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Final Master Thesis

Structure and Stability Theorems in Topological Data Analysis

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Abstract

This thesis presents a unified treatment of structure and stability theorems in Topological Data Analysis (TDA), focusing on persistent homology. The Structure Theorem establishes that persistence modules decompose into interval modules, enabling representation via barcodes or persistence diagrams. Four stability pillars are rigorously developed: Interleaving Stability proves an isometry between the algebraic interleaving distance for persistence modules and the combinatorial bottleneck distance for barcodes; Hausdorff Stability bounds the bottleneck distance between persistence diagrams of functions by their L^∞-distance; Gromov-Hausdorff Stability extends this to metric spaces, linking the bottleneck distance of Vietoris-Rips or Čech filtrations to the Gromov-Hausdorff distance between spaces; Vectorization Stability guarantees robustness for practical summaries like persistence landscapes, images, and Euler curves, enabling integration with machine learning. By synthesizing tools from algebraic topology, category theory, and metric geometry, this work fortifies the mathematical foundations of TDA and facilitates reliable applications in scientific domains and shape quantification in high-dimensional data.

Resumen

Esta tesis presenta un tratamiento unificado de teoremas de estructura y estabilidad en el Análisis Topológico de Datos (TDA), centrándose en la homología persistente. El Teorema de Estructura establece que los módulos de persistencia se descomponen en módulos de intervalo, permitiendo su representación mediante códigos de barras o diagramas de persistencia. Se desarrollan rigurosamente cuatro pilares de estabilidad: la Estabilidad de Entrelazado demuestra una isometría entre la distancia de entrelazado algebraico para módulos de persistencia y la distancia de cuello de botella para códigos de barras; la Estabilidad de Hausdorff acota la distancia bottleneck entre diagramas de persistencia de funciones por su norma L^∞; la Estabilidad de Gromov-Hausdorff extiende esto a espacios métricos, vinculando la distancia bottleneck de filtraciones de Vietoris-Rips o Čech con la distancia de Gromov-Hausdorff entre espacios; la Estabilidad de Vectorizaciones garantiza robustez para resúmenes prácticos como paisajes de persistencia, imágenes de persistencia y curvas de Euler, facilitando su integración con aprendizaje automático. Al sintetizar herramientas de topología algebraica, teoría de categorías y geometría métrica, este trabajo fortalece los fundamentos matemáticos del TDA y posibilita aplicaciones confiables en dominios científicos y la cuantificación de formas en datos de alta dimensión.

Key words

Topological Data Analysis, Persistent homology, Topological summary, Stability.

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Introduction

Topological Data Analysis (TDA) is an emerging field at the intersection of mathematics, statistics, and computer science that leverages tools from algebraic topology to extract robust, interpretable features from complex, high-dimensional data. Traditional data analysis techniques often struggle to capture the intrinsic shape and connectivity of modern datasets—ranging from biological networks and sensor arrays to financial time series and high-dimensional point clouds. TDA addresses this gap by focusing on the topological invariants of data, which provide a mathematically rigorous description of its global structure. Central to this approach is the idea that the "shape" of data—characterized by holes, voids, and connected components—reveals critical insights impervious to noise and geometric distortions.

The cornerstone of TDA is persistent homology, a multiscale extension of classical homology. While homology groups capture topological features (e.g., connected components, loops, cavities) of a space at a fixed scale, persistent homology tracks the birth, persistence, and death of these features across a continuum of scales. This process is formalized through filtrations: nested sequences of topological spaces (e.g., simplicial complexes) built from data, parameterized by a scale parameter (e.g., distance thresholds). As the scale evolves, homology groups induce a structure known as a persistence module—an algebraic object encoding how topological features evolve. The Structure Theorem (Chapter 2) classifies these modules, decomposing them into elementary interval modules. This decomposition underpins two powerful visual and analytical summaries: barcodes and persistence diagrams.

A fundamental challenge in TDA is ensuring that topological summaries are stable: small perturbations in input data should induce only controlled changes in the output. Stability theorems form the bedrock of reliable applications, guaranteeing that TDA-based inferences are robust to noise, discretization artifacts, and measurement errors. This thesis provides a unified treatment of four pillars of TDA stability:

- Interleaving Stability (Chapter 3): Establishes an isometry between the algebraic interleaving distance for persistence modules and the combinatorial bottleneck distance for barcodes.
- Hausdorff Stability (Chapter 4): Bounds the bottleneck distance between persistence diagrams of functions by their L^∞-distance, crucial for functional data.

- Gromov-Hausdorff Stability (Chapter 5): Extends stability to metric spaces, linking the bottleneck distance of Vietoris–Rips or Čech complexes with the Gromov–Hausdorff distance between spaces.
- Vectorization Stability (Chapter 6): Proves robustness for practical summaries like persistence landscapes, images, and Euler curves, enabling integration with machine learning workflows (following the results of [2], [1], [7] and [15]).

This work synthesizes foundational results from algebraic topology, category theory, and metric geometry to present a cohesive theoretical framework for TDA. By elucidating the stability properties of topological summaries, we aim to fortify the mathematical foundations of data analysis and empower applications across scientific domains—from understanding protein folding to quantifying shape in high-dimensional datasets. The subsequent chapters develop these ideas rigorously, beginning with the preliminaries of persistent homology and culminating in modern vectorization techniques.

Chapter 1

Preliminaries

Topological Data Analysis is a wide topic, where results from all Mathematical branches are put together in order to compute topological invariants among real world data. A basic understanding of algebra, analysis, and geometry is required to approach the theoretical results that underpin the core theory, which establishes the necessary structures and their stability. Therefore, to establish a common foundation and unify the diverse notations and procedures found in the literature, this chapter introduces the key definitions and basic results needed for the rest of the thesis.

Section 1.1 makes a brief recall of simplicial homology in order to define persistent homology modules. Section 1.2 then introduces persistence modules, the algebraic tool used to study the structure of persistence homology groups. Section 1.3 introduces a tool to summarize the data from the persistence homology groups, the barcodes. With this summary it is possible to compute how homologically different are two datasets, or how different are two filtrations among the same data. To make this measure, the interleaving and bottleneck distance are introduced. The first one measures distances between persistence modules, while the latter one measures distance between the barcodes of those modules. One key aspect of them, is that they are actually equivalent, as its later proved in Chapter 3.

A visual way of presenting barcodes is through persistence diagrams. Both barcodes and persistence diagrams are equivalent ways of presenting the homological features of data. To give a different perspective about how to measure distances between persistence diagrams, Section 1.4 introduces the Wasserstein distance and redefines the bottleneck distance as a particular case.

Finally, Section 1.5 recalls the Hausdorff distance to measure distance between subsets of metric spaces, and generalices it presenting the Gromov-Hausdorff distance, which allows to measure distances between sets located in different metric spaces. This will be useful to delimit bottleneck distance in order to proof its stability.

Section 1.1 primarily follows [18]. Sections 1.2 and 1.3 are based on the presentation at [16]. Section 1.4 takes the definitions given at [3] and Section 1.5 follows [4].

1.1 Persistent homology

In most Algebraic Topology introductory texts, homology is first presented in the form of homology groups (see [13], [11]). We are going to generalize homology groups to homology R-modules based on the notes by Wang [18]. This makes possible to take coefficients over an arbitrary commutative ring R and will lead us to introduce persistent homology and the persistence modules. Lets star recalling what an R-module is, to be followed by a brief presentation of simplicial homology.

