Exercise 7 (Gathmann 6.21.).

Proof. We define, following [1, Construction 3.13], the ring homomorphism between the coordinate ring $A(F) = K[x,y]/\langle F\rangle$ and the homogeneous coordinate ring of degree d, $S(F)$, formed by the homogeneous elements of degree d of $S(F) = K[x,y,z]/\langle F\rangle$, as

$$\begin{aligned} \phi : A(F) &\longrightarrow S_d(F) \\ f^i = \sum_{i+j\le d} a_{i,j}x^iy^j &\longmapsto \phi(f^i) = f^h = \sum_{i+j\le d} a_{i,j}x^iy^jz^{d-i-j}. \end{aligned}$$

If $f, g \in A(F)$, ϕ and $d = \max\{\deg f, \deg g\}$, ϕ is an homomorphism as

$$\begin{aligned} \phi(1) &= \phi(1x^0y^0) = 1x^0y^0z^0 = 1, \\ \phi(f+g) &= \phi\left(\sum_{i+j\le d} a_{i,j}x^iy^j + \sum_{i+j\le d} b_{i,j}x^iy^j\right) = \phi\left(\sum_{i+j\le d}(a_{i,j}+b_{i,j})x^iy^j\right) \\ &= \sum_{i+j\le d}(a_{i,j}+b_{i,j})x^iy^jz^{d-i-j} = \sum_{i+j\le d} a_{i,j}x^iy^jz^{d-i-j} + \sum_{i+j\le d} b_{i,j}x^iy^jz^{d-i-j} \text{ and} \\ &= \phi(f)+\phi(g) \\ \phi(f\cdot g) &= \phi\left(\sum_{i+j\le d} a_{i,j}x^iy^j \cdot \sum_{k+l\le d} b_{k,l}x^ky^l\right) = \phi\left(\sum_{\substack{i+j\le d\\ k+l\le d}} a_{i,j}b_{k,l}x^{i+k}y^{j+l}\right) \\ &= \sum_{\substack{i+j\le d\\ k+l\le d}} a_{i,j}b_{k,l}x^{i+k}y^{j+l}z^{2d-i-j-k-l} = \sum_{i+j\le d} a_{i,j}x^iy^jz^{d-i-j} \cdot \sum_{k+l\le d} b_{k,l}x^ky^lz^{d-k-l} \\ &= \phi(f)\cdot\phi(g). \end{aligned}$$

Actually, ϕ is an isomorphism, as we can define its inverse as

$$\begin{aligned} \phi^{-1} : S_d(F) &\longrightarrow A(F) \\ f^h = \sum_{i+j+k=d} a_{i,j,k}x^iy^jz^k &\longmapsto \phi^{-1}(f^h) = f^i = \sum_{i+j\le d} a_{i,j,k}x^iy^j. \end{aligned}$$

Now, we can define the desired isomorphism between $K(F) = \left\{\frac{f}{g} : f, g \in A(F)\right\}$ and $K(F^h) = \left\{\frac{f}{g} : f, g \in S_d(F)\right\}$ as

$$\begin{aligned} \Phi : K(F) &\longrightarrow K(F^h) \\ \frac{f^i}{g^i} &\longmapsto \Phi\left(\frac{f^i}{g^i}\right) = \frac{\phi(f^i)}{\phi(g^i)} = \frac{f^h}{g^h}, \end{aligned}$$

witch has inverse

$$\begin{aligned} \Phi^{-1} : K(F^h) &\longrightarrow K(F) \\ \frac{f^h}{g^h} &\longmapsto \Phi^{-1}\left(\frac{f^h}{g^h}\right) = \frac{\phi^{-1}(f^h)}{\phi^{-1}(g^h)} = \frac{f^i}{g^i}, \end{aligned}$$

