mathematics teacher as the reader of such information have any sense of how best to work with that student further?

While the critique that follows is not the first time some educator has spoken out against letter and number grades, it seems a moral commitment that the conversation continue. For such a form of evaluation is surely not in the best interests of those being educated or those who are to make judgements based on the absence of various critical information. This section of the book explores another direction. The focus will be on constructing evaluations after rethinking exams, homework,

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and considering a lens dedicated to understanding students in their mathematics classroom experience. But, first, let us look more closely at why letter and number grading should become as extinct as so many other unsuccessful means that have been tried and failed.

* * *

A number or letter grade, and worse an average, not only distorts the educational process as a learning experience by providing a surface reading of the experience, it necessarily misrepresents what has happened for a host of reasons. Quarterly, semester, or yearly assessments expressed as such do not recognize the nature of each student's personal, social, and intellectual experience. Those grading formats, regardless of how "sensitive" the measures, just make the evaluation process simplistic, remaining at a distance from the experience itself. For example, suppose a student gets 86 in an exam involving factoring—what does that exact value tell you about that student's mathematical abilities in factoring? Was the score the result of a student oversight—not reading carefully? Was it attributable to a conceptual misunderstanding or was it due to some minor efforts in calculations? See if you can write even one meaningful sentence knowing that score. Now imagine that same student receives 72 in a quiz that focused on the $distance = rate \times time$ relation. Easily we calculate the student's average from both assessments as 79. What does that grade represent exactly? Is averaging over disparate sets of data to determine a value an acceptable exception to the caveat given to students that "you cannot add apples and oranges"? What is it that we can say about that student's mathematical ability knowing that average? Furthermore, as both grades were apparently out of 100 points, does that mean the same amount of time and consideration was given to those learning experiences? If not, has one of the two scores not been given undeserved greater or lesser weight than the other? What if the student's grades had been 67 and 91—still a 79 average; still an "average" student? How would that student's next year's mathematics teacher be informed regarding any difficulties continuing to get in the student's way? What about the student's persistence relative to earlier in the year—have they been making progress? Are they aware they have? Are we aware of any particular strength(s) newly demonstrated that deserve(s) noting? Is a facility with algebra or analyzing data, or with particular problem-clarifying strategies, developing? How would the student know? What *was* the student's take on their experience? Would such an understanding not be of great value in following the student's development?

Unfortunately, the number or letter represents an exactness that can actually entrance us to believe we have made an objective evaluation, especially when we take it to the thousandths place! As if our subjectivity, as found in assigning varied points for different parts of problems depending upon what we think, or in the thinking that "there is no partial credit—it is either right or wrong in mathematics", that gain objectivity when expressed as a number or letter grade. Such evaluation formats are far from the meaning of the learning experience, of the intellectual, social, or personal/professional nature of the student's mathematics experience—all we have is a surface read to represent a complex situation. And if students see their assessment as whether they *are* an "A," "82," etc. or that they did or did not make the grade, then it has been a wasted opportunity from the perspective of being a socially responsible assessment; such practice would seem to deserve an F. Such labeling clearly says more about the system than it does about the student.

Even rubric-scoring can be problematic. For rubric descriptions provide behavioral qualities meaning there is room for further descriptions in the in-between. So quantifying and averaging a set of rubric scores to get an overall evaluation can be expected to create a total mess of the educational experience rather than provide any insight, except of course if there was no variation in the individual category scores. The primary benefit of using a rubric is that it gives students a descriptive framework from which they can decide if they want to act differently in the future toward securing a more fulfilling description or continue as they have with regard to a particular situation. That is to say, it is an excellent opportunity for a conversation between teacher and student regarding the latter's development.

Of course, good rubrics can help mathematics teachers gain greater understanding regarding their own practice. The Teaching for Robust Understanding of Mathematics rubric (Schoenfeld 2014) available at http://ats.berkeley.edu/tools.html and http://map.mathshell.org/materials/trumath.php includes a framework that provides valuable perspective for teaching the whole class and small groups.

* * *

A lot has to be in place to claim that an assessment of student's capacities has integrity—that it ultimately serves the student, the school, and the public well. The traditional narrow focus of school assessments is no secret. In the Executive Summary of "Tough Choices or Tough Times", the report of the ***new*** commission on the skills of the American workforce by the National Center on Education and the Economy (2007; boldface in original), the authors write that with regard to standardized exams: "On balance, designed to measure the acquisition of discipline-based knowledge in the core subjects in the curriculum,…,more often than not, little or nothing is done to measure many of the other qualities that…spell the difference between success and failure for the students who will grow up to be the workers of twenty-first century America: creativity and innovation, facility with the use of ideas and abstractions, the self-discipline and organization needed to manage one's work and drive it through to a successful conclusion, the ability to function well as a member of a team, and so on" (p. 15).

Those concerns reflect the three-dimensional perspective taken in this book. The question is how to make those considerations part of an evaluation schema. In this way students would become aware of how their more complete development is progressing toward making valuable gains in how they think, relate, and act in the process of learning mathematics. As well, it would seem that if teacher observations of student's personal/professional, social, and cognitive development would be supplemented by student reflections, our mathematics students could rightfully be expected to participate in their learning experience more thoughtfully, and in effect be more cognizant of and more alert to developing the twenty-first century concerns just considered.

