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Final Master Thesis

Structure and Stability Theorems in Topological Data Analysis Draft

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Abstract

This is a draft version of a in-work Master's thesis in TDA.

Key words

Contents

Chapter 1

Preliminaries

Topological Data Analysis is a wide topic, were results from all Mathematical branches are put together in order to compute topological invariants among real world data. Basic knowledge in both, Algebra and Analysis, as well as Geometry is needed to approach the theoretical results that strengthen the main theory that defines proves the needed structures and their stability. Therefore, in order to settle a common base and notation unifying the diverse procedures among bibliography, this chapter introduces the main definitions and basic results needed to tackle the rest of the thesis.

Section 1.1 introduces the main structure needed to compute persistence homology, the persistence modules. Section 1.2 introduces a tool to summarize the data from the persistence homology groups, the barcodes. With this summary it is possible to compute how homologically different are two datasets. To make this measure, the interleaving and bottleneck distance are introduced. The first one measures distances between persistence modules, while the later one measures distance between the barcodes of those modules. One key aspect of them, is that they are actually equivalent, as its later proved in Chapter 3.

A visual way of presenting barcodes is through persistence diagrams, which are presented in Section 1.3. Both barcodes and persistence diagrams are equivalent manners of presenting the homological features of some datasets. To give a different perspective about how to measure distances between persistence diagrams, 1.3 introduces the Wasserstein distance and redefines the bottleneck distance as a particular case.

Finally, 1.4 recalls the Hausdorff distance to measure distance between subsets of metric spaces, and generalices it presenting the Gromov-Hausdorff distance, which allows to measure distance between sets located in different metric spaces. This will be useful to delimit bottleneck distance in order to proof its stability.

The contents of this chapter are based on [3], [10], [11] and [13].

1.1 Persistence modules and interleaving distance

Definition 1.1.1 (Graded ring)**.** Let R be a ring. It is said that R is a **graded ring** if it can be decomposed into a direct sum of additive groups

$$R = \bigoplus_{n=1}^{\infty} R_n = R_1 \oplus R_2 \oplus \dots$$

such that for all $n, m \geq 0$,

$$R_n R_m = R_{n+m}.$$

Definition 1.1.2 (Graded ideal)**.** Let R be a graded ring. A **graded ideal** is a two sided ideal $I \subseteq R$ that can be decomposed into a direct sum

$$I = \bigoplus_{n=1}^{\infty} I_n$$

where each $n \geq 0$, $I_n = I \cap R_n$.

Definition 1.1.3 (Left module, Definition IV.1.1.1 [9])**.** Let R be a ring. A **left R-module** is an abelian group $(M, +)$ with an operation $\cdot \colon R \times M \to M$ such that for all $r, s \in R$ and for all $x, y \in M$,

(i) $(rs) \cdot x = r(s \cdot x)$,

(ii) $(r + s) = r \cdot x + s \cdot x$,

(iii) $r \cdot (x + y) = r \cdot x + r \cdot y$.

If R has a multiplicative identity 1, then M is said to be a **unitary R-module** and

(iv) $1 \cdot x = x$.

If R is a division ring, that is, a ring with identity where every non cero element is a unit, then a unitary R-module is called a **left R-vector space**. Note that in this case, R is in fact a field.

Definition 1.1.4 (Graded module, Definition 4.7 [13])**.** Let M be a left module over a graded ring R. It is said that M is a **left graded module** if it can be decomposed into a direct sum

$$M = \bigoplus_{n=1}^{\infty} M_n$$

if for each $n, m \geq 0$, $R_n M_m \subseteq M_{n+m}$.

Definition 1.1.5 (Persistence module)**.** Let $/F$ be a field and let T be a totally ordered set. Let $V = \{V_t\}_{t\in T}$ be a collection of F-vector spaces. A T-indexed **persistence module** is a pair (V, π) such that $\pi = \{\pi_{s\leq t}\}$ is a collection of linear maps $\pi_{s\leq t}\colon V_s \to V_t$ that verifies that for all $r, s, t \in T$,

$$\pi_{r\leq s} \circ \pi_{s\leq t} = \pi_{r\leq t}.$$

Definition 1.1.6 (Morphism between persistence modules)**.** Let T be a totally ordered set. Let $(V, \pi), (W, \theta)$ be two persistence modules. A **morphism** between persistence modules $p\colon (V, \pi) \to (W, \theta)$ is a family of linear maps $p_t\colon V_t \to W_t$ such that for all $s \leq t$ the following diagram commutes:

$$\begin{tikzcd} V_s \arrow[r, "\pi_{s\leq t}"] \arrow[d, "p_s"'] & V_t \arrow[d, "p_t"] \\ W_s \arrow[r, "\theta_{s\leq t}"'] & W_t \end{tikzcd}$$

If a morphism i verifies that for all $t \in T$, $i_t\colon V_t \to V_t$ is the identity, then i is the **identity morphism**. If there exists two morphisms $p\colon (V, \pi) \to (W, \theta)$ and $q\colon (W, \theta) \to (V, \pi)$ such that the compositions $p \circ q$ and $q \circ p$ are both the identity morphism, then p and q are **isomorphisms** of persistence modules. In this case, (V, π) and (W, θ) are said to be **isomorphic** persistence modules.

For now on, to simplify notation, we will limit our totally order set to be the real numbers, $T = \mathbb{R}$. Also, when there is no possible confusion, we might denote the persistence module (V, π) by just is collection of vector spaces V.

Definition 1.1.7 (Persistence module shift)**.** Let (V, π) be a persistence module and let $\delta \in \mathbb{R}$. The **δ-shift** of (V, π) is the persistence module (V_δ, π_δ) defined by taking

$$(V_\delta)_t \coloneqq V_{t+\delta}, \qquad (\pi_\delta)_{s\leq t} \coloneqq \pi_{s+\delta\leq t+\delta}.$$

Proposition 1.1.8 (Exercise 1.2.3, [11])**.** *Let $\delta > 0$. Let $(V, \pi), (V_\delta, \pi_\delta)$ be a persistence module and its shift. The map $\phi_\delta\colon (V, \pi) \to (V_\delta, \pi_\delta)$, defined as*

$$\phi_\delta(V_t) \coloneqq \pi_{t\leq t+\delta}(V_t) = V_{t+\delta},$$

is a persistence module morphism.

Proof. As $\delta > 0$, then $t \leq t + \delta$. Hence

$$\phi_\delta \circ \pi_{t\leq t+\delta}(V_t) = \phi_\delta(V_{t+\delta}) = V_{t+\delta+\delta} = V_{t+2\delta},$$
$$\pi_{t+\delta\leq t+2\delta} \circ \phi_\delta(V_t) = \pi_{t+\delta\leq t+2\delta}(V_{t+\delta}) = V_{t+2\delta}.$$

∎

Definition 1.1.9 (Shift morphism)**.** The persistence module morphism ϕ_δ defined as in Proposition 1.1.8 is named **δ-shift morphism**.

Definition 1.1.10 (δ-interleaved modules)**.** Let $(V,\pi),(W,\theta)$ be two persistence modules and let $\delta > 0$. V and W are **δ-interleaved** if there exists two persistence module morphisms $\phi\colon V \to W_\delta$ and $\psi\colon W \to V_\delta$ such that the following diagrams commute:

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[rr, bend right, "\pi_{2\delta}"'] & W_\delta \arrow[r, "\psi_\delta"] & V_{2\delta} \end{tikzcd}, \qquad \begin{tikzcd} W \arrow[r, "\psi"] \arrow[rr, bend right, "\theta_{2\delta}"'] & V_\delta \arrow[r, "\phi_\delta"] & W_{2\delta} \end{tikzcd}.$$

Persistence modules are a vast abstract algebraic tool. In order to make it more manageable, we give it some more structure, restricting the dimension of the vector spaces, Also, we limit as the amount of different up to isomorphism vector spaces there are.

Definition 1.1.11 (Tame persistence module)**.** A a persistence module (V,π) over $\mathbb{R}$ is **tame** if

(i) For all $t \geq 0$, $\dim(V_t)$ is finite.

(ii) For any $\varepsilon > 0$, there exists a finite subset $K \subset \mathbb{R}$ such that for all $t \in \mathbb{R} \setminus K$, the map $\pi_{t-\varepsilon \leq t+\varepsilon}\colon V_{t-\varepsilon} \to V_{t+\varepsilon}$ is not an isomorphism.

Definition 1.1.12 (Interleaving distance)**.** Let (V,π) and (W,θ) be tame two persistence modules. The **interleaving distance** between them is defined as

$$d_{\text{int}}(V,W) \coloneqq \inf\{\delta > 0 \mid V \text{ and } W \text{ are } \delta\text{-interleaved}\}.$$

Proposition 1.1.13. *The interleaving distance between two tame persistence modules is actually a distance.*

Definition 1.1.14 (Interval module)**.** Let $I = (a,b]$ be an interval with $b \leq \infty$ and let $\mathbb{F}$ be a field. An **interval module** $\mathbb{F}(I)$ is a persistence module defined as

$$\mathbb{F}(I)_t \coloneqq \begin{cases} \mathbb{F} \text{ if } t \in I, \\ 0 \text{ else,} \end{cases} \qquad \pi_{s\leq t} = \begin{cases} \text{Id if } t \in I, \\ 0 \text{ else.} \end{cases}$$

Definition 1.1.15 (Direct sum of persistance modules)**.** Let (V,π) and (V',π') be two persistence modules. Their **direct sum** (W,θ) is a persistence module where

$$W_t \coloneqq V_t \oplus V'_t, \text{ the direct sum of both vector spaces, and}$$
$$\theta_{s\leq t} \coloneqq \pi_{s\leq t} \oplus \pi'_{s\leq t}.$$

1.2 Barcodes and the bottleneck distance

Definition 1.2.1 (Barcode)**.** A **barcode** B is a finite multiset of intervals. That is, a collection $\{(I_i, m_i)\}$ of intervals I_i with multiplicities $m_i \in N$, where each interval I_i is either finite of the form $(a, b]$ or infinite of the form (a, ∞). Each interval I_i is named to be a **bar** of B.

Given an interval $I = (a, b]$, and some $\delta \geq 0$, we will denote

$$I^\delta := (a - \delta, b + \delta].$$

We will denote the strict upper triangular region of the Euclidean plane as

$$\mathbb{R}^2_< := \{(x, y) \in \mathbb{R}^2 : x < y\},$$

and the diagonal of the plane as

$$\Delta := \{(x, y) \in \mathbb{R}^2 : x = y\}.$$

Definition 1.2.2 (Multiset matching)**.** Let X and Y be two multi-sets and let $X' \subseteq X$, $Y' \subseteq Y$. A **matching** between them is a bijection $\mu \colon X' \to Y'$. The elements in X' and Y' are said to be **matched** by μ.

Note that $X' = \mathrm{coim}(\mu)$ and $Y' = \mathrm{im}(\mu)$. Also note that as X and Y are multisets, it might happen that one same element appears several times in one of the multisets, ans that some, but not all of its copies are matched to some element in the other multiset.

Definition 1.2.3 (δ-matching barcodes)**.** A delta matching between two barcodes B and C is a multiset matching that verifies

1. $B_{2\delta} \subseteq \mathrm{coim}(\mu)$,
2. $C_{2\delta} \subseteq \mathrm{im}(\mu)$,
3. If $\mu(I) = J$, then $I \subseteq J^\delta$ and $J \subseteq I^\delta$.

There are various ways of defining the bottleneck distance, all of them equivalent to one an other. We first give the natural definition that comes up following the use of δ-matchings.

Definition 1.2.4 (Bottleneck distance)**.** The **bottleneck distance** between two barcodes B and C is the infimum over all $\delta \in \mathbb{R}$ such that there exists a δ-matching between B and C.

1.3 Persistence diagrams and the Wasserstein distance

We will denote the strict upper triangular region of the Euclidean plane as

$$\mathbb{R}^2_< := \{(x, y) \in \mathbb{R}^2 : x < y\},$$

and the diagonal of the plane as

$$\Delta := \{(x, y) \in \mathbb{R}^2 : x = y\}.$$

Definition 1.3.1 (Persistence diagram)**.** Let I be a countable set. A *persistence diagram* is a function $D : I \to \mathbb{R}^2_<$.

Persistence diagrams are just a way of presenting the out coming from computing the persistence homology groups of a set of data. This output comes in the so called *barcodes*, which are multisets of intervals. As every interval is given with its *birth* and *death* parameters, it can as well be seen as a point in $\mathbb{R}^2_<$.

Definition 1.3.2 (Partial matching)**.** Let $D_1 : I_1 \to \mathbb{R}^2_<$ and $D_2 : I_2 \to \mathbb{R}^2_<$ be persistence diagrams. A *partial matching* between D_1 and D_2 is the triple (I_1', I_2', f) such that $f : I_1' \to I_2'$ is a bijection with $I_1' \subseteq I_1$ and $I_2' \subseteq I_2$.

Instead of probability measures, now we are actually dealing with countable sets of points in $\mathbb{R}$. We will make use of the l^p norm at countable spaces to measure the distance between matched pairs and the distance between unmatched pairs and the diagonal Δ. For a more detailed explanation of Lebesgue measures check [12][Definition 3.7]. This norm is named after Pafnuty Chebyshev.

Definition 1.3.3 (Chebyshev distance)**.** Let $a, b \in \mathbb{R}^2$ with $a = (a_x, a_y)$ and $b = (b_x, b_y)$. The *Chebyshev distance* is defined as

$$d_\infty(a, b) := ||a - b||_\infty := \max\{|a_x - b_x|, |a_y - b_y|\}.$$

To define our adapted Wasserstein distance we need to check how Chebyshev distance measures distances between points of $\mathbb{R}^2_<$ and Δ.

Proposition 1.3.4. *If $a = (a_x, a_y) \in \mathbb{R}^2_<$, then $d_\infty(a, \Delta) = \inf_{t\in\Delta} d_\infty(a, t) = \frac{a_y - a_x}{2}$.*

Proof. The t which minimizes the distance is the midpoint of a_x and a_y, that is $t = \left(\frac{a_x+a_y}{2}, \frac{a_x+a_y}{2}\right)$. Then,

$$\left| a_x - \frac{a_x + a_y}{2} \right| = \left| \frac{a_x - a_y}{2} \right| = \left| \frac{a_y - a_x}{2} \right| = \left| a_y - \frac{a_x + a_y}{2} \right|,$$

and as $a_y > a_x$ we have
$$d_\infty(a,t) = \left|\frac{a_y - a_x}{2}\right| = \frac{a_y - a_x}{2}.$$
□

We now verify that the upper triangular region of the Euclidean plane with the Chebyshev distance adapted to measure distances in Δ is a metric space.