and, given $\frac{f}{g}, \frac{h}{k} \in K(F)$, verifies

$$\Phi(\frac{1}{1}) = \frac{\phi(1)}{\phi(1)} = \frac{1}{1},$$
$$\Phi(\frac{f}{g}+\frac{h}{k}) = \Phi(\frac{fk+gh}{gk}) = \frac{\phi(fk+gh)}{\phi(gk)} = \frac{\phi(f)\cdot\phi(k)+\phi(g)\cdot\phi(h)}{\phi(g)\cdot\phi(k)} = \frac{\phi(f)}{\phi(g)}+\frac{\phi(h)}{\phi(k)} = \Phi(\frac{f}{g})+\Phi(\frac{h}{k}),$$
$$\Phi(\frac{f}{g}\cdot\frac{h}{k}) = \Phi(\frac{f\cdot h}{g\cdot k}) = \frac{\phi(f\cdot h)}{\phi(g\cdot k)} = \frac{\phi(f)\cdot\phi(h)}{\phi(\phi(g)\cdot\phi(k)} = \Phi(\frac{f}{g})\cdot\Phi(\frac{h}{k}).$$

□

Exercise 8 (Gathmann 6.25.).

Let $F = y^2z - x^3 + xz^2$ and $\varphi = \frac{y}{z}$. $F = 0$ at the points $P_1 = (0:0:1)$, $P_2 = (1:0:1)$, $P_3 = (-1:0:1)$, $P_4 = (0:1:0)$. Hence, using [1, Construction 6.17] and [1, Algorithm 2.12], we compute the multiplicity at each P_i of φ at F.

$$\mu_{P_1}(y) = \mu_{(0,0)}(y, y^2 - x^3 + x) = \mu_{(0,0)}(y, x(1-x^2)) = 1$$
$$\mu_{P_1}(z) = \mu_{(0,0)}(1, y^2 - x^3 + x) = 0$$
$$\mu_{P_1}(\varphi) = \mu_{P_1}(y) - \mu_{P_1}(z) = 1 - 0 = 1$$

$$\mu_{P_2}(y) = \mu_{(1,0)}(y, y^2 - x^3 + x) = \mu_{(0,0)}(y, y^2 - (x+1)^3 + x + 1) = \mu_{(0,0)}(y, -x(x^2+3x-2)) = 1$$
$$\mu_{P_2}(z) = \mu_{(1,0)}(1, y^2 - x^3 + x) = \mu_{(0,0)}(1, y^2 - (x+1)^3 + x + 1) = 0$$
$$\mu_{P_2}(\varphi) = \mu_{P_2}(y) - \mu_{P_2}(z) = 1 - 0 = 1$$

$$\mu_{P_3}(y) = \mu_{(-1,0)}(y, y^2 - x^3 + x) = \mu_{(0,0)}(y, y^2 - (x-1)^3 + x - 1) = \mu_{(0,0)}(y, x(x^2-3x+4)) = 1$$
$$\mu_{P_3}(z) = \mu_{(-1,0)}(1, y^2 - x^3 + x) = \mu_{(0,0)}(1, y^2 - (x-1)^3 + x - 1) = 0$$
$$\mu_{P_3}(\varphi) = \mu_{P_3}(y) - \mu_{P_3}(z) = 1 - 0 = 1$$

$$\mu_{P_4}(y) = \mu_{(0,0)}(1, z - x^3 + xz^2) = 0$$
$$\mu_{P_4}(z) = \mu_{(0,0)}(z, z - x^3 + xz^2) = \mu_{(0,0)}(z, z(1+xz) - x^3) = \mu_{(0,0)}(z, -x^3) = 3$$
$$\mu_{P_4}(\varphi) = \mu_{P_4}(y) - \mu_{P_4}(z) = 0 - 3 = -3$$

Now, following [1, Construction 6.23], we have

$$\operatorname{div}\frac{y}{z} = 1\cdot(0:0:1) + 1\cdot(1:0:1) + 1\cdot(-1:0:1) - 3\cdot(0:1:0).$$

References

[1] Andreas Gathmann, *Plane Algebraic Curves*, Class Notes RPTU Kaiserslautern 2023.