In that direction, as mathematics educators, we have to ask ourselves are there practices we associate with thoughtful, responsible, concerned students that we believe ought to be given more attention in the mathematics classroom experience we are helping to shape. That with additional emphasis on our part we would be helping to promote and secure the development of human beings who handle their emotions well, work constructively with others, and think resourcefully. For the benefit of the students in front of us and ultimately society that would seem to be *our* socially responsible assessment work. If we are truly interested in promoting qualities of human thought and action that support a robust and life-enriching democracy, then coming to do things well—especially thinking, relating, and being emotionally resilient in the mathematics classroom, would be an integral part of our assessments, from both the teacher's and the student's perspective. And of course, it is complicated.

Naturally, embedded in student efforts are expressions of "creativity, curiosity, persistence, [and] critical thinking". Should we include such in our evaluations? That is Trevor Shaw's thinking regarding his eighth grade physical computing class: "these skills and habits of mind ARE the real content" (http://www.eschoolnews.Com/2014/12/10/letter-grades-app-873/; emphasis in original). While he considers some apps that he sees has potential in capturing such behaviors in action, that consideration needs attention if it is to have value to the mathematics student and their education.

Evaluations somehow need to capture the complex hopefully developing engagement of our mathematics students, not just present some obscure statement in the explicit form of an exam score or grade. What of their social engagement that impacted class discussions? And their personal efforts at working to take better notes, etc.? It would seem that development would deserve to be acknowledged in student reports, as it makes clearer and more comprehensive what is valued. After all, is there anything more important than pointing out student's personal, social, and intellectual growth? That complex of development needs to be acknowledged for each student's and society's further development, not lost in the shadow of the thin read of some number or letter grade.

* * *

Inasmuch as student grades are determined to a great if not complete extent by exam scores, another question we mathematics educators need to ask ourselves and answer is what is worth testing, and how that ought to be assessed with respect to students' developmental efforts. In that direction, we can appreciate Hugh Burkhardt's thinking that "good" tests focus on "problem-solving and modeling with mathematics, reasoning, and critiques of reasoning, alongside the concepts and skills needed to make these possible" ("Engineering Good Math Tests", *Education Week*, October 3, 2012)." He adds "multiple-choice tests cannot handle this." That is because it is really difficult to evaluate student thinking on such tests.

Consider the following multiple-choice mathematics problem: "Four of five dentists interviewed recommended Yukkey Gum. 'What percentage of the dentists interviewed did not recommend it?' A teacher asked one student who correctly answered 20% to explain their solution. The student responded, 'Of' means multiply, so I multiplied four times five and got 20%!'" (Thompson and Briars 1989, p. 24). Indeed, it is our students' mathematical thinking that matters. And we cannot find that in assessments of being wrong or right unless we are very clever at composing multiple-choice questions. Even if we are clever enough, students could well get "prepped" on how to analyze a multiple-choice question regardless of the content—a practice that seems to have many practitioners at this time.

It is surely true how necessarily incomplete assessment is that only measures student knowledge in one format. The National Council of Teachers of Mathematics (NCTM) Assessment Standards (1995) point out the need for multiple evaluation methods to be part of classroom practice, and Webb (1993) makes a convincing case that any one form does not recognize all the qualities a student could demonstrate regarding their understanding of mathematics. The variations are considerable, including: restricted or unrestricted-time written tests, take-home tasks, group projects, oral exams via presentations, portfolios, homework assignments, self-assessments in addition to explicit skills test. And the formats naturally make a difference.

As we would expect, different students, if not genders, do differently in different testing situations. For instance, de Lange reports (1987, p. 260) that in the mathematics program he is involved with "girls perform less well on restricted-time written tests [but] more or less the same as boys on oral-tasks or take-home tasks". He reports "[The mathematics program] is strongly process-oriented; the mathematization process needs time to develop, time to reflect, time to generate creative and constructive thoughts. These 'higher' goals are not easily operationalized with timed tests" (de Lange 1987, p. 260).

Jan de Lange (1993) presents ideas that seem worth thinking seriously about with regard to mathematics exams with special reference to students becoming valuable citizens. He captures the complexity of the situation succinctly: "We prepare our students for a very narrow set of skills; as soon as we cross the border to real problem solving, more process-oriented skills, and unknown context areas, we are in deep trouble" (264). With the new mathematics curriculum de Lange and others have put together, they created assessments that included the following four principles (copied verbatim):

1. Tests should be an integrated part of the learning process, and therefore tests should improve learning.
2. Tests should enable students to show what they know, rather than what they do not know. (We call this *positive testing*.)
3. Tests should operationalize all goals.
4. The quality of the test ought not to be dictated by its possibilities for objective scoring (266, 267).

The mathematics problems reflect the commitment to what de Lange and others refer to as "realistic mathematics education". That is, the problems have more than one solution path, and more than one answer, allowing for multiple interpretations so that students consider different interpretations leading to alternative answers. As such they are more open, promote "higher order thinking skills", and allow for students to respond at their own level (274). Making this perspective the way we think in creating mathematics exams is of course a real challenge. Especially with the dual focus of having problems that elicit particular problem-clarifying strategies that are and have been being worked with in class. It would suggest that rather than another national committee creating another list of curricular items to be practiced in mathematics classrooms, there would be a national effort to promote a variety of questions within which the mathematics content is incorporated but the questions are of much richer nature, including a focus of the means for engaging those problems so that all students have access to the "tools of the trade".

For school evaluations to have integrity the general goals of what it is understood to mean to be educated—the global intellectual, social, and personal practices associated with capable human beings, must be reflected and worked with in the classroom practice of each of the disciplines students study, and constitute the rationale that shapes students' exams. And that critical developmental process of engagement cannot be made sense of with a set of scores but requires a set of rich descriptions that focus on the essential elements that school and society value.