Proposition 1.3.5. *The function d_∞ is a distance in $\mathbb{R}^2_<$ with the diagonal Δ.*

Proof. For points $a, b \in \mathbb{R}^2_< \subset \mathbb{R}^2$, d_∞ is a distance as usual Lebesgue norms are well defined. See [12][Chapter 3]. To verify that the metric requirements are fulfilled for $d_\infty(a,\Delta)$, it is enough to consider $t = \frac{a_y - a_x}{2}$ as in Proposition 1.3.4. □

Definition 1.3.6 (p-cost)**.** Let $D_1 : I_1 \to \mathbb{R}^2_<$ and $D_2 : I_2 \to \mathbb{R}^2_<$ be persistence diagrams. Let (I_1', I_2', f) be a partial matching between them. If $p < \infty$, the *p-cost of f* is defined as

$$\begin{aligned}\operatorname{cost}_p(f) := \Bigg(&\sum_{i\in I_1'} d_\infty(D_1(i), D_2(f(i)))^p \\ &+ \sum_{i\in I_1\setminus I_1'} d_\infty(D_1(i),\Delta)^p \\ &+ \sum_{i\in I_2\setminus I_2'} d_\infty(D_2(i),\Delta)^p \Bigg)^{\frac{1}{p}}.\end{aligned}$$

For $p = \infty$, the *∞-cost of f* is defined as

$$\begin{aligned}\operatorname{cost}_\infty(f) := \max \Bigg\{ &\sup_{i\in I_1'} d_\infty(D_1(i), D_2(f_i)), \\ &\sup_{i\in I_1\setminus I_1'} d_\infty(D_1(i),\Delta), \\ &\sup_{i\in I_2\setminus I_2'} d_\infty(D_2(i),\Delta) \Bigg\}.\end{aligned}$$

Definition 1.3.7 (p-Wasserstein distance)**.** Let D_1, D_2 be persistence diagrams. Let $1 \le p \le \infty$. Define

$$\tilde{\omega}_p(D_1, D_2) = \inf\{\operatorname{cost}_p(f) : f \text{ is a partial matching between } D_1 \text{ and } D_2\}.$$

Let $\emptyset$ denote the unique persistence diagram with empty indexing set. Let $(\mathrm{Dgm}_p, \omega_p)$ be the space of persistence diagrams D that satisfy $\tilde{\omega}_p(D,\emptyset) < \infty$ modulo the equivalence relation $D_1 \sim D_2$ if $\tilde{\omega}_p(D_1, D_2) = 0$. The metric ω_p is called the *p-Wasserstein distance.*

Definition 1.3.8 (Bottleneck distance)**.** In the conditions of Definition 1.3.7, if $p = \infty$, the metric ω_∞ is called the *bottleneck distance*.

Proposition 1.3.9. *There is only one matching between $D : I \to \mathbb{R}^2_<$ and $\emptyset$. Hence, if $p \leq \infty$,*

$$\tilde{\omega}_p(D, \emptyset) = \left(\sum_{i \in I} d_\infty(D(i), \Delta)^p \right)^{\frac{1}{p}},$$

and, if $p = \infty$,

$$\tilde{\omega}_\infty(D, \emptyset) = \sup_{i \in I} d_\infty(D_1(i), \Delta)$$

Proof. Let $I' \subseteq D$. If f is a partial matching between D and $\emptyset$, means that $f(I') = \emptyset$ is a bijection. That is only posible if $I' = \emptyset$ too. Therefore $I \setminus I' = I \setminus \emptyset = I$ and following Definition 1.3.6 we conclude our proof. □

Next proposition will prove that, in indeed, the space of persistence diagrams with the p-Wasserstein distance $(\mathrm{Dgm}_p, \omega_p)$ is a metric space. Its proof is usually omitted in literature, as it based on the simple fact that d_∞ is a distance. We will give, however, an step by step version here.

Proposition 1.3.10. *ω_p is a distance on the space* $(\mathrm{Dgm}_p, \omega_p)$.

Proof. Let $D_1, D_2, D_3 \in \mathrm{Dgm}_p$, with $1 \leq p \leq \infty$. First of all, $\omega_p(D_1, D_2) \geq 0$ because $d_\infty \geq 0$. $\omega_p(D_1, D_2) = 0$ if and only if $\tilde{\omega}_p(D_1, D_2) = 0$. Thus, because of the equivalence relationship used to define ω_p, it has to be $D_1 \sim D_2$.

To check symmetry, note that every partial matching f is bijective, therefore f^{-1} is a partial matching. But, for all $i \in I'_1$, exists $j \in I'_2$ such that $f(i) = j$ and

$$d_\infty(D_1(i), D_2(f(i))) = d_\infty(D_2(f(i)), D_1(i)) = d_\infty(D_2(j), D_1(f^{-1}(j))).$$

Then, $\mathrm{cost}_p(f) = \mathrm{cost}_p(f^{-1})$ and we have

$$\begin{aligned} \omega_p(D_1, D_2) &= \inf\{\mathrm{cost}_p(f) : f \text{ is a partial matching between } D_1 \text{ and } D_2\} \\ &= \inf\{\mathrm{cost}_p(f^{-1}) : f^{-1} \text{ is a partial matching between } D_2 \text{ and } D_1\} \\ &= \omega_p(D_2, D_1). \end{aligned}$$

Finally, lets prove the triangle inequality. If $f : I'_1 \to I'_2$ is a partial matching between D_1 and D_2 and $g : I'_2 \to I'_3$ is a partial matching between D_2 and D_3, $g \circ f : I'_1 \to I'_3$ is a partial matching between D_1 and D_3 as both f and g are bijective. Computing the cost of the matchings for $p < \infty$, we notice that

$$\begin{aligned}
&\sum_{i\in I_1'} d_\infty(D_1(i), D_2(f(i))) + \sum_{i\in I_1\setminus I_1'} d_\infty(D_1(i),\Delta) + \sum_{i\in I_2\setminus I_2'} d_\infty(D_2(i),\Delta)\\
&+\sum_{i\in I_2'} d_\infty(D_2(i), D_3(g(i))) + \sum_{i\in I_2\setminus I_2'} d_\infty(D_2(i),\Delta) + \sum_{i\in I_3\setminus I_3'} d_\infty(D_3(i),\Delta)\\
&\geq \sum_{i\in I_1'} d_\infty(D_1(i), D_3(g\circ f(i))) + \sum_{i\in I_1\setminus I_1'} d_\infty(D_1(i),\Delta) + \sum_{i\in I_3\setminus I_3'} d_\infty(D_3(i),\Delta)
\end{aligned}$$

as $d_\infty(D_1(i), D_2(f(i))) + d_\infty(D_2(f(i)), D_2(g(f(i)))) \geq d_\infty(D_1(i), D_3(g\circ f(i)))$ using the triangle inequality of d_∞. Therefore, for all partial matchings f and g as described, we have $\mathrm{cost}_p(f) + \mathrm{cost}_p(g) \geq \mathrm{cost}_p(g\circ f)$. Using the same reasoning, por $p = \infty$ we also obtain $\mathrm{cost}_\infty(f) + \mathrm{cost}_\infty(g) \geq \mathrm{cost}_\infty(g\circ f)$. Hence, we have verified that

$$\omega_p(D_1, D_2) + \omega_p(D_2, D_3) \geq \omega_p(D_1, D_3).$$

□

1.4 The Hausdorff and Gromov-Hausdorff distances

The Hausdorff distance is a way of measuring distances of different sets contained into a same metric space. This concept can be generalized defining a metric which allow us to measure distances between different metric spaces.

Definition 1.4.1 (Hausdorff distance). Let (M, d) be a metric space, and let $A \subseteq M$, $B \subseteq M$ two compact subspaces of M. Define the **r-neighborhood** of a set $S \subset M$ as

$$U_r(S) \coloneqq \{x \in S \mid d(x, S) \leq r\}.$$

The **Hausdorff distance** can be defined as

$$d_{\mathrm{H}}(A, B) \coloneqq \inf\{r > 0 \mid A \subset U_r(B) \text{ and } B \subset U_r(A)\}.$$

Definition 1.4.2 (Isometric metric spaces). Let $(X, d_X), (Y, d_Y)$ be metric spaces. X and Y are said to be **isometric** if there exists a bijective map $f : X \to Y$ such that distances are preserved. That is, for all $x_1, x_2 \in X$,

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)).$$

Definition 1.4.3 (Gromov-Hausdorff distance). Let $(X, d_X), (Y, d_Y)$ be metric spaces. The **Gromov-Hausdorff** distance is defined as

$$d_{\mathrm{GH}} \coloneqq \inf\{r > 0 \mid \exists (Z, d_Z) \text{ metric space such that}, \exists X', Y' \subseteq Z, d_{\mathrm{H}}(X', Y') < r\},$$

where X', Y' are isometric spaces to X and Y respectively.

Lemma 1.4.4 (Proposition 7.3.16, [3])**.** *Gromov-Hausdorff distance satisfy the triangle inequality. That is, for any metric spaces X_1, X_2, X_3 it is verified that*

$$d_{\mathrm{GH}}(X_1, X_3) \leq d_{\mathrm{GH}}(X_1, X_2) + d_{\mathrm{GH}}(X_2, X_3).$$

Proof. Let d_{12} be a metric over $X_1 \cup X_2$ and let d_{23} be a metric over $X_2 \cup X_3$. Over $X_1 \cap X_3$, define

$$d_{13} := \begin{cases} d_{X_1}(x_1, x_3) \text{ if } x_1, x_3 \in X_1, \\ d_{X_2}(x_1, x_3) \text{ if } x_1, x_3 \in X_3, \\ \inf_{x_2 \in X_2}\{d_{12}(x_1, x_2) + d_{23}(x_2, x_3)\} \text{ if } x_1 \in X_1, x_3 \in X_3. \end{cases}$$

For the first two cases we clearly have a metric. For the third one observe that taking $x_1 \in X_1, x_3 \in X_3$ and some $x \in X_1$ we have

$$\begin{aligned} d_{13}(x_1, x) + d_{13}(x, x_3) &= d_{X_1}(x_1, x) + \inf_{x_2 \in X_2}\{d_{12}(x, x_2) + d_{23}(x_2, x_3)\} \\ &= \inf_{x_2 \in X_2}\{d_{12}(x_1, x) + d_{12}(x, x_2) + d_{23}(x_2, x_3)\} \\ &\geq \inf_{x_2 \in X_2}\{d_{12}(x_1, x_2) + d_{23}(x_2, x_3)\} \\ &= d_{13}(x_1, x_3). \end{aligned}$$

This implies, taking the corresponding metric d_{ij} where $i, j = 1, 2, 3$, that

$$d_{\mathrm{H}}(X_1, X_3) \leq d_{\mathrm{H}}(X_1, X_2) + d_{\mathrm{H}}(X_2, X_3),$$

and, taking the infimum over the metrics d_{12} and d_{23} we have

$$d_{\mathrm{GH}}(X_1, X_3) \leq d_{\mathrm{GH}}(X_1, X_2) + d_{\mathrm{GH}}(X_2, X_3).$$

□

To check that Gromov-Hausdorff distance is actually a distance we first give a useful characterization in 1.4.7. It is expressed in terms of correspondance distortions.

Definition 1.4.5 (Correspondance between sets)**.** Given two sets X and Y, a **correspondance** between them is a set $R \subseteq X \times Y$ verifying that for every $x \in X$, there exists at least one $y \in Y$ such that $(x, y) \in R$ and, for every $y \in Y$, there exists an $x \in X$ such that $(x, y) \in R$.

Definition 1.4.6 (Distortion of a correspondance)**.** Let $(X, d_X), (Y, d_Y)$ be two metric spaces, and let R be a correspondance between them. The **distortion** of R is defined as

$$\mathrm{dis}(R) := \sup\left\{|d_X(x, x') - d_Y(y, y')| : (x, y), (x', y') \in R\right\}.$$

Proposition 1.4.7 (Theorem 7.3.25, [3]). *Let $(X, d_X), (Y, d_Y)$ be two metric spaces. The Gromov-Hausdorff distance between them can be characterized as*

$$d_{\mathrm{GH}}((X, d_X), (Y, d_Y)) = \frac{1}{2} \inf_R \operatorname{dis}(R).$$

Proof. Given $r \geq d_{\mathrm{GH}}(X, Y)$, for some metric space (Z, d_Z), we can take $X', Y' \subseteq Z$ such that X' and Y' are isometric embeddings of X and Y respectively and $d_{\mathrm{H}}(X', Y') < r$ in Z. Thus, we can see every element of X and Y as elements od Z trough some isometry. Therefore, we can define the correspondance

$$R \coloneqq \{(x, y) \in X \times Y : d_Z(x, y) < r\}.$$

The set R is actually a correspondance because the fact that $d_{\mathrm{H}}(X', Y') < r$ implies that for every $x \in X$ and every $y \in Y$, $d_Z(x, y) < r$, so every x and every y have some correspondance. Now, let $(x, y), (x', y') \in R$. Using the triangle inequality of Z we have

$$\begin{aligned}
\operatorname{dis}(R) &\leq |d_X(x, x') - d_Y(y - y')| \\
&= |d_Z(x, x') - d_Z(y - y')| \\
&\leq |d_Z(x, y) + d_Z(y, x') - d_Z(y - y')| \\
&\leq d_Z(x, y) + d_Z(x', y) + d_Z(y - y') \\
&\leq d_Z(x, y) + d_Z(x', y') \leq 2r.
\end{aligned}$$

This shows

$$2 d_{\mathrm{GH}}(X, Y) \geq \inf_R \operatorname{dis}(R).$$

To see the reverse inequality, let R be any correspondance, and lets take $\operatorname{dis}(R) = 2r$. Lets construct a metric space (Z, d_Z) formed by the disjoint union of spaces $Z = X \cup Y$. For every $z_1, z_2 \in Z$, we define d_Z as

$$d_Z(z_1, z_2) \coloneqq \begin{cases} d_X(z_1, z_2) \text{ if } z_1, z_2 \in X, \\ d_Y(z_1, z_2) \text{ if } z_1, z_2 \in Y, \\ \inf\{d_X(z_1, x') + r + d_Y(z_2, y') : (x', y') \in R\} \text{ if } z_1 \in X, z_2 \in Y. \end{cases}$$

By definition, it is clear that d_Z respects isometrically both d_X and d_Y. By the same reason $d_Z(z_1, z_2) = d_Z(z_2, z_1)$ and $d_Z(z_1, z_2) \geq 0$ for every $z_1, z_2 \in Z$, where $d_Z(z_1, z_2) = 0$ only if either $z_1 = z_2$ or $r = 0$. To check the triangle inequality we take $z_1, z_2, z_3 \in Z$. If either all three are elements of X, or all three are elements of Y, the inequality is verified as it is granted in X and Y with d_X and d_Y respectively. In case $z_1, z_2 \in X$ and $z_3 \in Y$

we can take some $y \in Y$ such that $(z_2, y) \in R$. Thus, we have

$$\begin{aligned} d_Z(z_1, z_2) + d_Z(z_2, z_3) &\geq d_X(z_1, z_2) + d_X(z_2, z_2) + r + d_Y(z_3, y) \\ &\geq d_X(z_1, z_2) + r + d_Y(z_3, y) \\ &\geq d_Z(z_1, z_3). \end{aligned}$$

Analogously, the argument follows for $z_1 \in X$ and $z_2, z_3 \in Y$. Thus, all is left to prove is to check $d_{\mathrm{H}}(X, Y) < r$.

