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Exercise 1

Proof. Let X and Y path-connected spaces. If $[\gamma] \in \pi_1(X \times Y, (x, y))$, then $\gamma \colon I \to X \times Y$ is a loop in the direct product pace where $\gamma(0) = \gamma(1) = (x, y)$. We can write γ as $\gamma(t) = (\gamma_X(t), \gamma_Y(t))$ where $\gamma_X \colon I \to X$ and $\gamma_Y \colon I \to Y$ are loops in X and Y respectively with $\gamma_X(0) = \gamma_X(1) = x$ and $\gamma_Y(0) = \gamma_Y(1) = y$. Hence we can define the morphism

$$\begin{aligned} f \colon \pi_1(X \times Y, (x, y)) &\to \pi_1(X, x) \times \pi_1(Y, y) \\ [\gamma] &\mapsto ([\gamma_X], [\gamma_Y]). \end{aligned}$$

- Let $*$ denote the path concatenation operator and let $[\gamma_1], [\gamma_2] \in \pi_1(X \times Y, (x, y))$, then

  $$\begin{aligned} f([\gamma_1] \cdot [\gamma_2]) &= f([\gamma_1 * \gamma_2]) = ([(\gamma_1 * \gamma_2)_X], [(\gamma_1 * \gamma_2)_Y]) \\ &= ([\gamma_{1X}], [\gamma_{1Y}]) \cdot ([\gamma_{2X}], [\gamma_{2Y}]) = f([\gamma_1]) \cdot f([\gamma_2]) \end{aligned}$$

  and f is in fact an homomorphism.

- If $([\gamma_X], [\gamma_Y])$ is the identity in $\pi_1(X, x) \times \pi_1(Y, y)$ then $[\gamma_X]$ is the class of the constant path $[\gamma_X] = [x_0]$. The same for $[\gamma_Y] = [y_0]$. Therefore, if $f([\gamma]) = ([x_0], [y_0])$ then $[\gamma] = [(x_0, y_0)]$ is also the identity in $\pi_1(X \times Y, (x, y))$. Hence, f is injective.

- For any pair of path classes $[\alpha] \in \pi_1(X, x)$ and $[\beta] \in \pi_1(Y, y)$ we can take the path $\gamma(t) = (\alpha(t), \beta(t))$ for which $f([\gamma]) = ([\alpha], [\beta])$. Hence, f is also surjective.

This makes f a isomorphism and $\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y)$ □

Exercise 7

a. We have that $:= \mathbb{R}^3 \setminus \{\text{x-axis and y-axis}\} = \mathbb{R}^3 \setminus \{(x, 0, 0), (0, y, 0) : \forall x, y \in \mathbb{R}\}$. Defining the retraction r_1 such as

$$\begin{aligned} r_1 \colon X &\to S^2 \setminus \{\pm(1, 0, 0), \pm(0, 1, 0)\} \\ (x, y, z) &\mapsto \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}, \end{aligned}$$

we show that the space minus two axis is homotopic to the sphere without four points. Making a translation of one of the missing points of the sphere, we have that

$$S^2 \setminus \{\pm(1, 0, 0), \pm(0, 1, 0)\} \cong S^2 \setminus \{(1, 0, 0), \pm(0, 1, 0), (0, 0, 1)\}$$

and defining the retraction r_2 as the stereographic projection

$$\begin{aligned} r_2 \colon S^2 \setminus \{(1, 0, 0), \pm(0, 1, 0), (0, 0, 1)\} &\to \mathbb{R}^2 \setminus \{\pm(1, 0), (1, 0)\} \\ (x, y, z) &\mapsto \left(\frac{y}{1 - x}, \frac{z}{1 - x}\right), \end{aligned}$$

we have that $X \simeq \mathbb{R}^2 \setminus \{\pm(1, 0), (1, 0)\} \simeq S^1 \vee S^1 \vee S^1$ and therefore

$$\pi_1(X) = \pi_1(S^1 \vee S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}.$$

b. Without lose of generality, we can take $X := \mathbb{R}^3 \setminus \{(x, 0, 0), (0, 1, 0) : x \in \mathbb{R}\}$. The open sets

$$\begin{aligned} U &:= \{(x, y, z) \in \mathbb{R}^3 : x < 2/3, y \neq 0 \text{ or } z \neq 0\} \\ V &:= \{(x, y, z) \in \mathbb{R}^3 : x > 1/3, (x, y, z) \neq (0, 1, 0)\} \end{aligned}$$

are path-connected and $U \cup V = X$. On one hand, we have that V and $U \cap V$ are contractible, therefore $\pi_1(V) = \pi_1(U \cap V) = \{\cdot\}$. On the other hand, $V \simeq S^1$, and $\pi_1(V) = \pi_1(S^1) = \mathbb{Z}$. Applying Seifert-Van Kampen theorem we have that

$$\pi_1(X) = \pi_1(V) = \mathbb{Z}.$$

c. Let $X := \mathbb{R}^3 \setminus \{(x,0,0), (x,1,0) : x \in \mathbb{R}\}$. Using the retract

$$\begin{aligned} r \colon X &\to \mathbb{R}^2 \setminus \{(0,0),(1,0)\} \\ (x,y,z) &\mapsto (y,z) \end{aligned}$$

we have that $X \simeq \mathbb{R}^2 \setminus \{(0,0),(1,0)\} \simeq S^1 \vee S^1$ and

$$\pi_1(X) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}.$$

d.

Exercise 8

In Figure 1 we have chosen an orientation and numbered the undercrossings of the knot 6_3. Following the algorithm seen in class we compute the following Wirtinger relations. We denote (RH) the right hand crossings and (LH) the left hand ones. Note that we only need to compute the first 5, as the sixth can be computed as combinations of the previous ones.

1. (RH) $i_*(x_1) = aeb^{-1}e^{-1}$
2. (LH) $i_*(x_2) = bd^{-1}c^{-1}d$
3. (LH) $i_*(x_3) = cf^{-1}d^{-1}f$
4. (RH) $i_*(x_4) = dae^{-1}a^{-1}$
5. (LH) $i_*(x_5) = ec^{-1}b^{-1}c$

Hence, the Wirtinger presentation of $\pi_1(\mathbb{R}^3 \setminus 6_3)$ is

$$\langle a,b,c,d,e,f \mid aeb^{-1}e^{-1}, bd^{-1}c^{-1}d, cf^{-1}d^{-1}f, dae^{-1}a^{-1}, ec^{-1}b^{-1}c\rangle.$$

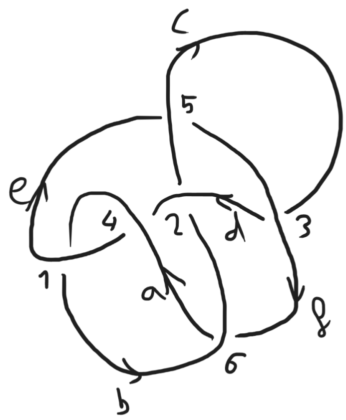

Figure 1: Knot 6_3 with the chosen orientation and undercrossings numbered.

References

[1] Allen Hatcher, *Algebraic Topology*, Allen Hatcher 2001.