If mathematics educators believe that "education is the fundamental method of social progress and reform" (Dewey 1897, p. 16), then every grade up to and including the last year of college ought to be assessing classroom practices and student thinking and behavior in the best interests of promoting, securing, and sustaining a vibrant democratic society. And it would seem clear that the evaluation provided by number or letter grades cannot possibly capture the richness and quality of the engagement and learning experience. Surely it is a complicated undertaking gaining a more socially responsible understanding. But in the best interests of everyone, it would seem evaluation needs to provide an informed perspective of each student's efforts and development as a thoughtful, collaborative, and self-reflective person, along with the goals they hold and how they are proceeding toward securing them. Hopefully how students would be best evaluated would be a collaborative expression reflecting a unified commitment to enabling the young to be mathematically able through the three-dimensional lens that informs how we think and act in the world.

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Chapter 10
Homework

While it is often the case that students' grades are the result of summative assessments, formative assessments have essential pedagogical value. For they are "designed to make students' thinking visible to both teachers and students. They permit the teacher to grasp the students' preconceptions, understand where the students are in the "developmental corridor" from informal to formal thinking, and design instruction accordingly. In the assessment-centered classroom environment, formative assessments help both teachers and students monitor progress" (Bransford et al. 2004, p. 24). And that is exactly the opportunity homework offers both the mathematics student and teacher.

Some educators question whether there should be homework at all. Yet, homework effort can make a real if not profound difference in what students know and are capable of dealing with and, as importantly, can be appreciated by the students themselves. Work done outside the confines of the classroom with respect to the daily considerations gives students the opportunity to experience their own thinking without the competing voices and arbitrary time constraints the mathematics classroom experience imposes. As such it provides an ideal opportunity for students to see, for example, not only if they have secured facility with certain mathematics techniques, procedures, or skills but whether they have become more facile with a valued inquiry practice, such as transforming a difficult problem into one that is easier to deal with.

Additionally, it is a perfect opportunity for students to build their resilience, flexibility, and appreciation of what it takes to become more thoughtful. For it gives them personal time to uncover an insightful question or create a well-reasoned argument that everyone can appreciate, including themselves. What needs oversight is how much time it could take. To help them make the right decision, it is our responsibility as mathematics teachers to make it worthwhile from their perspective.

That is, it is the student who should make the decisions regarding the quality and extent of their homework effort. They will have ample opportunity to learn from their decisions and be able to better decide if any changes need to be made in their decisions. This is not to say that it is okay if students who do not do the homework take class time asking for it to be explained, unless for some reason

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they were unable to get to it prior. But giving students a 0 or −5 or whatever other grade punishment if they do not do it (or points or candy or money for doing it) is antieducational, for it distorts the meaning of the effort. Homework needs to be seen as something of value in itself, not that of escaping a loss if not done, or getting something for doing it (like no homework the next night as a reward!). That latter mindset has homework as a chore or a punishment. It could be. But such a focus would seem to have questionable educational goals.

Hopefully, mathematics students would see that homework can help them develop their emotional and reasoning resilience when dealing with complexity, as working alone they have opportunity to talk to themselves in a constructive manner, especially if the teacher has made clear how valuable that practice is. Toward promoting mathematics students being more conscious of the thinking/inquiry process, and especially for shy students, they could be asked to bring questions to class reflecting where they became confused or unable to continue with their homework. Such focus directly promotes their talking to themselves by writing something down that they can share in class. And it also helps them appreciate the opportunities that doing homework provides.

Essentially, homework assignments could be shaped to fit each student's particular needs and acknowledge their interests with their making the selection. Very capable students may well find some homework questions tedious, as they are actually bored dealing with questions they find uninteresting. And students who are struggling could well find themselves drained of any positive energy in looking at a set of questions that they find too challenging. This suggests that students would need to become more able to prioritize their assessment opportunities—namely, what would be best to focus on. The naturally more capable or more determined may well want to spend their time working on a relatively more complicated mathematics problem that draws upon new or more sophisticated heuristics they have yet to become aware of (recall Steve Brown's framework for providing such questions). While students who are having a hard time may just want to gain more facility with applying a particular heuristic or procedure. This is to say that if homework is to be considered as a socially responsible assessment, then each student should have a decision-making role in the homework they do to build both their personal responsibility and formative mathematical judgement.

As they become more reflective regarding their own development as a more capable mathematics student, they become better educated as to where it would be best to place their energies. This can be made all the more possible by mathematics teachers having an eye toward what would help the homework experience be done thoughtfully. Such consideration would include ensuring there would be no need for rushing because of an excessive demand created by too many problems. Nor would students be required to do sets of problems night after night that only call on their demonstrating mechanical routines. For at bottom, students are in charge of their education regardless of what the teacher says or does; it is the dedicated energy they bring or not to classwork and homework that makes the real difference. And every homework assignment could provide a valuable opportunity for students seeing where they are, and what would be a valuable direction to continue, if it is thoughtfully offered and received.

Mathematics homework should be informative and rewarding, whether practicing a needed procedure or exploring some novel question(s), not oppressive timewise or emotionally. (Teachers who have students do extra mathematics problems as punishment for poor behavior make clear that the subject they are teaching should be found to be offensive—clearly a very questionable educational lesson.) There's every reason to believe that an experienced, aware, and thoughtful mathematics educator can put the assessment activity of doing homework in a good light for good reason. As a consequence, students would more likely provide the dedicated effort deserving of the wisdom of such a worthwhile endeavor. After all, they would be growing, intellectually and personally, from the experience. And of course, such an effort would be worth making note of so as to share with interested significant others, including the student.

