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Exercise 1

Exercise 2

Exercise 3

Exercise 4

Exercise 8

Let $G = \langle S \mid R \rangle = F(S)/\langle\langle R \rangle\rangle$ be a finite presentation. All words $w \in (S \sqcup S^{-1})^*$ such that $w = 1$ in G are the words $w \in \langle\langle R \rangle\rangle$ by the definition of group presentation. Recall that

$$\langle\langle R \rangle\rangle = \bigcup_{i=0}^{\infty} \left\{ \prod_{j=0}^{\infty} (g_j^{-1} r_j^{\epsilon_j} g_j) \mid g_j \in G, r_j \in R, \epsilon_j \in \{\pm 1\} \right\}.$$

To enumerate the words w we can proceed as follows:

1. As $|R|$ is finite, suppose $|R| = n$. We can enumerate all elements of R and R^{-1} numbering them as:

$$r_1, r_1^{-1}, r_2, r_2^{-1}, \dots, r_n, r_n^{-1}. \tag{1}$$

2. In the same manner, as $|S|$ is finite, suppose $|S| = m$, and enumerate all elements of S and S^{-1} as:

$$s_1, s_1^{-1}, s_2, s_2^{-1}, \dots, s_m, s_m^{-1}. \tag{2}$$

3. Finally, now we just need to enumerate the elements of $\langle\langle R \rangle\rangle$ in a sorted way without enumerating one same element more than once. For so, start enumerating the elements $g \in F(S)$ by making combinations of the elements of (2) in a lexicographic order and in increasing word length. As $|S|$ is finite, for each word length k, the amount of words of $F(S)$ of length k is going to be m^k minus the number of produced words that can be reduced. In any case, there is a finite number of words of length k in $F(S)$. Denote this set as $F(S)_k$.

   For each word length k, we can iterate over the elements of (1), and enumerate all the elements

$$\prod_{j=0}^{k} (g_j^{-1} r_j^{\epsilon_j} g_j) \text{ with } g_j \in F(S)_k, r_j \in R, \epsilon_j \in \{\pm 1\}.$$

Each k-th iteration of Step 3 of the previous procedure is finite as (1) is finite and $F(S)_k$ is finite. Therefore, on an input $w \in (S \sqcup S^{-1})^*$, if $w = 1$ in G, as w would have finite length, our procedure will find it in finite time. Else, our procedure may run forever.

References

[1] Allen Hatcher, *Algebraic Topology*, Allen Hatcher 2001.