Universidad Autónoma de Madrid

Final Master Thesis

Structure and Stability Theorems in Topological Data Analysis
Draft

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Abstract

This is a draft version of a in-work Master's thesis on TDA, compiled for the application to the PhD position in area of topological data analysis in VU University.

Key words

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1 Preliminaries

The contents of this chapter are based on [4], [5] and [6].

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2

Structure Theorem

2.1 Structure theorem for finitely generated modules over a principal ideal domain

2.2 Structure theorem for persistence diagrams

The Structure Theorem for persistence modules is referred to as the “first miracle” of persistence homology [4]. This algebraic property allows to express a persistence module of finite type as a direct sum of finitely many interval modules.

Its proof requires the algebraic structure theorem for finitely generated modules over a principal domain. Due to the lengthy concepts needed to prove it, we will refer to it as a well known fact. An introduction to module theory and a detailed proof of the theorem of Fact 2.2.1 can be found at [3, Chapter IV].

Fact 2.2.1 (Structure theorem for finitely generated modules over a principal ideal domain)**.** *[3, Chapter IV, Theorem 6.12] Let M be a finitely generated module over a principal ideal domain R. There exist a finite sequence of proper ideals* $(d_1) \supseteq (d_2) \supseteq \cdots \supseteq (d_n)$

such that

$$M \cong \bigoplus_{i=1}^{n} R/(d_i).$$

In addition to Fact 2.2.1, we will use the following simple algebraic statement.

Proposition 2.2.2 (Proposition 4.6, [6])**.** *An ideal $I \subseteq R$ is graded if and only if it is generated by homogeneous elements.*

Proof. First, if I is a graded ideal $I = \bigoplus_p I^p$ and is generated by $\bigcup_p I^p$. Then, each

$$I^p = I \cap R^p \subseteq R^p$$

is a subset of homogeneous elements. Therefore, I is generated by homogeneous elements.

Now, let I be generated by a set X of homogeneous elements. For sure, $I \cap R^p \subseteq I$, so we just need to prove the converse inclusion. As I is generated by X, its elements $u \in I$ are of the form

$$u = \sum_i r_i x_i s_i, \tag{2.1}$$

for $r_i, s_i \in R$ and $x_i \in X$. And as $I \subseteq R$, also,

$$u = \sum_p u_p,$$

for $u_p \in R^p$. For every term in (2.1), we have

$$r_i = \sum_j r_{i,j}, \qquad\qquad s_i = \sum_l s_{i,l},$$

with each $r_{i,j}$, $s_{i,l}$ being homogeneous. Therefore, combining all we have that

$$u = \sum_i \sum_{j,l} r_{i,j} x_i s_{i,l}. \tag{2.2}$$

Each term in (2.2) is homogeneous as is a product of homogeneous elements. Thus u_p is the sum of those terms, and u has degree p. Therefore $u_p \in I$ and $I \subseteq I \cap R^p$. □

Theorem 2.2.3 (Structure)**.** *[6, Proposition 4.8] Let (V, π) be a persistence module. There exist a barcode $\mathrm{Bar}(V, \pi)$, with $\mu\colon \mathrm{Bar}(V, \pi) \longrightarrow \mathbb{N}$, the multiplicity of the barcode intervals, such as there is a unique direct sum decomposition*

$$V \cong \bigoplus_{I \in \mathrm{Bar}(V)} \mathbb{F}(I)^{\mu(I)}. \tag{2.3}$$

Proof. V is of finite type, so it is a finite $\mathbb{F}[x]$-module. As $\mathbb{F}$ is a field, $\mathbb{F}[x]$ is a principal ideal domain, therefore, V is a finitely generated module over a principal ideal domain. Using Fact 2.2.1, V can be decompose in the direct sum of its free and torsion subgroups, $F \oplus T$. Thus, we have

$$F = \bigoplus_{i \geq q} x^i \cdot \mathbb{F}$$
$$T = \bigoplus_{i \geq q} R^i / I^i.$$

Each $x^i \cdot \mathbb{F}$ is isomorphic to ideals of the form (x^q). By Proposition 2.2.2, each R^i/I^i is isomorphic to some quotient of graded ideals of the form $(x^p)/(x^r)$. Note that the free subgroup can be seen as a particular case of the torsion group taking $r = 0$. Thus V can be decompose as described in (2.3). $\square$

3

Interleaving Stability Theorem

In this section we are going to give a detailed proof of the first stability theorem for persistence homology. This theorem is refer to as the “geometry miracle” of persistent homology, as it allows to describe an isometry between persistence modules and barcodes [4]. This shows that *small* changes in a data sets will perform *small* changes in their persistence modules, and therefore small changes in how persistent homology groups vary through time. The theorem claims that given two persistence modules, the distance between them using the interleaving distance, is the same as the distance between their barcodes using the bottleneck distance.

Theorem 3.0.1 (Stability)**.** *[5, Theorem 2.2.8] There is an isometry between a persistence module with the interleaving distance and its barcode with the bottleneck distance. That is, given two persistence modules V and W, it holds that*

$$d_{int}(V, W) = d_{bot}(\mathrm{Bar}(V), \mathrm{Bar}(W)).$$

For the presented proof we have followed [5]. Hence, we will divide the proof into proving two complementary inequalities separately. This implies checking that if there exists a δ-matching between two given barcodes, then there exists a δ-interleaving morphism between, Proposition 3.0.3. Also, we need to check that, if there exists a δ-interleaving morphism between two persistence modules, then there exists a δ-matching between their barcodes, 3.0.11.

The first claim can be deduced from the Structure Theorem in a rather direct way, proving first the case where our modules are just interval modules.