Definition 1.1.1 (R-module, Definition IV.1.1.1 [12])**.** Let R be a commutative ring. An **R-module** is an abelian group $(M, +)$ with an operation $\cdot\colon R \times M \to M$ such that for all $r, s \in R$ and for all $x, y \in M$,

(i) $(rs) \cdot x = r(s \cdot x)$,

(ii) $(r + s) \cdot x = r \cdot x + s \cdot x$,

(iii) $r \cdot (x + y) = r \cdot x + r \cdot y$.

If R has a multiplicative identity 1, then M is said to be a **unitary R-module** and

(iv) $1 \cdot x = x$.

If R is a division ring, that is, a ring with identity where every non cero element is a unit, then a unitary R-module is called a **left R-vector space**. Note that in this case, R is in fact a field. Also note that, when we take $R = (\mathbb{Z}, +, \cdot)$, a $\mathbb{Z}$ module is actually an abelian group.

Definition 1.1.2 (Affinely independent touple)**.** A $(k + 1)$-tuple of points $(x_0, \ldots, x_k)$ in $\mathbb{R}^n$, where each $x_i \in \mathbb{R}^n$ is said to be **affinely independent** if the set of vectors $\{x_j - x_0 \mid 1 \leq j \leq k\}$ is linearly independent.

Definition 1.1.3 (k-simplex)**.** A **k-simplex** is an ordered $(k + 1)$-tuple of affinely independent points $\sigma = (x_0, \ldots, x_k)$. The **vertices** of σ are the points of the set $V = \{x_0, \ldots, x_k\}$. Every simplex σ induces a total ordering $\preccurlyeq_\sigma$ on its set of vertices V, where if $p \leq q$, then $x_p \preccurlyeq_\sigma x_q$ in V. For $m \leq n$, an $(m + 1)$-tuple $(y_0, \ldots, y_m)$, where each $y_j \in V$ and if $j \leq k$, then $y_j \preccurlyeq_\sigma y_k$, is called an **$m$-face** of σ.

Definition 1.1.4 (Abstract simplicial complex)**.** An **abstract simplicial complex** K is a finite collection of simplices such that for every face of a simplex in K is also a simplex in K.

Definition 1.1.5 (Simplicial k-chain)**.** Let R be a commutative ring with additive identity 0 and multiplicative identity 1 and let K be a simplicial complex. A **simplicial k-chain** is a formal sum of k-simplices,

$$\sum_{i=1}^{N} r_i \sigma_i,$$

such that $r_i \in R$ and $\sigma_i \in K$.

Note that that set of simplicial k-chains with formal addition over R form a R-module. Denote that R-module as K_k.

Definition 1.1.6 (Boundary map)**.** Let K be a simplicial complex, and let K_k and K_{k-1} be two R-modules from K. For every k-simplex $\sigma = (x_0, \dots, x_k) \in K$, the **boundary** map $\partial_k \colon K_k \to K_{k-1}$ is defined as

$$\partial_k(\sigma) = \sum_{i=0}^{k} (-1)^i (x_0, \dots, \hat{x}_i, \dots, x_k),$$

where $(x_0, \dots, \hat{x}_i, \dots, x_k)$ denotes the $(k-1)$-face of σ obtained by removing the vertex x_i.

As the boundary map can be extends linearly to arbitrary k-chains, it forms an R-module homomorphism $\partial_k \colon K_k \to K_{k-1}$.

Definition 1.1.7 (Chain complex)**.** A **chain complex** (K, d) is a sequence of R-modules A_k with boundary maps $d_k \colon A_k \to A_{k-1}$, such that the composition $d_{k-1} \circ d_k = 0$ for all $k \leq 0$.

Lemma 1.1.8 (Lemma 2.1, [11])**.** *The collection of R-modules K_k connected by boundary maps ∂_k forms a chain complex (K, ∂).*

Proof. We just need to prove that for every k-simplex $\sigma = (x_0, \dots, x_k)$ for $k \geq 1$, we get $\partial_{k-1} \circ \partial_k(\sigma) = 0$. By linearity of the boundary map, we can just compute

$$\begin{aligned}
\partial_{k-1} \circ \partial_k(\sigma) &= \sum_{i=0}^{k} (-1)^i \partial_{k-1}(x_0, \dots, \hat{x}_i, \dots, x_k) \\
&= \sum_{j<i} (-1)^i (-1)^j (x_0, \dots, \hat{x}_j, \dots, \hat{x}_j, \dots, x_k) \\
&\quad + \sum_{j>i} (-1)^i (-1)^{j-1} (x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k).
\end{aligned}$$

The last two sums vanish as when changing i and j positions in the second, it becomes the negative of the first. $\square$

Corolary 1.1.9. *The boundary maps ∂_k satisfy that for every $k \leq 0$, $\operatorname{im} \partial_{k+1} \subseteq \ker \partial_k$.*

Proof. If there is some k-simplex $\sigma \in \operatorname{im}(\partial)$ then there exists some $(k+1)$-simplex τ such that $\partial_{k+1}\tau = \sigma$. Hence, as $\partial_k \circ \partial_k + 1 = 0$,we have that $\partial_k(\partial_{k+1}\tau) = \partial_k\sigma = 0$ and $\sigma \in \ker(\partial_k)$. □

Definition 1.1.10 (Homology module)**.** Let K be a simplicial complex. The **k-th homology module** of K is the R-module given by

$$H_k(K) := \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}}.$$

Homology can be extended to every topological space, not just those that are triangulable by introducing singular homology (see [13][Chapter 4]). However, in this work, we will restrict ourselves to simplicial homology. This choice is motivated by the nature of the data typically encountered in topological data analysis. The spaces involved are usually "sufficiently nice" and therefore triangulable. Moreover, real-world data intended for computational analysis is finite by nature, and hence, triangulable. Furthermore, simplicial homology of a finite simplicial complex K is computable (see [13][Theorem 11.5]) what makes it a prefect tool for computational studies.

Definition 1.1.11 (Filtration)**.** Let K be a simplicial complex and let T be a complete ordered index set. A **filtration** of K is a nested sequence of subcomplexes F_iK, $i \in T$, such that for every $i \in T$,

$$F_i \subseteq F_{i+1}.$$

We will denote the **inclusion simplicial maps** by $g_i \colon F_i \hookrightarrow F_{i+1}K$.

Filtrations are a natural way of obtaining simplicial complexes from datasets, as it is posible to set the first element of the filtration to be the points in the datasets, and then join in various steps each of the point to form a simplicial complex that connects every point by simplices of different dimensions. Naturally, depending on the procedure we follow we could get quite different filtration for one same dataset.

Example 1.1.12. Figure 1.1 depicts a posible filtration of a simplicial complex K in four steps. It starts with four 0-simplices and one 1-simplex at F_0K and adds components until it has six 0-simplices, six 1-simplices and one 2-simplex at F_3K.

