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Exercise 1

Exercise 2

Let $\mathbb{Q}^* = \mathbb{Q} - \{0\}$ be the multiplicative group of non-zero rationals. Every element $q \in \mathbb{Q}^*$ can be written as $q = \frac{a}{b}$ with $a, b \in \mathbb{Z}$. Every integer admits a prime factorization such that $a = p_1^{e_1} \dots p_n^{e_n}$, $b = q_1^{f_1} \dots q_m^{f_m}$ with p_i, q_i prime numbers and $e_i, f_k \in \mathbb{N}$. If $Q*$ were finitely generated, there would be a finite set S which generates $Q*$. Each element $\frac{a}{b} \in S$ could be decompose into a fraction of prime factorizations. But prime numbers are infinite, so piking a prime p not included in any of the factorizations of the elements in S would be a contradiction. As $p \in \mathbb{Q}^*$ but it can not be generated by elements of S as it is a prime not in S. Thus, $\mathbb{Q}^*$ must be infinitely generated.

Exercise 3

Exercise 4

Let $G_1 = \langle a, b \mid a^3b^5a^{-3}b^{-5} \rangle$. To show that G_1 is infinite we will construct a surjective homomorphism from G_1 to $\mathbb{Z}$. Define the homomorphism $\phi \colon G_1 \to \mathbb{Z}$ such that $\phi(a) = 5$ and $\phi(b) = -3$. From the restrictions of G_1 we know that $a^3b^5a^{-3}b^{-5} = 1$, the identity element in G_1. Indeed,

$$\phi(a^3b^5a^{-3}b^{-5}) = 3\phi(a) + 5\phi(b) - 3\phi(a) - 5\phi(b) = 0,$$

which is the identity element in $\mathbb{Z}$ with operation $+$. For $a^{n2}b^{n3}$, $n \in \mathbb{Z}$, we have that $\phi(a^{n2}b^{n3}) = n$. This makes any element of $\mathbb{Z}$ reachable from an element of G_1 by ϕ, making ϕ a surjective homomorphism, proving G_1 is infinite.

Let now $G_2 = \langle a, b \mid a^2b^3 \rangle$

Exercise 8

Let $G = \langle S \mid R \rangle = F(S)/\langle\langle R \rangle\rangle$ be a finite presentation. All words $w \in (S \sqcup S^{-1})^*$ such that $w = 1$ in G are the words $w \in \langle\langle R \rangle\rangle$ by the definition of group presentation. Recall that

$$\langle\langle R \rangle\rangle = \bigcup_{i=0}^{\infty} \left\{ \prod_{j=0}^{\infty} (g_j^{-1} r_j^{\epsilon_j} g_j) \mid g_j \in G, r_j \in R, \epsilon_j \in \{\pm 1\} \right\}.$$

To enumerate the words w we can proceed as follows:

1. As $|R|$ is finite, suppose $|R| = n$. We can enumerate all elements of R and R^{-1} numbering them as:
$$r_1, r_1^{-1}, r_2, r_2^{-1}, \dots, r_n, r_n^{-1}. \tag{1}$$
2. In the same manner, as $|S|$ is finite, suppose $|S| = m$, and enumerate all elements of S and S^{-1} as:
$$s_1, s_1^{-1}, s_2, s_2^{-1}, \dots, s_m, s_m^{-1}. \tag{2}$$
3. Finally, now we just need to enumerate the elements of $\langle\langle R \rangle\rangle$ in a sorted way without enumerating one same element more than once. For so, start enumerating the elements $g \in F(S)$ by making combinations of the elements of (2) in a lexicographic order and in increasing word length. As $|S|$ is finite, for each word length k, the amount of words of $F(S)$ of length k is going to be m^k minus the number of produced words that can be reduced. In any case, there is a finite number of words of length k in $F(S)$. Denote this set as $F(S)_k$.

For each word length k, we can iterate over the elements of (1), and enumerate all the elements

$$\prod_{j=0}^{k}(g_j^{-1}r_j^{\epsilon_j}g_j) \text{ with } g_j \in F(S)_k, r_j \in R, \epsilon_j \in \{\pm 1\}.$$

Each k-th iteration of Step 3 of the previous procedure is finite as (1) is finite and $F(S)_k$ is finite. Therefore, on an input $w \in (S \sqcup S^{-1})^*$, if $w = 1$ in G, as w would have finite length, our procedure will find it in finite time. Else, our procedure may run forever.

References

[1] Allen Hatcher, *Algebraic Topology*, Allen Hatcher 2001.