## Exercise 1

## Exercise 2

Let $\mathbb{Q}^* = \mathbb{Q} - \{0\}$ be the multiplicative group of non-zero rationals. Every element $q \in \mathbb{Q}^*$ can be written as $q = \frac{a}{b}$ with $a, b \in \mathbb{Z}$. Every integer admits a prime factorization such that $a = p_1^{e_1} \dots p_n^{e_n}$, $b = q_1^{f_1} \dots q_m^{f_m}$ with $p_i, q_i$ prime numbers and $e_i, f_k \in \mathbb{N}$. If $Q*$ were finitely generated, there would be a finite set $S$ which generates $Q*$. Each element $\frac{a}{b} \in S$ could be decompose into a fraction of prime factorizations. But prime numbers are infinite, so piking a prime $p$ not included in any of the factorizations of the elements in $S$ would be a contradiction. As $p \in \mathbb{Q}^*$ but it can not be generated by elements of $S$ as it is a prime not in $S$. Thus, $\mathbb{Q}^*$ must be infinitely generated.

## Exercise 3

## Exercise 4

Let $G_1 = \langle S_1 = \{a, b\} \mid R_1 = \{a^3b^5a^{-3}b^{-5}\}\rangle$. To show that $G_1$ is infinite we will construct a surjective homomorphism from $G_1$ to $\mathbb{Z}$. First, define $\theta\colon S_1 \to \mathbb{Z}$ as $\theta(a) = 2$ and $\theta(b) = -1$. By the universal property of free groups there is a unique homomorphism $\hat{\theta}\colon F(S) \to \mathbb{Z}$ extending $\theta$ such that $\hat{\theta}(a) = 2$ and $\hat{\theta}(b) = -1$. Thus, we have that

$$\hat{\theta}(a^3b^5a^{-3}b^{-5}) = 3\hat{\theta}(a) + 5\hat{\theta}(b) - 3\hat{\theta}(a) - 5\hat{\theta}(b) = 0.$$

Thus, $R_1 \subset \ker(\hat{\theta})$ and in fact $\langle\langle R_1\rangle\rangle \subset \ker(\hat{\theta})$. For $(ab)^n \in F(S_1)$, $n \in \mathbb{Z}$, we have that $\hat{\theta}((ab)^n) = n$. This makes any element of $\mathbb{Z}$ reachable from an element of $F(S_1)$ by $\hat{\theta}$, making $\hat{\theta}$ a surjective homomorphism. Hence, by the fundamental theorem on homomorphisms, there exists a unique surjective homomorphism $h_1 : F(S_1)/\langle\langle R_1\rangle\rangle = G_1 \to \mathbb{Z}$ proving $G_1$ is infinite.

Let now $G_2 = \langle S_2 = \{a, b\} \mid R_2 = \{a^2b^3\}\rangle$. We proceed as before. Define $\phi\colon S_2 \to \mathbb{Z}$ as $\phi(a) = 3$ and $\phi(b) = -2$. Then there exists a unique homomorphism $\hat{\phi}\colon F(S_2) \to \mathbb{Z}$ such that $\hat{\phi}(a) = 3$ and $\hat{\phi}(b) = -2$. Then

$$\hat{\phi}(a^2b^3) = 3\hat{\phi}(a) - 2\hat{\phi}(b) = 3 \cdot (-2) - 2 \cdot 3 = 0.$$

Hence, $\langle\langle R_2\rangle\rangle \subset \ker(\hat{\phi})$ and $\hat{\phi}((ab)^n) = n$, $\phi$ is surjective and there exists a unique surjective $h_2 : F(S_2)/\langle\langle R_2\rangle\rangle = G_2 \to \mathbb{Z}$. Thus, $G_2$ is also infinite.

## Exercise 8

Let $G = \langle S \mid R\rangle = F(S)/\langle\langle R\rangle\rangle$ be a finite presentation. All words $w \in (S \sqcup S^{-1})^*$ such that $w = 1$ in $G$ are the words $w \in \langle\langle R\rangle\rangle$ by the definition of group presentation. Recall that

$$\langle\langle R\rangle\rangle = \bigcup_{i=0}^{\infty} \left\{ \prod_{j=0}^{\infty} (g_j^{-1} r_j^{\epsilon_j} g_j) \mid g_j \in F(G), r_j \in R, \epsilon_j \in \{\pm 1\} \right\}.$$

To enumerate the words $w$ we can proceed as follows:

1. As $|R|$ is finite, suppose $|R| = n$. We can enumerate all elements of $R$ and $R^{-1}$ numbering them as:
$$r_1, r_1^{-1}, r_2, r_2^{-1}, \dots, r_n, r_n^{-1}. \tag{1}$$
2. In the same manner, as $|S|$ is finite, suppose $|S| = m$, and enumerate all elements of $S$ and $S^{-1}$ as:
$$s_1, s_1^{-1}, s_2, s_2^{-1}, \dots, s_m, s_m^{-1}. \tag{2}$$

3. Finally, now we just need to enumerate the elements of $\langle\langle R \rangle\rangle$ in a sorted way without enumerating one same element more than once. For so, start enumerating the elements $g \in F(S)$ by making combinations of the elements of (2) in a lexicographic order and in increasing word length. As $|S|$ is finite, for each word length $k$, the amount of words of $F(S)$ of length $k$ is going to be $m^k$ minus the number of produced words that can be reduced. In any case, there is a finite number of words of length $k$ in $F(S)$. Denote the set of words of length less than or equal to $k$ as $F(S)_k$ and note that it is finite too.

   For each word length $k$, we can iterate over the elements of (1), and enumerate all the elements

   $$\prod_{j=0}^{k} (g_j^{-1} r_j^{\epsilon_j} g_j) \text{ with } g_j \in F(S)_k, r_j \in R, \epsilon_j \in \{\pm 1\}.$$

   We reduce each obtained word and compare it with the finite number of words we had previously enumerated. If it is a new word, we enumerate it.

Each $k$-th iteration of Step 3 of the previous procedure is finite as (1) is finite and $F(S)_k$ is finite. Therefore, on an input $w \in (S \sqcup S^{-1})^*$, if $w = 1$ in $G$, as $w$ would have finite length, our procedure will find it in finite time. Else, our procedure may run forever.

## References

[1] Marco Linton, *Geometric group theory notes*, UAM Algebra Advance Course, 2025.