Universidad Autónoma de Madrid

Final Master Thesis

TDA
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Author:
Gonzalo Ortega Carpintero

Tutor:
Manuel Mellado Cuerno

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Abstract

Key words

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1 Preliminaries (to do work)

The contents of this chapter are based on [1], [2] and [3].

Definition 1.1 (Graded ring).

Definition 1.2 (Graded ideal).

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Definition 1.4 (Persistance module, finite type).

Definition 1.5 (Module morphism, shift).

Definition 1.6 (Interval module).

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Definition 1.8 (Barcode).

Definition 1.9 (δ-interleaving moudules).

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Definition 1.11 (Multiset matching).

Definition 1.12 (δ-matching barcodes).

Definition 1.13 (Bottleneck distance).

Definition 1.14 (Induced matching).

2 Structure Theorem

Fact 2.1 (Structure theorem for finitely generated modules over a principal ideal domain)**.** *Let M be a finitely generated module over a principal ideal domain. There exist a finite sequence of proper ideals $(d_1) \supseteq (d_2) \supseteq \cdots \supseteq (d_n)$ such that*

$$M \cong \bigoplus_{i=1}^{n} R/(d_i).$$

Proposition 2.2. *An ideal $I \subseteq R$ is graded if and only if it is generated by homogeneous elements.*

Theorem 2.3 (Structure)**.** *Let (V, π) be a persistence module. There exist a barcode* $\mathrm{Bar}(V, \pi)$*, with $\mu : \mathrm{Bar}(V, \pi) \longrightarrow \mathbb{N}$, the multiplicity of the barcode intervals, such as there is a unique direct sum decomposition*

$$V \cong \bigoplus_{I \in \mathrm{Bar}(V)} \mathbb{F}(I)^{\mu(I)}.$$

Proof. (INCOMPLETE) V is of finite type, so it is a finite $\mathbb{F}[x]$-module. As $\mathbb{F}$ is a field, $\mathbb{F}[x]$ is a principal ideal domain, therefore, V is a finitely generated module over a principal ideal domain. Using Fact 2.1 V can be decompose in the direct sum of its free and torsion subgroups, $F \oplus T$. Thus, we have

$$F =$$
$$T = .$$

$\square$

3 Stability Theorem

Lemma 3.1. *Let I, J be two δ-matched intervals. Then, their corresponding interval modules $(\mathbb{F}(I), \pi)$ and $(\mathbb{F}(J), \theta)$ are δ-interleaved.*

Proof. Let $I = (a, b]$, $J = (c, d]$. If ρ is the δ-matching between them, then $\rho(I) = J$ and, following Definition 1.12, $(a, b] \subseteq (c - \delta, d + \delta]$ and $(c, d] \subseteq (a - \delta, b + \delta]$, with $b - a > 2\delta$ and $d - c > 2\delta$. Then, the morphisms

$$\begin{array}{ccc} \phi_\delta : \mathbb{F}(I) \to \mathbb{F}(J)_\delta & \text{and} & \psi_\delta : \mathbb{F}(J) \to \mathbb{F}(I)_\delta \\ \phi_\delta(\mathbb{F}(I)_t) \mapsto \mathbb{F}(J)_{(t+\delta)} & & \psi_\delta(\mathbb{F}(J)_t) \mapsto \mathbb{F}(I)_{(t+\delta)} \end{array}$$

are well defined as for any $t \in (a, b]$, $t + \delta \in (c, d]$ as $a + \delta > c$ and $b + \delta \leq d$. In the same way, for any $t \in (c, d]$, $t + \delta \in (a, b]$. Thus, $\psi_\delta \circ \phi_\delta(\mathbb{F}(I)_t) = \psi_\delta(\mathbb{F}(J)_{(t+\delta)}) = \mathbb{F}(I)_{(t+2\delta)} = \pi_{t \leq t+2\delta}(\mathbb{F}(I)_t)$ and $\phi_\delta \circ \psi_\delta(\mathbb{F}(J)_t) = \phi_\delta(\mathbb{F}(I)_{(t+\delta)}) = \mathbb{F}(J)_{(t+2\delta)} = \theta_{t \leq t+2\delta}(\mathbb{F}(J)_t)$. Therefore, ϕ_δ and ψ_δ are a pair of δ-interleaving morphisms. $\square$

Proposition 3.2. *Given two persistence modules V, W, if there is a δ-matching between their barcodes, then there is a δ-interleaving morphism between them.*

