

Name: Gonzalo Ortega Carpintero

Exercise 4. Jacobian criterion (Gathmann 2.24)

Proof. Let f be a reduced polynomial and $P = (a, b) \in V(f)$ a smooth point. We have that

$$\begin{aligned}
 (a, b) \text{ is a smooth point} &\Leftrightarrow g(x, y) = f(x + a, y + b) \text{ is smooth in } (0, 0) \Leftrightarrow \\
 g_1(x, y) &\neq 0 \Leftrightarrow c_1x + c_2y \neq 0 \Leftrightarrow c_1 \neq 0 \text{ or } c_2 \neq 0 \Leftrightarrow \\
 \frac{\partial g}{\partial x} &= c_1 + \dots \neq 0 \text{ or } \frac{\partial g}{\partial y} = c_2 + \dots \neq 0 \Leftrightarrow \\
 \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right) \Big|_{(x,y)=(0,0)} &\neq (0, 0) \Leftrightarrow \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \Big|_{(x,y)=(a,b)} \neq (0, 0).
 \end{aligned}$$

□

Exercise 9. (Gathmann 2.7)

Let $f, g \in K[x, y]$ be two reduced polynomials without a common factor, that vanish at $(0, 0)$.

(a)

Proof. Because of Gathmann 1.12, exists $h(x) \neq 0 \in \langle f, g \rangle$. We have that $h = af + bg$ with $a, b \in \frac{K[x, y]_{(0,0)}}{\langle f, g \rangle}$ so evaluating at 0 we have $h(0) = 0$ as $f(0, 0) = g(0, 0) = 0$ and can assume that h has no constant coefficient. We can then factor out some x^{n_1} so that

$$h(x) = x^{n_1}t(x), \text{ with } t(x) = c_1 + c_2x + \dots + c_mx^{m-n_1}.$$

As $t(x)$ is a unit, we can divide by it having

$$x^{n_1} = \frac{h(x)}{t(x)} \in \langle f, g \rangle_{K[x, y]_{(0,0)}}.$$

That is to say $x^{n_1} = 0$. Analogously $y^{n_2} = 0$, and taking $n = \max(n_1, n_2)$ we have $x^n = y^n = 0$.

□

(b)

Proof. Lets recall the property of power series that says that

$$\frac{1}{1-z} = \sum_{i=0}^{\infty} z^i, \text{ for } |z| < 1.$$

Let now $\frac{f}{g} \in K[x, y]_{(0,0)}$. As K is a field, we can rewrite $g = 1 - h$ for some h , and as we are in the local ring around $(0, 0)$, $|h| < 1$. Thus

$$\frac{f}{g} = f \frac{1}{1-h} = f \sum_{i=0}^{\infty} h^i,$$

but because of (a), for some n all terms of degree greater or equal to n go to 0, so the previous sum is actually finite and every element of $\frac{K[x, y]_{(0,0)}}{\langle f, g \rangle}$ has a polynomial representative.

□

(c)

Proof. Because of (b), as every element of $\frac{K[x,y]_{(0,0)}}{\langle f,g \rangle}$ has a polynomial representative there are no infinite term elements in the local ring, therefore

$$\dim_K \frac{K[x,y]_{(0,0)}}{\langle f,g \rangle} < \infty.$$

□

Exercise 10. (Gathmann 2.8)

Let $f, g \in K[x, y]$ be two polynomials that vanish at the origin.

(a)

Proof. Let f, g have no common factor. We have that $K[V(g)]_{(0,0)} = \frac{K[x,y]_{(0,0)}}{\langle g \rangle}$. Suppose that there are some $a_i \neq 0 \in K, i \leq n$ such that $\sum_i a_i f^i = 0$. That is $\sum_i a_i f^i \in \langle g \rangle$, so for some $h \neq 0$ in $K[x, y]_{(0,0)}$, $\sum_i a_i f^i = hg$, and taking common factor we have

$$\sum_i a_i f^i = f^k (a_k + \sum_i a_i f^{i-k}) = hg.$$

As f and g have no common factor there has to be h' such as that

$$(a_k + \sum_i a_i f^{i-k}) = h'g,$$

but evaluating in $(0, 0)$ we have

$$(a_k + \sum_i a_i f^{i-k}(0, 0)) = h'(0, 0)g(0, 0) \Rightarrow (a_k + \sum_i a_i \cdot 0) = h'(0, 0) \cdot 0 \Rightarrow a_k = 0,$$

contradicting with $a_i \neq 0$ and proving that $\{f^n : n \in \mathbb{Z}_{\leq 0}\}$ is linearly independent. □

(b)

Proof. Let f, g have a common factor h that vanishes at the origin. We can write $f = f'h^a$ and $g = g'h^b$. We have

$$\dim_K \frac{K[x,y]_{(0,0)}}{\langle f,g \rangle} = \dim_K \frac{K[x,y]_{(0,0)}}{\langle f'h^a, g'h^b \rangle} \geq \dim_K \frac{K[x,y]_{(0,0)}}{\langle h \rangle},$$

and because of (a), as h and f' have no common factors (the same argument could be given taking g'), the infinite family $\{f'^n : n \in \mathbb{Z}_{\leq 0}\}$ is linearly independent in $\frac{K[x,y]_{(0,0)}}{\langle h \rangle}$, so $\dim_K \frac{K[x,y]_{(0,0)}}{\langle h \rangle} = \infty$, as polynomials of the form f'^n are linearly independent spanning an infinite-dimensional vector space. Thus

$$\dim_K \frac{K[x,y]_{(0,0)}}{\langle f,g \rangle} = \infty.$$

□

References

- [1] Andreas Gathmann, *Plane Algebraic Curves*, Class Notes RPTU Kaiserslautern 2023.