

Universidad Autónoma de Madrid

FINAL MASTER THESIS

TDA  
DRAFT 1

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**Abstract**

**Key words**

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# 1 Preliminaries (to do work)

The contents of this chapter are based on [1], [2] and [3].

**Definition 1.1** (Graded ring).

**Definition 1.2** (Graded ideal).

**Definition 1.3** (Graded module).

**Definition 1.4** (Persistence module, finite type).

**Definition 1.5** (Module morphism, shift).

**Definition 1.6** (Interval module).

**Definition 1.7** (Direct sum of persistence modules).

**Definition 1.8** (Barcode).

**Definition 1.9** ( $\delta$ -interleaving modules).

**Definition 1.10** (Interleaving distance).

**Definition 1.11** (Multiset matching).

**Definition 1.12** ( $\delta$ -matching barcodes).

**Definition 1.13** (Bottleneck distance).

**Definition 1.14** (Induced matching).

## 2 Structure Theorem

**Fact 2.1** (Structure theorem for finitely generated modules over a principal ideal domain). *Let  $M$  be a finitely generated module over a principal ideal domain. There exist a finite sequence of proper ideals  $(d_1) \supseteq (d_2) \supseteq \cdots \supseteq (d_n)$  such that*

$$M \cong \bigoplus_{i=1}^n R/(d_i).$$

**Proposition 2.2.** *An ideal  $I \subseteq R$  is graded if and only if it is generated by homogeneous elements.*

**Theorem 2.3** (Structure). *Let  $(V, \pi)$  be a persistence module. There exist a barcode  $\text{Bar}(V, \pi)$ , with  $\mu : \text{Bar}(V, \pi) \rightarrow \mathbb{N}$ , the multiplicity of the barcode intervals, such as there is a unique direct sum decomposition*

$$V \cong \bigoplus_{I \in \text{Bar}(V)} \mathbb{F}(I)^{\mu(I)}.$$

*Proof.* (INCOMPLETE)  $V$  is of finite type, so it is a finite  $\mathbb{F}[x]$ -module. As  $\mathbb{F}$  is a field,  $\mathbb{F}[x]$  is a principal ideal domain, therefore,  $V$  is a finitely generated module over a principal ideal domain. Using Fact 2.1  $V$  can be decompose in the direct sum of its free and torsion subgroups,  $F \oplus T$ . Thus, we have

$$F =$$

$$T = .$$

□

### 3 Stability Theorem

**Lemma 3.1.** *Let  $I, J$  be two  $\delta$ -matched intervals. Then, their corresponding interval modules  $(\mathbb{F}(I), \pi)$  and  $(\mathbb{F}(J), \theta)$  are  $\delta$ -interleaved.*

*Proof.* Let  $I = (a, b]$ ,  $J = (c, d]$ . If  $\rho$  is the  $\delta$ -matching between them, then  $\rho(I) = J$  and, following Definition 1.12,  $(a, b] \subseteq (c - \delta, d + \delta]$  and  $(c, d] \subseteq (a - \delta, b + \delta]$ , with  $b - a > 2\delta$  and  $d - c > 2\delta$ . Then, the morphisms

$$\begin{aligned} \phi_\delta : \mathbb{F}(I) &\rightarrow \mathbb{F}(J)_\delta & \text{and} & \quad \psi_\delta : \mathbb{F}(J) \rightarrow \mathbb{F}(I)_\delta \\ \phi_\delta(\mathbb{F}(I)_t) &\mapsto \mathbb{F}(J)_{(t+\delta)} & & \quad \psi_\delta(\mathbb{F}(J)_t) \mapsto \mathbb{F}(I)_{(t+\delta)} \end{aligned}$$

