Name: Gonzalo Ortega Carpintero

Exercise 8 (Gathmann 6.25.).

Let $F = y^2z - x^3 + xz^2$ and $\varphi = \frac{y}{z}$. $F = 0$ at the points $P_1 = (0 : 0 : 1)$, $P_2 = (1 : 0 : 1)$, $P_3 = (-1 : 0 : 1)$, $P_4 = (0 : 1 : 0)$. Hence, using [1, Construction 6.17] and [1, Algorithm 2.12], we compute the multiplicity at each P_i of φ at F.

$$\begin{aligned}
&\mu_{P_1}(y) = \mu_{(0,0)}(y, y^2 - x^3 + x) = \mu_{(0,0)}(y, x(1 - x^2)) = 1\\
&\mu_{P_1}(z) = \mu_{(0,0)}(1, y^2 - x^3 + x) = 0\\
&\mu_{P_1}(\varphi) = \mu_{P_1}(y) - \mu_{P_1}(z) = 1 - 0 = 1
\end{aligned}$$

$$\begin{aligned}
&\mu_{P_2}(y) = \mu_{(1,0)}(y, y^2 - x^3 + x) = \mu_{(0,0)}(y, y^2 - (x + 1)^3 + x + 1) = \mu_{(0,0)}(y, -x(x^2 + 3x - 2)) = 1\\
&\mu_{P_2}(z) = \mu_{(1,0)}(1, y^2 - x^3 + x) = \mu_{(0,0)}(1, y^2 - (x + 1)^3 + x + 1) = 0\\
&\mu_{P_2}(\varphi) = \mu_{P_2}(y) - \mu_{P_2}(z) = 1 - 0 = 1
\end{aligned}$$

$$\begin{aligned}
&\mu_{P_3}(y) = \mu_{(-1,0)}(y, y^2 - x^3 + x) = \mu_{(0,0)}(y, y^2 - (x - 1)^3 + x - 1) = \mu_{(0,0)}(y, x(x^2 - 3x + 4)) = 1\\
&\mu_{P_3}(z) = \mu_{(-1,0)}(1, y^2 - x^3 + x) = \mu_{(0,0)}(1, y^2 - (x - 1)^3 + x - 1) = 0\\
&\mu_{P_3}(\varphi) = \mu_{P_3}(y) - \mu_{P_3}(z) = 1 - 0 = 1
\end{aligned}$$

$$\begin{aligned}
&\mu_{P_4}(y) = \mu_{(0,0)}(1, z - x^3 + xz^2) = 0\\
&\mu_{P_4}(z) = \mu_{(0,0)}(z, z - x^3 + xz^2) = \mu_{(0,0)}(z, z(1 + xz) - x^3) = \mu_{(0,0)}(z, -x^3) = 3\\
&\mu_{P_4}(\varphi) = \mu_{P_4}(y) - \mu_{P_4}(z) = 0 - 3 = -3
\end{aligned}$$

Now, following [1, Construction 6.23], we have

$$\operatorname{div} \frac{y}{z} = 1 \cdot (0 : 0 : 1) + 1 \cdot (1 : 0 : 1) + 1 \cdot (-1 : 0 : 1) - 3 \cdot (0 : 1 : 0).$$

References

[1] Andreas Gathmann, *Plane Algebraic Curves*, Class Notes RPTU Kaiserslautern 2023.