... □

Definition 1.4.8 (Distortion of a map)**.** Let $(X, d_X), (Y, d_Y)$ be two metric spaces and let $f\colon X \to Y$ an arbitrary map. The **distortion** of f is defined as

$$\operatorname{dis}(f) := \sup_{x_1, x_2 \in X} |d_Y(d(x_1), f(x_2)) - d_X(x_1, x_2)|.$$

Definition 1.4.9 (ε-isometry)**.** Let X and Y be two metric spaces and let $\varepsilon > 0$. A **ε-isometry** between two metric spaces is a map $f\colon X \to Y$ such that $\operatorname{dis}(f) \leq \varepsilon$. The image $f(X)$ is called an **ε-net**.

Proposition 1.4.10 (Theorem 7.3.28.1, [3])**.** *Let X and Y be two metric spaces and let $\varepsilon > 0$. If $d_{\mathrm{GH}} < \varepsilon$, them there exists a $2\varepsilon -$ isometry from X to Y.*

Proof. Let R be a correspondance between X and Y. By 1.4.7 it holds that $\operatorname{dis}(R) < 2\varepsilon$. For every $x \in X$ we choose some $y \in Y$ such that $(x, y) \in R$ and define $f(x) := y$. This defines a map $f\colon X \to Y$. We then have

$$\operatorname{dis}(f) \leq \operatorname{dis}(R) < 2\varepsilon.$$

... □

Up to this moment we have seen that Gromov-Hausdorff distance defines a pseudo-metric over the set of metric spaces. Note that if X and Y are isometric, directly of the definition ge get, $d_{\mathrm{GH}}(X, Y) = 0$. To make Gromov-Hausdorff distance an actual metric we need to ask one more thing to our metric spaces. That is, to be compact. Denote

$$\mathcal{X} := \{(X, d_X) \colon (X, d_X) \text{ is a metric compact space}\}.$$

Theorem 1.4.11 (Theorem 7.3.30, [3])**.** *Gromov-Hausdorff distance is in fact a metric over the space of isometry clases of compact metric spaces.*

Proof. We just seen that if X and Y are isometric, directly of the definition ge get, $d_{\mathrm{GH}}(X,Y)=0$. By definition, Gromov-Hausdorff distance is nonnegative and symmetric and, by Lemma 1.4.4, it verifies the triangle inequality. It only remains to prove that given two metric spaces $X,Y\in\mathcal{X}$, if $d_{\mathrm{GH}}(X,Y)=0$ then X and Y are isometric.

Let $X,Y\in\mathcal{X}$ such that $d_{\mathrm{GH}}(X,Y)=0$. By Proposition 1.4.10, there exists a sequence of maps $f_n\colon X\to Y$ such that $\operatorname{dis}(f_n)\to 0$. As X is compact, we can fix a countable dense set $S\subset X$.

... □

In order to extend the scope of Gromov-Hausdorff distance we can endow our compact metric spaces with real-valued functions, which will still maintain good stability properties as we will see in Chapter 5.

Denote the collection of such spaces as

$$\mathcal{X}_1 \coloneqq \{(X,d_x,f)\colon (X,d_x)\in(X), f_X\colon X\to\mathbb{R}\text{ continuous}\}.$$

Definition 1.4.12. Let $X,Y\in\mathcal{X}_1$. We extend the **Gromov-Hausdorff distance over** X_1 as

$$d^1_{\mathrm{GH}}((X,d_X,f_X),(Y,d_Y,f_Y))=\inf_R\max\left\{\frac{1}{2}\operatorname{dis}(R),\|f_X-f_Y\|_{\ell^\infty}\right\}.$$

An analogous adaptation of Theorem 1.4.11 and the previous results proofs that d^1_{GH} defines a metric over the set of isomorphism clases od $\mathcal{X}_1$.

Chapter 2

Structure Theorem

2.1 Structure theorem for finitely generated modules over a principal ideal domain

Theorem 2.1.1 (Chapter IV, Theorem 6.12, [9])**.** *Let M be a finitely generated module over a principal ideal domain R. There exist a finite sequence of proper ideals $(d_1) \supseteq (d_2) \supseteq \cdots \supseteq (d_n)$ such that*

$$M \cong \bigoplus_{i=1}^{n} R/(d_i).$$

2.2 Structure theorem for persistence diagrams

The Structure Theorem for persistence modules is referred to as the "first miracle" of persistence homology [10]. This algebraic property allows to express a persistence module of finite type as a direct sum of finitely many interval modules. Its proof requires the algebraic structure theorem for finitely generated modules over a principal domain.

In addition to Theorem 2.1.1, we will use the following simple algebraic statement.

Proposition 2.2.1 (Proposition 4.6, [13])**.** *An ideal $I \subseteq R$ is graded if and only if it is generated by homogeneous elements.*

Proof. First, if I is a graded ideal $I = \bigoplus_p I^p$ and is generated by $\bigcup_p I^p$. Then, each

$$I^p = I \cap R^p \subseteq R^p$$

is a subset of homogeneous elements. Therefore, I is generated by homogeneous elements.

Now, let I be generated by a set X of homogeneous elements. For sure, $I \cap R^p \subseteq I$, so we just need to prove the converse inclusion. As I is generated by X, its elements $u \in I$ are of the form

$$u = \sum_i r_i x_i s_i, \tag{2.1}$$

for $r_i, s_i \in R$ and $x_i \in X$. And as $I \subseteq R$, also,

$$u = \sum_p u_p,$$

for $u_p \in R^p$. For every term in (2.1), we have

$$r_i = \sum_j r_{i,j}, \qquad s_i = \sum_l s_{i,l},$$

with each $r_{i,j}$, $s_{i,l}$ being homogeneous. Therefore, combining all we have that

$$u = \sum_i \sum_{j,l} r_{i,j} x_i s_{i,l}. \tag{2.2}$$

Each term in (2.2) is homogeneous as is a product of homogeneous elements. Thus u_p is the sum of those terms, and u has degree p. Therefore $u_p \in I$ and $I \subseteq I \cap R^p$. $\square$

Theorem 2.2.2 (Proposition 4.8, [13]). *Let (V, π) be a persistence module. There exist a barcode $\mathrm{Bar}(V, \pi)$, with $\mu \colon \mathrm{Bar}(V, \pi) \longrightarrow \mathbb{N}$, the multiplicity of the barcode intervals, such as there is a unique direct sum decomposition*

$$V \cong \bigoplus_{I \in \mathrm{Bar}(V)} \mathbb{F}(I)^{\mu(I)}. \tag{2.3}$$

Proof. V is of finite type, so it is a finite $\mathbb{F}[x]$-module. As $\mathbb{F}$ is a field, $\mathbb{F}[x]$ is a principal ideal domain, therefore, V is a finitely generated module over a principal ideal domain. Using Fact 2.1.1, V can be decompose in the direct sum of its free and torsion subgroups, $F \oplus T$. Thus, we have

$$F = \bigoplus_{i \geq q} x^i \cdot \mathbb{F}$$
$$T = \bigoplus_{i \geq q} R^i / I^i.$$

Each $x^i \cdot \mathbb{F}$ is isomorphic to ideals of the form (x^q). By Proposition 2.2.1, each R^i/I^i is isomorphic to some quotient of graded ideals of the form $(x^p)/(x^r)$. Note that the free subgroup can be seen as a particular case of the torsion group taking $r = 0$. Thus V can be decompose as described in (2.3). $\square$

Chapter 3

Interleaving Stability Theorem

This Chapter gives a detailed proof of the first stability theorem for persistence homology. This theorem is referred to as the "Geometry miracle" of persistent homology, as it allows to describe an isometry between persistence modules and barcodes [10]. This shows that *small* changes in a data sets will perform *small* changes in their persistence modules, and therefore small changes in how persistent homology groups vary through time. The theorem claims that given two persistence modules, the distance between them using the interleaving distance, is the same as the distance between their barcodes using the bottleneck distance.

Theorem 3.0.1 (Interleaving Stability, Theorem 2.2.8, [11]). *There is an isometry between a persistence module with the interleaving distance and its barcode with the bottleneck distance. That is, given two persistence modules V and W, it holds that*

$$d_{\text{int}}(V, W) = d_{\text{bot}}(\text{Bar}(V), \text{Bar}(W)).$$

For the presented proof we have followed [11]. Hence, we will divide the proof into proving two complementary inequalities separately. This implies checking that if there exists a δ-matching between two given barcodes, then there exists a δ-interleaving morphism between their persistence modules, Proposition 3.1.2. Also, we need to check that, if there exists a δ-interleaving morphism between two persistence modules, then there exists a δ-matching between their barcodes, 3.2.8.

3.1 First inequality

The first claim can be deduced from the Structure Theorem in a rather direct way, proving first the case where our modules are just interval modules.

Lemma 3.1.1 (Exercise 2.2.7, [11]). *Let I, J be two δ-matched intervals. Then, their corresponding interval modules $(\mathbb{F}(I), \pi)$ and $(\mathbb{F}(J), \theta)$ are δ-interleaved.*

Proof. Let $I = (a, b]$, $J = (c, d]$. If ρ is the δ-matching between them, then $\rho(I) = J$ and, following Definition 1.2.3, $(a, b] \subseteq (c - \delta, d + \delta]$ and $(c, d] \subseteq (a - \delta, b + \delta]$, with $b - a > 2\delta$ and $d - c > 2\delta$. Then, the morphisms

$$\begin{aligned} \phi_\delta \colon \mathbb{F}(I) &\to \mathbb{F}(J)_\delta \\ \phi_\delta(\mathbb{F}(I)_t) &\mapsto \mathbb{F}(J)_{(t+\delta)} \end{aligned} \quad \text{and} \quad \begin{aligned} \psi_\delta \colon \mathbb{F}(J) &\to \mathbb{F}(I)_\delta \\ \psi_\delta(\mathbb{F}(J)_t) &\mapsto \mathbb{F}(I)_{(t+\delta)} \end{aligned}$$

are well defined as for any $t \in (a, b]$, $t + \delta \in (c, d]$, as $a + \delta > c$ and $b + \delta \leq d$. In the same way, for any $t \in (c, d]$, $t + \delta \in (a, b]$. Thus,

$$\psi_\delta \circ \phi_\delta(\mathbb{F}(I)_t) = \psi_\delta(\mathbb{F}(J)_{(t+\delta)}) = \mathbb{F}(I)_{(t+2\delta)} = \pi_{t \leq t+2\delta}(\mathbb{F}(I)_t).$$

and

$$\phi_\delta \circ \psi_\delta(\mathbb{F}(J)_t) = \phi_\delta(\mathbb{F}(I)_{(t+\delta)}) = \mathbb{F}(J)_{(t+2\delta)} = \theta_{t \leq t+2\delta}(\mathbb{F}(J)_t).$$

Therefore, ϕ_δ and ψ_δ are a pair of δ-interleaving morphisms. □

Once we are able to build a δ-interleaving between two δ-matched interval modules, we will use the Structure Theorem for persistence modules to generalice the construction for arbitrary persistence modules. This will be useful to prove the first inequality needed to prove Theorem 3.0.1.

Proposition 3.1.2 (Theorem 3.0.1, [11])**.** *Given two persistence modules V, W, if there is a δ-matching between their barcodes, then there is a δ-interleaving morphism between them.*

Proof. Suppose that $\rho\colon \operatorname{Bar}(V) \to \operatorname{Bar}(W)$ is a δ-matching between the barcodes of V and W. By the Structure Theorem 2.1.1, V and W decompose in a finite direct sum of interval modules

$$V \cong \bigoplus_{I \in \operatorname{Bar}(V)} \mathbb{F}(I), \qquad W \cong \bigoplus_{J \in \operatorname{Bar}(W)} \mathbb{F}(W).$$

We can express $V = V_Y \oplus V_N$, $W = W_Y \oplus W_N$ denoting

$$V_Y \cong \bigoplus_{I \in \operatorname{coim} \rho} \mathbb{F}(I), \qquad V_N \cong \bigoplus_{I \in \operatorname{Bar}(V) \setminus \operatorname{coim} \rho} \mathbb{F}(I),$$

$$W_Y \cong \bigoplus_{J \in \operatorname{im} \rho} \mathbb{F}(J), \qquad W_N \cong \bigoplus_{J \in \operatorname{Bar}(J) \setminus \operatorname{im} \rho} \mathbb{F}(J).$$

The V_Y, W_Y modules separate the “yes, matched” intervals, from the V_N, W_N “not matched” intervals. For every interval $I \in \operatorname{Bar}(V_Y)$, I is δ-matched to an interval

$J \in \mathrm{Bar}(W_Y)$ by $\rho(I) = J$. Thus, by Lemma 3.1.1, for all pair I, J of matched intervals, there exist a par of δ-interleaved morphisms

$$\phi_\delta \colon \mathbb{F}(I) \to \mathbb{F}(J)_\delta \quad \text{and} \quad \psi_\delta \colon \mathbb{F}(J) \to \mathbb{F}(I)_\delta$$

which induce the pair of δ-interleaved morphisms

$$\phi_\delta \colon V_Y \to W_{Y\,\delta} \quad \text{and} \quad \psi_\delta \colon W_Y \to V_{Y\,\delta}.$$

Not matched intervals are of length smaller than 2δ, therefore both, V_N and V_Y are δ-interleaved with the empty set. We can now construct the δ-interleaving morphism $\phi \colon V \to W$ such as $\phi|_{V_Y} = \phi_Y$ and $\phi|_{V_N} = 0$. In a similar way, we also construct $\psi \colon W \to V$. $\square$

3.2 Second inequality

With Proposition 3.1.2 we have proven the first half of Stability Theorem 3.0.1. Now we need to prove that we can build a δ-interleaving morphism from a δ-matching. To verify this claim we need several previous lemmas that will lead us to prove Proposition 3.1.2.

