Mathematics and Applications. Master's Degree Thesis. 2024-2025.
Draft
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1 Preliminaries (to do work)

The contents of this chapter are based on [1], [2] and [3].

Definition 1.1 (Graded ring).

Definition 1.2 (Graded ideal).

Definition 1.3 (Graded moudule).

Definition 1.4 (Persistance module, finite type).

Definition 1.5 (Module morphism, shift).

Definition 1.6 (Interval module).

Definition 1.7 (Direct sum of persistance modules).

Definition 1.8 (Barcode).

Definition 1.9 (δ-interleaving moudules).

Definition 1.10 (Interleaving distance).

Definition 1.11 (Multiset matching).

Definition 1.12 (δ-matching barcodes).

Definition 1.13 (Bottleneck distance).

Definition 1.14 (Induced matching).

2 Structure Theorem

Fact 2.1 (Structure theorem for finitely generated modules over a principal ideal domain)**.** *Let M be a finitely generated module over a principal ideal domain. There exist a finite sequence of proper ideals $(d_1) \supseteq (d_2) \supseteq \cdots \supseteq (d_n)$ such that*

$$M \cong \bigoplus_{i=1}^{n} R/(d_i).$$

Proposition 2.1. *An ideal $I \subseteq R$ is graded if and only if it is generated by homogeneous elements.*

Theorem 2.1 (Structure)**.** *Let (V, π) be a persistence module. There exist a barcode $\mathcal{B}(V, \pi)$, with $\mu : \mathcal{B}(V, \pi) \longrightarrow \mathbb{N}$, the multiplicity of the barcode intervals, such as there is a unique direct sum decomposition*

$$V \cong \bigoplus_{I \in \mathcal{B}(V)} \mathbb{F}(I)^{\mu(I)}.$$

Proof. (INCOMPLETE) V is of finite type, so it is a finite $\mathbb{F}[x]$-module. As $\mathbb{F}$ is a field, $\mathbb{F}[x]$ is a principal ideal domain, therefore, V is a finitely generated module over a principal ideal domain. Using Fact 2.1 V can be decompose in the direct sum of its free and torsion subgroups, $F \oplus T$. Thus, we have

$$F =$$
$$T = .$$

□

3 Stability Theorem

Lemma 3.1. *Let I, J be two δ-matched intervals. Then, their corresponding interval modules $(\mathbb{F}(I), \pi)$ and $(\mathbb{F}(J), \theta)$ are δ-interleaved.*

Proof. (NOT SURE) Let $I = (a, b]$, $J = (c, d]$. If ρ is the δ-matching between them, then $\rho(I) = J$ and, following Definition 1.12, $(a, b] \subseteq (c + \delta, d + \delta]$ and $(c, d] \subseteq (a + \delta, b + \delta]$, with $b - a > 2\delta$ and $d - c > 2\delta$. Then, the morphisms

$$\begin{array}{ccc} \phi_\delta : \mathbb{F}(I) \to \mathbb{F}(J)_\delta & \multirow{2}{*}{and} & \psi_\delta : \mathbb{F}(J) \to \mathbb{F}(I)_\delta \\ \phi_\delta(\mathbb{F}(I)_t) \mapsto \mathbb{F}(J)_{(t+\delta)} & & \psi_\delta(\mathbb{F}(J)_t) \mapsto \mathbb{F}(I)_{(t+\delta)} \end{array}$$

are well defined as for any $t \in (a, b]$, $t + \delta \in (c, d]$ as $a + \delta > c$ and $b + \delta \leq d$. In the same way, for any $t \in (c, d]$, $t + \delta \in (a, b]$. Thus, $\psi_\delta \circ \phi_\delta(\mathbb{F}(I)_t) = \psi_\delta(\mathbb{F}(J)_{(t+\delta)}) = \mathbb{F}(I)_{(t+2\delta)} = \pi_{t \leq t+2\delta}(\mathbb{F}(I)_t)$ and $\phi_\delta \circ \psi_\delta(\mathbb{F}(J)_t) = \phi_\delta(\mathbb{F}(I)_{(t+\delta)}) = \mathbb{F}(J)_{(t+2\delta)} = \theta_{t \leq t+2\delta}(\mathbb{F}(J)_t)$. Therefore, ϕ_δ and ψ_δ are a pair of δ-interleaving morphisms. $\square$

Proposition 3.1. *Given two persistence modules V, W, if there is a δ-matching between their barcodes, then there is a δ-interleaving morphism between them.*

Proof. (INCOMPLETE) Suppose there is a δ-matching between the barcodes of V and W, $\rho : \mathcal{B}(V) \to \mathcal{B}(W)$. By the Structure Theorem 2.1, V and W decompose in a finite direct sum of interval modules

$$V \cong \bigoplus_{I \in \mathcal{B}(V)} \mathbb{F}(I), \quad W \cong \bigoplus_{J \in \mathcal{B}(W)} \mathbb{F}(W).$$