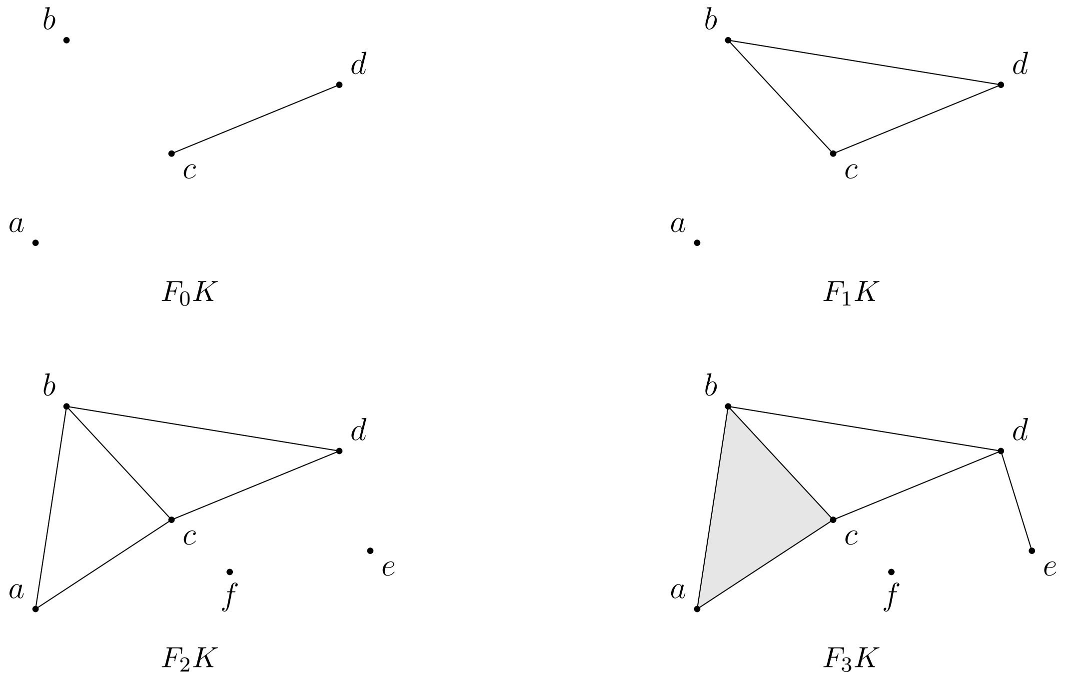


Figure 1.1: Four step filtration of a simplicial complex K.

Considering a filtration of a simplicial complex, we can compute the homology groups at each step of the filtration. Hence, for each $k \geq 0$, there are induced linear maps $H_k g_i \colon H_k(F_i K) \to H_k(F_{i+1}K)$ that fit together into a sequence of vector spaces

$$H_k(F_0K) \xrightarrow{H_k g_0} H_k(F_1K) \xrightarrow{H_k g_1} \cdots \xrightarrow{H_k g_{n-2}} H_k(F_{n-1}K) \xrightarrow{H_k g_{n-1}} H_k(F_nK).$$

As homology is functorial, when we consider the inclusion map formed by the composition of several others, the induced map at homology level also becomes a composition of the corresponding maps. That is, for any pair $i \leq j$ of indices, where $i, j = 1, \ldots, n$, the map induced on homology by the inclusion map $g_{i \leq j} \colon F_i K \hookrightarrow F_j K$, is given by the composition

$$H_k g_{i \leq j} = H_k g_{j-1} \circ H_k g_{j-2} \circ \cdots \circ H_k g_{i+1} \circ H_k g_i.$$

Such maps $H_k g_{i \leq j}$ give really relevant information about our data, as they permit us to relate the k-th homology groups of every subcomplex $F_i K$ that appear in a filtration of K.

Now if X is a triangulable topological space, and $f \colon X \to T$, we can extend the concept of filtration to be attained to X. For clarity, fix $T = \mathbb{R}$. Hence, for each $a \in \mathbb{R}$, the pre image of the interval $(-\infty, a]$ by f is some subspace X_a of X. That is,

$$f^{-1}((-\infty, a]) = X_a.$$

As X is triangulable, we will look for functions that satisfy that each X_a is triangulable too. An therefore we can compute its homology module.

Definition 1.1.13 (Homological critical value)**.** Let X be a topological space and let $f\colon X \to \mathbb{R}$. A **homological critical value** of f is a number $a \in \mathbb{R}$ such that there exists $k \in \mathbb{Z}$ such that for all $\varepsilon > 0$, the morphism $H_k(f^{-1}(-\infty, a-\varepsilon]) \to H_k(f_{-1}(-\infty, a+\varepsilon])$ is not an isomorphism.

Definition 1.1.14 (Tame function)**.** A function $f\colon X \to \mathbb{R}$ is said to be **tame** if it has a finite number of homological critical values, and for all $z \in \mathbb{Z}$, and for all $a \in \mathbb{R}$, $\dim F_a < \infty$.

Example 1.1.15 (Piecewise linear functions)**.** The first example of tame functions, and the one we will most make use of, are piecewise linear functions.

Proof. To prove that piecewise linear functions are tame, we need to verify the two conditions in Definition 1.1.14:

First we will check that piecewise linear functions have a finite number of homological critical values. A piecewise linear function $f\colon X \to \mathbb{R}$ is defined by its values on the vertices of a simplicial complex and is linearly interpolated on the simplices. The function f can only change the homology of its sublevel sets at the values it takes on the vertices. Since a simplicial complex has finitely many vertices, f has finitely many such values. These are the only candidates for homological critical values. Thus, f has finitely many homological critical values.

Now, we need to check that every sublevel sets has finite-dimensional homology. For any $a \in \mathbb{R}$, the sublevel set $f^{-1}(-\infty, a]$ is a subcomplex of the simplicial complex X (or a subset thereof). Because f is linear on each simplex, $f^{-1}(-\infty, a]$ consists of all simplices where $f \leq a$ on all vertices. The homology groups of a finite simplicial complex are finitely generated (and thus finite-dimensional when using field coefficients). Hence, $\dim H_k(f^{-1}(-\infty, a]) < \infty$ for all $k \in \mathbb{Z}$. □

Example 1.1.16 (Morse functions)**.** A real function defined over a differential manifold, $f\colon M \to \mathbb{R}$ it is said to be a **Morse function** if it has no degenerate analytical critical points. Recall that analytical critical points are those were the differential of f vanishes. A critical point $p \in \mathbb{R}$ is degenerate if the Hessian matrix determinant vanish.

By how Morse functions are defined, if they are taken over a compact manifold, then Morse functions are tame.

Example 1.1.17 (Non-tame function)**.** Of course, not every function is tame. A classic counterexample comes from the topologist's sine curve $f:[0,1]\in\mathbb{R}$ defined as

$$f(x)=\begin{cases}\sin(\frac{1}{x}) & \text{if } x\neq 0,\\ 0 & \text{if } x=0,\end{cases}$$

Note that the interval $[0,1]$ is isomorphic to a simplicial complex formed by one 1-simplex and two 0-simplices. As x tends to 0, $\sin(\frac{1}{x})$ oscillates between 1 and -1. Hence, each local extremum a of $\sin(\frac{1}{x})$ introduces a change in the topology of the sublevel sets $f^{-1}(-\infty,a]$, creating infinitely many homological critical values.