Lemma 3.0.2. *[5, Exercise 2.2.7] Let I, J be two δ-matched intervals. Then, their corresponding interval modules $(\mathbb{F}(I), \pi)$ and $(\mathbb{F}(J), \theta)$ are δ-interleaved.*

Proof. Let $I = (a, b]$, $J = (c, d]$. If ρ is the δ-matching between them, then $\rho(I) = J$ and, following Definition 1.0.12, $(a, b] \subseteq (c-\delta, d+\delta]$ and $(c, d] \subseteq (a-\delta, b+\delta]$, with $b-a > 2\delta$ and $d-c > 2\delta$. Then, the morphisms

$$\begin{aligned} \phi_\delta \colon \mathbb{F}(I) &\to \mathbb{F}(J)_\delta \\ \phi_\delta(\mathbb{F}(I)_t) &\mapsto \mathbb{F}(J)_{(t+\delta)} \end{aligned} \quad \text{and} \quad \begin{aligned} \psi_\delta \colon \mathbb{F}(J) &\to \mathbb{F}(I)_\delta \\ \psi_\delta(\mathbb{F}(J)_t) &\mapsto \mathbb{F}(I)_{(t+\delta)} \end{aligned}$$

are well defined as for any $t \in (a, b]$, $t + \delta \in (c, d]$, as $a + \delta > c$ and $b + \delta \leq d$. In the same way, for any $t \in (c, d]$, $t + \delta \in (a, b]$. Thus,

$$\psi_\delta \circ \phi_\delta(\mathbb{F}(I)_t) = \psi_\delta(\mathbb{F}(J)_{(t+\delta)}) = \mathbb{F}(I)_{(t+2\delta)} = \pi_{t \leq t+2\delta}(\mathbb{F}(I)_t)$$

and

$$\phi_\delta \circ \psi_\delta(\mathbb{F}(J)_t) = \phi_\delta(\mathbb{F}(I)_{(t+\delta)}) = \mathbb{F}(J)_{(t+2\delta)} = \theta_{t \leq t+2\delta}(\mathbb{F}(J)_t).$$

Therefore, ϕ_δ and ψ_δ are a pair of δ-interleaving morphisms. □

Once we are able to build a δ-interleaving between two δ-matched interval modules, we will use the Structure Theorem for persistence modules to generalice the construction for arbitrary persistence modules. This will prove useful to prove the first inequality needed to prove Theorem 3.0.1.

Proposition 3.0.3. *[5, Theorem 3.0.1] Given two persistence modules V, W, if there is a δ-matching between their barcodes, then there is a δ-interleaving morphism between them.*

Proof. Suppose that $\rho \colon \operatorname{Bar}(V) \to \operatorname{Bar}(W)$ is a δ-matching between the barcodes of V and W. By the Structure Theorem 2.2.3, V and W decompose in a finite direct sum of interval modules

$$V \cong \bigoplus_{I \in \operatorname{Bar}(V)} \mathbb{F}(I), \qquad W \cong \bigoplus_{J \in \operatorname{Bar}(W)} \mathbb{F}(W).$$

We can express $V = V_Y \oplus V_N$, $W = W_Y \oplus W_N$ denoting

$$V_Y \cong \bigoplus_{I \in \operatorname{coim} \rho} \mathbb{F}(I), \qquad V_N \cong \bigoplus_{I \in \operatorname{Bar}(V) \setminus \operatorname{coim} \rho} \mathbb{F}(I),$$

$$W_Y \cong \bigoplus_{J \in \operatorname{im} \rho} \mathbb{F}(J), \qquad W_N \cong \bigoplus_{J \in \operatorname{Bar}(J) \setminus \operatorname{im} \rho} \mathbb{F}(J).$$

The V_Y, W_Y modules separate the "yes, matched" intervals, from the V_N, W_N "not matched" intervals. For every interval $I \in \mathrm{Bar}(V_Y)$, I is δ-matched to an interval $J \in \mathrm{Bar}(W_Y)$ by $\rho(I) = J$. Thus, by Lemma 3.0.2, for all pair I, J of matched intervals, there exist a par of δ-interleaved morphisms

$$\phi_\delta \colon \mathbb{F}(I) \to \mathbb{F}(J)_\delta \quad \text{and} \quad \psi_\delta \colon \mathbb{F}(J) \to \mathbb{F}(I)_\delta$$

which induce the pair of δ-interleaved morphisms

$$\phi_\delta \colon V_Y \to W_{Y\delta} \quad \text{and} \quad \psi_\delta \colon W_Y \to V_{Y\delta}.$$

Not matched intervals are of length smaller than 2δ, therefore both, V_N and V_Y are δ-interleaved with the empty set. We can now construct the δ-interleaving morphism $\phi \colon V \to W$ such as $\phi|_{V_Y} = \phi_Y$ and $\phi|_{V_N} = 0$. In a similar way, we also construct $\psi \colon W \to V$. □

With Proposition 3.0.3 we have proven the first halve of Stability Theorem 3.0.1. Now we need to prove that we can build a δ-interleaving morphism from a δ-matching. To verify this claim we need several previous lemmas that will lead us to prove Proposition 3.0.3.

First, we will introduce some notation. Let (V, π), (W, θ) be two persistence modules and let $I = (b, d]$ be an interval with $d \in \mathbb{R} \cup \{+\infty\}$. Denote the set of bars of $\mathrm{Bar}(V)$ that start before b end exactly at d as

$$\mathrm{Bar}_{I^-}(V) \coloneqq \{(a, d] \in \mathrm{Bar}(V) \colon a \leq b\}.$$

Analogously, denote the set of bars that start at b and end after d as

$$\mathrm{Bar}_{I^+}(V) \coloneqq \{(b, c] \in \mathrm{Bar}(V) \colon c \geq d\}.$$

Lemma 3.0.4. *[5, Proposition 3.1.1] Let $I = (b, d]$ be an interval. It exists an injective morphism $\iota \colon (V, \pi) \in (W, \theta)$, then $\#(\mathrm{Bar}_{I^-}(V)) \leq \#(\mathrm{Bar}_{I^-}(W))$. Where $\#(\cdot)$ denotes the cardinal operator.*

Proof. For $b < s < d < r$, denote the set of elements in V_d witch come from all V_s and disappear in all V_r as

$$E_{I^-} = \bigcap_{b<s<d} \operatorname{im} \pi_{s \leq d} \cap \bigcap_{r>d} \ker \pi_{d \leq r} \subseteq V_d.$$

It holds that $\dim E_{I^-}(V) = \#(\mathrm{Bar}_{I^-}(V))$. For every morphism $p \colon (V, \pi) \to (W, \theta)$ the following diagram conmutes

$$\begin{tikzcd} V_s \arrow[r, "\pi_{s\leq r}"] \arrow[d, "p_s"'] & V_r \arrow[d, "p_r"] \\ W_s \arrow[r, "\theta_{s\leq r}"'] & W_r \end{tikzcd}$$