We can express $V = V_Y \oplus V_N$, $W = W_Y \oplus W_N$ denoting

$$\begin{aligned} SV_Y &\cong \bigoplus_{I \in \operatorname{coim} \rho} \mathbb{F}(I), & W_Y &\cong \bigoplus_{J \in \operatorname{im} \rho} \mathbb{F}(I), \\ V_N &\cong \bigoplus_{I \in \mathcal{B}(V) \setminus \operatorname{coim} \rho} \mathbb{F}(I), & W_N &\cong \bigoplus_{J \in \mathcal{B}(J) \setminus \operatorname{im} \rho} \mathbb{F}(J). \end{aligned}$$

The V_Y, W_Y modules separate the "yes, matched" intervals, from the V_N, W_N "not matched" intervals. For every interval I matched to J by $\rho(I) = J$, Lemma 3.1 $\square$

Lemma 3.2. *Let $I = (b, d]$ be an interval. It exists an injective morphism $i : (V, \pi) \in (W, \theta)$, then $\#\mathcal{B}(V)_I^- \leq \#\mathcal{B}(W)_I^-$. Where $\#$ denotes the cardinal operator.*

Lemma 3.3. *Let $I = (b, d]$ be an interval. It exists a surjective morphism $s : (V, \pi) \in (W, \theta)$, then $\#\mathcal{B}(V)_I^+ \geq \#\mathcal{B}(W)_I^+$. Where $\#$ denotes the cardinal operator.*

Lemma 3.4. *If there exists an injection $i : (V, \pi) \in (W, \theta)$, then the induced matching $\mu_{inj} : \mathcal{B}(V) \to \mathcal{B}(W)$ satisfies:*

1. $\operatorname{coim} \mu_{inj} = \mathcal{B}(V)$,
2. $\mu_{inj}(b, d] = (c, d], \ \forall c \leq b, \ \forall (b, d] \in \mathcal{B}(V)$.

Lemma 3.5. *If there exists a surjection $s : (V, \pi) \in (W, \theta)$, then the induced matching $\mu_{inj} : \mathcal{B}(V) \to \mathcal{B}(W)$ satisfies:*

1. $\operatorname{im} \mu_{inj} = \mathcal{B}(W)$,
2. $\mu_{inj}(b, d] = (b, e], \ \forall b \geq e, \ \forall (b, e] \in \mathcal{B}(V)$.

Lemma 3.6. *Let $(V, \pi), (W, \theta)$ are δ-interleaved persistence modules, with δ-interleaving morphisms $\phi : V \to W_\delta$ and $\psi : W \to V_\delta$. Let $\phi : V \to \operatorname{im} \phi$ be a surjection and $\mu_{sur} : \mathcal{B}(V) \to \mathcal{B}(\operatorname{im} \phi)$ the induced matching. Then*

1. $\operatorname{coim} \mu_{sur} \supseteq \mathcal{B}(V)_{\geq 2\delta}$,
2. $\operatorname{im} \mu_{sur} = \mathcal{B}(\operatorname{im} \phi)$ *and*
3. $\mu_{sur}(b, d] = (b, d'],\ (b, d'] \in \operatorname{coim} \mu_{sur},\ d' \in [d - 2\delta, d]$.

Lemma 3.7. *Let $(V, \pi), (W, \theta)$ are δ-interleaved persistence modules, with δ-interleaving morphisms $\phi : V \to W_\delta$ and $\psi : W \to V_\delta$. Let $\phi : V \to \operatorname{im} \phi$ be a injection and $\mu_{inj} : \mathcal{B}(\operatorname{im} \phi) \to \mathcal{B}(W_\delta)$ the induced matching. Then*

1. $\operatorname{coim} \mu_{sur} = \mathcal{B}(\operatorname{im} \phi)$,
2. $\operatorname{im} \mu_{inj} \supseteq \mathcal{B}(W_\delta)_{\geq 2\delta}$ *and*
3. $\mu_{inj}(b, d'] = (b', d'],\ (b, d'] \in \operatorname{coim} \mu_{inj},\ b' \in [b - 2\delta, b]$.

Proposition 3.2. *Given two persistence modules V, W, if there is a δ-interleaving morphism between them, then there is a δ-matching between their barcodes.*

Theorem 3.1 (Stability)**.** *There is an isometry between a persistence module with the interleaving distance and its barcode with the bottleneck distance. This means that, given two persistence modules V, W,*

$$d_{int}(V, W) = d_{bot}(\mathcal{B}(V), \mathcal{B}(W)).$$

Proof. Suppose $d_{int}(V, W) = \delta$. Proposition 3.2 asures there exist a δ-matching between $\mathcal{B}(V)$ and $\mathcal{B}(W)$. As $d_{bot}(V, W)$ is the infimum δ for witch exists a δ-matching, $d_{bot}(V, W) \leq d_{int}(V, W)$. On the other hand, Proposition 3.1 leads, with the same reasoning, to $d_{int}(V, W) \leq d_{bot}(V, W)$. Thus, it has to be $d_{int}(V, W) = d_{bot}(\mathcal{B}(V), \mathcal{B}(W))$. $\square$

References

[1] V. Nanda, "Computational algebraic topology, lecture notes," *University of Oxford*, 2020.

[2] L. Polterovich, D. Rosen, K. Samvelyan, and J. Zhang, *Topological Persistence in Geometry and Analysis.* American Mathematical Society, 2020.

[3] K. G. Wang, "The basic theory of persistent homology," *University of Chicago*, 2012.