Definition 1.1.18 (Persistent homology module)**.** Let X be a triangulable topological space and let $f\colon X\to\mathbb{R}$. For every fixed $k\in\mathbb{Z}$, we denote

$$F_x \coloneqq H_k(f^{-1}(-\infty,x]),$$

and for every $x\leq y$, we denote the induced inclusion map from X_x to X_y as

$$f_x^y\colon F_x\to F_y.$$

The **persistent homology module** associated to x and y is the subspace of X_y given by

$$F_x^y \coloneqq \operatorname{im} f_x^y.$$

The **persistent Betti numbers** are defined as

$$\beta_x^y \coloneqq \dim(F_x^y).$$

Example 1.1.19. We can compute persistent homology modules at each step of the filtration of Example 1.1.12. For simplicity, We will fix the R-module homology, taking $R=\mathbb{Z}$, to compute classic group homology. Also note that for every $k\neq 0,1$, homology groups will be the trivial ones, as there are no posible chain complexes of dimensions different from $0,1$ or 2. Recall that 0-dimensional homology measures the number of path components of a space, and that 1-dimensional homology counts the number of 1-dimensional "holes" (see [15][Section 3]).

Hence for each step of the filtration F_iK we have:

$$\begin{aligned} H_0(F_0K)&=\mathbb{Z}^3, & H_1(F_0K)&=\{1\},\\ H_0(F_1K)&=\mathbb{Z}^2, & H_1(F_1K)&=\mathbb{Z},\\ H_0(F_2K)&=\mathbb{Z}^3, & H_1(F_2K)&=\mathbb{Z}^2,\\ H_0(F_3K)&=\mathbb{Z}^2, & H_1(F_3K)&=\mathbb{Z}.\end{aligned}$$

Note that as this particular filtration is discrete, and there is not any isomorphism in each of the changes, every $i=0,1,2,3$ is a critical point both for $k=1$ and for $k=2$.

1.2 Persistence modules and interleaving distance

To enable a more rigorous and structured study, we introduce the category of persistence modules along with their morphisms. This categorical framework provides the foundation for analyzing the stability and structure of persistent homology.

Definition 1.2.1 (Persistence module)**.** Let $\mathbb{F}$ be a field and let T be a totally ordered set. Let $V = \{V_t\}_{t\in T}$ be a collection of F-vector spaces. A T-indexed **persistence module** is a pair (V, π) such that $\pi = \{\pi_{s\leq t}\}$ is a collection of linear maps $\pi_{s\leq t}\colon V_s \to V_t$ that verifies that for all $r, s, t \in T$,

$$\pi_{r\leq s} \circ \pi_{s\leq t} = \pi_{r\leq t}.$$

Definition 1.2.2 (Morphism between persistence modules)**.** Let T be a totally ordered set. Let $(V, \pi), (W, \theta)$ be two persistence modules. A **morphism** between persistence modules $p\colon (V, \pi) \to (W, \theta)$ is a family of linear maps $p_t\colon V_t \to W_t$ such that for all $s \leq t$ the following diagram commutes:

$$\begin{tikzcd} V_s \arrow[r, "\pi_{s\leq t}"] \arrow[d, "p_s"'] & V_t \arrow[d, "p_t"] \\ W_s \arrow[r, "\theta_{s\leq t}"'] & W_t \end{tikzcd}$$

If a morphism i verifies that for all $t \in T$, $i_t\colon V_t \to V_t$ is the identity, then i is the **identity morphism**. If there exists two morphisms $p\colon (V, \pi) \to (W, \theta)$ and $q\colon (W, \theta) \to (V, \pi)$ such that the compositions $p \circ q$ and $q \circ p$ are both the identity morphism, then p and q are **isomorphisms** of persistence modules. In this case, (V, π) and (W, θ) are said to be **isomorphic** persistence modules.

For now on, to simplify notation, we will limit our totally ordered set to be the real numbers, $T = \mathbb{R}$. Also, when there is no possible confusion, we might denote the persistence module (V, π) by just is collection of vector spaces V.

This section defines a distance between persistence modules that will allow us to differentiate how far are two modules from each other.

Definition 1.2.3 (Persistence module shift)**.** Let (V, π) be a persistence module and let $\delta \in \mathbb{R}$. The **δ-shift** of (V, π) is the persistence module (V_δ, π_δ) defined by taking

$$(V_\delta)_t \coloneqq V_{t+\delta}, \qquad (\pi_\delta)_{s\leq t} \coloneqq \pi_{s+\delta\leq t+\delta}.$$

Proposition 1.2.4 (Exercise 1.2.3, [16])**.** *Let $\delta > 0$. Let $(V, \pi), (V_\delta, \pi_\delta)$ be a persistence module and its shift. The map $\phi_\delta\colon (V, \pi) \to (V_\delta, \pi_\delta)$, defined as*

$$\phi_\delta(V_t) \coloneqq \pi_{t\leq t+\delta}(V_t) = V_{t+\delta},$$

is a persistence module morphism.

Proof. As $\delta > 0$, then $t \leq t + \delta$. Hence

$$\phi_\delta \circ \pi_{t \leq t+\delta}(V_t) = \phi_\delta(V_{t+\delta}) = V_{t+\delta+\delta} = V_{t+2\delta},$$
$$\pi_{t+\delta \leq t+2\delta} \circ \phi_\delta(V_t) = \pi_{t+\delta \leq t+2\delta}(V_{t+\delta}) = V_{t+2\delta}.$$

□

Definition 1.2.5 (Shift morphism)**.** The persistence module morphism ϕ_δ defined as in Proposition 1.2.4 is named **δ-shift morphism**.

Definition 1.2.6 (δ-interleaved modules)**.** Let $(V, \pi), (W, \theta)$ be two persistence modules and let $\delta > 0$. V and W are **δ-interleaved** if there exists two persistence module morphisms $\phi \colon V \to W_\delta$ and $\psi \colon W \to V_\delta$ such that the following diagrams commute:

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[rr, bend right, "\pi_{2\delta}"'] & W_\delta \arrow[r, "\psi_\delta"] & V_{2\delta} \end{tikzcd}, \qquad \begin{tikzcd} W \arrow[r, "\psi"] \arrow[rr, bend right, "\theta_{2\delta}"'] & V_\delta \arrow[r, "\phi_\delta"] & W_{2\delta} \end{tikzcd}.$$

Persistence modules are a vast abstract algebraic tool. In order to make it more manageable, we give it some more structure, restricting the dimension of the vector spaces, Also, we limit as the amount of different up to isomorphism vector spaces there are. Note that this is analogous to the tame functions we used in order to define persistent homology.

Definition 1.2.7 (Tame persistence module)**.** A a persistence module (V, π) over $\mathbb{R}$ is **tame** if

(i) For all $t \geq 0$, $\dim(V_t)$ is finite.

(ii) For any $\varepsilon > 0$, there exists a finite subset $K \subset \mathbb{R}$ such that for all $t \in \mathbb{R} \setminus K$, the map $\pi_{t-\varepsilon \leq t+\varepsilon} \colon V_{t-\varepsilon} \to V_{t+\varepsilon}$ is not an isomorphism.

Definition 1.2.8 (Interleaving distance)**.** Let (V, π) and (W, θ) be two tame persistence modules. The **interleaving distance** between them is defined as

$$d_{\text{int}}(V, W) := \inf\{\delta > 0 \mid V \text{ and } W \text{ are } \delta\text{-interleaved}\}.$$

Tameness property ensures that the persistence modules have manageable and well-behaved structures, which are essential for the infimum in the definition of the interleaving distance to exist and be meaningful. If there was some infinite dimensional vector space V_t or W_t, they could lead to some pathological behavior, making it difficult to define or compute distances meaningfully. On the other hand, if there were infinite many different isomorphisms, then the infimum in the definition of the interleaving distance might not exist or could be zero for modules that are not “close” in any intuitive sense.