Proof. Suppose there is a δ-matching between the barcodes of V and W, $\rho : \mathrm{Bar}(V) \to \mathrm{Bar}(W)$. By the Structure Theorem 2.3, V and W decompose in a finite direct sum of interval modules

$$V \cong \bigoplus_{I \in \mathrm{Bar}(V)} \mathbb{F}(I), \quad W \cong \bigoplus_{J \in \mathrm{Bar}(W)} \mathbb{F}(W).$$

We can express $V = V_Y \oplus V_N$, $W = W_Y \oplus W_N$ denoting

$$\begin{array}{ll} V_Y \cong \bigoplus_{I \in \operatorname{coim} \rho} \mathbb{F}(I), & W_Y \cong \bigoplus_{J \in \operatorname{im} \rho} \mathbb{F}(J), \\ V_N \cong \bigoplus_{I \in \mathrm{Bar}(V) \setminus \operatorname{coim} \rho} \mathbb{F}(I), & W_N \cong \bigoplus_{J \in \mathrm{Bar}(J) \setminus \operatorname{im} \rho} \mathbb{F}(J). \end{array}$$

The V_Y, W_Y modules separate the "yes, matched" intervals, from the V_N, W_N "not matched" intervals. For every interval $I \in \mathrm{Bar}(V_Y)$, I is δ-matched to an interval $J \in \mathrm{Bar}(W_Y)$ by $\rho(I) = J$. Thus, by Lemma 3.1, for all pair I, J of matched intervals, there exist a par of δ-interleaved morphisms

$$\phi_\delta : \mathbb{F}(I) \to \mathbb{F}(J)_\delta \quad \text{and} \quad \psi_\delta : \mathbb{F}(J) \to \mathbb{F}(I)_\delta$$

which induce the pair of δ-interleaved morphisms

$$\phi_\delta : V_Y \to W_{Y\delta} \quad \text{and} \quad \psi_\delta : W_Y \to V_{Y\delta}.$$

Not matched intervals are of length smaller than 2δ, therefore both, V_N and V_Y are δ-interleaved with the empty set. We can now construct the δ-interleaving morphism $\phi : V \to W$ such as $\phi|_{V_Y} = \phi_Y$ and $\phi|_{V_N} = 0$ and, in a similar way, we also construct $\psi : W \to V$. $\square$

Let (V, π), (W, θ) be two persistence modules. If $I = (b, d]$ is an interval with $d \in \mathbb{R} \cup \{+\infty\}$, denote $\mathrm{Bar}_{I-}(V) = \{(a, b] \in \mathrm{Bar}(V) : a \leq b\}$. Analogously, we can denote $\mathrm{Bar}_{I+}(V) = \{(b, c] \in \mathrm{Bar}(V) : c \geq d\}$. Let $\#$ denote the cardinal operator.

Lemma 3.3. *Let $I = (b, d]$ be an interval. It exists an injective morphism $i : (V, \pi) \in (W, \theta)$, then $\# \mathrm{Bar}_{I-}(V) \leq \# \mathrm{Bar}_{I-}(W)$.*

Proof. Let $E_{I-} = \bigcap_{b<s<d} \operatorname{im} \pi_{s\leq d} \cap \bigcap_{r>d} \ker \pi_{d\leq r} \subseteq V_d$ de the set of elements in V_d witch come from all V_s and disappear in all V_r, for $b < s < d < r$. Thus $\dim E_{I-}(V) = \# \mathrm{Bar}_{I-}(V)$. For every morphism $p : (V, \pi) \to (W, \theta)$ the following diagram conmutes

$$\begin{tikzcd} V_s \arrow[r, "\pi_{s\leq r}"] \arrow[d, "p_s"'] & V_r \arrow[d, "p_r"] \\ W_s \arrow[r, "\theta_{s\leq r}"'] & W_r \end{tikzcd}$$

This implies that $p_r(\operatorname{im} \pi_{s\leq r}) \subseteq \operatorname{im} \theta_{s\leq r}$ and $p_r(\ker \pi_{s\leq r}) \subseteq \ker \theta_{s\leq r}$. Taking $r = d$, $b < s < d$ in the first inclusion, and $s = d$, $r > d$ in the second, it happens that $p_d(E_{I-}(V)) \subseteq E_{I-}(W)$. If we now take $p = i$, the injective morphism of the hypothesis, we get $\dim E_{I-}(V) \leq \dim E_{I-}(W)$. $\square$

Lemma 3.4. *Let $I = (b, d]$ be an interval. It exists a surjective morphism $s : (V, \pi) \to (W, \theta)$, then $\# \mathrm{Bar}_{I+}(V) \geq \# \mathrm{Bar}_{I+}(W)$.*

Proof. $\square$

Lemma 3.5. *If there exists an injection $i : (V, \pi) \in (W, \theta)$, then the induced matching $\mu_{inj} : \mathrm{Bar}(V) \to \mathrm{Bar}(W)$ satisfies:*

1. $\operatorname{coim} \mu_{inj} = \mathrm{Bar}(V)$,

2. $\mu_{inj}(b,d] = (c,d],\ \forall c \leq b,\ \forall (b,d] \in \mathrm{Bar}(V)$.

