

1 Preliminaries (to do work)

The contents of this chapter are based on [1], [2] and [3].

Definition 1.1 (Graded ring).

Definition 1.2 (Graded ideal).

Definition 1.3 (Graded module).

Definition 1.4 (Persistence module, finite type). (V, π) , where $V = \{V_t\}_{t \in \mathbb{R}}$ is a collection of finite dimensional vector spaces over a field \mathbb{F} , and $\pi = \{\pi_{s \leq t}\}$ is a collection of linear maps $\pi_{s \leq t} : V_s \rightarrow V_t$.

Definition 1.5 (Barcode).

Definition 1.6 (δ -interleaving).

Definition 1.7 (Interleaving distance).

Definition 1.8 (δ -matching).

Definition 1.9 (Bottleneck distance).

2 Structure Theorem

Fact 2.1 (Structure theorem for finitely generated modules over a principal ideal domain). *Let M be a finitely generated module over a principal ideal domain. There exist a finite sequence of proper ideals $(d_1) \supseteq (d_2) \supseteq \cdots \supseteq (d_n)$ such that*

$$M \cong \bigoplus_{i=1}^n R/(d_i).$$

Proposition 2.1. *An ideal $I \subseteq R$ is graded if and only if it is generated by homogeneous elements.*

Theorem 2.1 (Structure). *Let (V, π) be a persistence module. There exist a finite set $\text{bar}(V, \pi)$ of intervals and a function $\mu : \text{bar}(V, \pi) \rightarrow \mathbb{N}$ and there is a unique direct sum decomposition*

$$(V, \pi) \cong \bigoplus_{i=1}^N (I_i, c_i)^{m_i}.$$

Proof. (V, π) is of finite type, so it is a finite $\mathbb{F}[x]$ -module. As \mathbb{F} is a field, $\mathbb{F}[x]$ is a principal ideal domain, therefore, V is a finitely generated module over a principal ideal domain. Using Fact 2.1 V can be decompose in the direct sum of its free and torsion subgroups, $F \oplus T$. Thus, we have

$$\begin{aligned} F &= \\ T &= . \end{aligned}$$

□

3 Stability Theorem

Lemma 3.1.

Theorem 3.1 (Stability). *Given two persistence modules (V, π) , (W, ϕ) , we have*

$$d_{int}((V, \pi), (W, \phi)) = d_{bot}(\text{bar}(V, \pi), \text{bar}(W, \phi)).$$

References

- [1] V. Nanda, “Computational algebraic topology, lecture notes,” *University of Oxford*, 2020.
- [2] L. Polterovich, D. Rosen, K. Samvelyan, and J. Zhang, *Topological Persistence in Geometry and Analysis*. American Mathematical Society, 2020.
- [3] K. G. Wang, “The basic theory of persistent homology,” *University of Chicago*, 2012.