are well defined as for any  $t \in (a, b]$ ,  $t + \delta \in (c, d]$  as  $a + \delta > c$  and  $b + \delta \leq d$ . In the same way, for any  $t \in (c, d]$ ,  $t + \delta \in (a, b]$ . Thus,  $\psi_\delta \circ \phi_\delta(\mathbb{F}(I)_t) = \psi_\delta(\mathbb{F}(J)_{(t+\delta)}) = \mathbb{F}(I)_{(t+2\delta)} = \pi_{t \leq t+2\delta}(\mathbb{F}(I)_t)$  and  $\phi_\delta \circ \psi_\delta(\mathbb{F}(J)_t) = \phi_\delta(\mathbb{F}(I)_{(t+\delta)}) = \mathbb{F}(J)_{(t+2\delta)} = \theta_{t \leq t+2\delta}(\mathbb{F}(J)_t)$ . Therefore,  $\phi_\delta$  and  $\psi_\delta$  are a pair of  $\delta$ -interleaving morphisms.  $\square$

**Proposition 3.2.** *Given two persistence modules  $V, W$ , if there is a  $\delta$ -matching between their barcodes, then there is a  $\delta$ -interleaving morphism between them.*

*Proof.* Suppose there is a  $\delta$ -matching between the barcodes of  $V$  and  $W$ ,  $\rho : \text{Bar}(V) \rightarrow \text{Bar}(W)$ . By the Structure Theorem 2.3,  $V$  and  $W$  decompose in a finite direct sum of interval modules

$$V \cong \bigoplus_{I \in \text{Bar}(V)} \mathbb{F}(I), \quad W \cong \bigoplus_{J \in \text{Bar}(W)} \mathbb{F}(J).$$

We can express  $V = V_Y \oplus V_N$ ,  $W = W_Y \oplus W_N$  denoting

$$\begin{aligned} V_Y &\cong \bigoplus_{I \in \text{coim } \rho} \mathbb{F}(I), & W_Y &\cong \bigoplus_{J \in \text{im } \rho} \mathbb{F}(J), \\ V_N &\cong \bigoplus_{I \in \text{Bar}(V) \setminus \text{coim } \rho} \mathbb{F}(I), & W_N &\cong \bigoplus_{J \in \text{Bar}(W) \setminus \text{im } \rho} \mathbb{F}(J). \end{aligned}$$

The  $V_Y, W_Y$  modules separate the “yes, matched” intervals, from the  $V_N, W_N$  “not matched” intervals. For every interval  $I \in \text{Bar}(V_Y)$ ,  $I$  is  $\delta$ -matched to an interval  $J \in \text{Bar}(W_Y)$  by  $\rho(I) = J$ . Thus, by Lemma 3.1, for all pair  $I, J$  of matched intervals, there exist a pair of  $\delta$ -interleaved morphisms

$$\phi_\delta : \mathbb{F}(I) \rightarrow \mathbb{F}(J)_\delta \quad \text{and} \quad \psi_\delta : \mathbb{F}(J) \rightarrow \mathbb{F}(I)_\delta$$

which induce the pair of  $\delta$ -interleaved morphisms

$$\phi_\delta : V_Y \rightarrow W_{Y\delta} \quad \text{and} \quad \psi_\delta : W_Y \rightarrow V_{Y\delta}.$$

Not matched intervals are of length smaller than  $2\delta$ , therefore both,  $V_N$  and  $V_Y$  are  $\delta$ -interleaved with the empty set. We can now construct the  $\delta$ -interleaving morphism  $\phi : V \rightarrow W$  such as  $\phi|_{V_Y} = \phi_Y$  and  $\phi|_{V_N} = 0$  and, in a similar way, we also construct  $\psi : W \rightarrow V$ .  $\square$

Let  $\#$  denote the cardinal operator.

