An introduction to the Bottleneck distance
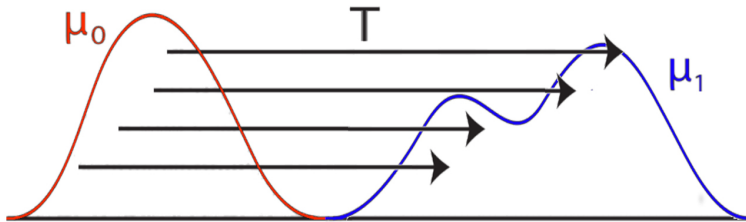
From optimal transport to the stability theorem

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N. Papadakis, Optimal Transport for Image Processing, habilitation à diriger des recherches, Université de Bordeaux, Dec. 2015

Definition (Push-forward measure)

Let $T : X \to Y$ be a Borel map, and $\mu \in \mathcal{P}(X)$. Let $A \in \text{Bar}$. The *push-forward measure* $T_\# \mu \in \mathcal{P}(Y)$ is defined as

$$T_\# \mu(A) := \mu(T^{-1}(A)).$$

Definition (Transport map)

Given $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$, a *transport map from μ to ν* is a Borel map $T : X \to Y$ that satisfies $T_\# \mu = \nu$.

Definition (Transport plan)

Let $\pi_X : (X \times Y) \to X$ and $\pi_Y : (X \times Y) \to Y$ such that for every $(x, y) \in (X, Y)$, $\pi_X(x, y) = x$ and $\pi_Y(x, y) = y$. A *transport plan between μ and ν* is a probability measure $\gamma \in \mathcal{P}(X \times Y)$ where

$$(\pi_X)_\# \gamma = \mu \quad \text{and} \quad (\pi_Y)_\# \gamma = \nu.$$

The set of all couplings between μ and ν is denoted $\Gamma(\mu, \nu)$.

Monge and Kantorovich transport problems

Definition (Transport problems)

Fix $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$ and consider a lower semicontinous map $c : X \times Y \to [0, \infty]$. Then

$$C_M(\mu, \nu) := \inf \left\{ \int_X c(x, T(x)) d\mu(x) : T_{\#}\mu = \nu \right\},$$

$$C_K(\mu, \nu) := \inf \left\{ \int_{X \times Y} c(x, y) d\gamma(x, y) : \gamma \in \Gamma(\mu, \nu) \right\}.$$

Definition (Probability measures with finite *p*-moment)

Let (X, d) be a locally compact and separable, metric space. Let $1 \leq p < \infty$. The *set of probability measures with finite p-moment* is defined As

$$\mathcal{P}_p(X) := \left\{ \sigma \in \mathcal{P}(X) : \int_X d(x, x_0)^p d\mu(x) < \infty \text{ for some } x_0 \in X \right\}.$$

Definition (*p*-Wasserstein distance)

Given $u, v \in \mathcal{P}_p(X)$, the *p-Wasserstein distance* is defined as

$$W_p(u, v) := \left(\inf_{\gamma \in \Gamma(u,v)} \int_{X \times X} d(x, y)^p d\gamma(x, y) \right)^{\frac{1}{p}}.$$

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Baic definitions

Definition (Persistence diagram)

Let I be a countable set. A *persistence diagram* is a function $D : I \to \mathbb{R}^2_<$.

Definition (Partial matching)

Let $D_1 : I_1 \to \mathbb{R}^2_<$ and $D_2 : I_2 \to \mathbb{R}^2_<$ be persistence diagrams. A *partial matching* between D_1 and D_2 is the triple (I'_1, I'_2, f) such that $f : I'_1 \to I'_2$ is a bijection with $I'_1 \subseteq I_1$ and $I'_2 \subseteq I_2$.