First, we will introduce some notation. Let (V, π), (W, θ) be two persistence modules and let $I = (b, d]$ be an interval with $d \in \mathbb{R} \cup \{+\infty\}$. Denote the set of bars of $\mathrm{Bar}(V)$ that start before b and end exactly at d as

$$\mathrm{Bar}_{I-}(V) \coloneqq \{(a, d] \in \mathrm{Bar}(V) \colon a \leq b\}\,.$$

Analogously, denote the set of bars that start at b and end after d as

$$\mathrm{Bar}_{I+}(V) \coloneqq \{(b, c] \in \mathrm{Bar}(V) \colon c \geq d\}\,.$$

Lemma 3.2.1 (Proposition 3.1.1, [11]). *Let $I = (b, d]$ be an interval. It exists an injective morphism $\iota \colon (V, \pi) \in (W, \theta)$, then $\#(\mathrm{Bar}_{I-}(V)) \leq \#(\mathrm{Bar}_{I-}(W))$. Where $\#(\cdot)$ denotes the cardinal operator.*

Proof. For $b < s < d < r$, denote the set of elements in V_d witch come from all V_s and disappear in all V_r as

$$E_{I-} = \bigcap_{b<s<d} \operatorname{im} \pi_{s \leq d} \cap \bigcap_{r>d} \ker \pi_{d \leq r} \subseteq V_d.$$

It holds that $\dim E_{I-}(V) = \#(\mathrm{Bar}_{I-}(V))$. For every morphism $p \colon (V, \pi) \to (W, \theta)$ the following diagram conmutes

$$\begin{tikzcd} V_s \arrow[r, "\pi_{s\leq r}"] \arrow[d, "p_s"'] & V_r \arrow[d, "p_r"] \\ W_s \arrow[r, "\theta_{s\leq r}"'] & W_r \end{tikzcd}$$

This implies that

$$p_r(\operatorname{im}\pi_{s\le r}) \subseteq \operatorname{im}\theta_{s\le r}, \qquad p_s(\ker\pi_{s\le r}) \subseteq \ker\theta_{s\le r}.$$

Taking $r = d$, $b < s < d$ in the first inclusion, and $s = d$, $r > d$ in the second, we have that

$$p_d(\operatorname{im}\pi_{s\le d}) \subseteq \operatorname{im}\theta_{s\le d}, \qquad p_d(\ker\pi_{d\le r}) \subseteq \ker\theta_{d\le r},$$

and

$$p_d(E_{I-}(V)) \subseteq E_{I-}(W).$$

If we now take p as the injective morphism of the hypothesis, $p = \iota$, we get

$$\dim E_{I-}(V) \le \dim E_{I-}(W).$$

□

Lemma 3.2.2 (Exercise 3.1.3, [11]). *Let $I = (b, d]$ be an interval. It exists a surjective morphism $s\colon (V, \pi) \to (W, \theta)$, then $\#(\mathrm{Bar}_{I+}(V)) \ge \#(\mathrm{Bar}_{I+}(W))$.*

Proof. Analogously to the proof of Lemma 3.2.1 we now define

$$E_{I+}(V) = \bigcap \operatorname{im}\pi_{d\le r}.$$

Therefore $\dim E_{I+}(V) = \#(\mathrm{Bar}_{I+}(V))$, and recalling the diagram used for the previous proof, and using the fact that is commutative, we have that

$$p_r(\operatorname{im}\pi_{s\le r}) \supseteq \operatorname{im}\theta_{s\le r}.$$

Taking $s = d$ we then have that

$$p_d(E_{I+}(V)) \supseteq E_{I+}(W).$$

And finally, taking the surjective morphism $p = s$ we have that

$$\dim E_{I-}(V) \ge \dim E_{I-}(W).$$

□

To build our δ-matching we first define two induced matchings, by an injection and by a surjection respectively. First, suppose that there exists an injection $\iota\colon V \to W$. For every $c \in \mathbb{R} \cup \{\infty\}$, sort the bars $(a_i, c] \in \mathrm{Bar}(V)$, $i \in \{1, \ldots, k\}$ by decreasing length order,

$$(a_1, c] \supseteq (a_2, c] \supseteq \cdots \supseteq (a_k, c], \text{ with } a_1 \le a_2 \le \cdots \le a_k.$$

Sort in the same manner the bars $(b_j, c] \in \mathrm{Bar}(V)$, $j \in \{1, \dots, l\}$,

$$(b_1, c] \supseteq (b_2, c] \supseteq \cdots \supseteq (a_k, c], \text{ with } b_1 \leq b_2 \leq \cdots \leq b_k.$$

As there is an injection between V and W, Lemma 3.2.1 asures that the amount of bars in $\mathrm{Bar}(V)$ is lower that the amount in $\mathrm{Bar}(V)$, i.e., $k \leq l$. We define the *injective induced matching* $\mu_{inj}\colon \mathrm{Bar}(V) \to \mathrm{Bar}(W)$ matching, for each $c \in \mathbb{R} \cup \{\infty\}$, the intervals from both lists by decreasing length.

Lemma 3.2.3 (Proposition 3.1.5, [11])**.** *If there exists an injection* $\iota\colon (V, \pi) \in (W, \theta)$*, then the induced matching* $\mu_{inj}\colon \mathrm{Bar}(V) \to \mathrm{Bar}(W)$ *satisfies:*

1. $\operatorname{coim} \mu_{inj} = \mathrm{Bar}(V)$,

2. $\mu_{inj}(a, c] = (b, c],\ \forall b \leq a,\ \forall (a, d] \in \mathrm{Bar}(V)$.

Proof. Applying Lemma 3.2.1 with the interval $(a_k, c]$, we have that for each $c \in \mathbb{R} \cup \{\infty\}$, $\#\,\mathrm{Bar}_{(a_k,c]-}(V) \leq \#\,\mathrm{Bar}_{(a_k,c]-}(W)$, having $k \leq l$ as we note earlier. This means that every bar in $\mathrm{Bar}(V)$ is matched to some bar in $\mathrm{Bar}(W)$. Hence $\operatorname{coim} \mu_{inj} = \mathrm{Bar}(V)$. Moreover, as the matching is carried out in length descending order, for each $i \in \{1, \dots, k\}$, $\mu_{inj}(a_i, c] = (b_i, c]$, and applying Lemma 3.2.1, now with the interval $(a_i, c]$, and making the same reasoning, $a_i \leq b_i$. □

Now we suppose that there exists a surjection $\sigma\colon V \to W$. For every $a \in \mathbb{R}$, sort the intervals $(a, c_i] \in \mathrm{Bar}(V)$, $i \in \{1, \dots, k\}$ by decreasing length order as before,

$$(a, c_1] \supseteq (a, c_2] \supseteq \cdots \supseteq (a, c_k], \text{ with } c_1 \geq c_2 \geq \cdots \geq a_k,$$

and again in the same manner, sort the intervals $(a, d_j] \in \mathrm{Bar}(V)$, $j \in \{1, \dots, l\}$,

$$(a, d_1] \supseteq (a, d_2] \supseteq \cdots \supseteq (a, d_l], \text{ with } d_1 \geq d_2 \geq \cdots \geq d_l.$$

We define the *surjective induced matching* $\mu_{sur}\colon \mathrm{Bar}(V) \to \mathrm{Bar}(W)$ matching, for each $a \in \mathbb{R}$, the intervals from both lists by decreasing length.

Lemma 3.2.4 (Exercise 3.1.8, [11])**.** *If there exists a surjection* $s\colon (V, \pi) \to (W, \theta)$*, then the induced matching* $\mu_{sur}\colon \mathrm{Bar}(V) \to \mathrm{Bar}(W)$ *satisfies:*

1. $\operatorname{im} \mu_{sur} = \mathrm{Bar}(W)$,

2. $\mu_{sur}(a, c] = (a, d],\ \forall c \geq d,\ \forall (a, d] \in \mathrm{Bar}(V)$.

Proof. Using Lemma 3.2.2 with the interval $(b, d_k]$ for each $b \in \mathbb{R}$, we get that, as there exists a surjection between the modules, now $k \geq l$. Therefore, every bar in $\mathrm{Bar}(W)$ is matched to some bar in $\mathrm{Bar}(V)$ and $\mathrm{im}\,\mu_{sur} = \mathrm{Bar}(W)$. In an analogue way to the previous lemma, as the intervals in both lists are matched in a decreasing manner, for every $j \in \{1, \ldots, l\}$, $\mu_{sur}(a, c_j] = (a, d_j]$, and if we now apply Lemma 3.2.2, we get that $c_j \geq d_j$. □

Hence, with the injective and the surjective induced matchings at hand, for a general morphism f, we can define the *induced matching* $\mu(f)\colon \mathrm{Bar}(V) \to \mathrm{Bar}(W)$, as the composition $\mu_{inj} \circ \mu_{sur}$, defined as $\mathrm{im}\,\mu_{sur} = \mathrm{Bar}(\mathrm{im}\, f) = \mathrm{coim}\,\mu_{inj}$.

The following lemma verifies that, in fact, the mapping between persistence modules with its morphisms and barcodes with induced matchings (either the injective or the surjective versions) has functorial properties between the two categories.

Lemma 3.2.5 (Claim 3.1.13, [11]). *Let U, V and W persistence diagrams and f, g, h morphisms between them defined as in the following diagram:*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend right, "h"'] & V \arrow[r, "g"] & W \end{tikzcd} .$$

If all f, g, h are all injections, or all surjections, then the corresponding diagram formed by the barcodes of the modules, and their respective matchings conmutes as well.

$$\begin{tikzcd} \mathrm{Bar}(U) \arrow[r, "\mu_*(f)"] \arrow[rr, bend right, "\mu_*(h)"'] & \mathrm{Bar}(V) \arrow[r, "\mu_*(g)"] & \mathrm{Bar}(W) \end{tikzcd} .$$

Where μ_* denotes μ_{inj} or μ_{sur} accordingly.

Proof. Let f, g, h injective morphisms, by the definition of the injective induced matching and Lemma 3.2.1 for any $d \in \mathbb{R} \cup \{+\infty\}$, there exist $k \leq l \leq q$ such that the barcodes of U, V, W consist on the following bars:

$$\begin{aligned} \mathrm{Bar}(U) :&(a_1, d] \supset \cdots \supset (a_k, d] \\ \mathrm{Bar}(V) :&(b_1, d] \supset \cdots \supset (b_k, d] \supset \cdots \supset (b_l, d] \\ \mathrm{Bar}(V) :&(c_1, d] \supset \cdots \supset (c_k, d] \supset \cdots \supset (c_l, d] \supset \cdots \supset (c_q, d]. \end{aligned}$$

Therefore, for any d the diagram conmutes as

$$\mu_{inj}(f)(a_i, d] = (b_i, d], \;\; \mu_{inj}(g)(b_i, d] = (c_i, d], \;\; \mu_{inj}(h)(a_i, d] = (c_i, d]$$

for $1 \leq i \leq k$. If f, g, h were surjective morphisms, an analogue reasoning using the surjective induced matching definition and Lemma 3.2.2 completes the proof. □

Finally, we can claim the two main lemmas from which we will construct our desired δ-matching.

Lemma 3.2.6 (Lemma 3.2.1, [11]). *Let $(V,\pi),(W,\theta)$ be δ-interleaved persistence modules, with δ-interleaving morphisms $\phi\colon V \to W_\delta$ and $\psi\colon W \to V_\delta$. Let $\phi\colon V \to \operatorname{im}\phi$ be a surjection and $\mu_{sur}\colon \operatorname{Bar}(V) \to \operatorname{Bar}(\operatorname{im}\phi)$ the induced matching. Then*

1. $\operatorname{coim}\mu_{sur} \supseteq \operatorname{Bar}(V)_{\geq 2\delta}$,
2. $\operatorname{im}\mu_{sur} = \operatorname{Bar}(\operatorname{im}\phi)$ *and*
3. $\mu_{sur}(b,d] = (b,d'],\ \forall (b,d] \in \operatorname{coim}\mu_{sur},\ d' \in [d-2\delta, d]$.

Proof. 1. To check the first part, we observe that, in the following diagram, the three morphisms are surjective as ϕ and $\pi_{t\leq t+2\delta}$ are defined onto their images, and the diagram conmutes,

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[rr, bend right, "\pi_{t\leq t+2\delta}"'] & \operatorname{im}\phi \arrow[r, "\psi_\delta"] & \operatorname{im}\pi_{t\leq t+2\delta} \end{tikzcd}.$$

Therefore, because of Lemma 3.2.5 the barcode diagram also conmutes:

$$\begin{tikzcd} \operatorname{Bar}(V) \arrow[r, "\mu_{sur}(\phi)"] \arrow[rr, bend right, "\mu_{sur}(\pi_{t\leq t+2\delta})"'] & \operatorname{Bar}(\operatorname{im}\phi) \arrow[r, "\mu_{sur}(\psi_\delta)"] & \operatorname{Bar}(\operatorname{im}\pi_{t\leq t+2\delta}) \end{tikzcd}.$$

By the definition of the surjective induced matching,

$$\operatorname{coim}\mu_{sur}(\pi_{t\leq t+2\delta}) = \operatorname{Bar}(V)_{\geq 2\delta}.$$

For each starting point $a \in \mathbb{R}$, we have that

$$\operatorname{Bar}(\operatorname{im}\pi_{t\leq t+2\delta}) = \{(a, b-2\delta]\colon (a,b] \in \operatorname{Bar}(V), b-a > 2\delta\}.$$

Sorting all bars of $\operatorname{Bar}(V)$ and of $\operatorname{Bar}(\operatorname{im}\pi_{t\leq t+2\delta})$ in length-not-increasing order and matching the bars though the longest-first order, each bar $(a,b] \in \operatorname{Bar}(V)$ is matched with the bar $(a, b-2\delta] \in \operatorname{Bar}(\operatorname{im}\pi_{t\leq t+2\delta})$ while $b-a > 2\delta$. The smaller bars are not matched. Thus,

$$\operatorname{coim}\mu_{sur}(phi) \supseteq \operatorname{coim}\mu_{sur}(\operatorname{im}\pi_{t\leq t+2\delta}) = \operatorname{Bar}(V)_{\geq 2\delta}.$$

2. The second part is just a reformulation of Lemma 3.2.1.

3. Let $(b,d] \in \operatorname{coim}$. There are two cases:

   On one hand, if $d-b \leq 2\delta$, $(b,d]$ is matched to $(b,d']$ where $d \geq d'$, by definition of μ_{sur}. Also, $d' > b$ and, as in this case we have $b \geq d-2\delta$, we have $d' > d-2\delta$. Therefore, $d' \in [d-2\delta, d]$.

