Exercise 4 (Gathmann 5.7.).

(a)

Proof. Let $F = F_1 \cdots F_n$ be a reduced curve of degree d with its decomposition into F_i irreducible components. Lets proof the statement by induction over the number n of irreducible components. For $n = 1$, $F = F_1$ is an irreducible curve, so using [1, Proposition 5.6], we have that F has at most $\binom{d-1}{2} \leq \binom{d}{2}$ singular points. Assuming the hypothesis for $n = m$, lets check it for $n = m + 1$. We have $F = F_1 \cdots F_m F_{m+1}$, were $\deg F = d$ and $\deg F_1 \cdots F_m = d - \deg F_{m+1}$. Denote $d' = \deg F_{m+1}$. Using the induction hypothesis we know that $F_1 \dots F_m$ has at most $\binom{d-d'}{2}$ singular points and, because [1, Proposition 5.6], F_{m+1} has $\binom{d-1}{2}$. The Bézout's theorem says that the intersection multiplicity of both curves is $(d - d')d'$ so the singular points F has are at most

$$\begin{aligned}\binom{d-d'}{2} + \binom{d'-1}{2} + (d-d')d' &= \frac{(d-d')(d-d'-1) + (d'-1)(d'-2) + 2dd' - 2d'^2}{2} \\ &= \frac{d(d-1)}{2} - d' + 1 \leq \frac{d(d-1)}{2} = \binom{d}{2}\end{aligned}$$

noting that $1 \leq d'$. □

(b)

To find an example for each d where F has exactly $\binom{d}{2}$ singular points, its enough to take d linear independent lines L_i, $i \in [1, d]$ where for each $i \neq j \in [1, d]$, if $P = L_i \cup L_j$, then $L_i \cup L_k \neq P$, $\forall k \neq i, j \in [1, d]$. A concrete example could be formed by the lines of the form $\{d(x + d)\}_{d\in\mathbb{N}}$. (Considering that the characteristic of the field K is not a multiple of d).

Exercise 7 (Gathmann 6.21.).

Proof. We define, following [1, Construction 3.13], the ring homomorphism between the coordinate ring $A(F) = K[x, y]/\langle F\rangle$ and the homogeneous coordinate ring of degree d, $S(F)$, formed by the homogeneous elements of degree d of $S(F) = K[x, y, z]/\langle F\rangle$, as

$$\begin{aligned}\phi : A(F) &\longrightarrow S_d(F) \\ f^i = \sum_{i+j\leq d} a_{i,j} x^i y^j &\longmapsto \phi(f^i) = f^h = \sum_{i+j\leq d} a_{i,j} x^i y^j z^{d-i-j}.\end{aligned}$$

If $f, g \in A(F)$, ϕ and $d = \max\{\deg f, \deg g\}$, ϕ is an homomorphism as

$$\begin{aligned}\phi(1) &= \phi(1x^0y^0) = 1x^0y^0z^0 = 1, \\ \phi(f+g) &= \phi\left(\sum_{i+j\leq d} a_{i,j}x^iy^j + \sum_{i+j\leq d} b_{i,j}x^iy^j\right) = \phi\left(\sum_{i+j\leq d}(a_{i,j}+b_{i,j})x^iy^j\right) \\ &= \sum_{i+j\leq d}(a_{i,j}+b_{i,j})x^iy^jz^{d-i-j} = \sum_{i+j\leq d} a_{i,j}x^iy^jz^{d-i-j} + \sum_{i+j\leq d} b_{i,j}x^iy^jz^{d-i-j} \text{ and} \\ &= \phi(f) + \phi(g)\end{aligned}$$

$$
\begin{aligned}
\phi(f\cdot g) &= \phi\left(\sum_{i+j\le d} a_{i,j}x^i y^j \cdot \sum_{k+l\le d} b_{k,l}x^k y^l\right) = \phi\left(\sum_{\substack{i+j\le d\\ k+l\le d}} a_{i,j}b_{k,l}x^{i+k}y^{j+l}\right)\\
&= \sum_{\substack{i+j\le d\\ k+l\le d}} a_{i,j}b_{k,l}x^{i+k}y^{j+l}z^{2d-i-j-k-l} = \sum_{i+j\le d} a_{i,j}x^i y^j z^{d-i-j} \cdot \sum_{k+l\le d} b_{k,l}x^k y^l z^{d-k-l}\\
&= \phi(f)\cdot\phi(g).
\end{aligned}
$$

Actually, ϕ is an isomorphism, as we can define its inverse as

$$
\begin{aligned}
\phi^{-1} : S_d(F) &\longrightarrow A(F)\\
f^h = \sum_{i+j+k=d} a_{i,j,k}x^i y^j z^k &\longmapsto \phi^{-1}(f^h) = f^i = \sum_{i+j\le d} a_{i,j,k}x^i y^j.
\end{aligned}
$$