Definition (Chebyshev distance)

Let $a, b \in \mathbb{R}^2$ with $a = (a_x, a_y)$ and $b = (b_x, b_y)$. The *Chebyshev distance* is defined as

$$d_\infty(a, b) := ||a - b||_\infty := \max\{|a_x - b_x|, |a_y - b_y|\}.$$

Definition (*p*-cost)

Let $D_1 : I_1 \to \mathbb{R}^2_<$ and $D_2 : I_2 \to \mathbb{R}^2_<$ be persistence diagrams. Let (I_1', I_2', f) be a partial matching between them. If $p < \infty$, the *p-cost of f* is defined as

$$
\mathrm{cost}_p(f) := \left(\sum_{i \in I_1'} d_\infty(D_1(i), D_2(f(i)))^p + \sum_{i \in I_1 \setminus I_1'} d_\infty(D_1(i), \Delta)^p + \sum_{i \in I_2 \setminus I_2'} d_\infty(D_2(i), \Delta)^p \right)^{\frac{1}{p}}.
$$

For $p = \infty$, the ∞-cost of f is defined as

$$
\mathrm{cost}_\infty(f) := \max \left\{ \sup_{i \in I_1'} d_\infty(D_1(i), D_2(f_i)), \sup_{i \in I_1 \setminus I_1'} d_\infty(D_1(i), \Delta), \sup_{i \in I_2 \setminus I_2'} d_\infty(D_2(i), \Delta) \right\}.
$$

## Bottleneck distance

Definition (p-Wasserstein distance)

Let D_1, D_2 be persistence diagrams. Let $1 \leq p \leq \infty$. Define

$$\tilde{\omega}_p(D_1, D_2) = \inf\{\text{cost}_p(f) : f \text{ is a partial matching between } D_1 \text{ and } D_2\}.$$

Let $\emptyset$ denote the unique persistence diagram with empty indexing set. Let (Dgm_p, ω_p) be the space of persistence diagrams D that satisfy $\tilde{\omega}_p(D, \emptyset) < \infty$ modulo the equivalence relation $D_1 \sim D_2$ if $\tilde{\omega}_p(D_1, D_2) = 0$. The metric ω_p is called the *p-Wasserstein distance*.

Definition (Bottleneck distance)

If $p = \infty$, the metric ω_∞ is called the *bottleneck distance*.

Wasserstein distance
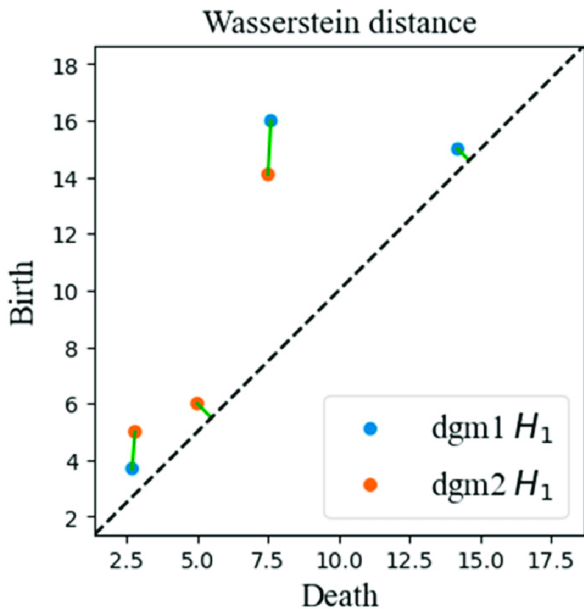
Definition (Isometric embedding)

Let $(X, d_X), (Y, d_Y)$ be metric spaces. An *isometric embedding* $\eta : (X, d_X) \to (Y, d_Y)$ is a mapping that satisfies

$$d_X(x_1, x_2) = d_Y(\eta(x_1), \eta(x_2))$$

for all $x_1, x_2 \in X$.

