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Draft
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1 Preliminaries (to do work)

The contents of this chapter are based on [1], [2] and [3].

Definition 1.1 (Graded ring)**.**

Definition 1.2 (Graded ideal)**.**

Definition 1.3 (Graded moudule)**.**

Definition 1.4 (Persistance module, finite type)**.**

Definition 1.5 (Barcode)**.**

Definition 1.6 (δ-interleaving)**.**

Definition 1.7 (Interleaving distance)**.**

Definition 1.8 (δ-matching)**.**

Definition 1.9 (Bottleneck distance)**.**

2 Structure Theorem

Fact 2.1 (Structure theorem for finitely generated modules over a principal ideal domain)**.** *Let M be a finitely generated module over a principal ideal domain. There exist a finite sequence of proper ideals $(d_1) \supseteq (d_2) \supseteq \cdots \supseteq (d_n)$ such that*

$$M \cong \bigoplus_{i=1}^{n} R/(d_i).$$

Theorem 2.1 (Structure)**.** *Let (V, π) be a persistence module. There exist a finite set* $\mathrm{bar}(V, \pi)$ *of intervals and a function $\mu : \mathrm{bar}(V, \pi) \longrightarrow \mathbb{N}$ and there is a unique direct sum decomposition*

$$(V, \pi) \cong \bigoplus_{i=1}^{N} (I_i, c_i)^{m_i}.$$

Proof. (V, π) is of finite type, so it is a finite $\mathbb{F}[x]$-module. As $\mathbb{F}$ is a field, $\mathbb{F}[x]$ is a principal ideal domain, therefor, (V, π) is a finitely generated module over a principal ideal domain. Fact 2.1 $\square$

3 Stability Theorem

Lemma 3.1.

Theorem 3.1 (Stability)**.** *Given two persistence modules (V, π), (W, ϕ), we have*

$$d_{int}((V, \pi), (W, \phi)) = d_{bot}(\mathrm{bar}(V, \pi), \mathrm{bar}(W, \phi)).$$

References

[1] V. Nanda, "Computational algebraic topology, lecture notes," *University of Oxford*, 2020.

[2] L. Polterovich, D. Rosen, K. Samvelyan, and J. Zhang, *Topological Persistence in Geometry and Analysis.* American Mathematical Society, 2020.

[3] K. G. Wang, "The basic theory of persistent homology," *University of Chicago*, 2012.