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Exercise 9. (Gathmann 2.7)

Proof. □

Exercise 10. (Gathmann 2.8) Let f, g $\in K[x,y]$ be two polynomials that vanish at the origin.

(a) If f, g have no common factor, then the family $\{f^n : n \in \mathbb{Z}_{\leq 0}\}$ is linearly independent in $K[V(g)]_{(0,0)}$.

Proof. We have that $K[V(g)]_{(0,0)} = \frac{D[x,y]_{(0,0)}}{<g>}$. Suppose that there are some $a_i \neq 0 \in K, i \leq n$ such that

$$\sum_i a_i f^i = 0.$$

That is $\sum_i a_i f^i \in < g >$, so for some $h \neq 0$ in $D[x,y]_{(0,0)}$, $\sum_i a_i f^i = hg$, and taking common factor we have

$$\sum_i a_i f^i = f^k(a_k + \sum_i a_i f^{i-k}) = hg.$$

As f and g have no common factor there has to be h' such as that

$$(a_k + \sum_i a_i f^{i-k}) = h'g,$$

but evaluating in $(0,0)$ we have

$$(a_k + \sum_i a_i f^{i-k}(0,0)) = h'(0,0)g(0,0) \Rightarrow (a_k + \sum_i a_i 0) = h'(0,0)0 \Rightarrow a_k = 0,$$

making a contradiction with $a_i \neq 0$ and proving that $\{f^n : n \in \mathbb{Z}_{\leq 0}\}$ is linearly independent. □