Proposition 1.2.9. *The interleaving distance between two tame persistence modules is actually a distance.*

Proof. We will not make this proof directly as, in Chapter 3, we will prove that the interleaving distance between two persistence modules is the same as the bottleneck distance between their respective barcodes. As each module is uniquely described my their barcodes as we will see in Chapter 2, and bottleneck is actually a distance as we will see in Proposition 1.4.12, this prove will be completed. □

Definition 1.2.10 (Interval module)**.** Let $I = (a, b]$ be an interval with $b \leq \infty$ and let $\mathbb{F}$ be a field. An **interval module** $\mathbb{F}(I)$ is a persistence module defined as

$$\mathbb{F}(I)_t := \begin{cases} \mathbb{F} \text{ if } t \in I, \\ 0 \text{ else,} \end{cases} \qquad \pi_{s\leq t} = \begin{cases} \mathrm{Id} \text{ if } t \in I, \\ 0 \text{ else.} \end{cases}$$

Definition 1.2.11 (Direct sum of persistance modules)**.** Let (V, π) and (V', π') be two persistence modules. Their **direct sum** (W, θ) is a persistence module where

$$W_t := V_t \oplus V'_t, \text{ the direct sum of both vector spaces, and}$$
$$\theta_{s\leq t} := \pi_{s\leq t} \oplus \pi'_{s\leq t}.$$

1.3 Barcodes and the bottleneck distance

Barcodes are the main representation element that will allow us to resume the information given by the persistence homology groups of a dataset.

Definition 1.3.1 (Barcode)**.** A **barcode** B is a finite multiset of intervals. That is, a collection $\{(I_i, m_i)\}$ of intervals I_i with multiplicities $m_i \in N$, where each interval I_i is either finite of the form $(a, b]$ or infinite of the form (a, ∞). Each interval I_i is named to be a **bar** of B. The first number, a is named the **birth** of the barcode and is second number is its **death**.

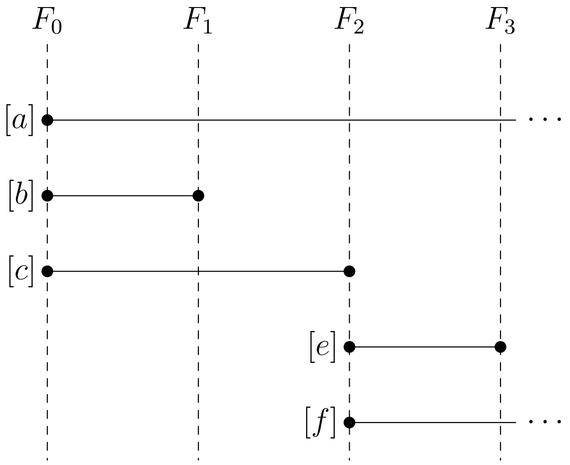
Example 1.3.2 (Barcodes)**.** We can compute the barcodes associated to the filtrations of Example 1.1.12. Figure 1.2 depicts the barcodes associated to the 0-dimensional and 1-dimensional homology of the filtration. The barcode of $H_0(K)$ is the multiset

$$\{(0, \infty], (0, 1], (0, 2], (2, 3], (2, \infty]\},$$

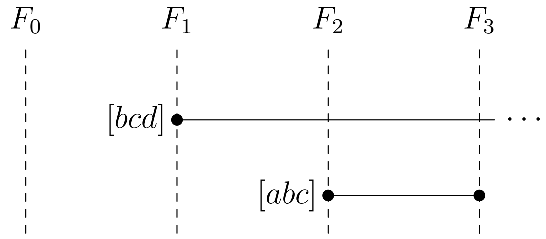
and the barcode of $H_1(K)$ is the multiset

$$\{(1, \infty], (2, 3]\}.$$

Note that in this particular example, the multiplicity of each bar is just one.



(a) Bars of $H_0(K)$.



(b) Bars of $H_1(K)$.

Figure 1.2: Barcodes associated to the filtration of Example 1.1.12.

Chapter 2 proves the Structure Theorem of persistence modules, that asserts that it is possible to express any persistence module as a direct sum of the interval modules given by the bars of a barcode. This is a key result in our purpose of introduce measures between the homological features of datasets.

Given an interval $I = (a, b]$, and some $\delta \geq 0$, we will denote

$$I^\delta := (a - \delta, b + \delta].$$

Definition 1.3.3 (Multiset matching)**.** Let X and Y be two multi-sets and let $X' \subseteq X$, $Y' \subseteq Y$. A **matching** between them is a bijection $\mu\colon X' \to Y'$. The elements in X' and Y' are said to be **matched** by μ.

Note that $X' = \mathrm{coim}(\mu)$ and $Y' = \mathrm{im}(\mu)$. Also note that as X and Y are multisets, it might happen that one same element appears several times in one of the multisets, and that some, but not all of its copies are matched to some element in the other multiset.

Definition 1.3.4 (δ-matching barcodes)**.** A **δ-matching** between two barcodes B and C is a multiset matching that verifies

1. $B_{2\delta} \subseteq \mathrm{coim}(\mu)$,

2. $C_{2\delta} \subseteq \mathrm{im}(\mu)$,

3. If $\mu(I) = J$, then $I \subseteq J^\delta$ and $J \subseteq I^\delta$.

There are various ways of defining the bottleneck distance, all of them equivalent to one an other. We first give the natural definition that comes up following the use of δ-matchings.

Definition 1.3.5 (Bottleneck distance)**.** The **bottleneck distance** between two barcodes B and C is the infimum over all $\delta \in \mathbb{R}$ such that there exists a δ-matching between B and C.

The fact that bottleneck distance defines a proper metric space within the set of barcodes, will be proved in Proposition 1.4.12, after we generalice it as a case of the Wasserstein distance.

1.4 Persistence diagrams and the Wasserstein distance

Barcodes can be seen as multisets of pair of points (a, b), where $b > a$. Hence we could treat each interval $(a, b]$ in a barcode as a point (a, b) in the Euclidean plane. We will denote the strict upper triangular region of the Euclidean plane as

$$\mathbb{R}^2_< := \{(x, y) \in \mathbb{R}^2 : x < y\},$$

and the diagonal of the plane as

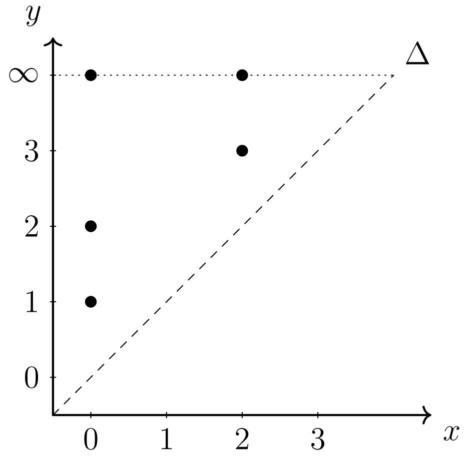
$$\Delta := \{(x, y) \in \mathbb{R}^2 : x = y\}.$$

Definition 1.4.1 (Persistence diagram)**.** Let I be a countable multiset. A *persistence diagram* is a function $D : I \to \mathbb{R}^2_<$.