**Lemma 3.3.** *Let  $I = (b, d]$  be an interval. It exists an injective morphism  $i : (V, \pi) \in (W, \theta)$ , then  $\# \text{Bar}(V)_I^- \leq \# \text{Bar}(W)_I^-$ .*

**Lemma 3.4.** *Let  $I = (b, d]$  be an interval. It exists a surjective morphism  $s : (V, \pi) \in (W, \theta)$ , then  $\# \text{Bar}(V)_I^+ \geq \# \text{Bar}(W)_I^+$ .*

**Lemma 3.5.** *If there exists an injection  $i : (V, \pi) \in (W, \theta)$ , then the induced matching  $\mu_{inj} : \text{Bar}(V) \rightarrow \text{Bar}(W)$  satisfies:*

1.  $\text{coim } \mu_{inj} = \text{Bar}(V)$ ,
2.  $\mu_{inj}(b, d] = (c, d], \forall c \leq b, \forall (b, d] \in \text{Bar}(V)$ .

**Lemma 3.6.** *If there exists a surjection  $s : (V, \pi) \in (W, \theta)$ , then the induced matching  $\mu_{inj} : \text{Bar}(V) \rightarrow \text{Bar}(W)$  satisfies:*

1.  $\text{im } \mu_{inj} = \text{Bar}(W)$ ,
2.  $\mu_{inj}(b, d] = (b, e], \forall b \geq e, \forall (b, e] \in \text{Bar}(V)$ .

**Lemma 3.7.** *Let  $(V, \pi), (W, \theta)$  are  $\delta$ -interleaved persistence modules, with  $\delta$ -interleaving morphisms  $\phi : V \rightarrow W_\delta$  and  $\psi : W \rightarrow V_\delta$ . Let  $\phi : V \rightarrow \text{im } \phi$  be a surjection and  $\mu_{sur} : \text{Bar}(V) \rightarrow \text{Bar}(\text{im } \phi)$  the induced matching. Then*

1.  $\text{coim } \mu_{sur} \supseteq \text{Bar}(V)_{\geq 2\delta}$ ,
2.  $\text{im } \mu_{sur} = \text{Bar}(\text{im } \phi)$  and
3.  $\mu_{sur}(b, d] = (b, d'], (b, d'] \in \text{coim } \mu_{sur}, d' \in [d - 2\delta, d]$ .

**Lemma 3.8.** *Let  $(V, \pi), (W, \theta)$  are  $\delta$ -interleaved persistence modules, with  $\delta$ -interleaving morphisms  $\phi : V \rightarrow W_\delta$  and  $\psi : W \rightarrow V_\delta$ . Let  $\phi : V \rightarrow \text{im } \phi$  be a injection and  $\mu_{inj} : \text{Bar}(\text{im } \phi) \rightarrow \text{Bar}(W_\delta)$  the induced matching. Then*

1.  $\text{coim } \mu_{sur} = \text{Bar}(\text{im } \phi)$ ,
2.  $\text{im } \mu_{inj} \supseteq \text{Bar}(W_\delta)_{\geq 2\delta}$  and
3.  $\mu_{inj}(b, d'] = (b', d']$ ,  $(b, d'] \in \text{coim } \mu_{inj}$ ,  $b' \in [b - 2\delta, b]$ .

**Proposition 3.9.** *Given two persistence modules  $V, W$ , if there is a  $\delta$ -interleaving morphism between them, then there is a  $\delta$ -matching between their barcodes.*

**Theorem 3.10** (Stability). *There is an isometry between a persistence module with the interleaving distance and its barcode with the bottleneck distance. This means that, given two persistence modules  $V, W$ ,*

$$d_{int}(V, W) = d_{bot}(\text{Bar}(V), \text{Bar}(W)).$$

*Proof.* Suppose  $d_{int}(V, W) = \delta$ . Proposition 3.9 assures there exist a  $\delta$ -matching between  $\text{Bar}(V)$  and  $\text{Bar}(W)$ . As  $d_{bot}(V, W)$  is the infimum  $\delta$  for which exists a  $\delta$ -matching,  $d_{bot}(V, W) \leq d_{int}(V, W)$ . On the other hand, Proposition 3.2 leads, with the same reasoning, to  $d_{int}(V, W) \leq d_{bot}(V, W)$ . Thus, it has to be  $d_{int}(V, W) = d_{bot}(\text{Bar}(V), \text{Bar}(W))$ .  $\square$

## References

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