Definition (Ball in persistence diagrams)

Let $1 \leq p \leq \infty$. Let $D_0 \in \mathrm{Dgm}_p$. The *ball* at the space of persistence diagrams is defined as $B_p(D_0, r) := \{D \in \mathrm{Dgm}_p : w_p(D, D_0) < r\}$.
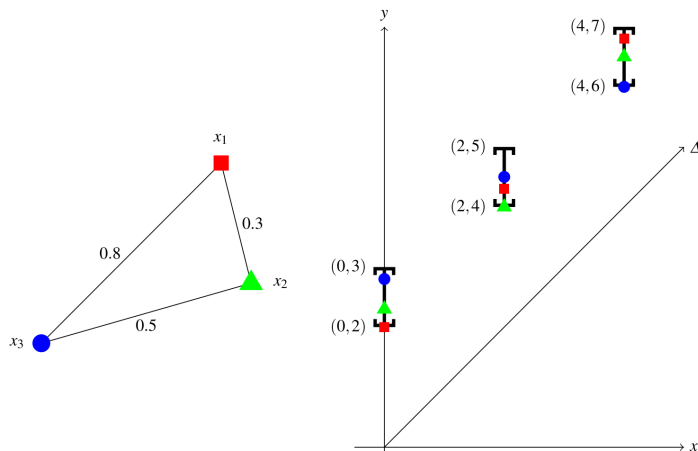
Theorem (Isometric embeeding of metric spaces into persistance diagrams)

Let (X, d) be a separable, bounded metric space. Then there exists an isometric embedding to the space of persistence diagrams $\eta : (X, d) \to (\mathrm{Dgm}_\infty, \omega_\infty)$ such that $\eta(X) \subseteq B(\emptyset, \frac{3c}{c}) \backslash B(\emptyset, c)$.

As (X, d) is bounded, we can let $c > \sup\{d(x, y) : x, y \in X\}$. As (X, d) is separable, we can take $\{x_k\}_{k=1}^{\infty}$, a countable, dense subset of (X, d). Consider

$$\eta : (X, d) \rightarrow (\text{Dgm}_{\infty}, \omega_{\infty})$$
$$x \mapsto \{(2c(k-1), 2ck + d(x, x_k))\}_{k=1}^{\infty}$$

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DEFINITION 6.17. For $\epsilon \geq 0$, an **ϵ-matching** between two multi-sets B and B' of intervals is a bijection $\rho : B_0 \rightarrow B_0'$ between a pair of multi-subsets $B_0 \subset B$ and $B_0' \subset B'$ subject to the following constraints:

    (1) Every $[s,t]$ in $(B - B_0) \cup (B' - B_0')$ has length $t - s \leq 2\epsilon$, and
    (2) If $\rho[s,t] = [s',t']$ for some $[s,t]$ in B_0, then $|s - s'| \leq \epsilon \geq |t - t'|$.

DEFINITION 6.18. The **bottleneck distance** $d_{\text{Bot}}(B, B')$ between multi-sets of intervals B and B' is the infimum over all $\epsilon \geq 0$ for which there exists an ϵ-matching between them.
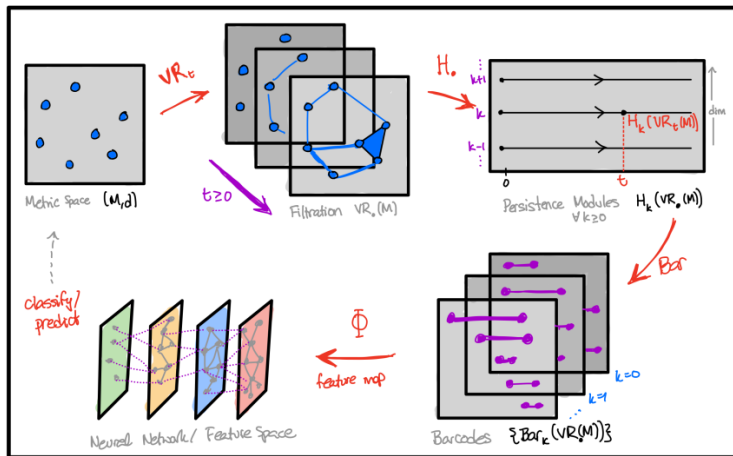
Bottleneck distance for the stability theorem

Theorem (Stability)

There is an isometry between a persistence module with the interleaving distance and its barcode with the bottleneck distance. This means that, given two persistence modules V, W,

$$d_{int}(V, W) = d_{bot}(\text{Bar}(V), \text{Bar}(W)).$$

But...

Where does the name Bottleneck come from? Who knows.