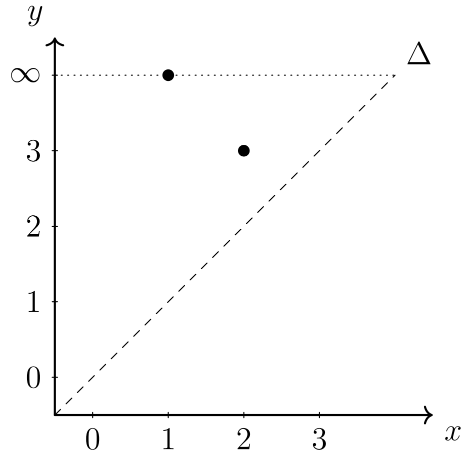
Persistence diagrams are just a way of presenting barcodes as subsets of the plane, so we could introduce more distances between them. Note that while formally a persistence diagram is a function, usually it is used to also denote the set of points and their multiplicities in the plane. Note that when we do so, it is equivalent to name a barcode than a persistence diagram. Hence, the bottleneck distance as defined in Definition **??** can be also used to measure distances between persistence diagrams.

Example 1.4.2 (Persistence diagrams)**.** In order to express an arbitrary barcode B as a persistence diagram, we just take the bars in B as if they were coordinates. To deal with the bars of the form (a, ∞) we need to fix some point grater than every other finite death of the barcode. Picking this upper bound is possible as long as we are working with a barcode given by a tame filtration function or a tame persistence module, thanks to the finite number of change of isomorphism type.

In Figure 1.3 we can see the persistence diagram representation of the barcodes of Example 1.3.2.



(a) Persistence diagram of $H_0(K)$.



(b) Persistence diagram of $H_1(K)$.

Figure 1.3: Persistence diagrams associated to the filtration of Example 1.1.12.

While definition **??** is purely algebraic, this section gives an alternative analytical definition for the bottleneck distance making an adaptation of the classic Wasserstein distance, used originally to measure distances between probability measures (see [10]).

Instead of probability measures, now we are actually dealing with countable sets of points in $\mathbb{R}$. We will make use of the l^p norm at countable spaces to measure the distance between matched pairs and the distance between unmatched pairs and the diagonal Δ. For a more detailed explanation of Lebesgue measures see [17][Definition 3.7]. This norm is named after Pafnuty Chebyshev.

Definition 1.4.3 (Chebyshev distance)**.** Let $a, b \in \mathbb{R}^2$ with $a = (a_x, a_y)$ and $b = (b_x, b_y)$. The *Chebyshev distance* is defined as

$$d_\infty(a, b) := ||a - b||_\infty := \max\{|a_x - b_x|, |a_y - b_y|\}.$$

To define our adapted Wasserstein distance we need to check how Chebyshev distance measures distances between points of $\mathbb{R}^2_<$ and Δ.

Proposition 1.4.4. *If $a = (a_x, a_y) \in \mathbb{R}^2_<$, then $d_\infty(a, \Delta) = \inf_{t\in\Delta} d_\infty(a, t) = \frac{a_y - a_x}{2}$.*

Proof. The t which minimizes the distance is the midpoint of a_x and a_y, that is $t =$

$\left(\frac{a_x+a_y}{2}, \frac{a_x+a_y}{2}\right)$. Then,

$$\left|a_x - \frac{a_x + a_y}{2}\right| = \left|\frac{a_x - a_y}{2}\right| = \left|\frac{a_y - a_x}{2}\right| = \left|a_y - \frac{a_x + a_y}{2}\right|,$$

and as $a_y > a_x$ we have

$$d_\infty(a, t) = \left|\frac{a_y - a_x}{2}\right| = \frac{a_y - a_x}{2}.$$

□

We now verify that the upper triangular region of the Euclidean plane with the Chebyshev distance adapted to measure distances in Δ is a metric space.

Proposition 1.4.5. *The function d_∞ is a distance in $\mathbb{R}^2_<$ with the diagonal Δ.*

Proof. For points $a, b \in \mathbb{R}^2_< \subset \mathbb{R}^2$, d_∞ is a distance as usual Lebesgue norms are well defined. See [17][Chapter 3]. To verify that the metric requirements are fulfilled for $d_\infty(a, \Delta)$, it is enough to consider $t = \frac{a_y - a_x}{2}$ as in Proposition 1.4.4. □

Definition 1.4.6 (Partial matching)**.** Let $D_1 : I_1 \to \mathbb{R}^2_<$ and $D_2 : I_2 \to \mathbb{R}^2_<$ be persistence diagrams. A *partial matching* between D_1 and D_2 is the triple (I'_1, I'_2, f) such that $f : I'_1 \to I'_2$ is a bijection with $I'_1 \subseteq I_1$ and $I'_2 \subseteq I_2$.

Definition 1.4.7 (p-cost)**.** Let $D_1 : I_1 \to \mathbb{R}^2_<$ and $D_2 : I_2 \to \mathbb{R}^2_<$ be persistence diagrams. Let (I'_1, I'_2, f) be a partial matching between them. If $p < \infty$, the *p-cost of f* is defined as

$$\text{cost}_p(f) := (\sum_{i \in I'_1} d_\infty(D_1(i), D_2(f(i)))^p + \sum_{i \in I_1 \setminus I'_1} d_\infty(D_1(i), \Delta)^p + \sum_{i \in I_2 \setminus I'_2} d_\infty(D_2(i), \Delta)^p)^{\frac{1}{p}}.$$

For $p = \infty$, the *∞-cost of f* is defined as

$$\text{cost}_\infty(f) := \max\{\sup_{i \in I'_1} d_\infty(D_1(i), D_2(f_i)), \sup_{i \in I_1 \setminus I'_1} d_\infty(D_1(i), \Delta), \sup_{i \in I_2 \setminus I'_2} d_\infty(D_2(i), \Delta)\}.$$

Definition 1.4.8 (p-Wasserstein distance)**.** Let D_1, D_2 be persistence diagrams. Let $1 \leq p \leq \infty$. Define

$$\tilde{\omega}_p(D_1, D_2) = \inf\{\text{cost}_p(f) : f \text{ is a partial matching between } D_1 \text{ and } D_2\}.$$

Let $\emptyset$ denote the unique persistence diagram with empty indexing set. Let (Dgm_p, ω_p) be the space of persistence diagrams D that satisfy $\tilde{\omega}_p(D, \emptyset) < \infty$ modulo the equivalence relation $D_1 \sim D_2$ if $\tilde{\omega}_p(D_1, D_2) = 0$. The metric ω_p is called the *p-Wasserstein distance*.

Definition 1.4.9 (Bottleneck distance)**.** In the conditions of Definition 1.4.8, if $p = \infty$, the metric ω_∞ is called the *bottleneck distance*.