Proof. The first part 1 $\square$

Lemma 3.6. *If there exists a surjection $s : (V,\pi) \in (W,\theta)$, then the induced matching $\mu_{inj} : \mathrm{Bar}(V) \to \mathrm{Bar}(W)$ satisfies:*

1. $\mathrm{im}\,\mu_{inj} = \mathrm{Bar}(W)$,
2. $\mu_{inj}(b,d] = (b,e],\ \forall b \geq e,\ \forall (b,e] \in \mathrm{Bar}(V)$.

Lemma 3.7. *Let $(V,\pi),(W,\theta)$ are δ-interleaved persistence modules, with δ-interleaving morphisms $\phi : V \to W_\delta$ and $\psi : W \to V_\delta$. Let $\phi : V \to \mathrm{im}\,\phi$ be a surjection and $\mu_{sur} : \mathrm{Bar}(V) \to \mathrm{Bar}(\mathrm{im}\,\phi)$ the induced matching. Then*

1. $\mathrm{coim}\,\mu_{sur} \supseteq \mathrm{Bar}(V)_{\geq 2\delta}$,
2. $\mathrm{im}\,\mu_{sur} = \mathrm{Bar}(\mathrm{im}\,\phi)$ *and*
3. $\mu_{sur}(b,d] = (b,d'],\ (b,d'] \in \mathrm{coim}\,\mu_{sur},\ d' \in [d-2\delta, d]$.

Lemma 3.8. *Let $(V,\pi),(W,\theta)$ are δ-interleaved persistence modules, with δ-interleaving morphisms $\phi : V \to W_\delta$ and $\psi : W \to V_\delta$. Let $\phi : V \to \mathrm{im}\,\phi$ be a injection and $\mu_{inj} : \mathrm{Bar}(\mathrm{im}\,\phi) \to \mathrm{Bar}(W_\delta)$ the induced matching. Then*

1. $\mathrm{coim}\,\mu_{sur} = \mathrm{Bar}(\mathrm{im}\,\phi)$,
2. $\mathrm{im}\,\mu_{inj} \supseteq \mathrm{Bar}(W_\delta)_{\geq 2\delta}$ *and*
3. $\mu_{inj}(b,d'] = (b',d'],\ (b,d'] \in \mathrm{coim}\,\mu_{inj},\ b' \in [b-2\delta, b]$.

Proposition 3.9. *Given two persistence modules V, W, with a δ-interleaving morphism between them, then there is a δ-matching between their barcodes.*

Proof.

$$\begin{tikzcd}
\mathcal{B}(V) & \mathcal{B}(W[\delta])_{2\delta} & \cap & \mathcal{B}(W)_{2\delta} \\
\cup & & & \\
\mathcal{B}(V)_{2\delta} \arrow[r, "\mu_{\mathrm{sur}}"] \arrow[d, "\psi"'] & \mathcal{B}(\mathrm{im}\, f) \arrow[r, "\mu_{\mathrm{inj}}"] \arrow[d, "\psi"'] & \mathrm{im}\,\mu_{\mathrm{inj}} \arrow[r, "\Psi_\delta"] \arrow[d, "\psi"'] & \mathcal{B}(W) \arrow[d, "\psi"'] \\
(b,d) \arrow[r, mapsto] & (b,d') \arrow[r, mapsto] & (b',d') \arrow[r, mapsto] & (b'+\delta, d'+\delta)
\end{tikzcd}$$

□

Theorem 3.10 (Stability)**.** *There is an isometry between a persistence module with the interleaving distance and its barcode with the bottleneck distance. This means that, given two persistence modules* V*,* W*,*

$$d_{int}(V, W) = d_{bot}(\mathrm{Bar}(V), \mathrm{Bar}(W)).$$

Proof. Suppose $d_{int}(V, W) = \delta$. Proposition 3.9 asures there exist a δ-matching between $\mathrm{Bar}(V)$ and $\mathrm{Bar}(W)$. As $d_{bot}(V, W)$ is the infimum δ for witch exists a δ-matching, $d_{bot}(V, W) \leq d_{int}(V, W)$. On the other hand, Proposition 3.2 leads, with the same reasoning, to $d_{int}(V, W) \leq d_{bot}(V, W)$. Thus, it has to be $d_{int}(V, W) = d_{bot}(\mathrm{Bar}(V), \mathrm{Bar}(W))$. □

References

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