This implies that

$$p_r(\operatorname{im}\pi_{s\leq r}) \subseteq \operatorname{im}\theta_{s\leq r}, \qquad\qquad p_s(\ker\pi_{s\leq r}) \subseteq \ker\theta_{s\leq r}.$$

Taking $r = d$, $b < s < d$ in the first inclusion, and $s = d$, $r > d$ in the second, we have that

$$p_d(\operatorname{im}\pi_{s\leq d}) \subseteq \operatorname{im}\theta_{s\leq d}, \qquad\qquad p_d(\ker\pi_{d\leq r}) \subseteq \ker\theta_{d\leq r},$$

and

$$p_d(E_{I-}(V)) \subseteq E_{I-}(W).$$

If we now take p as the injective morphism of the hypothesis, $p = \iota$, we get

$$\dim E_{I-}(V) \leq \dim E_{I-}(W).$$

□

Lemma 3.0.5. *[5, Exercise 3.1.3] Let $I = (b, d]$ be an interval. It exists a surjective morphism $s\colon (V, \pi) \to (W, \theta)$, then $\#(\mathrm{Bar}_{I+}(V)) \geq \#(\mathrm{Bar}_{I+}(W))$.*

Proof. Analogously to the proof of Lemma 3.0.4 we now define

$$E_{I+}(V) = \bigcap \operatorname{im}\pi_{d\leq r}.$$

Therefore $\dim E_{I+}(V) = \#(\mathrm{Bar}_{I+}(V))$, and recalling the diagram used for the previous proof, and using the fact that is commutative, we have that

$$p_r(\operatorname{im}\pi_{s\leq r}) \supseteq \operatorname{im}\theta_{s\leq r}.$$

Taking $s = d$ we then have that

$$p_d(E_{I+}(V)) \supseteq E_{I+}(W).$$

And finally, taking the surjective morphism $p = s$ we have that

$$\dim E_{I-}(V) \geq \dim E_{I-}(W).$$

□

To build our δ-matching we first define two induced matchings, by an injection and by a surjection respectively. First, suppose that there exists an injection $\iota\colon V \to W$. For every $c \in \mathbb{R} \cup \{\infty\}$, sort the bars $(a_i, c] \in \mathrm{Bar}(V)$, $i \in \{1, \ldots, k\}$ by decreasing length order,

$$(a_1, c] \supseteq (a_2, c] \supseteq \cdots \supseteq (a_k, c], \text{ with } a_1 \leq a_2 \leq \cdots \leq a_k.$$

Sort in the same manner the bars $(b_j, c] \in \mathrm{Bar}(V)$, $j \in \{1, \dots, l\}$,

$$(b_1, c] \supseteq (b_2, c] \supseteq \cdots \supseteq (a_k, c], \text{ with } b_1 \leq b_2 \leq \cdots \leq b_k.$$

As there is an injection between V and W, Lemma 3.0.4 asures that the amount of bars in $\mathrm{Bar}(V)$ is lower that the amount in $\mathrm{Bar}(V)$, i.e., $k \leq l$. We define the *injective induced matching* $\mu_{inj}\colon \mathrm{Bar}(V) \to \mathrm{Bar}(W)$ matching, for each $c \in \mathbb{R} \cup \{\infty\}$, the intervals from both lists by decreasing length.

Lemma 3.0.6. *[5, Proposition 3.1.5] If there exists an injection $\iota\colon (V, \pi) \in (W, \theta)$, then the induced matching $\mu_{inj}\colon \mathrm{Bar}(V) \to \mathrm{Bar}(W)$ satisfies:*

1. $\operatorname{coim} \mu_{inj} = \mathrm{Bar}(V)$,
2. $\mu_{inj}(a, c] = (b, c], \ \forall b \leq a, \ \forall (a, d] \in \mathrm{Bar}(V)$.

Proof. Applying Lemma 3.0.4 with the interval $(a_k, c]$, we have that for each $c \in \mathbb{R} \cup \{\infty\}$, $\# \mathrm{Bar}_{(a_k,c]-}(V) \leq \# \mathrm{Bar}_{(a_k,c]-}(W)$, having $k \leq l$ as we note earlier. This means that every bar in $\mathrm{Bar}(V)$ is matched to some bar in $\mathrm{Bar}(W)$. Hence $\operatorname{coim} \mu_{inj} = \mathrm{Bar}(V)$. Moreover, as the matching is carried out in length descending order, for each $i \in \{1, \dots, k\}$, $\mu_{inj}(a_i, c] = (b_i, c]$, and applying Lemma 3.0.4, now with the interval $(a_i, c]$, and making the same reasoning, $a_i \leq b_i$. $\square$

Now we suppose that there exists a surjection $\sigma\colon V \to W$. For every $a \in \mathbb{R}$, sort the intervals $(a, c_i] \in \mathrm{Bar}(V)$, $i \in \{1, \dots, k\}$ by decreasing length order as before,

$$(a, c_1] \supseteq (a, c_2] \supseteq \cdots \supseteq (a, c_k], \text{ with } c_1 \geq c_2 \geq \cdots \geq a_k,$$

and again in the same manner, sort the intervals $(a, d_j] \in \mathrm{Bar}(V)$, $j \in \{1, \dots, l\}$,

$$(a, d_1] \supseteq (a, d_2] \supseteq \cdots \supseteq (a, d_l], \text{ with } d_1 \geq d_2 \geq \cdots \geq d_l.$$

We define the *surjective induced matching* $\mu_{sur}\colon \mathrm{Bar}(V) \to \mathrm{Bar}(W)$ matching, for each $a \in \mathbb{R}$, the intervals from both lists by decreasing length.

Lemma 3.0.7. *[5, Exercise 3.1.8] If there exists a surjection $s\colon (V, \pi) \to (W, \theta)$, then the induced matching $\mu_{sur}\colon \mathrm{Bar}(V) \to \mathrm{Bar}(W)$ satisfies:*

1. $\operatorname{im} \mu_{sur} = \mathrm{Bar}(W)$,
2. $\mu_{sur}(a, c] = (a, d], \ \forall c \geq d, \ \forall (a, d] \in \mathrm{Bar}(V)$.