Proposition 1.4.10. *There is only one matching between $D : I \to \mathbb{R}^2_<$ and $\emptyset$. Hence, if $p \leq \infty$,*

$$\tilde{\omega}_p(D, \emptyset) = \left(\sum_{i \in I} d_\infty(D(i), \Delta)^p \right)^{\frac{1}{p}},$$

and, if $p = \infty$,

$$\tilde{\omega}_\infty(D, \emptyset) = \sup_{i \in I} d_\infty(D(i), \Delta)$$

Proof. Let $I' \subseteq D$. If f is a partial matching between D and $\emptyset$, means that $f(I') = \emptyset$ is a bijection. That is only posible if $I' = \emptyset$ too. Therefore $I \setminus I' = I \setminus \emptyset = I$ and following Definition 1.4.7 we conclude our proof. □

Proposition 1.4.11. *Let B and C be two barcodes, and let D_1 and D_2 be the corresponding persistence diagrams (each interval $[b, d)$ represented as the point (b, d) in $\mathbb{R}^2$). Then the bottleneck distance defined via δ-matchings (Definition 1.3.5) is equal to the ∞-Wasserstein distance between D_1 and D_2 (Definition 1.4.9).*

$$d_{\text{bot}}(B, C) = \omega_\infty(D_1, D_2).$$

Proof. We will show the equality by proving the inequalities in both directions.

($\Rightarrow$) Suppose $d_{\text{bot}}(B, C) \leq \delta$. Then $\omega_\infty(D_1, D_2) \leq \delta$. Let $\mu : B \to C$ be a δ-matching between B and C, as in Definition 1.3.5. Then the condition $I \subseteq J^\delta$ and $J \subseteq I^\delta$ implies:

$$|b_1 - b_2| \leq \delta \quad \text{and} \quad |d_1 - d_2| \leq \delta,$$

which means the L^∞ distance between the corresponding points in the persistence diagrams is at most δ:

$$d_\infty((b_1, d_1), (b_2, d_2)) \leq \delta.$$

Construct a partial matching f from D_1 to D_2 by setting $f((b_1, d_1)) = (b_2, d_2)$ whenever $\mu([b_1, d_1)) = [b_2, d_2)$. For unmatched intervals in B or C, observe that the conditions $B_{2\delta} \subseteq \text{coim}(\mu)$ and $C_{2\delta} \subseteq \text{im}(\mu)$ imply that any unmatched interval must have length at most 2δ. Consequently, the corresponding point in the persistence diagram lies within δ of the diagonal Δ, since:

$$d_\infty((b, d), \Delta) = \frac{d - b}{2} \leq \delta.$$

Thus, for this partial matching f, we have

$$\mathrm{cost}_\infty(f) \leq \delta.$$

Taking the infimum over all such f, we obtain

$$\omega_\infty(D_1, D_2) \leq \delta.$$

Since this holds for all δ such that a δ-matching exists, we conclude:

$$\omega_\infty(D_1, D_2) \leq d_{\mathrm{bot}}(B, C).$$

($\Leftarrow$) Suppose $\omega_\infty(D_1, D_2) \leq \delta$. Then $d_B(B, C) \leq \delta$. Let $f : I_1' \to I_2'$ be a partial matching between D_1 and D_2 such that:

$$\mathrm{cost}_\infty(f) \leq \delta.$$

That is,

$$\max \left\{ \sup_{i \in I_1'} d_\infty(D_1(i), D_2(f(i))),\ \sup_{i \in I_1 \setminus I_1'} d_\infty(D_1(i), \Delta),\ \sup_{j \in I_2 \setminus I_2'} d_\infty(D_2(j), \Delta) \right\} \leq \delta.$$

Now define a multiset matching μ between barcodes B and C as follows:

- For each matched pair $(b_1, d_1) \in D_1$, $(b_2, d_2) \in D_2$ with $f((b_1, d_1)) = (b_2, d_2)$, set $\mu([b_1, d_1)) = [b_2, d_2)$.
- For any unmatched interval $[b, d) \in B$ such that $(b, d) \in D_1 \setminus I_1'$, we have $d - b \leq 2\delta$, so $[b, d) \in B_{2\delta}$.
- Similarly, unmatched intervals in C are in $C_{2\delta}$.

Hence, μ defines a δ-matching:

- $B_{2\delta} \subseteq \mathrm{coim}(\mu)$,
- $C_{2\delta} \subseteq \mathrm{im}(\mu)$,
- For any matched intervals, $d_\infty((b_1, d_1), (b_2, d_2)) \leq \delta$ implies

  $$|b_1 - b_2| \leq \delta \quad \text{and} \quad |d_1 - d_2| \leq \delta,$$

  hence,

  $$[b_1, d_1) \subseteq [b_2, d_2)^\delta \quad \text{and} \quad [b_2, d_2) \subseteq [b_1, d_1)^\delta.$$

Therefore, there exists a δ-matching between B and C, and taking infimum over all such δ, we obtain

$$d_{\text{bot}}(B, C) \leq \omega_\infty(D_1, D_2).$$

□

Next proposition will prove that, in indeed, the space of persistence diagrams with the p-Wasserstein distance (Dgm_p, ω_p) is a metric space. Its proof is usually omitted in literature, as it based on the simple fact that d_∞ is a distance. We will give, however, an step by step version here.

Proposition 1.4.12. *ω_p is a distance on the space* (Dgm_p, ω_p).

Proof. Let $D_1, D_2, D_3 \in \text{Dgm}_p$, with $1 \leq p \leq \infty$. First of all, $\omega_p(D_1, D_2) \geq 0$ because $d_\infty \geq 0$. $\omega_p(D_1, D_2) = 0$ if and only if $\tilde{\omega}_p(D_1, D_2) = 0$. Thus, because of the equivalence relationship used to define ω_p, it has to be $D_1 \sim D_2$.

To check symmetry, note that every partial matching f is bijective, therefore f^{-1} is a partial matching. But, for all $i \in I'_1$, exists $j \in I'_2$ such that $f(i) = j$ and

$$d_\infty(D_1(i), D_2(f(i))) = d_\infty(D_2(f(i)), D_1(i)) = d_\infty(D_2(j), D_1(f^{-1}(j))).$$

Then, $\text{cost}_p(f) = \text{cost}_p(f^{-1})$ and we have

$$\begin{aligned}
\omega_p(D_1, D_2) &= \inf\{\text{cost}_p(f) : f \text{ is a partial matching between } D_1 \text{ and } D_2\} \\
&= \inf\{\text{cost}_p(f^{-1}) : f^{-1} \text{ is a partial matching between } D_2 \text{ and } D_1\} \\
&= \omega_p(D_2, D_1).
\end{aligned}$$

Finally, lets prove the triangle inequality. If $f : I'_1 \to I'_2$ is a partial matching between D_1 and D_2 and $g : I'_2 \to I'_3$ is a partial matching between D_2 and D_3, $g \circ f : I'_1 \to I'_3$ is a partial matching between D_1 and D_3 as both f and g are bijective. Computing the cost of the matchings for $p < \infty$, we notice that

$$\begin{aligned}
&\sum_{i \in I'_1} d_\infty(D_1(i), D_2(f(i))) + \sum_{i \in I_1 \setminus I'_1} d_\infty(D_1(i), \Delta) + \sum_{i \in I_2 \setminus I'_2} d_\infty(D_2(i), \Delta) \\
&+ \sum_{i \in I'_2} d_\infty(D_2(i), D_3(g(i))) + \sum_{i \in I_2 \setminus I'_2} d_\infty(D_2(i), \Delta) + \sum_{i \in I_3 \setminus I'_3} d_\infty(D_3(i), \Delta) \\
&\geq \sum_{i \in I'_1} d_\infty(D_1(i), D_3(g \circ f(i))) + \sum_{i \in I_1 \setminus I'_1} d_\infty(D_1(i), \Delta) + \sum_{i \in I_3 \setminus I'_3} d_\infty(D_3(i), \Delta)
\end{aligned}$$