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Chapter 11
Classroom Observations

Regardless of the classroom format and activity—whether it is the whole class, a small group, or individual students working on a typical mathematics textbook problem or an extended investigation, observation tools (including apps) can surely be valuable. Classroom observational tools provide opportunities for further conversations with students and ourselves as mathematics educators in support of promoting more productive mathematics classroom practices and students becoming more capable individuals. The checklists presented here were formulated to provide the mathematics teacher with means to capture significant aspects of the personal, social, and intellectual dimensions of students' mathematics classroom engagement that may go unnoticed, as fleeting as many of those moments are. The three frameworks focus on student dispositions, both productive and otherwise, to support teacher and student conversations directed toward enabling students to become appreciated members of society.

11.1 Students' Psychological Development

Experience confirms that our emotions infuse our attitudes, dispositions, and behaviors, including our thinking. This is to say that students' psychological/emotional state is critically important in shaping what they do in the mathematics classroom. While this is clearly not a new realization, the connection is beginning to be recognized globally with regard to students' mathematics exam assessments. In "U.S. Math, Science Achievement Exceeds World Average," Erick W. Robelen in his "Curriculum Matters" (*Education Week,* December 11, 2012, Vol. 32, Issue 15) relates some of the recent *Trends in International Mathematics and Science Study* (TIMMS) findings. The 2011 TIMMS included a number of new measures to better help put student achievement in context. "'One thing we've worked on is [getting] better indicators of what goes on in classrooms,' Mr. Martin of the International Study Center said. 'We've sharpened our focus on student engagement. [One] measure is based on asking students how engaged they feel in their classroom. That

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makes a very nice scale that relates to achievement.'" (In addition to being "very nice" from the psychometrician's read, it seems we can add *fundamentally important* from the mathematics teacher's perspective.)

Another scale they developed helped them uncover that students across nations seem to lose enthusiasm for mathematics as they get older. Although less than half (48 %) of fourth graders said they "like learning mathematics," that slipped to one quarter (26 %) by the time they reached the eighth grade. And at both levels, that attitude has a correlation with test scores. That is, the less students like mathematics, the lower their achievement, on average, which could well be expected. In that direction, almost three quarters of fourth graders around the world (69 %) reported having mathematics teachers who made efforts to use instructional practices to interest students and reinforce learning, such as posing questions to elicit reasons and explanations, and bringing interesting items to class. At the eighth-grade level, however, only 39 % of students internationally reported that their teachers frequently related lessons to their daily lives, and just 18 % said they had mathematics teachers who routinely brought interesting materials to class.

These findings strongly corroborate what John Dewey and all of us know: Interest promotes effort. So, any assessment of a student's mathematics knowledge without an understanding and appreciation of how engaged that student was is ultimately incomplete. For, if we like something, we are naturally more interested in knowing more about it and find it relatively easy to focus our energies there. From that perspective, poor scores on a mathematics exam raise the question of how emotionally disconnected students were to the material on the exam, as opposed to how mathematically able they are.

That's why "interested" is first on the list of qualities on the psychological dimension observation checklist included here (Fig. 11.1). Like the rest of us, a student's level of persistence including flexibility of thought might falter in 2 s, 2 min, or extend to 2 days or weeks or more depending on their level of interest. That temporal spectrum makes the point that judgements about student's problem-solving behavior, including their resilience and patience, must first include consideration of the extent to which the student was engaged with the topic or problem under discussion, or is otherwise motivated (as in getting a high grade). As noted earlier,

Psychological Disposition Checklist												
Students	Interested	Resilient	Focuses well	Loses focus	Impulsive	Manages impulsivity	Patient	Confident	Lacks Confidence	Takes initiative	Uncomfortable taking a position	Uncomfortable changing position
Arturo												
Bobbie												
Carmin												
Dwayne												

Fig. 11.1 Psychological Disposition Checklist

a socially responsible assessment would have to begin there, for making any statements regarding a student's demonstration or absence of problem-solving capacities depends on how connected they are to their mathematics experience. Their commitment quite naturally and directly would be the concrete manifestation of their felt motivation.

Yet, it could be the case that students could well be interested in the topic but find the classroom experience stressful. Possible explanations for the discomfort could be a function of the group the student is in, or it could be that they do not feel they know enough to take an active role or are confident enough to ask a question and participate in the classroom discussion. It may even be a long-standing issue. In any case, every student's psychological energies deserve attention toward their having a more emotionally productive engagement.

What follows is an observational tool whose categories are representative of students' intellectual energies as a direct expression of their emotional state of mind. You may well think that there should be others included or excluded, for the emotional spectrum associated with intellectual efforts surely contains a rich set. And of course that is the object of this offering to provide considerations for mathematics educators' decision-making. However, for the selected set to be of value, here or in any observation tool, a criterion worth considering is that it must satisfy the Goldilocks test of not too many (hot) or too few (cold). That qualification may well be a function of how many students are in the class or what particular concerns are the focus at a particular time, etc. Also, it can well be expected, as with anything else taken on that is complex, that it would take time and practice to make it an efficient observation tool.

In operation, the "psychological disposition checklist" wording that headlines the emotional dispositions would be replaced with the date and the nature of the mathematics engagement, along with whether it is a whole-class, small-group, or individual effort. In this way, a number of observations over time would help generate a read of each student's psychological development or lack thereof in the context of their working on a variety of mathematics problems in a variety of settings. Data would become available to help promote teacher–student discussions, student reports, and teacher reflections.