Proof. Using Lemma 3.0.5 with the interval $(b, d_k]$ for each $b \in \mathbb{R}$, we get that, as there exists a surjection between the modules, now $k \geq l$. Therefore, every bar in $\mathrm{Bar}(W)$ is matched to some bar in $\mathrm{Bar}(V)$ and $\mathrm{im}\,\mu_{sur} = \mathrm{Bar}(W)$. In an analogue way to the previous lemma, as the intervals in both lists are matched in a decreasing manner, for every $j \in \{1, \dots, l\}$, $\mu_{sur}(a, c_j] = (a, d_j]$, and if we now apply Lemma 3.0.5, we get that $c_j \geq d_j$. □

Hence, with the injective and the surjective induced matchings at hand, for a general morphism f, we can define the *induced matching* $\mu(f)\colon \mathrm{Bar}(V) \to \mathrm{Bar}(W)$, as the composition $\mu_{inj} \circ \mu_{sur}$, defined as $\mathrm{im}\,\mu_{sur} = \mathrm{Bar}(\mathrm{im}\, f) = \mathrm{coim}\,\mu_{inj}$.

The following lemma verifies that, in fact, the mapping between persistence modules with its morphisms and barcodes with induced matchings (either the injective or the surjective versions) has functorial properties between the two categories.

Lemma 3.0.8. *[5, Claim 3.1.13] Let U, V and W persistence diagrams and f, g, h morphisms between them defined as in the following diagram:*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend right, "h"'] & V \arrow[r, "g"] & W \end{tikzcd} .$$

If all f, g, h are all injections, or all surjections, then the corresponding diagram formed by the barcodes of the modules, and their respective matchings conmutes as well.

$$\begin{tikzcd} \mathrm{Bar}(U) \arrow[r, "\mu_*(f)"] \arrow[rr, bend right, "\mu_*(h)"'] & \mathrm{Bar}(V) \arrow[r, "\mu_*(g)"] & \mathrm{Bar}(W) \end{tikzcd} .$$

Where μ_* denotes μ_{inj} or μ_{sur} accordingly.

Proof. Let f, g, h injective morphisms, by the definition of the injective induced matching and Lemma 3.0.4 for any $d \in \mathbb{R} \cup \{+\infty\}$, there exist $k \leq l \leq q$ such that the barcodes of U, V, W consist on the following bars:

$$\begin{aligned} \mathrm{Bar}(U) :& (a_1, d] \supset \cdots \supset (a_k, d] \\ \mathrm{Bar}(V) :& (b_1, d] \supset \cdots \supset (b_k, d] \supset \cdots \supset (b_l, d] \\ \mathrm{Bar}(V) :& (c_1, d] \supset \cdots \supset (c_k, d] \supset \cdots \supset (c_l, d] \supset \cdots \supset (c_q, d]. \end{aligned}$$

Therefore, for any d the diagram conmutes as

$$\mu_{inj}(f)(a_i, d] = (b_i, d], \ \mu_{inj}(g)(b_i, d] = (c_i, d], \ \mu_{inj}(h)(a_i, d] = (c_i, d]$$

for $1 \leq i \leq k$. If f, g, h were surjective morphisms, an analogue reasoning using the surjective induced matching definition and Lemma 3.0.5 completes the proof. □

Finally, we can claim the two main lemmas from which we will construct our desired δ-matching.

Lemma 3.0.9. *[5, Lemma 3.2.1] Let $(V, \pi), (W, \theta)$ be δ-interleaved persistence modules, with δ-interleaving morphisms $\phi\colon V \to W_\delta$ and $\psi\colon W \to V_\delta$. Let $\phi\colon V \to \operatorname{im}\phi$ be a surjection and $\mu_{sur}\colon \operatorname{Bar}(V) \to \operatorname{Bar}(\operatorname{im}\phi)$ the induced matching. Then*

1. $\operatorname{coim}\mu_{sur} \supseteq \operatorname{Bar}(V)_{\geq 2\delta}$,
2. $\operatorname{im}\mu_{sur} = \operatorname{Bar}(\operatorname{im}\phi)$ *and*
3. $\mu_{sur}(b, d] = (b, d'],\ \forall (b, d] \in \operatorname{coim}\mu_{sur},\ d' \in [d - 2\delta, d]$.

Proof. 1. To check the first part, we observe that, in the following diagram, the three morphisms are surjective as ϕ and $\pi_{t\leq t+2\delta}$ are defined onto their images, and the diagram conmutes,

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[rr, bend right, "\pi_{t\leq t+2\delta}"'] & \operatorname{im}\phi \arrow[r, "\psi_\delta"] & \operatorname{im}\pi_{t\leq t+2\delta} \end{tikzcd}.$$

Therefore, because of Lemma 3.0.8 the barcode diagram also conmutes:

$$\begin{tikzcd} \operatorname{Bar}(V) \arrow[r, "\mu_{sur}(\phi)"] \arrow[rr, bend right, "\mu_{sur}(\pi_{t\leq t+2\delta})"'] & \operatorname{Bar}(\operatorname{im}\phi) \arrow[r, "\mu_{sur}(\psi_\delta)"] & \operatorname{Bar}(\operatorname{im}\pi_{t\leq t+2\delta}) \end{tikzcd}.$$

By the definition of the surjective induced matching,

$$\operatorname{coim}\mu_{sur}(\pi_{t\leq t+2\delta}) = \operatorname{Bar}(V)_{\geq 2\delta}.$$

For each starting point $a \in \mathbb{R}$, we have that

$$\operatorname{Bar}(\operatorname{im}\pi_{t\leq t+2\delta}) = \{(a, b - 2\delta]\colon (a, b] \in \operatorname{Bar}(V), b - a > 2\delta\}.$$

Sorting all bars of $\operatorname{Bar}(V)$ and of $\operatorname{Bar}(\operatorname{im}\pi_{t\leq t+2\delta})$ in length-not-increasing order and matching the bars though the longest-first order, each bar $(a, b] \in \operatorname{Bar}(V)$ is matched with the bar $(a, b - 2\delta] \in \operatorname{Bar}(\operatorname{im}\pi_{t\leq t+2\delta})$ while $b - a > 2\delta$. The smaller bars are not matched. Thus,

$$\operatorname{coim}\mu_{sur}(phi) \supseteq \operatorname{coim}\mu_{sur}(\operatorname{im}\pi_{t\leq t+2\delta}) = \operatorname{Bar}(V)_{\geq 2\delta}.$$

2. The second part is just a reformulation of Lemma 3.0.4.

3. Let $(b, d] \in \operatorname{coim}$. There are two cases:

   On one hand, if $d - b \leq 2\delta$, $(b, d]$ is matched to $(b, d']$ where $d \geq d'$, by definition of μ_{sur}. Also, $d' > b$ and, as in this case we have $b \geq d - 2\delta$, we have $d' > d - 2\delta$. Therefore, $d' \in [d - 2\delta, d]$.

