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Exercise 1.

Proof. Let X be a set, and let U_i for $i \in I$ be topological spaces, where $X = \bigcup U_i$. Let τ_i be the topology of each U_i. For every $i, j \in I$, the topologies τ_i, τ_j on U_i and U_j restricted to $U_i \cap U_j$ agree. A definition of restricted topology, also known as the relative topology can be found in Section 6 of [2]. We can define a topology over X by considering the union of all open sets of all U_i's. That is

$$\mathcal{T} = \{V \subseteq X : V \cap U_i \in \tau_i, \ \forall i \in I\}.$$

Let's check that $\mathcal{T}$ is actually a topology. We have $\emptyset \in \mathcal{T}$ as for all $i \in I$, $\emptyset \cap U_i = \emptyset \in \tau_i$. The same happens with X, as for all $i \in I$, $X \cap U_i = U_i \in \tau_i$. Therefore $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

Suppose $V_1, V_2 \in \mathcal{T}$, then

$$(V_1 \cap V_2) \cap U_i = (V_1 \cap U_i) \cap (V_2 \cap U_i) \in \tau_1,$$

as $V_1 \cap U_i$ and $V_2 \cap U_i$ are open sets of U_i, and the intersection of open sets is open. Thus $V_1 \cap V_2 \in \mathcal{T}$. In the same manner, if $\{V_\alpha\}_{\alpha \in A}$ is a collection of sets of $\mathcal{T}$, we have

$$\left(\bigcup_\alpha V_\alpha\right) = \bigcup_\alpha (V_\alpha \cup U_i) \in \tau_i$$

as each $V_\alpha \cup U_i$ is open in U_i. We have then $\bigcup_\alpha V_\alpha \in \mathcal{T}$, and we can conclude that $\mathcal{T}$ is a topology of X.

We have proved existence, lets prove now the uniqueness of $\mathcal{T}$. Suppose we have two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ of X that restrict to the topology of each U_i. Let τ_i be the restricted topology of $\mathcal{T}_1$ on each U_i and τ_j be the restricted topology of $\mathcal{T}_2$ on each U_j. And let τ_i', τ_j' be the restricted topologies of τ_i and τ_j on $U_i \cap U_j$. We then have that

$$V \in \mathcal{T}_1 \Leftrightarrow V \cap U_i \in \tau_i \Leftrightarrow V \cap U_i \cap U_j \in \tau_i' \underset{hypothesis}{\Leftrightarrow} V \cap U_i \cap U_j \in \tau_j' \Leftrightarrow V \cap U_j \in \tau_j \Leftrightarrow V \in \mathcal{T}_2.$$

Therefore, $T_1 = T_2$, so there is exactly one topology on X that restricts to the topology on each U_i. □

Exercise 3.

If $f \in K[x_0, x_1, x_2]$ be a non-constant homogeneous polynomial and let $f = \prod_{i=1}^n g_i^{m_i}$ be the decomposition of f in irreducible polynomials.

(a)

Proof.

($\Rightarrow$) If $V(f)$ is irreducible, suppose that we can not write f as a power of a irreducible polynomial. Then, there exist $g_1, g_2 \in K[x_0, x_1, x_2]$, with $g_1 \neq g_2$, such that $f = g_1 g_2$. But then we have $V(f) = V(g_1 g_2) = V(g_1) \cup V(g_2)$ as seen in Remark 3.9 of [1]. This is a contradiction as $V(f)$ is irreducible so it need to be $f = g^m$ for some g irreducible.

($\Leftarrow$) If $f = g^m$, with g irreducible, then we have $V(f) = V(g^m) = V(g)$. $V(g)$ needs to be irreducible, because if not it could be expressed as the union of two curves $V(g_1)$ and $V(g_2)$, having $V(g) = V(g_1) \cup V(g_2) = V(g_1 g_2)$, and it would be $g = g_1^m g_2^m$ contradicting the fact of g being irreducible. Thus, $V(g^m)$ is irreducible. □

(b)

Proof. For $n = 1$, $f = g_1^{m_1}$, so $g_1^{m_1}$ is homogeneous and so g_1, as f is homogeneous. Suppose $h = \prod_{i=1}^{n-1} g_i^{m_i}$ homogeneous. If g_n was not homogeneous, $g_n^{m_n}$ neither would be, and the product $hg_n^{m_n}$ would not be homogeneous as the product of an homogeneous and a non homogeneous polynomials can not be homogeneous. But $hg_n^{m_n} = \prod_{i=1}^{n} g_i^{m_i} = f$ contradicting the fact of f been homogeneous. Thus g_n must be homogeneous and, because of induction, each g_i is homogeneous.

As each g_i is an homogeneous irreducible polynomial, because of **(a)**, $V(g_i)$ is a irreducible curve. Therefore, $V(f) = \cup_i V(g_i)$ is a decomposition in irreducible curves of $V(f)$. □

Exercise 4.

