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Exercise 4. Jacobian criterion (Gathmann 2.24)

Proof. Let f be a reduced polynomial and $P = (a, b) \in V(f)$ a smooth point. We have that

$$\begin{gathered} (a,b) \text{ is a smooth point } \Leftrightarrow g(x,y) = f(x+a, x+b) \text{ is smooth in } (0,0) \Leftrightarrow \\ g_1(x,y) \neq 0 \Leftrightarrow c_1x + c_2y \neq 0 \Leftrightarrow c_1 \neq 0 \text{ or } c_2 \neq 0 \Leftrightarrow \\ \frac{\partial g}{\partial x} = c_1 + \ldots \neq 0 \text{ or } \frac{\partial g}{\partial y} = c_2 + \ldots \neq 0 \Leftrightarrow \\ \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)\Bigg|_{(x,y)=(0,0)} \neq (0,0) \Leftrightarrow \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\Bigg|_{(x,y)=(a,b)} \neq (0,0). \end{gathered}$$

□

Exercise 9. (Gathmann 2.7)

Let $f, g \in K[x, y]$ be two reduced polynomials without a common factor, that vanish at $(0, 0)$.

(a)

Proof. Because of Gathmann 1.12, exists $h(x) \neq 0 \in < f, g >$. We have that $h = af + bg$ with $a, b \in \frac{K[x,y]_{(0,0)}}{\langle f,g \rangle}$ so evaluating at 0 we have $h(0) = 0$ as $f(0,0) = g(0,0) = 0$ and can asume that h has no constant coefficient. We can then factor out some x^{n_1} so that

$$h(x) = x^{n_1}t(x), \text{ with } t(x) = c_1 + c_2x + \cdots + c_mx^{m-n_1}.$$

As $t(x)$ is a unit, we can divide by it having

$$x^{n_1} = \frac{h(x)}{t(x)} \in < f, g >_{K[x,y]_{(0,0)}} .$$

That is to say $x^{n_1} = 0$. Analogously $y^{n_2} = 0$, and taking $n = \max(n_1, n_2)$ we have $x^n = y^n = 0$.

□

(b)

Proof. Lets recall the property of power series that says that

$$\frac{1}{1-z} = \sum_{i=0}^{\infty} z^i, \text{ for } |z| < 1.$$

Let now $\frac{f}{g} \in K[x, y]_{(0,0)}$. As K is a field, we can rewrite $g = 1 - h$ for some h, and as we are in the local ring around $(0, 0)$, $|h| < 1$. Thus

$$\frac{f}{g} = f\frac{1}{1-h} = f\sum_{i=0}^{\infty} h^i,$$

but because of **(a)**, for some n all terms of degree grater or equal to n go to 0, so the previous sum is actually finite and every element of $\frac{K[x,y]_{(0,0)}}{<f,g>}$ has a polynomial representative. □

(c)

Proof. Because of **(b)**, as every element of $\frac{K[x,y]_{(0,0)}}{<f,g>}$ has a polynomial representative there are no infinite term elements in the local ring, therefore

$$\dim_K \frac{K[x,y]_{(0,0)}}{< f,g >} < \infty.$$

□

Exercise 10. (Gathmann 2.8)

Let $f, g \in K[x,y]$ be two polynomials that vanish at the origin.

(a)

Proof. Let f, g have no common factor. We have that $K[V(g)]_{(0,0)} = \frac{K[x,y]_{(0,0)}}{<g>}$. Suppose that there are some $a_i \neq 0 \in K, i \leq n$ such that $\sum_i a_i f^i = 0$. That is $\sum_i a_i f^i \in < g >$, so for some $h \neq 0$ in $K[x,y]_{(0,0)}$, $\sum_i a_i f^i = hg$, and taking common factor we have

$$\sum_i a_i f^i = f^k(a_k + \sum_i a_i f^{i-k}) = hg.$$

As f and g have no common factor there has to be h' such as that

$$(a_k + \sum_i a_i f^{i-k}) = h'g,$$

but evaluating in $(0,0)$ we have

$$(a_k + \sum_i a_i f^{i-k}(0,0)) = h'(0,0)g(0,0) \Rightarrow (a_k + \sum_i a_i \cdot 0) = h'(0,0) \cdot 0 \Rightarrow a_k = 0,$$

contradicting with $a_i \neq 0$ and proving that $\{f^n : n \in \mathbb{Z}_{\leq 0}\}$ is linearly independent. □

(b)

Proof. Let f, g have a common factor h that vanishes at the origin. We can write $f = f'h^a$ and $g = g'h^b$. We have

$$\dim_K \frac{K[x,y]_{(0,0)}}{< f,g >} = \dim_K \frac{K[x,y]_{(0,0)}}{< f'h^a, g'h^b >} \geq \dim_K \frac{K[x,y]_{(0,0)}}{< h >},$$

and because of **(a)**, as h and f' have no common factors (the same argument could be given taking g'), the infinite family $\{f'^n : n \in \mathbb{Z}_{\leq 0}\}$ is linearly independent in $\frac{K[x,y]_{(0,0)}}{<h>}$, so $\dim_K \frac{K[x,y]_{(0,0)}}{<h>} = \infty$, as polynomials of the form f'^n are linearly independent spanning an infinite-dimensional vector space. Thus

$$\dim_K \frac{K[x,y]_{(0,0)}}{< f,g >} = \infty.$$

□

References

[1] Andreas Gathmann, *Plane Algebraic Curves*, Class Notes RPTU Kaiserslautern 2023.