   On the other hand, if $d - b > 2\delta$, $(b, d]$ is matched to $(b, d']$ by $\mu_{sur}(\phi)$, with $(b, d'] \in W_{\leq 2\delta}$. We can therefore use Lemma 3.0.7 to check that $d' \geq d$. In the same manner, $(b, d']$ is matched to $(b, d'']$ by $\mu_{sur}(\psi)_\delta$ with $d'' \geq d'$. Finally, using the commutativity of the following diagram, we have that $(b, d''] = (b, d - 2\delta]$, making $d' \in [d - 2\delta, d]$.

$$\begin{tikzcd}
\operatorname{Bar}(V)_{\geq 2\delta} & \operatorname{Bar}(\operatorname{im}\phi) & \operatorname{Bar}(\operatorname{im}\pi_{t\leq t+2\delta}) \\
\rotatebox[origin=c]{90}{$\in$} & \rotatebox[origin=c]{90}{$\in$} & \rotatebox[origin=c]{90}{$\in$} \\
(b, d] \arrow[r, mapsto, "\mu_{sur}(\phi)"] \arrow[rrd, mapsto, bend right, "\mu_{sur}(\pi_{t\leq t+2\delta})"'] & (b, d'] \arrow[r, mapsto, "\mu_{sur}(\psi_\delta)"] & (b, d''] \arrow[d, equal] \\
 & & (b, d - 2\delta]
\end{tikzcd}$$

$\square$

Lemma 3.0.10. *[5, Proposition 3.2.2] Let $(V, \pi), (W, \theta)$ be δ-interleaved persistence modules, with δ-interleaving morphisms $\phi\colon V \to W_\delta$ and $\psi\colon W \to V_\delta$. Let $\phi\colon V \to \operatorname{im}\phi$ be a injection and $\mu_{inj}\colon \operatorname{Bar}(\operatorname{im}\phi) \to \operatorname{Bar}(W_\delta)$ the induced matching. Then*

1. $\operatorname{coim}\mu_{sur} = \operatorname{Bar}(\operatorname{im}\phi)$,

2. $\operatorname{im}\mu_{inj} \supseteq \operatorname{Bar}(W_\delta)_{\geq 2\delta}$ *and*

3. $\mu_{inj}(b, d'] = (b', d']$, $(b, d'] \in \operatorname{coim}\mu_{inj}$, $b' \in [b - 2\delta, b]$.

Proof. 1. Immediate using Lemma 3.0.6.

2. As $\phi_\delta \circ \psi = \theta_{t\leq t+2\delta}$ the following diagram conmutes:

$$\begin{tikzcd}
W \arrow[r, "\psi"] \arrow[rr, bend right, "\theta_{t\leq t+2\delta}"] & \operatorname{im}\psi \arrow[r, "\phi_\delta"] & W_{2\delta}
\end{tikzcd}.$$

   This implies that $\operatorname{im}\theta_{t\leq t+2\delta} \subseteq \operatorname{im}\phi_\delta \subseteq W_{2\delta}$, so there are some injections j and i which make the following diagram commute as well:

$$\begin{tikzcd}
\operatorname{im}\theta_{t\leq t+2\delta} \arrow[r, "j"] \arrow[rr, bend right, "\theta_{t\leq t+2\delta}"'] & \operatorname{im}\phi_\delta \arrow[r, "i"] & W_{2\delta}
\end{tikzcd}.$$

As all morphisms in the diagram abobe are injections, we can use the functorial properties of Lemma 3.0.8 having a commutative diagram of the barcodes of each of the previous persistence modules:

$$\begin{tikzcd}
\mathrm{Bar}(\mathrm{im}\,\theta_{t\leq t+2\delta}) \arrow[r, "\mu_{inj}(j)"] \arrow[rr, bend right, "\mu_{inj}(\theta_{t\leq t+2\delta})"'] & \mathrm{Bar}(\mathrm{im}\,\phi_\delta) \arrow[r, "\mu_{inj}(i)"] & \mathrm{Bar}(W_{2\delta})
\end{tikzcd}.$$

We have that

$$\begin{aligned}
\mathrm{Bar}(\mathrm{im}\,\theta_{t\leq t+2\delta}) &= \{(b, d-2\delta)\colon (b,d] \in \mathrm{Bar}(W), b < d-2\delta\}, \\
\mathrm{Bar}(W_{2\delta}) &= \{(b-2\delta, d-2\delta)\colon (b,d] \in \mathrm{Bar}(W)\} \text{ and} \\
\mu_{inj}(\theta_{t\leq t+2\delta})((b, d-2\delta)) &= (b-2\delta, 2-2\delta)
\end{aligned}$$

Therefore $\mathrm{im}_\mu\, inj(i) \supseteq \mathrm{im}\,\mu_{inj}(\psi_{t\leq t+2\delta}) = \mathrm{Bar}(W_{2\delta})_{2\delta}$. This, undoing the shifts made, make the prove.

3. Let $(b,d] \in \mathrm{Bar}(\mathrm{im}\, f_\delta)$ such as for some b', $\mu_{inj}(b,d] = (b',d] \in \mathrm{Bar}(W)$. Because of Lemma 3.0.6, $b' \leq b$. There are again two cases:

   If $d-b \leq 2\delta$, then $b' \geq d-2\delta \geq b > b-2\delta$ and $b' \in [b-2\delta, b]$.