   On the other hand, if $d-b > 2\delta$, $(b,d]$ is matched to $(b,d']$ by $\mu_{sur}(\phi)$, with $(b,d'] \in W_{\leq 2\delta}$. We can therefore use Lemma 3.2.4 to check that $d' \geq d$. In the same manner, $(b,d']$ is matched to $(b,d'']$ by $\mu_{sur}(\psi)_\delta$ with $d'' \geq d'$. Finally, using the commutativity of the following diagram, we have that $(b,d''] = (b,d-2\delta]$, making $d' \in [d-2\delta, d]$.

$$\begin{tikzcd}
\mathrm{Bar}(V)_{\geq 2\delta} & \mathrm{Bar}(\operatorname{im}\phi) & \mathrm{Bar}(\operatorname{im}\pi_{t\leq t+2\delta}) \\
\Cup & \Cup & \Cup \\
(b,d] \arrow[r, mapsto, "\mu_{sur}(\phi)"] \arrow[rrd, bend right, "\mu_{sur}(\pi_{t\leq t+2\delta})"] & (b,d'] \arrow[r, mapsto, "\mu_{sur}(\psi_\delta)"] & (b,d''] \arrow[d, equal] \\
 & & (b,d-2\delta]
\end{tikzcd}$$

□

Lemma 3.2.7 (Proposition 3.2.2, [11]). *Let $(V,\pi),(W,\theta)$ be δ-interleaved persistence modules, with δ-interleaving morphisms $\phi\colon V \to W_\delta$ and $\psi\colon W \to V_\delta$. Let $\phi\colon V \to \operatorname{im}\phi$ be a injection and $\mu_{inj}\colon \mathrm{Bar}(\operatorname{im}\phi) \to \mathrm{Bar}(W_\delta)$ the induced matching. Then*

1. $\operatorname{coim}\mu_{sur} = \mathrm{Bar}(\operatorname{im}\phi)$,

2. $\operatorname{im}\mu_{inj} \supseteq \mathrm{Bar}(W_\delta)_{\geq 2\delta}$ *and*

3. $\mu_{inj}(b,d'] = (b',d'],\ (b,d'] \in \operatorname{coim}\mu_{inj},\ b' \in [b-2\delta, b]$.

Proof. 1. Immediate using Lemma 3.2.3.

2. As $\phi_\delta \circ \psi = \theta_{t\leq t+2\delta}$ the following diagram conmutes:

$$\begin{tikzcd}
W \arrow[r, "\psi"] \arrow[rr, bend right, "\theta_{t\leq t+2\delta}"'] & \operatorname{im}\psi \arrow[r, "\phi_\delta"] & W_{2\delta}
\end{tikzcd}.$$

   This implies that $\operatorname{im}\theta_{t\leq t+2\delta} \subseteq \operatorname{im}\phi_\delta \subseteq W_{2\delta}$, so there are some injections j and i which make the following diagram commute as well:

$$\begin{tikzcd}
\operatorname{im}\theta_{t\leq t+2\delta} \arrow[r, "j"] \arrow[rr, bend right, "\theta_{t\leq t+2\delta}"'] & \operatorname{im}\phi_\delta \arrow[r, "i"] & W_{2\delta}
\end{tikzcd}.$$

As all morphisms in the diagram above are injections, we can use the functorial properties of Lemma 3.2.5 having a commutative diagram of the barcodes of each of the previous persistence modules:

$$\begin{tikzcd}
\mathrm{Bar}(\mathrm{im}\,\theta_{t\leq t+2\delta}) \arrow[r, "\mu_{inj}(j)"] \arrow[rr, bend right, "\mu_{inj}(\theta_{t\leq t+2\delta})"'] & \mathrm{Bar}(\mathrm{im}\,\phi_\delta) \arrow[r, "\mu_{inj}(i)"] & \mathrm{Bar}(W_{2\delta})
\end{tikzcd}.$$

We have that

$$\begin{aligned}
\mathrm{Bar}(\mathrm{im}\,\theta_{t\leq t+2\delta}) &= \{(b, d-2\delta)\colon (b,d] \in \mathrm{Bar}(W), b < d-2\delta\},\\
\mathrm{Bar}(W_{2\delta}) &= \{(b-2\delta, d-2\delta)\colon (b,d] \in \mathrm{Bar}(W)\} \text{ and}\\
\mu_{inj}(\theta_{t\leq t+2\delta})((b, d-2\delta)) &= (b-2\delta, 2-2\delta)
\end{aligned}$$

Therefore $\mathrm{im}_\mu\, inj(i) \supseteq \mathrm{im}\,\mu_{inj}(\psi_{t\leq t+2\delta}) = \mathrm{Bar}(W_{2\delta})_{2\delta}$. Thus, undoing the shifts, the proof is complete.

3. Let $(b,d] \in \mathrm{Bar}(\mathrm{im}\, f_\delta)$ such as for some b', $\mu_{inj}(b,d] = (b',d] \in \mathrm{Bar}(W)$. Because of Lemma 3.2.3, $b' \leq b$. There are again two cases:

   If $d - b \leq 2\delta$, then $b' \geq d - 2\delta \geq b > b - 2\delta$ and $b' \in [b-2\delta, b]$.

   Else, if $d - b > 2\delta$, there exists an interval $(b'+2\delta, d] \in \mathrm{Bar}(\mathrm{im}\,\theta_{t\leq t+2\delta})$ such that

$$\mu_{inj}(\theta_{t\leq t+2\delta})(b'+2\delta, d] = \mu_{inj}(i)(b,d] = (a,d].$$

   Thus, $b \leq b' + 2\delta$ and $b' \in [b-2\delta, b]$.

□

At last, we can now prove the other part of the Stability theorem. For so, we will construct a δ-matching out of a δ-interleaving morphism.

Proposition 3.2.8 (Theorem 3.0.2, [11])**.** *Given two persistence modules V, W, with a δ-interleaving morphism between them, then there is a δ-matching between their barcodes.*

Proof. Let $\mu(\phi) = \mu_{inj} \circ \mu_{sur}$ and let $\Phi_\delta\colon \mathrm{Bar}(W_\delta) \to \mathrm{Bar}(W)$ be the *shift map* that carries each bar $(a,b]$ into $(a+\delta, b+\delta]$. The composition $\Phi_\delta \circ \mu(\phi)$ is a matching between

Bar(V) and Bar(W). Hence, using Lemma 3.2.7 and 3.2.6, we get the following diagram:

$$\begin{tikzcd}
\mathrm{Bar}(V) & & \mathrm{Bar}(W_\delta)_{\geq 2\delta} & \mathrm{Bar}(W)_{\geq 2\delta} \\
\cup\!\mid & & \cap\!\mid & \cap\!\mid \\
\mathrm{Bar}(V)_{\geq 2\delta} \arrow[r, "\mu_{sur}"] & \mathrm{Bar}(\mathrm{im}\, f) \arrow[r, "\mu_{inj}"] & \mathrm{im}\, \mu_{inj} \arrow[r, "\Psi_\delta"] & \mathrm{Bar}\, B(W) \\
\uplus & \uplus & \uplus & \uplus \\
(b, d] \arrow[r, mapsto] & (b, d'] \arrow[r, mapsto] & (b', d'] \arrow[r, mapsto] & (b' + \delta, d' + \delta]
\end{tikzcd}$$

The diagram shows that, by Lemma 3.2.6, a bar $(b, d] \in \mathrm{Bar}(V)_{\geq 2\delta}$ is sent to $\mu_{sur}(b, d] = (b, d'] \in \mathrm{Bar}(\mathrm{im}\,\phi)$ with $d' \in [d - 2\delta, d]$. Then, by Lemma 3.2.6, it is sent to $\mu_{sur}(b, d'] = (b', d']$ with $b' \in [b - 2\delta, b]$. At last, using the shift morphism Φ_δ it is carried to $(b' + \delta, d' + \delta]$.

This shows that any bar in $\mathrm{Bar}(V)_{\geq 2\delta}$ is matched. In the same manner it can be seen that any bar in $\mathrm{Bar}(W)_{\geq 2\delta}$ is matched. Thus, we have that

$$\begin{cases} d - 2\delta \leq d' \leq d \\ b - 2\delta \leq b' \leq b \end{cases} \Rightarrow \begin{cases} d - \delta \leq d' + \delta \leq d + \delta \\ b - \delta \leq b' + \delta \leq b + \delta \end{cases},$$

and therefore, $\Phi_\delta \circ \mu(\phi)$ is a δ-matching between Bar(V) and Bar(W). □

The constructions made by Proposition 3.1.2 and Proposition 3.2.8 asure that given a δ-interleaving morphism we can build a δ-matching, and conversely, given a δ-matching we can build a δ-interleaving morphism. This means that if one of the two exists, it fixes a δ. Both the interleaving distance and the bottleneck distance try to minimice this δ, so once fixed for one of them, the other needs an smaller or equal δ'. Thus, with each of the propositions we can prove one of the inequalities needed to reach the isomorphism between the space of persistence diagrams and the space of their barcodes.

Theorem 3.0.1 (Interleaving Stability, Theorem 2.2.8, [11])**.** *There is an isometry between a persistence module with the interleaving distance and its barcode with the bottleneck distance. That is, given two persistence modules V and W, it holds that*

$$d_{\mathrm{int}}(V, W) = d_{\mathrm{bot}}(\mathrm{Bar}(V), \mathrm{Bar}(W)).$$

Proof. Suppose $d_{int}(V, W) = \delta$. Proposition 3.2.8 asures there exist a δ-matching between Bar(V) and Bar(W). As $d_{bot}(V, W)$ is the infimum δ for witch exists a δ-matching, $d_{bot}(V, W) \leq d_{int}(V, W)$. On the other hand, Proposition 3.1.2 leads, with the same reasoning, to $d_{int}(V, W) \leq d_{bot}(V, W)$. Thus, it has to be $d_{int}(V, W) = d_{bot}(\mathrm{Bar}(V), \mathrm{Bar}(W))$. □

Chapter 4

Edelsbrunner & Harer's (Hausdorff) Stability Theorem

Persistence diagrams help summarize the information given by the homology groups of a filtration over a certain data set. They represent the birth and death of every feature in an easy to analice format scattering points over the upper half of the plane $\mathbb{R}^2$ and its diagonal Δ. Once we have computed the diagrams given by two datasets we can measure distances between them using the bottleneck distance, enabling us to decide wether two diagrams are close to each other. However, this would be kind of useless if the bottleneck distance were not stable. If minor differences in original data would cause great changes in the bottleneck distance between the corresponding persistence diagrams, then this data summary method would be as good as any other random method.

Fortunately for us, David Cohen-Steiner, Herbert Edelsbrunner and John Harer, proved in their 2005 paper that the bottleneck distance over persistence diagrams is indeed stable [7]. This means that, when comparing the bottleneck distance between the persistence diagrams formed by the pre images of two tame functions, see Definition 4.0.2, the first will be always lower or equal that the Lebesgue L_∞ norm between the two functions.

Along this chapter we will follow Edelsbrunner et al. paper [7] to prove Theorem 4.0.4. To simplify notation, along this chapter we will adopt the following. Let $H_k(X)$ be the k-th singular homology group of a topological space X. The dimension of $H_k(x)$ is denoted by the k-th Betti number $\beta_k(X) := \dim H_k(x)$.

Let $f\colon X \to \mathbb{R}$, $x < y \in \mathbb{R}$. Denote the k-th homology group of the pre-image by f of an interval $(-\infty, x]$ as $F_x := H_k(f^{-1}(-\infty, x])$. Denote the inclusion map from the k-th homology group F_x to the k-th homology group F_y as $f_x^y\colon F_x \to F_y$. Finally, denote $F_x^y := \operatorname{im} f_x^y$ and name the persistent Betti numbers as $\beta_x^y := \dim(F_x^y)$.

Note that if $y = \infty$, F_x^y is the trivial group. Also, if $x = \infty$, then $y = \infty$ too.

Definition 4.0.1 (Homological critical value)**.** Let X be a topological space and let

$f\colon X \to \mathbb{R}$. A **homological critical value** of f is a number $a \in \mathbb{R}$ such that there exists $k \in \mathbb{Z}$ such that for all $\varepsilon > 0$, the morphism $H_k(f^{-1}(-\infty, a-\infty)) \to H_k(f_{-1}(-\infty, a+\varepsilon))$ is not an isomorphism.

Definition 4.0.2 (Tame function)**.** A function $f\colon X \to \mathbb{R}$ is said to be **tame** if it has a finite number of homological critical values, and for all $z \in \mathbb{Z}$, and for all $a \in \mathbb{R}$, $\dim F_a < \infty$.

Definition 4.0.3 (Multiplicity)**.** Let $f\colon X \to \mathbb{R}$ be tame, and $(a_i)_{i=1,\dots,n}$ be its homological critical values. Take $(b_i)_{i=1,\dots,n}$ be an interleaved sequence of non critical values such that $b_{i-1} < a_i < b_i$ for all $i = 1, \dots, n$. Define $b_{-1} = a_0 = -\infty$, $b_{n+1} = a_{n+1} = \infty$. The **multiplicity** of $(a_i, a_j) \in D(f)$, denoted μ_i^j is

$$\mu_i^j \coloneqq \beta_{b_{i-1}}^{b_j} - \beta_{b_i}^{b_j} + \beta_{b_i}^{b_{j-1}} - \beta_{b_{i-1}}^{b_{j-1}}.$$

The **total multiplicity** of a multiset A, denoted $\#(A)$ is the sum of the multiplicities of every element in A.

Note that the total multiplicity of a multiset is the the generalized concept of cardinality of a normal set. While the cardinality of a set counts the number of elements in the set, the multiplicity of a multiset counts how many elements, different or not, are there in the multiset.

Note that when a function f is tame, we can form a finite multiset of barcodes taking each pair of critical points (a_i, a_j) with their corresponding multiplicity μ_i^j, for each $0 \le i < j \le n+1$. Hence we can form a persistence diagram $D(f)$. The main theorem of this chapter states that the distance of two of those persistence diagrams is never grater that the L^∞ norm between the functions that form them.

Theorem 4.0.4 (Main Theorem, [7])**.** *Let X be a triangulable space, and $f, g\colon X \to \mathbb{R}$ continuous tame functions. Then,*

$$d_{\mathrm{bot}}(D(f), D(g)) \le \|f - g\|_\infty$$

4.1 Hausdorff Stability

Before approaching the proof of Theorem 4.0.4, this section shows how the claim of the theorem is true when the bottleneck distance is replaced by the Hausdorff distance. On Section 4.2 this assertion is used to give an upper limit of the bottleneck distance by Hausdorff distance, proving the main theorem.