11.2 Social Development

Students' social development would also seem integral to their learning, for that includes behaviors that promote or prevent productive collaborative efforts. The National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (2000) and the *Professional Standards for Teaching Mathematics* (1991) discuss the importance of students listening and establishing a learning environment in which they can work together. With opportunity and practice, they can see how respect is expressed by listening carefully and taking care of how they respond. Hopefully all students will have the opportunity to develop their communicative competence, have positive interactions, and appreciate the gains of working together, which includes clarifying and changing one's thinking as a consequence of others'.

Behaviors such as students sharing their thinking with the class and in groups, and listening carefully to one another, are not part of the written mathematics curriculum and so often are outside the teacher's evaluation focus, except in cases of student's problematic behavior. However, such practices are of fundamental pedagogical importance as they are essential to the individual's learning mathematics and social growth, along with classroom cohesion. Also, such considerations could well have consequences for the future society at large.

Inasmuch as many students, like many of us, do not know what they are thinking until they hear themselves speak, their social development requires a mathematics curriculum that would promote conversations and collaborations on a regular basis. They would need to be frequent or intensive enough so that students have considerable opportunity to develop those social practices that gain the appreciation of others as well themselves when working together. And, as noted earlier, were the conversations enriched as the result of student questioning and their drawing on problem-clarifying strategies, in contrast with being teacher directed, those conversations could be all the more valuable and valued.

An additional reason for including collaborative experiences is found in former-student responses to what they remember most about school: It is often the projects they were involved in. That makes good sense of course as those activities represent the extended classroom efforts they made that required relatively greater emotional, social, and intellectual commitment. Project investigations usually contain a number of considerations and as such ensure conjectures will be offered and rejected along with arguments for and against as students learn to distinguish assumptions from evidence, and evidence from proof, and work together to make good decisions. Such collaborative mathematics experiences need to include time for reflecting on one's role in the engagement, of course; and such opportunities could well yield long-lasting positive personal and social outcomes.

In that direction, there needs to be space for those students who wish to play out ideas themselves, who appreciate the chance to continue trying to make sense of things alone, and listening rather than speaking. Given that opportunity, hopefully they will feel comfortable at some point to join a group to share their thinking as well.

Yet, recall the criticism of group work by one of the readers who wrote about the third-grade mathematics class in Ontario that students can just sit back and do nothing. Surely that can happen. But what could be the cause for a student's lack of social involvement is not always easy to discern. It may be all the more likely if the student is in general shy, or the particular material under investigation does not have much appeal, or the group conversation is for some reason uninviting—all legitimate reasons for some student(s) to refrain from participating. Also, some students could be so invested in their own thinking that they do not listen well and instead are or appear to be dismissive, and may often be argumentative. So, there can surely be mathematics students reluctant to participate but for different reasons. This suggests that while the extent of student involvement can be categorically expressed in the form of a number or letter grade, it could well be a surface reading that requires further investigation and conversations for social growth to be possible.

Observing that some students or group members are having a hard time communicating productively suggests of course the need for intervention. Discussion may uncover the need for the participants to come to appreciate that disagreement should not only be expected but even welcomed. For, gains are made when hypotheses are ruled out, and thinking is clarified in the dialectical process provided by challenging conversations. But of course, their tone matters. This is an important lesson to be learned for students who tend to be rather immediately convinced of their own thinking.

Additionally, students' learning to separate criticism of their thinking from criticism of themselves as persons is essential for working productively with others. This may be promoted through class discussions of the expectations of group work and how group members might support each other in developing those characteristics of effective group participation. Their future interactions at work, as parents, and in other settings will provide in essence an ongoing assessment of how successful classroom conversations and activities actually were in developing their social intelligence.

This focus on the nature of group engagement is not to dismiss whole-class discussions. Whole-class discussions can surely provide opportunity for students to learn from each other and question/challenge what is said in a constructive manner. Yet, the richer the collaborative opportunities, the more opportunities students will have in shaping the conversation and the decisions made. So, it seems a good idea to have group work as a common classroom activity along with there being individual time to consider a problem that the whole class is working on. Also, project engagement requiring an extended learning experience could be very valuable for every mathematics student. Such complexity demands considerable integration and organization, including the capacities to deal with various threads of varying degrees of information, and being able to bounce back when one's "wonderful" idea is found wanting, and developing the capacity to reflect on the situation amidst all the distractions.

Students working collaboratively does take time away from including additional mathematics content that the teacher could present more efficiently. But given a broader, deeper, and more realistic commitment to students' personal and social development and intellectual experience, and an observational schema that helps locate significant social elements of the learning experience worth discussing, that time could be well spent educationally, despite the complexity it introduces to assessment and the omission of some mathematics content.

Hopefully, such opportunities will benefit students who initially have a hard time dealing with multiple inputs, or who have been uncomfortable sharing their thinking for fear of being wrong, and be so constructed that all students can add to the mathematics experience and none continue to be dispirited when finding their thinking goes unappreciated. These concerns suggest that grouping need not be random, or rigid as in being determined by role playing, or too concerned with being

Social Disposition Checklist										
Students	Participates Constructively	Listens Actively	Difficulty Listening	Takes Critique Well/Poorly	Critiques Carefully/Poorly	Inclined to Lead	Easily Dissuaded	Offers Suggestions	Builds on Others Ideas	Inclined To Work Alone
Arturo										
Bobbie										
Carmin										
Dwayne										

Fig. 11.2 Social Disposition Checklist

politically correct (as balanced by gender), but actually organized to help every student progress in their social development. As noted earlier, being grouped by interest helps establish a shared positive energy with which to begin. Also, grouping where some students' heightened interest could inspire others might be the essential spark needed to create a dedicated collaborative effort.

