Name: Gonzalo Ortega Carpintero

Exercise 1

Proof. Let X and Y path-connected spaces. If $[\gamma] \in \pi_1(X \times Y, (x, y))$, then $\gamma\colon I \to X \times Y$ is a loop in the direct product pace where $\gamma(0) = \gamma(1) = (x, y)$. We can write γ as $\gamma(t) = (\gamma_X(t), \gamma_Y(t))$ where $\gamma_X\colon I \to X$ and $\gamma_Y\colon I \to Y$ are loops in X and Y respectively with $\gamma_X(0) = \gamma_X(1) = x$ and $\gamma_Y(0) = \gamma_Y(1) = y$. Hence we can define the morphism

$$\begin{aligned} f\colon \pi_1(X \times Y, (x, y)) &\to \pi_1(X, x) \times \pi_1(Y, y) \\ [\gamma] &\mapsto ([\gamma_X], [\gamma_Y]). \end{aligned}$$

- Let $*$ denote the path concatenation operator and let $[\gamma_1], [\gamma_2] \in \pi_1(X \times Y, (x, y))$, then

  $$f([\gamma_1]\cdot[\gamma_2]) = f([\gamma_1 * \gamma_2]) = ([(\gamma_1 * \gamma_2)_X], [(\gamma_1 * \gamma_2)_Y]) = ([\gamma_{1X}], [\gamma_{1Y}])\cdot([\gamma_{2X}], [\gamma_{2Y}]) = f([\gamma_1])\cdot f([\gamma_2])$$

  and f is in fact an homomorphism.
- If $([\gamma_X], [\gamma_Y])$ is the identity in $\pi_1(X, x) \times \pi_1(Y, y)$ then $[\gamma_X]$ is the class of the constant path $[\gamma_X] = [x_0]$. The same for $[\gamma_Y] = [y_0]$. Therefore, if $f([\gamma]) = ([x_0], [y_0])$ then $[\gamma] = [(x_0, y_0)]$ is also the identity in $\pi_1(X \times Y, (x, y))$. Hence, f is injective.
- For any pair of path classes $[\alpha] \in \pi_1(X, x)$ and $[\beta] \in \pi_1(Y, y)$ we can take the path $\gamma(t) = (\alpha(t), \beta(y))$ for which $f([\gamma]) = ([\alpha], [\beta])$. Hence, f is also surjective.

This makes f a isomorphism and $\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y)$ $\square$

References

[1] Allen Hatcher, *Algebraic Topology*, Allen Hatcher 2001.