**Mathematics and Applications. Master's Degree Thesis. 2024-2025.**

**Draft**

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# 1 Preliminaries (to do work)

The contents of this chapter are based on [1], [2] and [3].

**Definition 1.1** (Graded ring).

**Definition 1.2** (Graded ideal).

**Definition 1.3** (Graded moudule).

**Definition 1.4** (Persistance module, finite type). $(V, \pi)$, where $V = \{V_t\}_{t\in\mathbb{R}}$ is a collection of finite dimensional vector spaces over a field $\mathbb{F}$, and $\pi = \{\pi_{s\leq t}\}$ is a collection of linear maps $\pi_{s\leq t} : V_s \to V_t$.

**Definition 1.5** (Barcode).

**Definition 1.6** ($\delta$-interleaving).

**Definition 1.7** (Interleaving distance).

**Definition 1.8** ($\delta$-matching).

**Definition 1.9** (Bottleneck distance).

## 2 Structure Theorem

**Fact 2.1** (Structure theorem for finitely generated modules over a principal ideal domain)**.** *Let $M$ be a finitely generated module over a principal ideal domain. There exist a finite sequence of proper ideals $(d_1) \supseteq (d_2) \supseteq \cdots \supseteq (d_n)$ such that*

$$M \cong \bigoplus_{i=1}^{n} R/(d_i).$$

**Proposition 2.1.** *An ideal $I \subseteq R$ is graded if and only if it is generated by homogeneous elements.*

**Theorem 2.1** (Structure)**.** *Let $(V, \pi)$ be a persistence module. There exist a finite set* $\operatorname{bar}(V, \pi)$ *of intervals and a function $\mu : \operatorname{bar}(V, \pi) \longrightarrow \mathbb{N}$ and there is a unique direct sum decomposition*

$$(V, \pi) \cong \bigoplus_{i=1}^{N} (I_i, c_i)^{m_i}.$$

*Proof.* $(V, \pi)$ is of finite type, so it is a finite $\mathbb{F}[x]$-module. As $\mathbb{F}$ is a field, $\mathbb{F}[x]$ is a principal ideal domain, therefore, $V$ is a finitely generated module over a principal ideal domain. Using Fact 2.1 $V$ can be decompose in the direct sum of its free and torsion subgroups. ☐

# 3 Stability Theorem

**Lemma 3.1.**

**Theorem 3.1** (Stability)**.** *Given two persistence modules* $(V, \pi)$*,* $(W, \phi)$*, we have*

$$d_{int}((V, \pi), (W, \phi)) = d_{bot}(\mathrm{bar}(V, \pi), \mathrm{bar}(W, \phi)).$$

## References


[1] V. Nanda, "Computational algebraic topology, lecture notes," *University of Oxford*, 2020.

[2] L. Polterovich, D. Rosen, K. Samvelyan, and J. Zhang, *Topological Persistence in Geometry and Analysis.* American Mathematical Society, 2020.

[3] K. G. Wang, "The basic theory of persistent homology," *University of Chicago*, 2012.