   Else, if $d-b > 2\delta$, there exists an interval $(b'+2\delta, d] \in \mathrm{Bar}(\mathrm{im}\,\theta_{t\leq t+2\delta})$ such that

   $$\mu_{inj}(\theta_{t\leq t+2\delta})(b'+2\delta, d] = \mu_{inj}(i)(b,d] = (a,d].$$

   Thus, $b \leq b'+2\delta$ and $b' \in [b-2\delta, b]$.

□

At last, we can now prove the other part of the Stability theorem. For so, we will construct a δ-matching out of a δ-interleaving morphism.

Proposition 3.0.11. *[5, Theorem 3.0.2] Given two persistence modules V, W, with a δ-interleaving morphism between them, then there is a δ-matching between their barcodes.*

Proof. Let $\mu(\phi) = \mu_{inj} \circ \mu_{sur}$ and let $\Phi_\delta\colon \mathrm{Bar}(W_\delta) \to \mathrm{Bar}(W)$ be the *shift map* that carries each bar $(a,b]$ into $(a+\delta, b+\delta]$. The composition $\Phi_\delta \circ \mu(\phi)$ is a matching between

$\mathrm{Bar}(V)$ and $\mathrm{Bar}(W)$. Hence, using Lemma 3.0.10 and 3.0.9, we get the following diagram:

$$\begin{tikzcd}
\mathrm{Bar}(V) & & \mathrm{Bar}(W_\delta)_{\geq 2\delta} & \mathrm{Bar}(W)_{\geq 2\delta} \\
\cup\!\mid & & \cap\!\mid & \cap\!\mid \\
\mathrm{Bar}(V)_{\geq 2\delta} \arrow[r, "\mu_{sur}"] & \mathrm{Bar}(\mathrm{im}\, f) \arrow[r, "\mu_{inj}"] & \mathrm{im}\,\mu_{inj} \arrow[r, "\Psi_\delta"] & \mathrm{Bar}\, B(W) \\
\uplus & \uplus & \uplus & \uplus \\
(b,d] \arrow[r, mapsto] & (b,d'] \arrow[r, mapsto] & (b',d'] \arrow[r, mapsto] & (b'+\delta, d'+\delta]
\end{tikzcd}$$

The diagram shows that, by Lemma 3.0.9, a bar $(b,d] \in \mathrm{Bar}(V)_{\geq 2\delta}$ is sent to $\mu_{sur}(b,d] = (b,d'] \in \mathrm{Bar}(\mathrm{im}\,\phi)$ with $d' \in [d-2\delta, d]$. Then, by Lemma 3.0.9, it is sent to $\mu_{sur}(b,d'] = (b',d']$ with $b' \in [b-2\delta, b]$. At last, using the shift morphism Φ_δ it is carried to $(b'+\delta, d'+\delta]$.

This shows that any bar in $\mathrm{Bar}(V)_{\geq 2\delta}$ is matched. In the same manner it can be seen that any bar in $\mathrm{Bar}(W)_{\geq 2\delta}$ is matched. Thus, we have that

$$\begin{cases} d-2\delta \leq d' \leq d \\ b-2\delta \leq b' \leq b \end{cases} \Rightarrow \begin{cases} d-\delta \leq d'+\delta \leq d+\delta \\ b-\delta \leq b'+\delta \leq b+\delta \end{cases},$$

and therefore, $\Phi_\delta \circ \mu(\phi)$ is a δ-matching between $\mathrm{Bar}(V)$ and $\mathrm{Bar}(W)$. □

The constructions made by Proposition 3.0.3 and Proposition 3.0.11 asure that given a δ-interleaving morphism we can build a δ-matching, and conversely, given a δ-matching we can build a δ-interleaving morphism. This means that if one of the two exists, it fixes a δ. Both the interleaving distance and the bottleneck distance try to minimice this δ, so once fixed for one of them, the other needs an smaller or equal δ'. Thus, with each of the propositions we can prove one of the inequalities needed to reach the isomorphism between the space of persistence diagrams and the space of their barcodes.

Theorem 3.0.1 (Stability). *[5, Theorem 2.2.8] There is an isometry between a persistence module with the interleaving distance and its barcode with the bottleneck distance. That is, given two persistence modules V and W, it holds that*

$$d_{int}(V,W) = d_{bot}(\mathrm{Bar}(V), \mathrm{Bar}(W)).$$

Proof. Suppose $d_{int}(V,W) = \delta$. Proposition 3.0.11 asures there exist a δ-matching between $\mathrm{Bar}(V)$ and $\mathrm{Bar}(W)$. As $d_{bot}(V,W)$ is the infimum δ for witch exists a δ-matching, $d_{bot}(V,W) \leq d_{int}(V,W)$. On the other hand, Proposition 3.0.3 leads, with the same reasoning, to $d_{int}(V,W) \leq d_{bot}(V,W)$. Thus, it has to be $d_{int}(V,W) = d_{bot}(\mathrm{Bar}(V), \mathrm{Bar}(W))$. □

4

Edelsbrunner & Harer's (Hausdorff) Stability Theorem

In this chapter we will follow [2]. To simplify notation, along this chapter we will adopt the following. Let $H_k(X)$ be the k-th singular homology group of a topological space X. The dimension of $H_k(x)$ is denoted by the k-th Betti number $\beta_k(X) \coloneqq \dim H_k(x)$.

Let $f\colon X \to \mathbb{R}$, $x < y \in \mathbb{R}$. Denote the k-th homology group of the pre-image by f of an interval $(-\infty, x]$ as $F_x \coloneqq H_k(f^{-1}(-\infty, x])$. Denote the inclusion map from the k-th homology group F_x to the k-th homology group F_y as $f_x^y\colon F_x \to F_y$. Finally, denote $F_x^y \coloneqq \operatorname{im} f_x^y$.

Note that if $y = \infty$, F_x^y is the trivial group. Also, if $x = \infty$, then $y = \infty$ too.

Definition 4.0.1 (Homological critical value)**.** Let X be a topological space and let $f\colon X \to \mathbb{R}$. A **homological critical value** of f a number $a \in \mathbb{R}$ such that there exists $k \in \mathbb{Z}$ such that for all $\varepsilon > 0$, the morphism $H_k(f^{-1}(-\infty, a - \infty)) \to H_k(f_{-1}(-\infty, a + \varepsilon))$ is not an isomorphism.