Of course, there could well be students who wish to work alone, as noted earlier, and it is the insightful mathematics teacher who can determine whether those students are making such a choice for good benefit or to escape being part of a group and so provide good reason for a private conversation. Helping them come to appreciate that working to negotiate a unifying understanding could well be valuable throughout their lives is an extremely important lesson, both for society's future and one's personal and professional future. Overall, it would seem that youngsters who are in mathematics classrooms where collaborative interactions are "the way things are" would become more comfortable and capable in the ways they act toward each other and themselves, given all the trial-and-error opportunities and especially with the assistance of an observant mathematics teacher. That would seem a worthy institutional goal across mathematics classrooms in grades 6–12 (and of course earlier).

In that direction, here is a social disposition framework checklist (Fig. 11.2—that is also open to being worked with). The object would be here as well to have ongoing conversations with those students who find interactions hard going, for whatever reasons. Here too the "social disposition checklist" label would be replaced with the date and the focus of the students' mathematics engagement and the class format.

11.3 Cognitive Development

Whatever we choose as educators and a society that the young learn in mathematics is what we believe they *ought* to know. For when we decide to know something—how to drive a car, how to make an omelet, etc.—it is because we have decided that "it is *good* to know *that.*" So we direct our mental, emotional, and physical energies to that learning, for in essence there is a moral imperative underlying that commitment. With this perspective, we can question the often-made claim in mathematics classrooms that the content *should* be taught/learned because "students will need to know it later." That students ought to learn it now seems questionable given that their need to knowing it later is in essence an argument that students do not need to know it now. It seems the material ought to be left for the future where it apparently fits. Surely we can appreciate that "We always live at the time we live and not at some other time, and only by extracting at each present time the full meaning of each present experience are we prepared for doing the same thing in the future. This is the only preparation which in the long run amounts to anything" (Dewey 1938/1969, p. 49).

That is, as mentioned on more than one occasion, especially with regard to a student's psychological state, the mathematics content must somehow connect to the student. If it does not, it could be experienced as a chore or an obligation—in essence, it is offered as a gift but experienced as an imposition. Is the thinking that inasmuch as the young are obligated to listen to their elders that is sufficient reason for them to do what they are told in mathematics class? As students have obligations around the house, ought they not do the same in school? But those activities around the house are in the interests of the family. Are we thinking that a mathematics curriculum is formed by being in the best interests of the public good—even if there is little, if any, student interest shown or real connection made? This is not to say that sharing "interesting" mathematics, as pleasant as that can be, should be the focus of our work—that is not what makes for an education. An education naturally requires productive energies, discipline, and special practices. In that essential direction, it seems mathematics curriculum writers and mathematics educators ought to make it a moral imperative to create mathematically engaging experiences all the more available.

Otherwise, if students do not recognize a real connection or cannot be shown a way to see why mathematics is of value to them, then we can expect that the kind and degree of focus would likely not be as we would hope. As a consequence, any quantitative or letter grade of students' mathematical ability could well be thought to represent their intellectual distance from that experience. In essence, the cognitive assessment measure in this instance is not so much about what mathematics students came to know as that they did not care (as would be reflected in the classroom assessments mentioned in the psychological and social sections). It is the dedicated focus brought to the engagement and the educational quality of the interaction that makes the mathematics experience intellectually and morally valid. It is there that we find the real measure of school and society's civic promise.

In the absence of that connection, those who are less dutiful will likely find the content assessment to be a punitive activity. Namely, the student will be given a poor grade for the lack of knowledge, when in fact it is a lack of commitment. In Dewey's time "If the pupil…engaged in physical truancy, or in the mental truancy of mind-wandering and finally built up an emotional revulsion against the subject, he was held to be at fault. No question was raised whether the trouble might not lie in the subject-matter or in the way in which it was offered" (1963, p. 46) and, as Dewey also recognized, how open the student was to connecting to the material.

The two checklists that follow provide descriptors mathematics educators can use to gain a more complete sense of students' cognitive strengths and weaknesses (Fig. 11.3) and their intellectual capacities drawing upon valuable problem-clarifying strategies (Fig. 11.4). Over time, these lists should make evident what intellectual development is or is not taking place. However, observations always contain assumptions, and determining causality is not an easy matter. For example, imagine after a number of days the check-off list of problem-clarifying strategies is rather blank. Would we say that the teacher is not modeling the practices and so that is the cause for its absence in students' behavior? Or, could it be that the mathematics problems the text provides do not require anything more than a direct technique for solution? Only the practitioner knows. In either case, the absence of mathemati-

Intellectual Disposition Checklist												
Students	Flexible Thinking	Persistent	Resilient	Raises Good Questions	Interested in Applications	Interested in Abstraction	Generalizes Hastily	Seeks Accuracy	Seeks Explanation	Reflective	Explains Well	Makes If-Then Connection
Arturo												
Bobbie												
Carmin												
Dwayne												

Fig. 11.3 Intellectual Disposition Checklist

Intellectual Practices Checklist															
Students	Looks for patterns	Visualizes	Tinkers	Describes	Represents Symbolically	Proves	Checks for Plausibility	Takes Things Apart	Conjectures	Simplifies the Problem	Works Backwards	Re-examines the Problem	Changes Representations	Creates
Arturo														
Bobbie														
Carmin														
Dwayne														

Fig. 11.4 Intellectual Practices Checklist

cal heuristics, problem-clarifying strategies, from the conversation would hopefully suggest an adjustment in the environment was needed.

Partitioning the intellectual scale into two sets seemed to make sense. They are offered here too as suggestions and reasons for future conversations and may well need to be reconstituted to best fit each mathematics classroom. Again, the heading provided would best be replaced by the date and focus of student engagement and the class format.