We will denote the closed upper left quadrant of a point $(x, y) \in \mathbb{R}^2$ as

$$Q_x^y \coloneqq [-\infty, x] \times [y, \infty].$$

Lemma 4.1.1 (k-Triangle Lemma, [7]). *Let $f: X \to \mathbb{R}$ be a tame function, $x < y \in \mathbb{R}$ be non critical values of f. Then the multiplicity μ of the persistence diagram of f in the closed upper left quadrant is*

$$\mu = \#(D(f) \cap Q_x^y) = \beta_x^y.$$

Proof. Let $x = b_i$, $y = b_{j-1}$.

$$\mu = \sum_{k \leq i \leq j \leq l} \mu_k^l = \sum_{k \leq i \leq j \leq l} \beta_{b_{k-1}}^{b_l} - \beta_{b_k}^{b_l} + \beta_{b_k}^{b_{l-1}} - \beta_{b_{k-1}}^{b_{l-1}} \tag{4.1}$$

$$= \beta_{b_{-1}}^{b_{n+1}} - \beta_{b_i}^{b_{n+1}} + \beta_{b_i}^{b_{j-1}} - \beta_{b_{j-1}}^{b_{-1}} = \beta_{b_k}^{b_{l-1}} = \beta_x^y. \tag{4.2}$$

The fist two equalities in (4.1) are just the definition of total multiplicity. In (4.2), note that every other term in the sum cancels. Then note that $\beta_{b_{-1}}^{b_{n+1}} = \dim F_{-\infty}^{\infty}$, $\beta_{b_i}^{b_{n+1}} = \dim F_x^{\infty}$ and $\beta_{b_{j-1}}^{b_{-1}} = \dim F_{-\infty}^{y}$. All of them are the dimension of the trivial group, therefore, equal to 0. This leaves only one remaining term and completes the proof. □

Denote the **upper left quadrants** $Q \coloneqq Q_b^c = [-\infty, b] \times [c, \infty]$, $Q_\varepsilon \coloneqq Q_{b-\varepsilon}^{c+\varepsilon} = [-\infty, b-\varepsilon] \times [c+\varepsilon, \infty]$.

Lemma 4.1.2 (Quadrant Lemma, [7]). *Let $f, g: X \to \mathbb{R}$ be two tame functions. With the notation abobe, the following inequality holds,*

$$\#(D(f), \cap Q_\varepsilon) \leq \#(D(g) \cap Q).$$

Proof. Let $\varepsilon \coloneqq \|f - g\|_\infty$. Hence, considering the pre-image of the functions, we have the following inclusions

$$f^{-1}((-\infty, x]) \subseteq g^{-1}((-\infty, x+\varepsilon)), \tag{4.3}$$

$$g^{-1}((-\infty, x]) \subseteq f^{-1}((-\infty, x+\varepsilon)). \tag{4.4}$$

Name $\varphi_x: F_x \to G_{x+\varepsilon}$ to the inclusion map induced by (4.3) and $\psi_x: G_x \to F_{x+\varepsilon}$ to the inclusion map induced by (4.4). Let $b < c \in \mathbb{R}$. With the described maps, we can form commutative diagram (4.5) where we observe that

$$\operatorname{im}(f_{c-\varepsilon}^{c+\varepsilon}) = F_{c-\varepsilon}^{c+\varepsilon} \subseteq \psi_c \circ g_b^c(G_b) = \psi_c(G_b^c).$$

$$\begin{tikzcd} F_{b-\varepsilon} \arrow[r, "f_{b-\varepsilon}^{c+\varepsilon}"] \arrow[d, "\varphi_{b-\varepsilon}"'] & F_{c+\varepsilon} \\ G_b \arrow[r, "g_b^c"] & G_c \arrow[u, "\psi_c"'] \end{tikzcd} \tag{4.5}$$

Last inclusion is enough for the requirements of this proof, never the less, we will make one more note that will be useful latter on, through the proof of Lemma 4.1.3. Fit the maps so that they describe commutative diagram (4.6), showing that

$$\psi_c(G_b^c) = \psi_c \circ g_b^c(G_b) = f_{b+\varepsilon}^{c+\varepsilon} \circ \psi_b(G_b) \subseteq F_{b+\varepsilon}^{c+\varepsilon}.$$

$$\begin{tikzcd} F_{b+\varepsilon} \arrow[r, "f_{b+\varepsilon}^{c+\varepsilon}"] & F_{c+\varepsilon} \\ G_b \arrow[u, "\varphi_b"] \arrow[r, "g_b^c"] & G_c \arrow[u, "\psi_c"'] \end{tikzcd} \tag{4.6}$$

From both diagrams we finally obtain the inclusion chain

$$F_{c-\varepsilon}^{c+\varepsilon} \subseteq \psi_c(G_b^c) \subseteq F_{b+\varepsilon}^{c+\varepsilon}. \tag{4.7}$$

By Lemma 4.1.1, we are able to count the elements in the intersection of the diagrams with the upper left quadrants. Hence

$$\#(D(f) \cap Q_\varepsilon) = \beta_{b-\varepsilon}^{c+\varepsilon} = \dim F_{b-\varepsilon}^{c+\varepsilon},$$
$$\#(D(g) \cap Q) = \beta_b^c = \dim G_b^c.$$

As if one homology group is contain in an other, the dimension of the first must be lower or equal to the one of the second. Also, the dimension is invariant under inclusion maps. Thus, the first inclusion of (4.7) asserts that $F_{c-\varepsilon}^{c+\varepsilon} \subseteq \psi_c(G_b^c)$ and therefore we have proven that $\dim F_{c-\varepsilon}^{c+\varepsilon} \leq \dim G_b^c$. □

Before next lemma we will introduce some new notation. Let $f\colon X \to \mathbb{R}$ fe a tame function. Let $w < x < y < z \in \mathbb{R}$ be numbers different from critical values of f. Recall that $F_x = H_k(f^{-1}(-\infty, x])$, $f_x^y\colon F_x \to F_y$ and $F_x^y = \dim f_x^y$. We denote

$$f_x^{y,z} := f_y^z|_{F_x^y}, \qquad F_x^{y,z} := \dim f_x^{y,z}.$$

Note, from linear algebra, that $\dim F_x^{y,z} = \dim F_x^y - \dim F_x^z$. Note too that $F_w^y \subseteq F_x^y$. Therefore, $\ker F_w^y \subseteq \ker F_x^y$ and we can define the quotient

$$F_{w,x}^{y,z} := F_x^{y,z} / F_w^{y,z}.$$

Let $a < b < c < d \in \mathbb{R}$. Denote the **rectangles** $R := [a,b] \times [c,d]$, $R_\varepsilon := [a+\varepsilon, b-\varepsilon] \times [c+\varepsilon, d-\varepsilon]$.

Lemma 4.1.3 (Box Lemma, [7])**.** *With the notation abobe, the following inequality holds,*

$$\#(D(f), \cap R_\varepsilon) \leq \#(D(g) \cap R).$$

Proof. Note that we can asume that $a+\varepsilon < b-\varepsilon$ and $c+\varepsilon < d-\varepsilon$. Otherwise there would not be rectangle R_ε. Also note that

$$\begin{aligned}\#(D(f)\cap R_\varepsilon) &= \dim F^{c+\varepsilon,d-\varepsilon}_{a+\varepsilon,b-\varepsilon}\\ \#(D(g)\cap R) &= \dim G^{c,d}_{a,b}\end{aligned}$$

To make our prove we draw diagram 4.1. Lets analice every element of the diagram.

$$\begin{tikzcd}
G_a^d \arrow[rrr, "r_1"] & & & G_b^d \\
& F^{d-\varepsilon}_{a+\varepsilon} \arrow[r, "r_2"] & F^{d-\varepsilon}_{b-\varepsilon} \arrow[ur, "s_1"] & \\
& F^{c+\varepsilon}_{a+\varepsilon} \arrow[u, "u_2"] \arrow[r, "r_3"] & F^{c+\varepsilon}_{b-\varepsilon} \arrow[u, "u_3"'] & \\
G_a^c \supseteq E_a^c \arrow[uuu, "u_1"] \arrow[ur, "s_2"] \arrow[rrr, "r_4"] & & & E_b^c \subseteq G_b^c \arrow[uuu, "u_4"'] \arrow[ul, "s_3"']
\end{tikzcd}$$

First of all, the middle upside arrows, are

$$u_2 = f^{c+\varepsilon,d-\varepsilon}_{a+\varepsilon}, \qquad u_3 = f^{c+\varepsilon,d-\varepsilon}_{b-\varepsilon}.$$

Right arrows r_1, r_2, r_3, r_4 represent the inclusions from its respective vector spaces to their destination. The objetive is to define que respective quotients to define $G^{c,d}_{a,b}$. Recall the inclusion maps defined in the proof of Lemma 4.1.2, $\varphi_x\colon F_x \to G_{x+\varepsilon}$ and $\psi_x\colon G_x \to F_{x+\varepsilon}$. We define

$$E_b^c := \psi_c^{-1}(F^{c+\varepsilon,d-\varepsilon}_{b-\varepsilon}) \cap G_b^c, \qquad E_a^c := G_a^c \cap E_b^c.$$

We have then that the outer upside arrows are the respective restrictions

$$u_1 = g_a^{c,d}|_{E_a^c}, \qquad u_4 = g_b^{c,d}|_{E_b^d}.$$

We denote

$$s_1 := \varphi_{d-\varepsilon}|_{F^{d-\varepsilon}_{b-\varepsilon}}, \qquad s_2 := \psi_c|_{E_a^c}, \qquad s_3 := \psi_c(G_b^c)|_{E_b^c}.$$

By the inclusions (4.7), we have that

$$\varphi_{d-\varepsilon}(F^{d-\varepsilon}_{b-\varepsilon}) \subseteq G_b^d, \qquad \psi_c(G_a^c), \subseteq F^{c+\varepsilon}_{a+\varepsilon} \qquad F^{c+\varepsilon}_{b-\varepsilon} \subseteq \psi_c(G_b^c).$$

Note that by the manner we have define the diagram, it need to be that

$$\operatorname{im}(s_3) = \ker(u_3), \qquad \operatorname{im}(s_1) \subseteq G_b^d.$$

Also, we can observe that $u_4 = s_1 \circ u_3 \circ s_3$. As $u_3 \circ s_3 = 0$, then $E^c_b = \ker(u_4)$. Also, as $r_1 \circ u_1 = u_4 \circ r_4 = 0$ and r_1 is the inclusion, then $E^c_a = \ker(u_1)$. Hence, we can write

$$E^c_b = E^{c,d}_b \subseteq G^{c,d}_b, \qquad\qquad E^c_a = E^{c,d}_a \subseteq G^{c,d}_a.$$

As $E^{c,d}_a = E^{c,d}_b \cap G^{c,d}_a$, the following quotient inclusion holds

$$E^{c,d}_{a,b} = E^{c,d}_b / E^{c,d}_a \subseteq G^{c,d}_b / G^{c,d}_a = G^{c,d}_{a,b}.$$

Therefore, we have

$$\dim(E^{c,d}_{a,b}) \leq \dim(G^{c,d}_{a,b}).$$

Now note that

$$E^{c,d}_{a,b} = \ker(u_4)/\ker(u_1), \qquad\qquad F^{c+\varepsilon,d+\varepsilon}_{a+\varepsilon,b+\varepsilon} = \ker(u_3/\ker(u_2)).$$

By construction $s_3(\ker(u_4)) = \ker(u_3)$. As for every $x \in \ker(u_1)$, $r_2 \circ u_2 \circ s_2(x) = u_3 \circ s_3 \circ r_4(x) = 0$, and r_2 is an injection, then $s_3(\ker(u_1)) = s_2(\ker(u_1) \subseteq \ker(u_2))$, we get

$$\dim(F^{c+\varepsilon,d+\varepsilon}_{a+\varepsilon,b+\varepsilon}) \leq \dim(E^{c,d}_{a,b}).$$

Hence, the desired inequality is hold as we have seen that

$$\#(D(f) \cap R_\varepsilon) = \dim F^{c+\varepsilon,d-\varepsilon}_{a+\varepsilon,b-\varepsilon} \leq \dim(E^{c,d}_{a,b}) \leq \dim(G^{c,d}_{a,b}) = \#(D(g) \cap R).$$

□

Theorem 4.1.4 (Hausdorff Stability)**.**

$$d_{\mathrm{H}}(D(f), D(g)) \leq \|f - g\|_\infty.$$

Proof. As a direct consequence of Lemma 4.1.3 if $(x, y) \in D(f)$ then there must exist some point at $D(g)$ at distance less than or equal to $\varepsilon = \|f - g\|_\infty$ from (x, y) since the total multiplicity of $D(g) \cap R_\varepsilon$ is at least one. □

4.2 Bottleneck Stability

In this section we are going to prove Theorem 4.0.4.

Definition 4.2.1 (Very close tame functions)**.** Let $f, g\colon X \to \mathbb{R}$ be tame functions. We define

$$\delta_f = \min\{\|p-q\|_\infty : p \in D(f) \setminus \Delta,\ q \in D(f),\ p \neq q\}.$$

We say that g is **very close** to f if $\|f-g\|_\infty < \delta_f/2$.

