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Exercise 3.

If $f \in K[x_0, x_1, x_2]$ be a non-constant homogeneous polynomial and let $f = \prod_{i=1}^{n} g_i^{m_i}$ be the decomposition of f in irreducible polynomials.

(a)

Proof.

($\Rightarrow$) If $V(f)$ is irreducible, suppose that we can not write f as a power of a irreducible polynomial. Then, there exist $g_1, g_2 \in K[x_0, x_1, x_2]$, with $g_1 \neq g_2$, such that $f = g_1g_2$. But then we have $V(f) = V(g_1g_2) = V(g_1) \cup V(g_2)$ as seen in Remark 3.9 of [1]. This is a contradiction as $V(f)$ is irreducible so it need to be $f = g^m$ for some g irreducible.

($\Leftarrow$) If $f = g^m$, with g irreducible, then we have $V(f) = V(g^m) = V(g)$. $V(g)$ needs to be irreducible, because if not it could be expressed as the union of two curves $V(g_1)$ and $V(g_2)$, having $V(g) = V(g_1) \cap V(g_2) = V(g_1g_2)$, and it would be $g = g_1^m g_2^m$ contradicting the fact of g being irreducible. Thus, $V(g^m)$ is irreducible. □

(b)

Proof. For $n = 1$, $f = g_1^{m_1}$, so $g_1^{m_1}$ is homogeneous and so g_1, as f is homogeneous. Suppose $h = \prod_{i=1}^{n-1} g_i^{m_i}$ homogeneous. If g_n was not homogeneous, $g_n^{m_n}$ neither would be, and the product $hg_n^{m_n}$ would not be homogeneous as the product of an homogeneous and a non homogeneous polynomials can not be homogeneous. But $hg_n^{m_n} = \prod_{i=1}^{n} g_i^{m_i} = f$ contradicting the fact of f been homogeneous. Thus g_n must be homogeneous and, because of induction, each g_i is homogeneous.

As each g_i is an homogeneous irreducible polynomial, because of **(a)**, $V(g_i)$ is a irreducible curve. Therefor, $V(f) = \cup_i V(g_i)$ is a decomposition in irreducible curves of $V(f)$. □

Exercise 4

Proof. Let $I < K[x_0, x_1, x_2]$ be a homogeneous ideal. Let $K[x_0, x_1, x_2]/I$ be finite dimensional. If $V(I) \neq \emptyset \subset \mathbb{P}^2$ then $\exists p = [p_0 : p_1 : p_2] \in \mathbb{P}^2$ such that $\forall f \in I$, $f(p) = 0$. Without loos of generality we can asume $p = [1 : p_1 : p_2]$ (the argument will follow analogously choosing any other coordinate to be different from 0). Thus $\forall n \in \mathbb{N}, x_0^n \notin I$.

Therefor we have an infinite family $\{x_0^n + I\}_{n \in \mathbb{N}}$ of elements of $K[x_0, x_1, x_2]/I$. If the family elements were linearly dependent it would exist $n_i \in \mathbb{N}$, and a family of indices $J \subset \mathbb{N}$ such that $x_0^{n_i} = \sum_{j \in J} a_j(x_0^{n_j} + I)$. Let $g = x_0^{n_i} - \sum_{j \in J} a_j(x_0^{n_j} + I) \in I$. Because of the equivalences of the homogeneous ideal definition seen in class, we have $g_i \in I$ for each homogeneous part g_i of g. In particular $x_0^{n_i} \in I$ forming a contradiction. Thus, $\{x_0^n + I\}_{n \in \mathbb{N}}$ is a linearly independent infinite family of $K[x_0, x_1, x_2]/I$ so the quotient can not be infinite dimensional contradicting the exercise hypothesis. It then need to be $V(I) = \emptyset$. □

References

[1] Andreas Gathmann, *Plane Algebraic Curves*, Class Notes RPTU Kaiserslautern 2023.