Definition 4.0.2 (Tame function)**.** A function $f\colon X \to \mathbb{R}$ is said to be **tame** if it has a finite number of homological critical values, and for all $z \in \mathbb{Z}$, and for all $a \in \mathbb{R}$, $\dim F_a < \infty$.

Definition 4.0.3 (Multiplicity)**.** Let $f\colon X \to \mathbb{R}$ be tame, and $(a_i)_{i=1,\ldots,n}$ be its homological critical values. Take $(b_i)_{i=1,\ldots,n}$ be an interleaved sequence of non critical values such that $b_{i-1} < a_i < b_i$ for all $i = 1,\ldots,n$. Define $b_{-1} = a_0 = -\infty$, $b_{n+1} = a_{n+1} = \infty$. The **multiplicity** of $(a_i, a_j) \in D(f)$, denoted μ_i^j is

$$\mu_i^j \coloneqq \beta_{b_{i-1}}^{b_j} - \beta_{b_i}^{b_j} + \beta_{b_i}^{b_{j-1}} - \beta_{b_{i-1}}^{b_{j-1}}.$$

.

The **total multiplicity** of a multiset A, denoted $\#(A)$ is the sum of the multiplicities of every element elements in A.

Note that the total multiplicity of a multiset is the the generalized concept of cardinality of a normal set. While the cardinality of a set counts the number of elements in the set, the multiplicity of a multiset counts how many elements, different or not, are there in the multiset.

Theorem 4.0.4 (Main Theorem, [2])**.** *Let X be a triangulable space, and $f, g\colon X \to \mathbb{R}$ continuous tame functions. Then,*

$$d_{\text{bot}}(D(f), D(g)) \leq \|f - g\|_\infty$$

4.1 Hausdorff Stability

We will denote the closed upper left quadrant of a point $(x, y) \in \mathbb{R}^2$ as $Q_x^y \coloneqq [-\infty, x] \times [y, \infty]$.

Lemma 4.1.1 (k-Triangle Lemma, [2])**.** *Let $f\colon X \to \mathbb{R}$ be a tame function, $x < y \in \mathbb{R}$ be non critical values of f. Then the multiplicity μ of the persistence diagram of f in the closed upper left quadrant is*

$$\mu = \#(D(f) \cap Q_x^y) = \beta_x^y.$$

Proof. Let $x = b_i$, $y = b_{j-1}$.

$$\mu = \sum_{k \leq i \leq j \leq l} \mu_k^l = \sum_{k \leq i \leq j \leq l} \beta_{b_{k-1}}^{b_l} - \beta_{b_k}^{b_l} + \beta_{b_k}^{b_{l-1}} - \beta_{b_{k-1}}^{b_{l-1}} \tag{4.1}$$

$$= \beta_{b_{-1}}^{b_{n+1}} - \beta_{b_i}^{b_{n+1}} + \beta_{b_i}^{b_{j-1}} - \beta_{b_{j-1}}^{b_{-1}} = \beta_{b_k}^{b_{l-1}} = \beta_x^y. \tag{4.2}$$

The fist two equalities in (4.1) are just the definition of total multiplicity. In (4.2), note that every other term in the sum cancels. Then note that $\beta_{b_{-1}}^{b_{n+1}} = \dim F_{-\infty}^{\infty}$, $\beta_{b_i}^{b_{n+1}} = \dim F_x^{\infty}$ and $\beta_{b_{j-1}}^{b_{-1}} = \dim F_{-\infty}^{y}$. All of them are the dimension of the trivial group, therefore, equal to 0. This leaves only one remaining term and completes the proof. $\square$

Denote the **upper left quadrants** $Q := Q_b^c = [-\infty, b] \times [c, \infty]$, $Q_\varepsilon := Q_{b-\varepsilon}^{c+\varepsilon} = [-\infty, b-\varepsilon] \times [c+\varepsilon, \infty]$.

Lemma 4.1.2 (Quadrant Lemma, [2])**.** *Let $f, g\colon X \to \mathbb{R}$ be two tame functions. With the notation abobe, the following inequality holds,*

$$\#(D(f), \cap Q_\varepsilon) \leq \#(D(g) \cap Q).$$

Proof. Let $\varepsilon := \|f - g\|_\infty$. Hence, considering the pre-image of the functions, we have the following inclusions

$$f^{-1}((-\infty, x]) \subseteq g^{-1}((-\infty, x + \varepsilon)), \tag{4.3}$$
$$g^{-1}((-\infty, x]) \subseteq f^{-1}((-\infty, x + \varepsilon)). \tag{4.4}$$

Name $\varphi_x\colon F_x \to G_{x+\varepsilon}$ to the inclusion map induced by (4.3) and $\psi_x\colon G_x \to F_{x+\varepsilon}$ to the inclusion map induced by (4.4). Let $b < c \in \mathbb{R}$. With the described maps, we can form commutative diagram (4.5) where we observe that

$$\operatorname{im}(f_{c-\varepsilon}^{c+\varepsilon}) = F_{c-\varepsilon}^{c+\varepsilon} \subseteq \psi_c \circ g_b^c(G_b) = \psi_c(G_b^c).$$

$$\begin{tikzcd}
F_{b-\varepsilon} \arrow[r, "f_{b-\varepsilon}^{c+\varepsilon}"] \arrow[d, "\varphi_{b-\varepsilon}"'] & F_{c+\varepsilon} \\
G_b \arrow[r, "g_b^c"] & G_c \arrow[u, "\psi_c"']
\end{tikzcd} \tag{4.5}$$

Last inclusion is enough for the requirements of this proof, never the less, we will make one more note that will be useful latter on, through the proof of Lemma 4.1.3. Fit the maps so that they describe commutative diagram (4.6), showing that

$$\psi_c(G_b^c) = \psi_c \circ g_b^c(G_b) = f_{b+\varepsilon}^{c+\varepsilon} \circ \psi_b(G_b) \subseteq F_{b+\varepsilon}^{c+\varepsilon}.$$

$$\begin{tikzcd}
F_{b+\varepsilon} \arrow[r, "f_{b+\varepsilon}^{c+\varepsilon}"] & F_{c+\varepsilon} \\
G_b \arrow[r, "g_b^c"] \arrow[u, "\varphi_b"] & G_c \arrow[u, "\psi_c"']
\end{tikzcd} \tag{4.6}$$