Lemma 4.2.2 (Easy Bijection Lemma, [7])**.** *Let $f, g\colon X \to \mathbb{R}$ be tame functions, where g is very close to f. Then, following holds,*

$$d_{\text{bot}}(D(f), D(g)) \leq \|f-g\|_\infty.$$

Proof. Let $p := (a_i, a_j) \in D(f) - \Delta$ be a point in the diagram of f that is not in the diagonal, and let $\mu := \beta_i^j$ denote its multiplicity. Let S_ε be the square of center p and radius $\varepsilon = \|f-g\|_\infty$. That is, the square of side 2ε. By definition of the square S_ε we have that the number of points of its intersection with the diagram of g must be grater or equal than the multiplicity at p. Hence, by the Box Lemma 4.1.3 we have

$$\mu \leq \#(D(g) \cap S_\varepsilon) \leq \#(D(f) \cap S_2\varepsilon).$$

As g is very close to f, we have $2\varepsilon \leq \delta_f$. Hence p is the only point of $D(f)$ in S_ε, and therefore the previous inequality is in fact an equivalence. If there was a point in the intersection which was not in μ then it would be inside the square S_ε and meaning the distance $\|f-g\|$ would be smaller. That is

$$\mu = \#(D(g) \cap S_\varepsilon).$$

Hence we can map every point in $D(g) \cap S_epsilon$ with p. We can then repeat this process for every other $p \in D(f) \setminus \Delta$. After this process, every point of $D(g)$ which have not been matched jet, must be at distance grater than ε from $D(f) \setminus \Delta$. By Theorem 4.1.4, every unmatched point must be at distance at most ε from the diagonal Δ. Hence, if we map each of this points to Δ, we have built a bijection between $D(f)$ and $D(g)$ that moves each point at most ε. □

Definition 4.2.3. Let $\hat{f}, \hat{g}$ be two picewise linear functions over a simplicial complex K. Let $\lambda \in [0,1]$. A **convex combination** of $\hat{f}$ and $\hat{g}$ is a function of the form

$$h_\lambda := (1-\lambda)\hat{f} + \lambda\hat{g}.$$

Lemma 4.2.4 (Interpolation Lemma, [7])**.** *Let K be a simplicial complex. Take two picewise linear functions $\hat{f}, \hat{g}\colon K \to \mathbb{R}$. Then, following holds,*

$$d_{\text{bot}}(D(\hat{f}), D(\hat{g})) \leq \|\hat{f}-\hat{g}\|_\infty.$$

Proof. Let $c := \|\hat{f} - \hat{g}\|_\infty$. For every $\lambda \in [0,1]$, define $\delta(\lambda) := \delta_{h_\lambda} > 0$. Let J_λ denote open intervals around each δ as follows, and consider the set C of all J_λ be

$$C := \left\{ J_\lambda := \left(\lambda - \frac{\delta(\lambda)}{4c}, \lambda + \frac{\delta(\lambda)}{4c} \right) \right\}.$$

The set C is an open cover of the interval $[0,1]$. Let C' be the minimal subcover of C. As $[0,1]$ is compact, the subcover C' must be finite. Consider then $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ the midpoints of the intervals in C'. As C' is minimal, the each intersection $J_{\lambda_i} \cap J_{\lambda_{i+1}}$ is not empty. Hence

$$\lambda_i + \lambda_{i+1} \leq \frac{\delta(\lambda_i) + \delta(\lambda_{i+1})}{4c} \leq \frac{\max\{\delta(\lambda_i), \delta(\lambda_{i+1})\}}{2c}.$$

Therefore, by definition of c and each h_{δ_i}, it holds

$$\|h_{\delta_i} - h_{\delta_{i+1}}\|_\infty = c(\lambda_{i+1} - \lambda_i) \leq \frac{\max\{\delta(\lambda_i), \delta(\lambda_{i+1})\}}{2}.$$

This implies that h_{δ_i} is very close to $h_{\delta_{i+1}}$ or viceversa. Then, by Lemma 4.2.2, for every $1 \leq i \leq n-1$,

$$d_{\text{bot}}(D(h_{\lambda_i}), D(h_{\lambda_{i+1}})) \leq \|h_{\lambda_i} - h_{\lambda_{i+1}}\|_\infty. \tag{4.8}$$

Let $\lambda_0 = 0$ and $\lambda_{n+1} = 1$. Then $h_{\lambda_0} = \hat{f}$ is very close to h_{λ_1} and $h_{\lambda_1} = \hat{g}$ is very close to h_{λ_n} and therefore (4.8) also holds for $i = 0$ and $i = n+1$. Finally, using the triangle inequality we have

$$d_{\text{bot}}(D(\hat{f}), D(\hat{g})) \leq \sum_{i=0}^{n} d_{\text{bot}}(D(h_{\lambda_i}), D(h_{\lambda_{i+1}})) \leq \sum_{i=0}^{n} \|h_{\lambda_i} - h_{\lambda_{i+1}}\|_\infty = \|\hat{f} - \hat{g}\|_\infty.$$

□

For the final prove lets recall

Definition 4.2.5 (Star of a simplicial complex)**.** Let σ be a simplex in a simplicial complex L. The **star** $\mathrm{St}(\sigma)$ of σ is the set of simplices in L which contain σ as a face. The **star of a subset** K of L, denoted $\mathrm{St}(K)$, is the union of the stars of each simplex of K.

Theorem 4.0.4 (Main Theorem, [7])**.** *Let X be a triangulable space, and $f, g \colon X \to \mathbb{R}$ continuous tame functions. Then,*

$$d_{\text{bot}}(D(f), D(g)) \leq \|f - g\|_\infty$$

Proof. As X is triangulable, there exists a finite simplicial complex L and a homeomorphism $\Phi\colon L \to X$. Hence a persistence diagram is invariant under this change of variables. That is, $f \circ \Phi\colon L \to \mathbb{R}$ is tame and $D(f \circ \Phi) = D(f)$. Since f and g are continuous and L is compact, for every $\delta < 0$ there exists a subdivision K of L such that for every u, v points of a common simplex $\sigma \in K$,

$$|f \circ \Phi(u) - f \circ \Phi(v)| \leq \delta,$$
$$|g \circ \Phi(u) - g \circ \Phi(v)| \leq \delta.$$

Let $\hat{f}, \hat{g}\colon \mathrm{St}(K) \to \mathbb{R}$ be the picewise linear interpolations of $f \circ \Phi$ and $g \circ \Phi$ on K. By construction of K and the definition of the L_∞-norm, this interpolations satisfy

$$\|\hat{f} - f \circ \Phi\|_\infty \leq \delta,$$
$$\|\hat{g} - g \circ \Phi\|_\infty \leq \delta.$$

Hence, by Lemma 4.2.4 and the triangle inequality

$$d_{\text{bot}}(D(\hat{f}), D(\hat{g})) \leq \|\hat{f} - \hat{g}\|_\infty \leq \|f \circ \Phi - g \circ \Phi\|_\infty + 2\delta \leq \|f - g\|_\infty + 2\delta.$$

Now, we can take some δ such that $\delta \leq \delta_f/2$ so $\hat{f}$ is very close to f. This allows to use Lemma 4.2.2 to make a bijection that satisfy

$$d_{\text{bot}}(D(f), D(\hat{f})) \leq d_{\text{bot}}(D(f \circ \Phi), D(\hat{f})) \leq \delta.$$

Analogously, also assuring $\delta < \delta_g$ we also have

$$d_{\text{bot}}(D(g), D(\hat{g})) \leq d_{\text{bot}}(D(g \circ \Phi), D(\hat{g})) \leq \delta,$$

and therefore, by triangle inequality again,

$$d_{\text{bot}}(D(f), D(g)) \leq d_{\text{bot}}(D(f), D(\hat{f})) + d_{\text{bot}}(D(\hat{f}), D(\hat{g})) + d_{\text{bot}}(D(\hat{g}), D(g)) \leq 4\delta.$$

As this holds for any δ smaller that δ_f and δ_g, taking the limit when δ tends to 0, we complete the proof. $\square$

Chapter 5

Gromov-Hausdorff's Stability Theorem

5.1 Gromov-Hausdorff stability

Definition 5.1.1 (Vietoris-Rips filtration)**.** Let (X, d) be a finite metric space and let $\alpha > 0$. The **Vietoris-Rips complex** associated with X of radius α, $\mathcal{R}_\alpha(X, d)$, is the simplicial complex whose 0-simplices are the elements of X and, for $k \geq 1$, its k-dimensional simplices are formed by every subset $\{x_0, x_1, \ldots, x_k\} \subseteq X$ such that $d(x_i, x_j) \leq \alpha$ for every $i, j = 1, \ldots, k$.

The family $\mathcal{R}(X, d) \coloneqq \{\mathcal{R}_\alpha(X, d)\}_{\alpha>0}$ is named **Vietoris-Rips filtration**.

Given a real function $f\colon X \to R$, let $X_\alpha \coloneqq f^{-1}((-\infty, \alpha]) \subseteq X$ be the **pre-image of** f **delimited by** α. We define the **Vietoris-Rips filtration associated with** f as $\mathcal{R}(X, d, f) \coloneqq \{\mathcal{R}_\alpha(X_\alpha, d)\}_{\alpha>0}$.

Definition 5.1.2 (Čech filtration)**.** Let (X, d) be a finite metric space and let $\alpha > 0$. The **Čech complex** associated with X of radius α, $\check{\mathcal{C}}_\alpha(Z, d)$, is the simplicial complex whose 0-simplices are the elements of X and, for $k \geq 1$, its k-dimensional simplices are formed by every subset $\{x_0, x_1, \ldots, x_k\} \subseteq X$ such that there exists some $x \in X$ such that $d(x, x_i) \leq \alpha$ for all $i = 1, \ldots, k$.

The family $\check{\mathcal{C}}(Z, d) \coloneqq \{\check{\mathcal{C}}_\alpha(Z, d)\}_{\alpha>0}$ is named **Čech filtration**.

Given a real function $f\colon X \to R$, let $X_\alpha \coloneqq f^{-1}((-\infty, \alpha]) \subseteq X$. We define the **Čech filtration associated with** f as $\check{\mathcal{C}}(X, d, f) \coloneqq \{\check{\mathcal{C}}_\alpha(X_\alpha, d)\}_{\alpha>0}$.

Lemma 5.1.3 (Exercise 3.5.4, [3])**.** *Any finite metric space of cardinality n can be isometrically embedded into* $(\mathbb{R}^n, \ell^n)$.

Proof. Let X be a compact metric space and let $C(X)$ be the space of all continuous functions from X to $\mathbb{R}$. Let $f, g \in C(X)$. Recall the uniform distance given by

$$d_\infty(f,g)\colon \sup |f(x) - g(x)|.$$

First, we will check that the pair $(C(X), d_\infty)$ is a metric space. Naturally nonnegativity holds and

$$d_\infty(f,f) = \sup |f(x) - f(x)| = 0.$$

Commutativity also holds as

$$d_\infty(f,g) = \sup |f(x) - g(x)| = \sup |g(x) - f(x)| = d_\infty(g,f).$$

Finally, if $h \in C(X)$, triangle inequality holds because

$$\begin{aligned} d_\infty(f,h) &= \sup |f(x) - h(x)| = \sup |f(x) + g(x) - g(x) - h(x)| \\ &\leq \sup |f(x) - g(x)| + \sup |g(x) - h(x)| = d_\infty(f,h) + d_\infty(h,g). \end{aligned}$$

Now we are going to verify that the map $E\colon X \to C(X)$ defined by $E(x) = d(x,\cdot)$ is an isometric embedding onto its image. Note that

$$d_\infty(d(x,\cdot), d(y,\cdot)) = \sup_z |d(x,z) - d(y,z)| \leq \sup_z |d(x,y)| = d(x,y).$$

On the other hand if we take $z = y$ we then have

$$|d(x,y) - d(y,y)| = d(x,y),$$

and therefore

$$\sup_z |d(x,z) - d(y,z)| \geq= d(x,y).$$

The proof of the lemma is just an analogous case taking $C_n(X)$ as the set of continuous functions $f\colon X \to \mathbb{R}^n$, and, for every $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $\ell^\infty(x,y) = \max_i |x_i - y_i|$. □

Lemma 5.1.4 (Lemma VII, [8])**.** *Let $X \subset \mathbb{R}^n$ and $\alpha > 0$. Then the α-Čech and the α-Rips complexes coincide when using the ℓ^∞-norm. That is*

$$\check{C}_\alpha(X, \ell^\infty) = R_\alpha(X, \ell^\infty).$$

Definition 5.1.5 (Paracompact space)**.** Let X be a topological space. It is said to be **paracompact** if for all covering U of X, there exists $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V}$ is a finite covering.

Definition 5.1.6 (Good cover)**.** Let S be a topological space and I a set of indexes. A **good cover** of S is a family $\mathcal{U} = U_i$ of open subsets covering S such that for every finite subset $J \subset I$, the common intersection

$$\bigcap_{j \in J} U_j$$

is either empty or contractible.

Lemma 5.1.7 ([5])**.** *Let $S \subset S'$ be two paracompact spaces. Let $\mathcal{U} = \{U_x\}_{x \in A}$, $\mathcal{U}' = \{U'_x\}_{x \in A'}$ be two good covers of S and S' respectively, based on finite parameter sets $A \subset A'$ such that $U_x \subset U'_x$ for all $x \in A$. Then the homotopy equivalences $\mathcal{N}\mathcal{U} \to S$ and $\mathcal{N}U' \to S'$ commute with the canonical inclusions $S \to S'$ and $\mathcal{N}\mathcal{U} \to \mathcal{N}\mathcal{U}'$ at homology level.*

Theorem 5.1.8 (Theorem 3.1, [4])**.** *Let (X, d_X), (Y, d_Y) be finite metric spaces. Then, for any $k \in \mathbb{N}$,*

$$d_{\text{bot}}((\mathcal{R}(X, d_X)), D_k(\mathcal{R}(Y, d_Y))) \leq d_{\text{GH}}((X, d_X), (Y, d_Y)).$$

Proof. Let $\varepsilon = d_{\text{GH}}((X, d_X), (Y, d_Y))$. As X and Y are finite, they are compact, and therefore the infimum when computing Gromov-Hausdorff distance using Definition 1.4.3 is in fact a minimum. That is, there exists a metric space (Z, d_Z) and two isometric embeddings $\gamma_X : X \to Z$ and $\gamma_Y : Y \to Z$ such that

$$d_{\text{H}}^Z(\gamma_X(X), \gamma_Y(Y)) = \varepsilon,$$

where d_{H}^Z denotes the Hausdorff distance respect the distance d_Z. Consider the subspace $\gamma_X(X) \cup \gamma_Y(Y) \subseteq Z$ with the induced metric from Z. As both X and Y are finite, let

$$n \coloneqq \#(X) + \#(Y).$$

Hence, by Lemma 5.1.3, there exists an isometric embedding

$$\gamma \colon (\gamma_X(X) \cup \gamma_Y(Y), d_Z) \to (\mathbb{R}^n, \ell^\infty).$$

Let d_{H}^∞ denote the Hausdorff distance respect the distance d_∞. We then have

$$d_{\text{H}}^\infty(\gamma \circ \gamma_X(X), \gamma \circ \gamma_Y(Y)) = d_{\text{H}}^Z(\gamma_X(X), \gamma_Y(Y)) = \varepsilon.$$

Let δ_X be the distance function from a point in $\mathbb{R}^n$ to X, and analogously, let δ_X be the distance function to Y. In ℓ^∞ norm, by hoy we defined ε, we have

$$\|\delta_X - \delta_Y\|_\infty = \max_{i=1,\dots,n} |\delta_{x_i} - \delta_{y_i}| \leq \varepsilon.$$

As distance functions are linear, both δ_X and δ_Y are lower envelopes of picewise-linear functions and therefore they are picewise-linear too. Hence, both δ_X and δ_Y are tame and continuous so by Theorem 4.0.4 we have

$$d_{\text{bot}}(D(\delta_X), D(\delta_Y)) \leq \|\delta_X - \delta_Y\|_\infty \leq \varepsilon.$$

Let $\alpha \in \mathbb{R}$. Define an off-set of radius α around the image of the embedding of X into $\mathbb{R}^n$ as

$$\gamma \circ \gamma_X(X)^\alpha := \bigcup_{x \in \gamma \circ \gamma_X(X)} B_\alpha^{\ell^\infty}(x),$$

where $B_\alpha^{\ell^\infty}(x)$ denotes de ball of radius α and center x using distance d_∞. As balls in ℓ^∞ are hypercubes, they are convex, and therefore their intersection is either empty or contractible. By Lemma 5.1.7 know that δ_X has the same persistence diagram as the Čech complex $\check{\mathcal{C}}(\gamma \circ \gamma_X, \ell^\infty)$. By Lemma 5.1.4, when using the ℓ^∞-norm, Čech and Rips complexes coincide and so do their filtrations. As $\gamma \circ \gamma_X$ is an isometric embedding, we then have

$$\check{\mathcal{C}}(\gamma \circ \gamma_X, \ell^\infty) = \mathcal{R}(\gamma \circ \gamma_X, \ell^\infty) = \mathcal{R}(X, d_X).$$

Hence, the persistence diagram of $\mathcal{R}(X, \ell^\infty)$ is the same as the persistence diagram of γ_X. The same is true taking Y and therefore we have

$$d_{\text{bot}}(D(\mathcal{R}(X, d_X)), D(\mathcal{R}(Y, d_Y))) = d_{\text{bot}}(D(\gamma_X), D(\gamma_Y)) \leq \varepsilon.$$

□

Proposition 5.1.9. *Let (X, d_X), (Y, d_Y) be finite metric spaces. Then, for any $k \in \mathbb{N}$, the bottleneck distance*

$$d_{\text{bot}}(D_k(\mathcal{R}(X, d_X)), D_k(\mathcal{R}(Y, d_Y))),$$

is a tight lower bound of

$$d_{\text{GH}}((X, d_X), (Y, d_Y)).$$

That is, it is the largest possible lower bound.