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Part V
In Conclusion

Education is a fostering, a nurturing, a cultivating process.
John Dewey

Chapter 12
Summing Up

The school's commitment to the development of students' mathematical, social, and emotional intelligence acknowledges that each has a significance that informs the others, and that a vibrant citizenry depends on all three. Most directly, the mathematics curricula for grades 6–12 should recognize that "A responsible and informed citizenship in a modern economic democracy depends on quantitative understanding and the ability to reason mathematically" (Ball 2003, p. 2). Quantitative understanding and reasoning mathematically makes clear that heuristics and mathematics problems that promote discussions need to be included along with whatever techniques and skills are considered necessary.

Of course, whatever mathematics is chosen for each grade, the problem-solving experiences would have the greatest chance of success if students were open-minded, capable of listening carefully, and responding constructively to each other. That suggests such development ought to be an integral concern of our work as teachers of mathematics as well. By providing opportunity for big and small investigations, time for reflecting, and discussions with students regarding how things are going, we would be helping our mathematics students as well as funding society with the prospect of productive conversations and collaborative decision-making.

To ensure the intellectual and social development, we naturally need to give attention to the psychological/professional dimension of each student's mathematics experience. Providing grades 6–12 mathematics curriculum that promotes the development of patient, resilient, and flexible-minded students when engaging problems makes good sense. Those concerns make clear such consideration are also an essential element of classroom discussions and evaluations. In that way, students would have many opportunities to learn how they can count on themselves to make constructive advances. With their having learned to work productively, including with their emotions, those students would surely be valuable members in their future work and participation in society. For while freedom is fundamental to a democratic society, "genuine freedom … is intellectual; it rests in the trained *power of thought*" (Dewey 1933/1936, p. 90; italics in original).

All in all, the personal, social, and intellectual qualities that a democratic society would want to be in evidence in future decision-makers would need to be

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made explicit and supported when making decisions with regard to constructing the mathematics classroom experience, including exams. Toward that end, our work as mathematics educators is “to shape the experiences of the young so that instead of reproducing current habits, better habits shall be formed, and thus the future adult society be an improvement to their own” (Dewey 1916/1944, p. 79). How the mathematics was offered and the learning dimension assessed so as to ensure students’ participation and developing agency would be the means by which the goals of a democratic society would be secured. Realizing this perspective may well require change in the professional practices of some mathematics teachers, as well the programs offered to prospective and practicing teachers of mathematics in universities and the customs of some educational institutions. But of course we would expect that our understanding is developing, especially of such a complex endeavor as the collective classroom experience, and change is the natural and essential way we acknowledge that development.

* * *

Any mathematics curriculum is at best a guide, as it is to be experienced by specific students in each classroom. While following a script may well help novice mathematics teachers to deal with the complexity of organizing a learning environment, doing so is quite unresponsive to the personal, social, and intellectual energies present. It risks promoting a “failure to take into account the instinctive or native powers of the young; secondly, failure to develop initiative in coping with novel situations; [and] thirdly, an undue emphasis upon drill and other devices which secure automatic skill at the expense of personal perception” (Dewey 1916/1944, p. 50).

Being a responsive mathematics teacher naturally includes giving space to the evolving development of students’ mathematical thinking. That would suggest deciding how time ought to be spent in class, and especially the time given to student’s thinking and the environments that best support their developing thoughtful behavior, need our careful consideration. If society is to prosper for the investment made in schools, it needs to be appreciated the investment is not only the funds that were brought to support the school, the faculty, textbooks, materials, computers, buildings, etc., but also the thought and energy of the student participants in the mathematics classroom. For, ultimately “…the principal agent in education, the primary dynamic factor or propelling force, is the internal vital principle in the one to be educated” (Maritain 1960, p. 31).

So, in the criticism of a mathematics curriculum that is a “mile wide, inch deep,” there is the potential for a promising future in mathematics classrooms. Especially with students participating wholeheartedly in their own mathematics education, we can count on their effort. And if we help them come to see that they can realistically count on themselves and each other in their educational experience, their mathematics learning could truly transform their lives and shape our shared future for the better.

That there is a educational movement toward covering less mathematics topics but digging deeper, with more group involvement engendering “21st century skills,” suggests there is real opportunity for students more fulfilling development as thoughtful, socially aware, and emotionally resilient human beings. After all, as John Ewing, who was executive director of the American Mathematical Society for 15 years, remarked, "The end goal of education isn't to get students to answer the right number of questions. The goal is to have curious and creative students who can function in life" (*The New Yorker*, July 21, 2014, p. 63).

What would the mathematics classroom look like to engender such development? With students more familiar and comfortable with the content of thinking mathematically along with what is involved in reflecting productively on their thinking, it would be fair to say all students would find their mathematics experience more rewarding. Over time they would be more able to make decisions based on an increasingly developed intuition informed by reason and experience, not as a consequence of rigid thinking, indifference, or impulsive guessing. They would be more emotionally able—more resilient, persistent, flexible—in engaging perplexing mathematics situations, rather than coming to ill-conceived conclusions based on stressful feelings. The evidence would be clear in their ability to work critically and imaginatively with their ideas and others', and their capacity to suspend judgment rather than accepting whatever comes to mind. In this way, they would be more open to considering alternatives, including another's thinking even if it challenges their own. In this way, the democratic community prospers.

After all, how to help ensure the mathematics classroom experience is in the best interests of a society committed to the personal, social, and intellectual developments of all of its citizens ought to give shape and purpose to our efforts as mathematics educators.

Best of luck with yours.

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What Is Your Thinking?

Hopefully questions have come to mind that you found worth thinking about as you read this book. The following are offered in that spirit.

