Exercise 1

Proof. Let X and Y path-connected spaces. If $[\gamma] \in \pi_1(X \times Y, (x, y))$, then $\gamma \colon I \to X \times Y$ is a loop in the direct product pace where $\gamma(0) = \gamma(1) = (x, y)$. We can write γ as $\gamma(t) = (\gamma_X(t), \gamma_Y(t))$ where $\gamma_X \colon I \to X$ and $\gamma_Y \colon I \to Y$ are loops in X and Y respectively with $\gamma_X(0) = \gamma_X(1) = x$ and $\gamma_Y(0) = \gamma_Y(1) = y$. Hence we can define the morphism

$$f \colon \pi_1(X \times Y, (x, y)) \to \pi_1(X, x) \times \pi_1(Y, y)$$
$$[\gamma] \mapsto ([\gamma_X], [\gamma_Y]).$$

- Let $*$ denote the path concatenation operator and let $[\gamma_1], [\gamma_2] \in \pi_1(X \times Y, (x, y))$, then

$$f([\gamma_1] \cdot [\gamma_2]) = f([\gamma_1 * \gamma_2]) = ([(\gamma_1 * \gamma_2)_X], [(\gamma_1 * \gamma_2)_Y])$$
$$= ([\gamma_{1X}], [\gamma_{1Y}]) \cdot ([\gamma_{2X}], [\gamma_{2Y}]) = f([\gamma_1]) \cdot f([\gamma_2])$$

and f is in fact an homomorphism.

- If $([\gamma_X], [\gamma_Y])$ is the identity in $\pi_1(X, x) \times \pi_1(Y, y)$ then $[\gamma_X]$ is the class of the constant path $[\gamma_X] = [x_0]$. The same for $[\gamma_Y] = [y_0]$. Therefore, if $f([\gamma]) = ([x_0], [y_0])$ then $[\gamma] = [(x_0, y_0)]$ is also the identity in $\pi_1(X \times Y, (x, y))$. Hence, f is injective.

- For any pair of path classes $[\alpha] \in \pi_1(X, x)$ and $[\beta] \in \pi_1(Y, y)$ we can take the path $\gamma(t) = (\alpha(t), \beta(t))$ for which $f([\gamma]) = ([\alpha], [\beta])$. Hence, f is also surjective.

This makes f a isomorphism and $\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y)$ $\qquad\qquad\square$

Exercise 7

a. We have that $:= \mathbb{R}^3 \setminus \{\text{x-axis and y-axis}\} = \mathbb{R}^3 \setminus \{(x, 0, 0), (0, y, 0) : \forall x, y \in \mathbb{R}\}$. Defining the retraction r_1 such as

$$r_1 \colon X \to S^2 \setminus \{\pm(1, 0, 0), \pm(0, 1, 0)\}$$
$$(x, y, z) \mapsto \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}},$$

we show that the space minus two axis is homotopic to the sphere without four points. Making a translation of one of the missing points of the sphere, we have that

$$S^2 \setminus \{\pm(1, 0, 0), \pm(0, 1, 0)\} \cong S^2 \setminus \{(1, 0, 0), \pm(0, 1, 0), (0, 0, 1)\}$$

and defining the retraction r_2 as the stereographic projection

$$r_2 \colon S^2 \setminus \{(1, 0, 0), \pm(0, 1, 0), (0, 0, 1)\} \to \mathbb{R}^2 \setminus \{\pm(1, 0), (1, 0)\}$$
$$(x, y, z) \mapsto \left(\frac{y}{1 - x}, \frac{z}{1 - x} \right),$$

we have that $X \simeq \mathbb{R}^2 \setminus \{\pm(1, 0), (1, 0)\} \simeq S^1 \vee S^1 \vee S^1$ and therefore

$$\pi_1(X) = \pi_1(S^1 \vee S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}.$$

b. Without lose of generality, we can take $X := \mathbb{R}^3 \setminus \{(x, 0, 0), (0, 1, 0) : x \in \mathbb{R}\}$. The open sets

$$U := \{(x, y, z) \in \mathbb{R}^3 : x < 2/3, y \neq 0 \text{ or } z \neq 0\}$$
$$V := \{(x, y, z) \in \mathbb{R}^3 : x > 1/3, (x, y, z) \neq (0, 1, 0)\}$$

are path-connected and $U \cup V = X$. On one hand, we have that V and $U \cap V$ are contractible, therefore $\pi_1(V) = \pi_1(U \cap V) = \{\cdot\}$. On the other hand, $V \simeq S^1$, and $\pi_1(V) = \pi_1(S^1) = \mathbb{Z}$. Applying Seifert-Van Kampen theorem we have that

$$\pi_1(X) = \pi_1(V) = \mathbb{Z}.$$

c. The space $\mathbb{R}^3$ without any two disjoint lines is homeomorphic to $\mathbb{R}^3$ without two disjoint *parallel* lines. Without lose of generality, let $X := \mathbb{R}^3 \setminus \{(x, 0, 0), (x, 1, 0) : x \in \mathbb{R}\}$. Using the retract

$$r \colon X \to \mathbb{R}^2 \setminus \{(0, 0), (1, 0)\}$$
$$(x, y, z) \mapsto (y, z)$$

we have that $X \simeq \mathbb{R}^2 \setminus \{(0, 0), (1, 0)\} \simeq S^1 \vee S^1$ and