Proof. Its enough to find an example were both distances are equal. For so, take the two point spaces $X = \{a, b\}$ with distance $d_X(a, b) = 2$, and $Y = \{c, d\}$ with $d_Y(c, d) = 2+2\varepsilon$. Both spaces can be isometrically mapped into the real line $\mathbb{R}$, with X mapped to $\{0, 2\}$ and Y mapped to $\{-\varepsilon, 2+\varepsilon\}$. Hence $d_{\text{GH}}(X, Y) \leq \epsilon$.

On the other hand, the 0-dimensional persistence diagram of the Rips filtration of (X, d_X) and (Y, d_Y) are

$$D_0(\mathcal{R}(X, d_X)) = \{(0, \infty), (0, 1)\},$$
$$D_0(\mathcal{R}(Y, d_Y)) = \{(0, \infty), (0, 1 = \varepsilon)\},$$

and therefore, $d_{\text{bot}}(D_k(\mathcal{R}(X, d_X)), D_k(\mathcal{R}(Y, d_Y))) = \varepsilon$. □

The following theorem generalizes Theorem 5.1.8.

Theorem 5.1.10 (Theorem 3.2, [4]). *Let (X, d_X), (Y, d_Y) be finite metric spaces endowed with the functions $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$. Then*

$$d_{\text{bot}}(D_k(\mathcal{R}(X, d_X, f)), D_k(\mathcal{R}(Y, d_Y, g))) \leq d^1_{\text{GH}}((X, d_X, f), (Y, d_Y, g)).$$

Proof. We follow a similar procedure to the proof of Theorem 5.1.8. Start setting

$$\varepsilon \coloneqq d^1_{\text{GH}}((X, d_X, f), (Y, d_Y, g)).$$

For every $\alpha \in \mathbb{R}$, recall the notation for the pre-images of f and g by α,

$$X_\alpha \coloneqq f^{-1}((-\infty, \alpha]) \subseteq X,$$
$$Y_\alpha \coloneqq g^{-1}((-\infty, \alpha]) \subseteq Y.$$

As before, as X and Y are finite, the infimum in d^1_{GH} of Definition 1.4.12 is actually a minimum realized by some correspondance $R \in (X \times Y)$. Also, the disjoint union $Z = X \cup Y$, can be endowed with a metric d_Z and a pair of inclusions $\gamma_X\colon X \to Z$, $\gamma_Y\colon Y \to Z$ such that for every $(x, y) \in R$,

$$d_Z(\gamma_X(X), \gamma_Y(Y)) \leq \frac{1}{2}\,\text{dis}(R) \leq \varepsilon, \text{ and}$$
$$|f(x) - g(y)| \leq \|f - g\|_{\ell^\infty} \leq \varepsilon.$$

By Lemma 5.1.3, $(\gamma_X(X) \cup \gamma_Y(Y), d_Z)$ can be isometrically embedded by some γ into $(\mathbb{R}^n, \ell^\infty)$, where

$$n \coloneqq \#(X) + \#(Y).$$

Hence, for every $(x, y) \in R$ we have

$$\|\gamma \circ \gamma_X(X) - \gamma \circ \gamma_Y(Y)\|_{\ell^\infty}.$$

Note that the filtrations given by the off-set defined in the proof of Theorem 5.1.8, can be seen as persistence modules $V \coloneqq \{\gamma \circ \gamma_X(X_\alpha)^\alpha\}_{\alpha>0}$ and $W \coloneqq \{\gamma \circ \gamma_Y(Y_\alpha)^\alpha\}_{\alpha>0}$ are ε-interleaved. That is, for all $\alpha > 0$,

$$\gamma \circ \gamma_X(X_\alpha)^\alpha \subseteq \gamma \circ \gamma_Y(Y_\alpha)^{\alpha+\varepsilon} \subseteq \gamma \circ \gamma_X(X_\alpha)^{\alpha+2\varepsilon}.$$

This is because for every element $p \in \gamma \circ \gamma_X(X_\alpha)^\alpha$, there exists some $x \in X$ such that

$$\|p - \gamma \circ \gamma_X(x)\|_{\ell^\infty} \leq \alpha.$$

Hence, taking some $y \in Y$ such that $(x, y) \in R$ we have that

$$\|\gamma \circ \gamma_X(x) - \gamma \circ \gamma_Y(y)\|_{\ell^\infty} \leq \varepsilon,$$

and as

$$g(y) \leq f(x) + \varepsilon \leq \alpha + \varepsilon,$$

we have that $y \in Y_{\alpha+\varepsilon}$ and therefore

$$\|p - \gamma \circ \gamma_Y(y)\|_{\ell^\infty} \leq \alpha + \varepsilon,$$

and so $p \in \gamma \circ \gamma_Y(Y_\alpha)^{\alpha+\varepsilon}$. The second inclusion follows analogously.

Note that as we have two ε-interleaved modules we can use 3.0.1 to expresses the bottleneck distance as the interleaving distance. Thus

$$d_{\text{bot}}(D_k(V), D_k(W)) = d_{\text{int}}(V, W) \leq \epsilon.$$

Analogously to the previous proof, Lemma 5.1.7 tells this inequality is also valid taking the Čech filtrations, and Lemma 5.1.4 let us take the Rips filtrations, completing the proof. □

Chapter 6

Vectorizations' Stability Theorems

6.1 Persistence landscapes

Lemma 6.1.1 (Lemma 1, [2])**.** *Let $a \leq b \leq c \leq d$ be real numbers. Then $\beta_b^c \geq \beta_a^d$.*

Proof. ... $\square$

Lemma 6.1.2 (Lemma 2, [2])**.** *Let $0 \leq h_1 \leq h_2$ be real numbers. Then $\beta_{t-h_1}^{t+h_1} \geq \beta_{t-h_2}^{t+h_2}$.*

Proof. ... $\square$

Let $\overline{\mathbb{R}} \coloneqq \mathbb{R} \cup \{-\infty, \infty\}$ denote the extended real numbers.

Definition 6.1.3 (Persistence landscape)**.** A **persistence landscape** is a function $\lambda\colon \mathbb{N} \times \mathbb{R} \to \overline{\mathbb{R}}$, defined as

$$\lambda(k,t) \coloneqq \sup\{m \geq 0 \mid \beta^{t-m,t+m} \geq k\}.$$

Note that this function can also be seen as a sequence of function $\lambda_k\colon \mathbb{R} \to \overline{\mathbb{R}}$, where $\lambda_k(t) = \lambda(k,t)$.

Definition 6.1.4 (K-Lipschitz)**.** Let (X, d_X), (Y, d_Y) be two metric spaces and let $K > 0$. A K-**Lipschitz** map is a map $f\colon (X, d_X) \to (Y, d_Y)$, such that for every $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2).$$

Lemma 6.1.5 (Lemma 3, [2])**.** *Let $\lambda_k\colon \mathbb{R} \to \overline{\mathbb{R}}$ be an element of a persistence diagram. The following properties are verified.*

1. $\lambda_k(t) \geq 0$*,*

2. $\lambda_k(t) \geq \lambda_{k+1}(t)$*,*

3. λ_k *is* 1*-Lipschitz, that is, for* $t, s \in \mathbb{R}$*,* $|\lambda_k(t) - \lambda_k(s)| \leq |t - s|$*.*

Proof. Properties 1. and 2. came directly from the definition. For 3., suppose $\lambda_k(s) \leq \lambda_k(t)$. If $\lambda_k(t) \leq |t-s|$ then, of course, $\lambda_k(t) - \lambda_k(s) \leq \lambda_k(t) \leq |t-s|$. Else, if $\lambda_k(t) > |t-s|$, we can take some $h \in (0, \lambda_k(t) - |t-s|)$ verifying

$$t - \lambda_k(t) < s - h < s + h < t + \lambda_k(t).$$

Hence, by Lemma 6.1.1 □

Definition 6.1.6 (p-landscape distance)**.** Let W and W be two persistence modules, and let λ and λ' its corresponding persistence landscapes. Let $1 \leq p \leq \infty$. The p**-landscape distance between persistence modules** V and W is defined as

$$\Lambda_p(V, W) \coloneqq \|\lambda - \lambda'\|_p.$$

Similarly, if D and D' are two persistence diagrams, and are λ and λ' its corresponding persistence landscapes, The p**-landscape distance between persistence diagrams** V and W is defined as

$$\Lambda_p(D, D') \coloneqq \|\lambda - \lambda'\|_p.$$

Theorem 6.1.7 (Theorem 12, Lemma 1, [2])**.** *Consider the persistence modules* $V = F_x$*,* $W = G_x$ *given by the maps* $f, g \colon X \to \mathbb{R}$*. Then*

$$\Lambda_\infty(V, W) \leq \|f - g\|_\infty$$

Proof. ... □

Theorem 6.1.8 (Theorem 13, Lemma 1, [2])**.** *Let* D *and* D' *be two persistence diagrams, then*

$$\Lambda_\infty(D, D') \leq d_{\mathrm{bot}}(D, D').$$

Proof. ... □

6.2 Persistence images

Definition 6.2.1 (Persistence surface)**.** Let D be a persistence diagram. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(x, y) = (x, y - x)$. Fix a nonnegative weighting function $f\colon \mathbb{R}^2 \to R$ that is zero along the horizontal axis, continuous and picewise differentiable. Fix a differentiable probability distribution $\phi_u\colon \mathbb{R}^2 \to \mathbb{R}$, with mean $u \in \mathbb{R}^2$. The **persistence surface** associated to D, by f and ϕ_u is a function $\rho_D\colon \mathbb{R}^2 \to \mathbb{R}$ defined as

$$\rho_D(z) \coloneqq \sum_{u \in T(D)} f(u)\phi_u(z).$$

Definition 6.2.2 (Persistence image)**.** Let D be a persistence diagram with an associated persistence surface ρ_D. The **persistence image** of D by ρ_D is the collection ρ of **pixels**

$$I(\rho_D)_p \coloneqq \iint_p \rho_B dydx.$$

Lemma 6.2.3 (Lemma 3, [1])**.** *Let $u, v \in \mathbb{R}^2$. The following inequality asserts.*

$$\|f(u)\phi_u - f(v)\phi_v\|_\infty \leq (\|f\|_\infty |\nabla\phi| + \|\phi\|_\infty |\nabla f|)\|u - v\|_2.$$

Proof. ... □

Persistence surfaces are stable with respect to the 1-Wasserstein distance.

Theorem 6.2.4 (Theorem 4, [1])**.** *Let D, D' be two persistent diagrams and $\rho_D, \rho_{D'}$ two persistence surfaces associated to each diagram respectively. Then*

$$\|\rho_B - \rho_{B'}\|_\infty \leq \sqrt{10}(\|f\|_\infty |\nabla\phi| + \|\phi\|_\infty |\nabla f|)\omega_1(D, D').$$

Proof. ... □

Persistence images are stable with respect to the 1-Wasserstein distance.

Theorem 6.2.5 (Theorem 5, [1])**.** *Let A be be the maximum area of any pixel in the image, A' the total area of the image, and n the number of pixels in the image. Then*

$$\begin{aligned}
\|I(\rho_B) - I(\rho_{B'})\|_\infty &\leq \sqrt{10}A(\|f\|_\infty |\nabla\phi| + \|\phi\|_\infty |\nabla f|)\omega_1(D, D'),\\
\|I(\rho_B) - I(\rho_{B'})\|_1 &\leq \sqrt{10}A'(\|f\|_\infty |\nabla\phi| + \|\phi\|_\infty |\nabla f|)\omega_1(D, D'),\\
\|I(\rho_B) - I(\rho_{B'})\|_2 &\leq \sqrt{10n}A(\|f\|_\infty |\nabla\phi| + \|\phi\|_\infty |\nabla f|)\omega_1(D, D').
\end{aligned}$$

Proof. ... □

6.3 Euler curves

Definition 6.3.1 (Euler characteristic)**.** Let K be a simplicial complex, and let K^p be its p-skeleton. The **Euler characteristic** of K is the alternating sum of the number of cells in its dimension

$$\chi(K) \coloneqq \sum_d (-1)^d \#(K^d).$$

Definition 6.3.2. Let K be a simplicial complex. Let $f \colon K \to \mathbb{R}$ be a filtration function. The **Euler characteristic curve** is a function that assign an Euler characteristic χ for each filtration level $t \in \mathbb{R}$.

$$\mathrm{ECC}(K, t) \coloneqq \chi(K_t),$$

where $K_t f^{-1}(-\infty, t]$.

Proposition 6.3.3 (Proposition 2, [6])**.** *Let X, Y be two filtered cell complexes and let $D(X), D(Y)$ be its respective persistence diagrams. Then,*

$$\| \mathrm{ECC}(X, t) - \mathrm{ECC}(Y, t)\|_1 \leq \sum_k 2\omega_1(D(X), D(Y)).$$

Proof. ... □

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