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Exercise 1

Proof. Let $f\colon X \to Y$ be a continuous map homotopic to an homotopy equivalence $g\colon X \to Y$, $f \simeq g$. As g is an homotopy equivalence, there exists $h\colon Y \to X$ such that $g \circ h \simeq \mathrm{id}_Y$ and $h \circ g \simeq \mathrm{id}_X$. Also, as $f \simeq g$, there exists an homotopy $H\colon X \times I \to Y$ such that

$$\begin{aligned} H(x,0) &= f(x), \\ H(x,1) &= g(x). \end{aligned}$$

Therefore, we can define the homotopies $H_1\colon Y \times I \to X$ and $H_2\colon X \times I \to Y$ as

$$H_1(x,t) \coloneqq h \circ H(x,t), \qquad\qquad H_2(y,t) \coloneqq H(h(x),t),$$

where

$$\begin{aligned} H_1(x,0) &= h \circ H(x,0) = h \circ f(x), & H_2(y,0) &= H(h(y),0) = f(h(y)) = f \circ h(y), \\ H_1(x,1) &= h \circ H(x,1) = h \circ g(x), & H_2(y,1) &= H(h(y),1) = g(h(y)) = g \circ h(y). \end{aligned}$$

Hence, $h \circ f \simeq h \circ g \simeq \mathrm{id}_X$ and $f \circ h \simeq g \circ h \simeq \mathrm{id}_Y$, proving that f is also a homotopy equivalence. □

Exercise 2

Proof. Let X be a topological space.

(a) ⇒ (b). Let X be contractible and let x_0 be a single point set, such as $X \simeq x_0$. Let Y be a topological space and $f\colon X \to Y$ a continuous function. Then there exists an homotopy $H\colon X \times I \to X$ such that

$$\begin{aligned} H(x,0) &= \mathrm{id}_X = x, \\ H(x,1) &= x_0. \end{aligned}$$

Defining $H_*\colon X \times I \to Y$ as $H_*(x,t) = f \circ H(x,t)$,

$$\begin{aligned} H_*(x,0) &= f \circ H(x,0) = f(x) \\ H_*(x,1) &= f \circ H(x,1) = f(x_0). \end{aligned}$$

Hence, f is nullhomotopic.

(b) ⇒ (c) If for every topological space Y and every continuous function $f\colon X \to Y$ is nullhomotopic, in particular the $\mathrm{id}_X\colon X \to X$ is nullhomotopic and there exist an homotopy $H\colon X \times I \to X$ such that

$$\begin{aligned} H(x,0) &= \mathrm{id}_X = x, \\ H(x,1) &= x_0. \end{aligned}$$

. If $g\colon Y \to X$ is continuous we can define $H_*\colon Y \times I \to X$ as $H_*(x,t) \coloneqq H(f(x),t)$,

$$\begin{aligned} H_*(x,0) &= H(g(x),0) = g(x) \\ H_*(x,1) &= H(g(x),1) = x_0. \end{aligned}$$

Hence, g is nullhomotopic

(c) ⇒ (a). If for every topological space Y and every continuous function $g\colon Y \to X$ is nullhomotopic, in particular, $\mathrm{id}_X\colon X \to X$ is nullhomotopic. That is $\mathrm{id}_X \simeq x_0$ and X is contractible. □

Exercise 3

Proof. Let $X = \{(x,y) \in \mathbb{R}^2 \colon x = t,\ y = t/n,\ t \in [0,1], n \in \mathbb{N}\} \cup \{(x,y) \in \mathbb{R}^2 \colon x = t,\ y = 0,\ t \in [0,1]\}$. Define the homotopy $H \colon X \times \to X$ such that

$$H((x,y),t) := \begin{cases} (1-2t)(x,y), & \text{if } 0 \le t < \frac{1}{2}, \\ (2t-1,0), & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then, $H((x,y),0) = (x,y) = \mathrm{id}_X$ and $H((x,y),1) = (1,0)$. As H is continuous, the set $\{(1,0)\}$ is a deformation retract of X.

Suppose $\{(1,0)\}$ is a strong deformation retract of X, then for all $t \in [0,1]$, $H((1,0),t) = H((1,0),0) = (1,0)$. Lets find a $t_0 \in [0,1]$ that contradicts this. For every $n \in \mathbb{N}$, there exists a $t_n \in [0,1]$ such that $H((1,\frac{1}{n}),t_n) = (0,0)$, as if not, H would not be continuous. As $\{t_n\}_{n\ge 0}$ is a subset of the compact set $[0,1]$, there exists a sub collection $\{t_{n_k}\}_{k\ge 0}$ such that when $k \to \infty$, $t_{n_k} \to t_0$ and $\frac{1}{t_{n_k}} \to 0$. Therefore

$$H((1,0),t_0) = \lim_{k\to\infty} H(t_{n_k}, t_{n_k}) = \lim_{k\to\infty} (0,0) = (0,0).$$

□

Exercise 7

Let X be a finite dimensional complex of dimension n. Then

$$X = X^n = \frac{X^{n-1} \sqcup_\alpha D^n_\alpha}{x \sim \phi_\alpha(x)},\ \forall \alpha \in I,\ \forall x \in S^{n-1}_\alpha,$$

is a topological space with the quotient topology, were ϕ_α is the gluing map.

Thus, a n-cell e^n_α is open in X if the set $\{x \in X^{n-1} \sqcup_\alpha D^n_\alpha \colon [x] \in e^n_\alpha\}$ is open in $X^{n-1} \sqcup_\alpha D^n_\alpha$. But this set is equal to e^n_α, and e^n_α is open in $X^{n-1} \sqcup_\alpha D^n_\alpha$ as it is a copy of B^n, witch is open in some D^n_α. Therefore, e^n_α is open in X.

References

[1] Allen Hatcher, *Algebraic Topology*, Allen Hatcher 2001.