Proof. Let $I < K[x_0, x_1, x_2]$ be a homogeneous ideal. Let $K[x_0, x_1, x_2]/I$ be finite dimensional. If $V(I) \neq \emptyset \subset \mathbb{P}^2$ then $\exists p = [p_0 : p_1 : p_2] \in \mathbb{P}^2$ such that $\forall f \in I$, $f(p) = 0$. Without lose of generality we can asume $p = [1 : p_1 : p_2]$ (the argument will follow analogously choosing any other coordinate to be different from 0). Thus $\forall n \in \mathbb{N}, x_0^n \notin I$.

Therefore we have an infinite family $\{x_0^n + I\}_{n\in\mathbb{N}}$ of elements of $K[x_0, x_1, x_2]/I$. If the family elements were linearly dependent it would exist $n_i \in \mathbb{N}$ and a finite family of indices $J \subset \mathbb{N}$ such that $x_0^{n_i} = \sum_{j\in J} a_j(x_0^{n_j} + I)$. Let $g = x_0^{n_i} + I - \sum_{j\in J} a_j(x_0^{n_j} + I) \in I$. Because of the equivalences of the homogeneous ideal definition seen in class, we have $g_i \in I$ for each homogeneous part g_i of g. In particular $x_0^{n_i} \in I$ forming a contradiction. Thus, $\{x_0^n + I\}_{n\in\mathbb{N}}$ is a linearly independent infinite family of $K[x_0, x_1, x_2]/I$ so the quotient can not be infinite dimensional contradicting the exercise hypothesis. It then need to be $V(I) = \emptyset$. □

Exercise 8 (Gathmann 4.9.).

(a)

Over $\mathbb{C}$, let $f = x + y^2$ and $g = x + y^2 - x^3$. Their respective homogenizations are $f^h = xz + y^2$ and $g^h = xz^2 + y^2z - x^3$. Making $z = 0$ to obtain their points at infinity we get $f^h(z = 0) = y^2$ and $g^h(z = 0) = -x^3$. So the only point at infinity of f is $(1 : 0 : 0)$ and of g, $(0 : 1 : 0)$. Thus, $V(f)$ and $V(g)$ do not intersect at infinity. In the affine part, their only intersection point is $p = (0 : 0 : 1)$, and as $\mathbb{C} = \bar{\mathbb{C}}$, because of Bézout's theorem

$$\mu_p(f, g) = \deg f \cdot \deg g = 2 \cdot 3 = 6.$$

(b)

Over $\mathbb{C}$, now let $f = y^2 - x_2$ and $g = (x+y+1)(y-x+1) = y^2 - x^2 + 2y + 1$. Their homogenizations are $f^h = y^2 - x^2 + z^2$ and $g^h = y^2 - x^2 + 2yz + z^2$. Then we have $f^h(z = 0) = y^2 - x^2$ and $g^h(z = 0) = y^2 - x^2$, so both curves have two common points at infinity, $p_1 = (1 : 1 : 0)$ and $p_2 = (1 : -1 : 0)$. In the affine part we have other two common points, $p_3 = (1 : 1 : 1)$ and $p_4 = (1 : -1 : 1)$ so using Bézout's theorem we have

$$\sum_{i=1}^{4} \mu_{p_i}(f, g) = \deg f \cdot \deg g = 2 \cdot 2 = 4,$$

so it could only be $\mu_{p_i}(f, g) = 1$ for each i.

Exercise 9 (Gathmann 4.10.).

Proof. If $f, g \in \mathbb{R}[x_0, x_1, x_2]$ then also $f, g \in \mathbb{C}[x_0, x_1, x_2]$ and because Bézout's theorem we have

$$\sum_{p \in V(f) \cap V(g) \subset \mathbb{C}} \mu_p(f, g) = \sum_{p \in V(f) \cap V(g) \subset \mathbb{R}} \mu_p(f, g) + \sum_{p \in V(f) \cap V(g) \subset \mathbb{C} \setminus \mathbb{R}} \mu_p(f, g) = \dim f \cdot \dim g, \tag{1}$$

separating the multiplicities at real points of the ones at strictly complex points. Given $p, \bar{p} \in \mathbb{C}$ a complex point and its conjugate, and $h \in \mathbb{C}[x, y]$ a complex polynomial, if $h(p) = 0$ then $h(\bar{p}) = 0$. So if $p \in V(f) \cap V(g) \subset \mathbb{C} \setminus \mathbb{R}$ then $\bar{p} \in V(f) \cap V(g) \subset \mathbb{C} \setminus \mathbb{R}$. Also, this make local rings in p and its conjugate the same ring, having then $\mu_p(f, g) = \mu_{\bar{p}}(f, g)$. Thus, $\sum_{p \in V(f) \cap V(g) \subset \mathbb{C} \setminus \mathbb{R}} \mu_p(f, g)$ is an even number $2n$, so using the modulo 2 relationship with Equation 1 we have

$$\sum_{p \in V(f) \cap V(g) \subset \mathbb{R}} \mu_p(f, g) + 2n \equiv \sum_{p \in V(f) \cap V(g) \subset \mathbb{R}} \mu_p(f, g) \equiv \dim f \cdot \dim g \mod 2.$$

□

References

[1] Andreas Gathmann, *Plane Algebraic Curves*, Class Notes RPTU Kaiserslautern 2023.

[2] Stephen Willard, *General Topology*, Addison-Wesley 1970.