Exercise 1

1. For each $n \geq 0$, define the set $S_n = \{pqp^{-1}, p^2qp^{-2}, \dots, p^2qp^{-n}\}$ and the function $\theta\colon S_n \to F_2 = \langle a, b \mid - \rangle$ that carries each p^iqp^{-i} to $\theta(p^iqp^{-i}) = a^iba^{-i}$. The set S_n has n elements so the free group generated by S_n, $F(S_n)$, is isomorphic to F_n. By the universal property of free groups, there is a unique homomorphism $\hat{\theta}\colon F(S_n) \to F_2$ extending θ. Clearly, the restriction to its image $\hat{\theta}\colon F(S_n) \to \hat{\theta}(F(S_n))$ is a bijection, so $F_n \cong \hat{\theta}(F(S_n))$. We just need to prove that $\hat{\theta}(F(S_n))$ is a subgroup of F_2.

   Let $x = \prod_{i=0}^{l}(p^{\alpha_i}q^{\epsilon_i}p^{-\alpha_i})$, $y = \prod_{j=0}^{m}(p^{\beta_j}q^{\delta_j}p^{-\beta_j})$, with $l, m \in \mathbb{N}$, $\alpha_i, \beta_j \in \mathbb{Z}$, $\epsilon_i, \delta_j \in \{\pm 1\}$ be two arbitrary elements of $\hat{\theta}(F(S_n))$. Then

$$xy^{-1} = \prod_{i=0}^{l}(a^{\alpha_i}b^{\epsilon_i}a^{-\alpha_i})\prod_{j=0}^{m}(a^{\beta_j}b^{-\delta_j}a^{-\beta_j}) = \prod_{k=0}^{l+m}(a^{\gamma_k}b^{\xi_k}a^{-\gamma_k})$$

   with $\gamma_k \in \mathbb{Z}$, $\xi_k \in \{\pm 1\}$. Thus $xy^{-1} \in \hat{\theta}(F(S_n))$ and $\hat{\theta}(F(S_n))$ is a subgroup of F_2.

2. To prove that F_2 contains a subgroup isomorphic to F_∞ its enough to take the set $S_\infty = \{p^iqp^{-i} : i \in \mathbb{N}\}$ and repeat the steps followed in 1.

Exercise 2

Let $\mathbb{Q}^* = \mathbb{Q} - \{0\}$ be the multiplicative group of non-zero rationals. Every element $q \in \mathbb{Q}^*$ can be written as $q = \frac{a}{b}$ with $a, b \in \mathbb{Z}$. Every integer admits a prime factorization such that $a = p_1^{e_1} \dots p_n^{e_n}$, $b = q_1^{f_1} \dots q_m^{f_m}$ with p_i, q_i prime numbers and $e_i, f_k \in \mathbb{N}$. If $Q*$ were finitely generated, there would be a finite set S which generates $Q*$. Each element $\frac{a}{b} \in S$ could be decompose into a fraction of prime factorizations. But prime numbers are infinite, so piking a prime p not included in any of the factorizations of the elements in S would be a contradiction. As $p \in \mathbb{Q}^*$ but it can not be generated by elements of S as it is a prime not in S. Thus, $\mathbb{Q}^*$ must be infinitely generated.

Exercise 3

Let $G = \langle S \mid R \rangle$ and $H = \langle T \mid Q \rangle$. Define $\theta : S \sqcup T \to G \times H$ such that $\theta(s) = (s, 1)$ and $\theta(s) = (1, t)$ for all $s \in S, t \in T$. By the universal property of free groups, there is a unique $\hat{\theta}\colon F(S \sqcup T) \to G \times H$. As $R, Q, [S, T] \subset \ker(\hat{\theta})$, also $\langle\langle R \sqcup Q \sqcup [S, T] \rangle\rangle \subset \ker(\hat{\theta})$. Thus, by the fundamental theorem on homomorphisms, there exists a unique surjective homomorphism $f\colon F(S \sqcup T)/\langle\langle R \sqcup Q \sqcup [S, T]\rangle\rangle \to G \times H$. Its easy to check that f is also injective, making

$$G \times H = \langle S \sqcup T \mid R \sqcup Q \sqcup [S, T] \rangle.$$

Now if G and H are finitely presented, then $|S|, |R|, |T|, |Q| < \infty$. Therefore $|S \sqcup T|, |R \sqcup Q|, [S, T] < \infty$ and $G \times H$ is finitely presented.

If $G \times H$ is finitely presented then it has a presentation $\langle X | Y \rangle$ such that $|X|, |Y| < \infty$. Taking the projections $\phi\colon G \times H \to G$ and $\psi\colon G \times H \to H$. Therefore, $G = \langle \phi(X) | R \rangle$ and $G = \langle \psi(X) | Q \rangle$ were R and Q are two unknown pair of restriction sets. By the first part of this exercise, $G \times H$ can also be presented as

$$G \times H = \langle \phi(X) \sqcup \psi(X) | R \sqcup Q \sqcup [\phi(X), \psi(X)] \rangle.$$

We have seen in class that Y is finite there must exist some $Y' \subseteq R \sqcup Q \sqcup [\phi(X), \psi(X)]$ which is also finite and

$$G \times H = \langle \phi(X) \sqcup \psi(X) | Y' \rangle.$$

But it needs to be $Y' \cap R = R$ and $Y' \cap Q = Q$ so G and H are also finitely presented.