Now, we can define the desired isomorphism between $K(F) = \left\{\frac{f}{g} : f,g \in A(F)\right\}$ and $K(F^h) = \left\{\frac{f}{g} : f,g \in S_d(F)\right\}$ as

$$
\begin{aligned}
\Phi : K(F) &\longrightarrow K(F^h)\\
\frac{f^i}{g^i} &\longmapsto \Phi\left(\frac{f^i}{g^i}\right) = \frac{\phi(f^i)}{\phi(g^i)} = \frac{f^h}{g^h},
\end{aligned}
$$

witch has inverse

$$
\begin{aligned}
\Phi^{-1} : K(F^h) &\longrightarrow K(F)\\
\frac{f^h}{g^h} &\longmapsto \Phi^{-1}\left(\frac{f^h}{g^h}\right) = \frac{\phi^{-1}(f^h)}{\phi^{-1}(g^h)} = \frac{f^i}{g^i},
\end{aligned}
$$

and, given $\frac{f}{g}, \frac{h}{k} \in K(F)$, verifies

$$
\begin{aligned}
\Phi(\frac{1}{1}) &= \frac{\phi(1)}{\phi(1)} = \frac{1}{1},\\
\Phi(\frac{f}{g}+\frac{h}{k}) &= \Phi(\frac{fk+gh}{gk}) = \frac{\phi(fk+gh)}{\phi(gk)} = \frac{\phi(f)\cdot\phi(k)+\phi(g)\cdot\phi(h)}{\phi(g)\cdot\phi(k)} = \frac{\phi(f)}{\phi(g)} + \frac{\phi(h)}{\phi(k)}\\
&= \Phi(\frac{f}{g}) + \Phi(\frac{h}{k}),\\
\Phi(\frac{f}{g}\cdot\frac{h}{k}) &= \Phi(\frac{f\cdot h}{g\cdot k}) = \frac{\phi(f\cdot h)}{\phi(g\cdot k)} = \frac{\phi(f)\cdot\phi(h)}{\phi(g)\cdot\phi(k)} = \Phi(\frac{f}{g})\cdot\Phi(\frac{h}{k}).
\end{aligned}
$$

□

Exercise 8 (Gathmann 6.25.).

Let $F = y^2z - x^3 + xz^2$ and $\varphi = \frac{y}{z}$. $F = 0$ at the points $P_1 = (0:0:1)$, $P_2 = (1:0:1)$, $P_3 = (-1:0:1)$, $P_4 = (0:1:0)$. Hence, using [1, Construction 6.17] and [1, Algorithm 2.12], we compute the multiplicity at each P_i of φ at F.

$$\begin{aligned}
\mu_{P_1}(y) &= \mu_{(0,0)}(y, y^2 - x^3 + x) = \mu_{(0,0)}(y, x(1-x^2)) = 1\\
\mu_{P_1}(z) &= \mu_{(0,0)}(1, y^2 - x^3 + x) = 0\\
\mu_{P_1}(\varphi) &= \mu_{P_1}(y) - \mu_{P_1}(z) = 1 - 0 = 1
\end{aligned}$$

$$\begin{aligned}
\mu_{P_2}(y) &= \mu_{(1,0)}(y, y^2 - x^3 + x) = \mu_{(0,0)}(y, y^2 - (x+1)^3 + x + 1) = \mu_{(0,0)}(y, -x(x^2+3x-2)) = 1\\
\mu_{P_2}(z) &= \mu_{(1,0)}(1, y^2 - x^3 + x) = \mu_{(0,0)}(1, y^2 - (x+1)^3 + x + 1) = 0\\
\mu_{P_2}(\varphi) &= \mu_{P_2}(y) - \mu_{P_2}(z) = 1 - 0 = 1
\end{aligned}$$

$$\begin{aligned}
\mu_{P_3}(y) &= \mu_{(-1,0)}(y, y^2 - x^3 + x) = \mu_{(0,0)}(y, y^2 - (x-1)^3 + x - 1) = \mu_{(0,0)}(y, x(x^2-3x+4)) = 1\\
\mu_{P_3}(z) &= \mu_{(-1,0)}(1, y^2 - x^3 + x) = \mu_{(0,0)}(1, y^2 - (x-1)^3 + x - 1) = 0\\
\mu_{P_3}(\varphi) &= \mu_{P_3}(y) - \mu_{P_3}(z) = 1 - 0 = 1
\end{aligned}$$

$$\begin{aligned}
\mu_{P_4}(y) &= \mu_{(0,0)}(1, z - x^3 + xz^2) = 0\\
\mu_{P_4}(z) &= \mu_{(0,0)}(z, z - x^3 + xz^2) = \mu_{(0,0)}(z, z(1+xz) - x^3) = \mu_{(0,0)}(z, -x^3) = 3\\
\mu_{P_4}(\varphi) &= \mu_{P_4}(y) - \mu_{P_4}(z) = 0 - 3 = -3
\end{aligned}$$

Now, following [1, Construction 6.23], we have

$$\operatorname{div}\frac{y}{z} = 1\cdot(0:0:1) + 1\cdot(1:0:1) + 1\cdot(-1:0:1) - 3\cdot(0:1:0).$$

References

[1] Andreas Gathmann, *Plane Algebraic Curves*, Class Notes RPTU Kaiserslautern 2023.