- Does your mathematics text begin pretty much every chapter with a short introduction, followed by a few demonstrations of a technique or a procedure, and then a set of problems for students that advance from easier to more complex? Have you considered introducing the topic or chapter by presenting one of the more challenging problems or letting students choose problems from the question set to see what they would choose and what they could do "from scratch"? Any experienced teacher of mathematics has seen how students can really surprise us. And if they have had background working with problem-clarifying strategies, it may well be more frequent. At the least, working on problems without having seen a set procedure would provide insight into student's thinking and create the opportunity for interest groups of those who selected the same problem.
- Do you agree with the thinking that if you want to develop students' resilience working with a mathematics problem, then having them work in groups is better than having a whole-class discussion because as soon as one student "gets it," they will likely share the answer and so end the time others would remain thinking? Are there times when whole-class discussion is more effective than groups and vice versa?
- Do you think the discussion of how to introduce arithmetic series by other than a demonstration of Gauss's approach would be worth doing with all students? Or would you be inclined to skip doing so with weaker mathematics students? If so, are you certain they could not look at the counting series from 1 to 10, say, and not be able to discover adding up opposite pairs in a reasonable amount of time? How much time would you consider giving them to try is surely the question.
- Do you think telling students the Pythagorean Theorem when presenting a right triangle along with showing them some triples that satisfy the relationship is a good idea? In the absence of sharing a proof as well, do you think it will inspire students to look for patterns beneath the surface of things? If so, do you follow up that presentation with other such opportunities for students to uncover other

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patterns? You might consider, if you do not already, sharing with students Bhaskara's "Behold!" proof—it will surely promote thinking as a consequence of their visualizing.

- Making time so problem-clarifying strategies can be introduced and worked with means of course less time to include other mathematics content, so a value judgement needs to be made. It has been argued in this book that providing students with the "tools of the trade" can develop mathematical habits of mind that would make them more capable mathematics students. If you found the argument compelling, do you have any ideas as how you might incorporate that perspective into your curriculum? Will you make it a conversation with your department members?
- If you have not already done so, do you think giving students' choice in their homework selection is something worth trying?
- Some mathematics teachers collect homework on a regular or random basis to get an idea of how and what students are doing. Did you think it is a good idea to ask students from time to time to write how their personal/professional development is going or if they have noticed their becoming more aware of using some problem-clarifying strategy(ies)?
- Is it part of your teaching practice to stop after having solved a challenging problem with students to have them make note of the crucial decision(s)/step(s)? Including the question(s) that helped make inroads to the solution? It might well be not that easy to do as students often want to celebrate the victory of the conceptual hunt and could feel energized to want to move on and engage another problem, rather than stop and reflect. It would seem it would have to become a learned practice, an acquired appreciation of their burgeoning connoisseurship.
- Informal assessment seems both a good idea and yet not an easy practice to include as part of class time. Are you giving any thought to using any of the disposition lists discussed? Surely it would take time to become more comfortable using them, but it would seem valuable toward securing a "thick description" of what each of your students is about.
- Is it a practice of yours to bring history into your mathematics lessons? If not, the National Council of Teachers of Mathematics (NCTM) has some really good books where mathematical concepts are considered from their origins and development.
- Are you giving any thought to including multiple-center investigations into your curriculum? If so, with regard to assessing such an activity, it is valuable to have students in each group write up their recollections of the experience, including what each of the participants did. This way you can have a fairly good idea of who was responsible for what aspect of the engagement, and over time students tend to become more observant, which is a good thing in itself. Project-based learning sites offer focus problems that perhaps could be developed further so as to establish follow-up interest-group investigations. Do you think other members of your department would be interested in working together to create such investigations?

- Students most likely have heard about Albert Einstein and how brilliant he was. But as he noted, what was critical to his making major findings was his patience. He spent 11 years thinking about one problem! Do you discuss with students the valuable qualities of persistence, patience, and resilience when you take time to talk about what is needed to be successful doing mathematics? How do you think those valuable qualities can be developed further as part of your students' mathematics experience? Or are you thinking that it will just happen as a natural consequence of students engaging mathematics problems? Are there mathematics problems in class where students get 11 min to think and investigate? In their homework?
- Does discussing signpost behaviors with students with regard to their emotional, social, and intellectual development seem a good idea? Would it seem worth a conversation with department colleagues?
- Has it been your practice to have new students in your class write about their past mathematics experience, and what they hope would happen in their present mathematic class? If not, has what you read in this book giving you any reason to change your mind?
- Did you find the discussion regarding students thinking of themselves as artists of their own lives and choosing behaviors to work with so as to become more capable a good idea? If so, there is the possibility here for student-interest groups so students can discuss among themselves how their development is going and what they have learned that has helped them in their efforts so that others could gain from that perspective.
- Does your grading scheme have room for comments about each student? Some mathematics teachers have so many students that it would take an extraordinary amount of time for them to write about each student's development each marking period. A checklist could make that possible. It would seem that discussions regarding what the most important qualities are and what attitudes and behaviors are valuable to include would seem worth a conversation in class and department.
- Did you think the disposition spectrums work as they are? What changes, if any, would you make? If you plan to use them is it your thinking to have whole-class discussions from time to time just to help remind them of the value such considerations provide?
- There were many allusions to a "vibrant," "pluralistic," "democratic," "robust" society as being the impetus for including the three dimensions of the educational experience, the cognitive, social, and psychological, as the essential focus of mathematics educators. Do you think that if that perspective was shared with students throughout their K-12 or K-16 school mathematics experience there would be more capable, cooperative, and confident citizens to support the workings of society?

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