Exercise 4

Let $G_1 = \langle S_1 = \{a, b\} \mid R_1 = \{a^3b^5a^{-3}b^{-5}\}\rangle$. To show that G_1 is infinite we will construct a surjective homomorphism from G_1 to $\mathbb{Z}$. First, define $\theta\colon S_1 \to \mathbb{Z}$ as $\theta(a) = 2$ and $\theta(b) = -1$. By the universal property of free groups there is a unique homomorphism $\hat{\theta}\colon F(S) \to \mathbb{Z}$ extending θ such that $\hat{\theta}(a) = 2$ and $\hat{\theta}(b) = -1$. Thus, we have that

$$\hat{\theta}(a^3b^5a^{-3}b^{-5}) = 3\hat{\theta}(a) + 5\hat{\theta}(b) - 3\hat{\theta}(a) - 5\hat{\theta}(b) = 0.$$

Thus, $R_1 \subset \ker(\hat{\theta})$ and in fact $\langle\langle R_1\rangle\rangle \subset \ker(\hat{\theta})$. For $(ab)^n \in F(S_1)$, $n \in \mathbb{Z}$, we have that $\hat{\theta}((ab)^n) = n$. This makes any element of $\mathbb{Z}$ reachable from an element of $F(S_1)$ by $\hat{\theta}$, making $\hat{\theta}$ a surjective homomorphism. Hence, by the fundamental theorem on homomorphisms, there exists a unique surjective homomorphism $h_1 : F(S_1)/\langle\langle R_1\rangle\rangle = G_1 \to \mathbb{Z}$ proving G_1 is infinite.

Let now $G_2 = \langle S_2 = \{a, b\} \mid R_2 = \{a^2b^3\}\rangle$. We proceed as before. Define $\phi\colon S_2 \to \mathbb{Z}$ as $\phi(a) = 3$ and $\phi(b) = -2$. Then there exists a unique homomorphism $\hat{\phi}\colon F(S_2) \to \mathbb{Z}$ such that $\hat{\phi}(a) = 3$ and $\hat{\phi}(b) = -2$. Then

$$\hat{\phi}(a^2b^3) = 3\hat{\phi}(a) - 2\hat{\phi}(b) = 3 \cdot (-2) - 2 \cdot 3 = 0.$$

Hence, $\langle\langle R_2\rangle\rangle \subset \ker(\hat{\phi})$ and $\hat{\phi}((ab)^n) = n$, ϕ is surjective and there exists a unique surjective $h_2 : F(S_2)/\langle\langle R_2\rangle\rangle = G_2 \to \mathbb{Z}$. Thus, G_2 is also infinite.

Exercise 8

Let $G = \langle S \mid R\rangle = F(S)/\langle\langle R\rangle\rangle$ be a finite presentation. All words $w \in (S \sqcup S^{-1})^*$ such that $w = 1$ in G are the words $w \in \langle\langle R\rangle\rangle$ by the definition of group presentation. Recall that

$$\langle\langle R\rangle\rangle = \bigcup_{i=0}^{\infty} \left\{ \prod_{j=0}^{\infty} (g_j^{-1} r_j^{\epsilon_j} g_j) \mid g_j \in F(G), r_j \in R, \epsilon_j \in \{\pm 1\} \right\}.$$

To enumerate the words w we can proceed as follows:

1. As $|R|$ is finite, suppose $|R| = n$. We can enumerate all elements of R and R^{-1} numbering them as:
$$r_1, r_1^{-1}, r_2, r_2^{-1}, \ldots, r_n, r_n^{-1}. \tag{1}$$

2. In the same manner, as $|S|$ is finite, suppose $|S| = m$, and enumerate all elements of S and S^{-1} as:
$$s_1, s_1^{-1}, s_2, s_2^{-1}, \ldots, s_m, s_m^{-1}. \tag{2}$$

3. Finally, now we just need to enumerate the elements of $\langle\langle R\rangle\rangle$ in a sorted way without enumerating one same element more than once. For so, start enumerating the elements $g \in F(S)$ by making combinations of the elements of (2) in a lexicographic order and in increasing word length. As $|S|$ is finite, for each word length k, the amount of words of $F(S)$ of length k is going to be m^k minus the number of produced words that can be reduced. In any case, there is a finite number of words of length k in $F(S)$. Denote the set of words of length less than or equal to k as $F(S)_k$ and note that it is finite too.

   For each word length k, we can iterate over the elements of (1), and enumerate all the elements
$$\prod_{j=0}^{k} (g_j^{-1} r_j^{\epsilon_j} g_j) \text{ with } g_j \in F(S)_k, r_j \in R, \epsilon_j \in \{\pm 1\}.$$

   We reduce each obtained word and compare it with the finite number of words we had previously enumerated. If it is a new word, we enumerate it.

Each k-th iteration of Step 3 of the previous procedure is finite as (1) is finite and $F(S)_k$ is finite. Therefore, on an input $w \in (S \sqcup S^{-1})^*$, if $w = 1$ in G, as w would have finite length, our procedure will find it in finite time. Else, our procedure may run forever.

References

[1] Marco Linton, *Geometric group theory notes*, UAM Algebra Advance Course, 2025.