From both diagrams we finally obtain the inclusion chain

$$F_{c-\varepsilon}^{c+\varepsilon} \subseteq \psi_c(G_b^c) \subseteq F_{b+\varepsilon}^{c+\varepsilon}. \tag{4.7}$$

By Lemma 4.1.1, we are able to count the elements in the intersection of the diagrams with the upper left quadrants. Hence

$$\#(D(f) \cap Q_\varepsilon) = \beta_{b-\varepsilon}^{c+\varepsilon} = \dim F_{b-\varepsilon}^{c+\varepsilon},$$
$$\#(D(g) \cap Q) = \beta_b^c = \dim G_b^c.$$

As if one homology group is contain in an other, the dimension of the first must be lower or equal to the one of the second. Also, the dimension is invariant under inclusion maps. Thus, the first inclusion of (4.7) asserts that $F_{c-\varepsilon}^{c+\varepsilon} \subseteq \psi_c(G_b^c)$ and therefore we have proven that $\dim F_{c-\varepsilon}^{c+\varepsilon} \leq \dim G_b^c$. $\square$

Before next lemma we will introduce some new notation. Let $f\colon X \to \mathbb{R}$ fe a tame function. Let $w < x < y < z \in \mathbb{R}$ be numbers different from critical values of f. Recall that $F_x = H_k(f^{-1}(-\infty, x])$, $f_x^y\colon F_x \to F_y$ and $F_x^y = \dim f_x^y$. We denote

$$f_x^{y,z} := f_y^z|_{F_x^y}, \qquad F_x^{y,z} := \dim f_x^{y,z}.$$

Note, from linear algebra, that $\dim F_x^{y,z} = \dim F_x^y - \dim F_x^z$. Note too that $F_w^y \subseteq F_x^y$. Therefore, $\ker F_w^y \subseteq \ker F_x^y$ and we can define the quotient

$$F_{w,x}^{y,z} := F_x^{y,z} / F_w^{y,z}.$$

Let $a < b < c < d \in \mathbb{R}$. Denote the **rectangles** $R := [a,b] \times [c,d]$, $R_\varepsilon := [a+\varepsilon, b-\varepsilon] \times [c+\varepsilon, d-\varepsilon]$.

Lemma 4.1.3 (Box Lemma, [2])**.** *With the notation abobe, the following inequality holds,*

$$\#(D(f), \cap R_\varepsilon) \leq \#(D(g) \cap R).$$

Proof. Note that we can asume that $a+\varepsilon < b-\varepsilon$ and $c+\varepsilon < d-\varepsilon$. Otherwise there would not be rectangle R_ε. Also note that

$$\#(D(f) \cap R_\varepsilon) = \dim F_{a+\varepsilon, b-\varepsilon}^{c+\varepsilon, d-\varepsilon}$$
$$\#(D(g) \cap R) = \dim G_{a,b}^{c,d}$$

To make our prove we draw diagram 4.1. Lets analice every element of the diagram.

$$\begin{tikzcd}
G_a^d \arrow[rrr, "r_1"] & & & G_b^d \\
& F_{a+\varepsilon}^{d-\varepsilon} \arrow[r, "r_2"] & F_{b-\varepsilon}^{d-\varepsilon} \arrow[ur, "s_1"] & \\
& F_{a+\varepsilon}^{c+\varepsilon} \arrow[u, "u_2"] \arrow[r, "r_3"] & F_{b-\varepsilon}^{c+\varepsilon} \arrow[u, "u_3"'] & \\
G_a^c \supseteq E_a^c \arrow[uuu, "u_1"] \arrow[ur, "s_2"] \arrow[rrr, "r_4"] & & & E_b^c \subseteq G_b^c \arrow[uuu, "u_4"'] \arrow[ul, "s_3"']
\end{tikzcd}$$

First of all, the middle upside arrows, are

$$u_2 = f_{a+\varepsilon}^{c+\varepsilon, d-\varepsilon}, \qquad u_3 = f_{b-\varepsilon}^{c+\varepsilon, d-\varepsilon}.$$

Right arrows r_1, r_2, r_3, r_4 represent the inclusions from its respective vector spaces to their destination, in the same manner. The objetive is to define que respective quotients to define $G_{a,b}^{c,d}$. Recall the inclusion maps defined in the proof of Lemma 4.1.2, $\varphi_x\colon F_x \to G_{x+\varepsilon}$ and $\psi_x\colon G_x \to F_{x+\varepsilon}$. We define

$$E_b^c := \psi_c^{-1}(F_{b-\varepsilon}^{c+\varepsilon,d-\varepsilon}) \cap G_b^c, \qquad\qquad E_a^c := G_a^c \cap E_b^c.$$

We have then that the outer upside arrows are the respective restrictions

$$u_1 = g_a^{c,d}|_{E_a^c} \qquad\qquad u_4 = g_b^{c,d}|_{E_b^d}$$

□

4.2 Bottleneck Stability

Lemma 4.2.1 (Easy Bijection Lemma, [2]). *Let $f, g\colon X \to \mathbb{R}$ be tame functions, where g is very close to f. Then, following holds,*

$$d_{\text{bot}}(D(f), D(g)) \leq \|f - g\|_\infty.$$

Lemma 4.2.2 (Interpolation Lemma, [2]). *Let K be a simplicial complex. Take two picewise linear functions $\hat{f}, \hat{g}\colon K \to \mathbb{R}$. Then, following holds,*

$$d_{\text{bot}}(D(\hat{f}), D(\hat{g})) \leq \|\hat{f} - \hat{g}\|_\infty.$$

Theorem 4.0.4 (Main Theorem, [2]). *Let X be a triangulable space, and $f, g\colon X \to \mathbb{R}$ continuous tame functions. Then,*

$$d_{\text{bot}}(D(f), D(g)) \leq \|f - g\|_\infty$$

5

Gromov-Hausdorff's Stability Theorem

[1]

6

Vectorizations' Stability Theorems

6.1 Persistence landscapes

6.2 Persistence images

6.3 Euler curves

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