1   Preliminaries (to do work)

The contents of this chapter are based on [1], [2] and [3].

Definition 1.1 (Graded ring)**.**

Definition 1.2 (Graded ideal)**.**

Definition 1.3 (Graded moudule)**.**

Definition 1.4 (Persistance module, finite type)**.**

Definition 1.5 (Module morphism, shift)**.**

Definition 1.6 (Interval module)**.**

Definition 1.7 (Direct sum of persistance modules)**.**

Definition 1.8 (Barcode)**.**

Definition 1.9 (δ-interleaving moudules)**.**

Definition 1.10 (Interleaving distance)**.**

Definition 1.11 (Multiset matching)**.**

Definition 1.12 (δ-matching barcodes)**.**

Definition 1.13 (Bottleneck distance)**.**

2 Structure Theorem

Fact 2.1 (Structure theorem for finitely generated modules over a principal ideal domain)**.** *Let M be a finitely generated module over a principal ideal domain. There exist a finite sequence of proper ideals $(d_1) \supseteq (d_2) \supseteq \cdots \supseteq (d_n)$ such that*

$$M \cong \bigoplus_{i=1}^{n} R/(d_i).$$

Proposition 2.1. *An ideal $I \subseteq R$ is graded if and only if it is generated by homogeneous elements.*

Theorem 2.1 (Structure)**.** *Let (V, π) be a persistence module. There exist a barcode $\mathcal{B}(V, \pi)$, with $\mu : \mathcal{B}(V, \pi) \longrightarrow \mathbb{N}$, the multiplicity of the barcode intervals, such as there is a unique direct sum decomposition*

$$V \cong \bigoplus_{I \in \mathcal{B}(V)} \mathbb{F}(I)^{\mu(I)}.$$

Proof. (INCOMPLETE) V is of finite type, so it is a finite $\mathbb{F}[x]$-module. As $\mathbb{F}$ is a field, $\mathbb{F}[x]$ is a principal ideal domain, therefore, V is a finitely generated module over a principal ideal domain. Using Fact 2.1 V can be decompose in the direct sum of its free and torsion subgroups, $F \oplus T$. Thus, we have

$$F =$$
$$T = .$$

$\square$

3 Stability Theorem

Lemma 3.1. *Let I, J be two δ-matched intervals. Then, their corresponding interval modules $(\mathbb{F}(I), \pi)$ and $(\mathbb{F}(J), \theta)$ are δ-interleaved.*

Proof. (NOT SURE) Let $I = (a, b]$, $J = (c, d]$. If ρ is the δ-matching between them, then $\rho(I) = J$ and, following Definition 1.12, $(a, b] \subseteq (c + \delta, d + \delta]$ and $(c, d] \subseteq (a + \delta, b + \delta]$, with $b - a > 2\delta$ and $d - c > 2\delta$. Then, the morphisms

$$\phi_\delta : \mathbb{F}(I) \to \mathbb{F}(J)_\delta \qquad \psi_\delta : \mathbb{F}(J) \to \mathbb{F}(I)_\delta$$
$$\phi_\delta(\mathbb{F}(I)_t) \mapsto \mathbb{F}(J)_{(t+\delta)} \quad \text{and} \quad \psi_\delta(\mathbb{F}(J)_t) \mapsto \mathbb{F}(I)_{(t+\delta)}$$

are well defined as for any $t \in (a, b]$, $t + \delta \in (c, d]$ as $a + \delta > c$ and $b + \delta \leq d$. In the same way, for any $t \in (c, d]$, $t + \delta \in (a, b]$. Thus, $\psi_\delta \circ \phi_\delta(\mathbb{F}(I)_t) = \psi_\delta(\mathbb{F}(J)_{(t+\delta)}) = \mathbb{F}(I)_{(t+2\delta)} = \pi_{t \leq t+2\delta}(\mathbb{F}(I)_t)$ and $\phi_\delta \circ \psi_\delta(\mathbb{F}(J)_t) = \phi_\delta(\mathbb{F}(I)_{(t+\delta)}) = \mathbb{F}(J)_{(t+2\delta)} = \theta_{t \leq t+2\delta}(\mathbb{F}(J)_t)$. Therefore, ϕ_δ and ψ_δ are a pair of δ-interleaving morphisms. $\square$

Proposition 3.1. *Given two persistence modules V, W, if there is a δ-matching between their barcodes, then there is a δ-interleaving morphism between them.*

Proof. (INCOMPLETE) Suppose there is a δ-matching between the barcodes of V and W, $\rho : \mathcal{B}(V) \to \mathcal{B}(W)$. By the Structure Theorem 2.1, V and W decompose in a finite direct sum of interval modules

$$V \cong \bigoplus_{I \in \mathcal{B}(V)} \mathbb{F}(I), \quad W \cong \bigoplus_{J \in \mathcal{B}(W)} \mathbb{F}(W).$$

We can express $V = V_Y \oplus V_N$, $W = W_Y \oplus W_N$ denoting

$$SV_Y \cong \bigoplus_{I \in \operatorname{coim} \rho} \mathbb{F}(I), \qquad W_Y \cong \bigoplus_{J \in \operatorname{im} \rho} \mathbb{F}(I),$$
$$V_N \cong \bigoplus_{I \in \mathcal{B}(V) \backslash \operatorname{coim} \rho} \mathbb{F}(I), \qquad W_N \cong \bigoplus_{J \in \mathcal{B}(J) \backslash \operatorname{im} \rho} \mathbb{F}(J).$$

The V_Y, W_Y modules separate the "yes, matched" intervals, from the V_N, W_N "not matched" intervals. For every interval I matched to J by $\rho(I) = J$, Lemma 3.1 $\square$

Proposition 3.2. *Given two persistence modules V, W, if there is a δ-interleaving morphism between them, then there is a δ-matching between their barcodes.*

Theorem 3.1 (Stability)**.** *There is an isometry between a persistence module with the interleaving distance and its barcode with the bottleneck distance. This means that, given two persistence modules V, W,*

$$d_{int}(V, W) = d_{bot}(\mathcal{B}(V), \mathcal{B}(W)).$$

Proof. Suppose $d_{int}(V, W) = \delta$. Proposition 3.2 asures there exist a δ-matching between $\mathcal{B}(V)$ and $\mathcal{B}(W)$. As $d_{bot}(V, W)$ is the infimum δ for witch exists a δ-matching, $d_{bot}(V, W) \leq d_{int}(V, W)$. On the other hand, Proposition 3.1 leads, with the same reasoning, to $d_{int}(V, W) \leq d_{bot}(V, W)$. Thus, it has to be $d_{int}(V, W) = d_{bot}(\mathcal{B}(V), \mathcal{B}(W))$. $\square$

References

[1] V. Nanda, "Computational algebraic topology, lecture notes," *University of Oxford*, 2020.

[2] L. Polterovich, D. Rosen, K. Samvelyan, and J. Zhang, *Topological Persistence in Geometry and Analysis*. American Mathematical Society, 2020.

[3] K. G. Wang, "The basic theory of persistent homology," *University of Chicago*, 2012.