$$\pi_1(X) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}.$$

d. Let $X := T^2 - \{p, q\}$ be the torus minus two points. In Figure 1 we make use of the diagrams seen in class to compute the fundamental group of the space. In step (1) we picture the diagram of the torus with two points removed. In (2) we contract one of the removed points to the boundary of the diagram while cutting one edge to the other point. Thus, in (3) we can see the obtained result. By deforming the diagram as in (4) and gluing the repeated edges a and b together we obtain the wedge of (5). That is, $X \simeq S^1 \vee S^1 \vee S^1$ and

$$\pi_1(X) = \pi_1(S^1 \vee S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}.$$
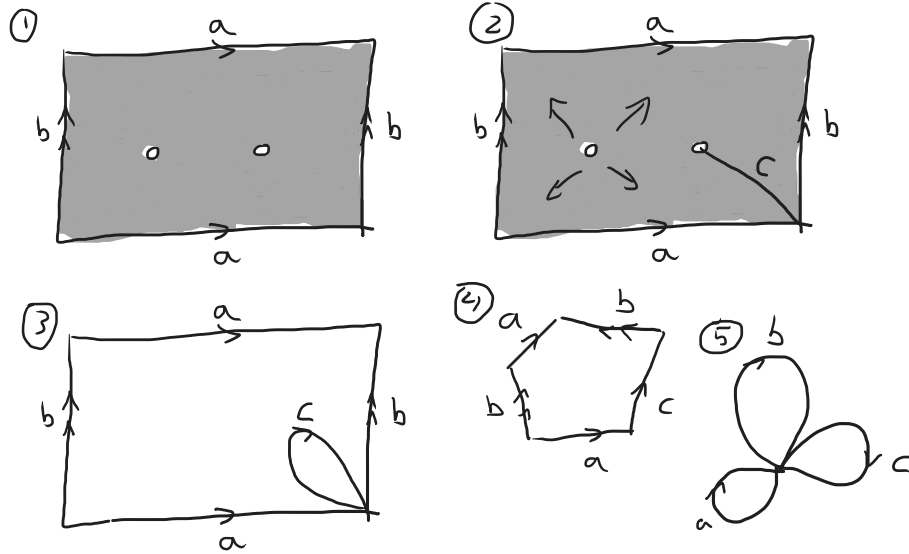

Figure 1: Diagrams of the steps followed to compute the fundamental group of the torus T^2 minus two points.

Exercise 8

In Figure 2 we have chosen an orientation and numbered the under-crossings of the knot 6_3. Following the algorithm seen in class we compute the following Wirtinger relations. We denote (RH) the right hand crossings and (LH) the left hand ones. Note that we only need to compute the first 5, as the sixth can be computed as combinations of the previous ones.

1. (RH) $i_*(x_1) = aeb^{-1}e^{-1}$

2. (LH) $i_*(x_2) = bd^{-1}c^{-1}d$

3. (LH) $i_*(x_3) = cf^{-1}d^{-1}f$

4. (RH) $i_*(x_4) = dae^{-1}a^{-1}$

5. (LH) $i_*(x_5) = ec^{-1}b^{-1}c$

Hence, the Wirtinger presentation of $\pi_1(\mathbb{R}^3 \setminus 6_3)$ is

$$\langle a, b, c, d, e, f \mid aeb^{-1}e^{-1}, bd^{-1}c^{-1}d, cf^{-1}d^{-1}f, dae^{-1}a^{-1}, ec^{-1}b^{-1}c \rangle.$$
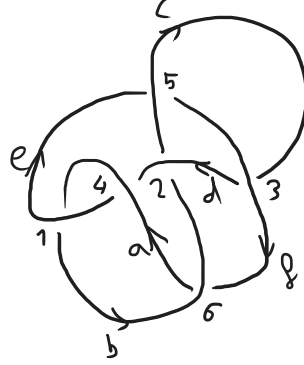

Figure 2: Knot 6_3 with the chosen orientation and under-crossings numbered.

Exercise 10

Let K be an arbitrary knot with $n \geq 3$ crossings. The Wirtinger presentation of the fundamental group of a knot complement in S^3 is a presentation of the from

$$\pi_1(S^3 \setminus K) \cong \left\langle S = \{a_1, \ldots, a_n\} \mid R = \left\{ a_i a_{k_i}^{\epsilon_i} a_{i+1}^{-1} a_{k_i}^{-\epsilon_i} : i \in \{1, \ldots, n-1\}, \epsilon_i \in \{\pm 1\} \right\} \right\rangle.$$

If we consider the abelianization of $\pi_1(S^3 \setminus K)$ we need to add the commutators of every pair of elements in S. That is add the set $C = \{[a_i, a_j] : a_i, a_j \in S\}$ to the presentation restrictions. Therefore

$$\mathrm{Ab}(\pi_1(S^3 \setminus K)) \cong \langle S \mid R \cup C \rangle.$$

This implies that each element of R can be permuted and

$$1 = a_i a_{k_i}^{\epsilon_i} a_{i+1}^{-1} a_{k_i}^{-\epsilon_i} = a_i a_{i+1}^{-1} a_{k_i}^{\epsilon_i} a_{k_i}^{-\epsilon_i} = a_i a_{i+1}^{-1}.$$

Thus, $a_i = a_{i+1}$ for all $i \in \{1, \ldots, n-1\}$. As all the generators in S are the same, all elements in $R \cup C$ can be removed, leaving

$$\mathrm{Ab}(\pi_1(S^3 \setminus K)) \cong \langle a_1 \mid - \rangle \cong F_1 \cong \mathbb{Z}.$$

References

[1] Allen Hatcher, *Algebraic Topology*, Allen Hatcher 2001.

[2] *In Exercise 7.d., the cut made to obtain vertex c was inspired by the one explained to me by Nicolas.*