as $d_\infty(D_1(i), D_2(f(i))) + d_\infty(D_2(f(i)), D_2(g(f(i)))) \geq d_\infty(D_1(i), D_3(g \circ f(i)))$ using the triangle inequality of d_∞. Therefore, for all partial matchings f and g as described, we have $\text{cost}_p(f) + \text{cost}_p(g) \geq \text{cost}_p(g \circ f)$. Using the same reasoning, por $p = \infty$ we also obtain $\text{cost}_\infty(f) + \text{cost}_\infty(g) \geq \text{cost}_\infty(g \circ f)$. Hence, we have verified that

$$\omega_p(D_1, D_2) + \omega_p(D_2, D_3) \geq \omega_p(D_1, D_3).$$

□

Example 1.4.13 (Wasserstein distance)**.** Figure 1.4 depicts a matching between the 1-dimensional persistence diagram of Example 1.4.2 and an other diagram composed by tree points. If the matching were optimal, minimizing its cost, it would be the Wasserstein distance between the diagrams.

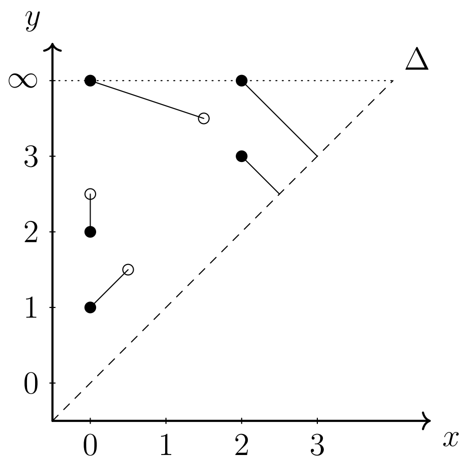


Figure 1.4: Wasserstein distance between two persistence diagrams.

1.5 The Hausdorff and Gromov-Hausdorff distances

The Hausdorff distance is a way of measuring distances of different compact, non-empty sets contained into a same metric space. This concept can be generalized defining a metric which allow us to measure distances between different metric spaces.

Definition 1.5.1 (Hausdorff distance)**.** Let (M, d) be a metric space, and let $A \subseteq M$, $B \subseteq M$ two compact, non-empty subspaces of M. Define the r**-neighborhood** of a set $S \subset M$ as

$$U_r(S) \coloneqq \{x \in S \mid d(x, S) \leq r\}.$$

The **Hausdorff distance** can be defined as

$$d_{\mathrm{H}}(A, B) \coloneqq \inf \{r > 0 \mid A \subset U_r(B) \text{ and } B \subset U_r(A)\}.$$

Definition 1.5.2 (Isometric metric spaces)**.** Let $(X, d_X), (Y, d_Y)$ be metric spaces. X and Y are said to be **isometric** if there exists a bijective map $f : X \to Y$ such that distances are preserved. That is, for all $x_1, x_2 \in X$,

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)).$$

Definition 1.5.3 (Gromov-Hausdorff distance)**.** Let $(X, d_X), (Y, d_Y)$ be metric spaces. The **Gromov-Hausdorff** distance is defined as

$$d_{\mathrm{GH}} \coloneqq \inf \left\{ r > 0 \mid \exists (Z, d_Z) \text{ metric space such that}, \exists X', Y' \subseteq Z, d_{\mathrm{H}}(X', Y') < r \right\},$$

where X', Y' are isometric spaces to X and Y respectively.

Lemma 1.5.4 (Proposition 7.3.16, [4])**.** *The Gromov-Hausdorff distance satisfies the triangle inequality. That is, for any metric spaces X_1, X_2, X_3 it is verified that*

$$d_{\mathrm{GH}}(X_1, X_3) \leq d_{\mathrm{GH}}(X_1, X_2) + d_{\mathrm{GH}}(X_2, X_3).$$

Proof. Let d_{12} be a metric over $X_1 \cup X_2$ and let d_{23} be a metric over $X_2 \cup X_3$. Over $X_1 \cap X_3$, define

$$d_{13} \coloneqq \begin{cases} d_{X_1}(x_1, x_3) \text{ if } x_1, x_3 \in X_1, \\ d_{X_2}(x_1, x_3) \text{ if } x_1, x_3 \in X_3, \\ \inf_{x_2 \in X_2}\{d_{12}(x_1, x_2) + d_{23}(x_2, x_3)\} \text{ if } x_1 \in X_1, x_3 \in X_3. \end{cases}$$

For the first two cases we clearly have a metric. For the third one observe that taking $x_1 \in X_1, x_3 \in X_3$ and some $x \in X_1$ we have

$$\begin{aligned} d_{13}(x_1, x) + d_{13}(x, x_3) &= d_{X_1}(x_1, x) + \inf_{x_2 \in X_2}\{d_{12}(x, x_2) + d_{23}(x_2, x_3)\} \\ &= \inf_{x_2 \in X_2}\{d_{12}(x_1, x) + d_{12}(x, x_2) + d_{23}(x_2, x_3)\} \\ &\geq \inf_{x_2 \in X_2}\{d_{12}(x_1, x_2) + d_{23}(x_2, x_3)\} \\ &= d_{13}(x_1, x_3). \end{aligned}$$

This implies, taking the corresponding metric d_{ij} where $i, j = 1, 2, 3$, that

$$d_{\mathrm{H}}(X_1, X_3) \leq d_{\mathrm{H}}(X_1, X_2) + d_{\mathrm{H}}(X_2, X_3),$$

and, taking the infimum over the metrics d_{12} and d_{23} we have

$$d_{\mathrm{GH}}(X_1, X_3) \leq d_{\mathrm{GH}}(X_1, X_2) + d_{\mathrm{GH}}(X_2, X_3).$$

□

To check that Gromov-Hausdorff distance is actually a distance we first give a useful characterization in 1.5.7. It is expressed in terms of correspondance distortions.

Definition 1.5.5 (Correspondance between sets)**.** Given two sets X and Y, a **correspondance** between them is a set $R \subseteq X \times Y$ verifying that for every $x \in X$, there exists at least one $y \in Y$ such that $(x, y) \in R$ and, for every $y \in Y$, there exists an $x \in X$ such that $(x, y) \in R$.

Definition 1.5.6 (Distortion of a correspondance)**.** Let $(X, d_X), (Y, d_Y)$ be two metric spaces, and let R be a correspondance between them. The **distortion** of R is defined as

$$\operatorname{dis}(R) \coloneqq \sup \{|d_X(x, x') - d_Y(y, y')| \colon (x, y), (x', y') \in R\} .$$

Proposition 1.5.7 (Theorem 7.3.25, [4])**.** *Let $(X, d_X), (Y, d_Y)$ be two metric spaces. The Gromov-Hausdorff distance between them can be characterized as*

$$d_{\mathrm{GH}}((X, d_X), (Y, d_Y)) = \frac{1}{2} \inf_R \operatorname{dis}(R).$$

Proof. Given $r \geq d_{\mathrm{GH}}(X, Y)$, for some metric space (Z, d_Z), we can take $X', Y' \subseteq Z$ such that X' and Y' are isometric embeddings of X and Y respectively and $d_{\mathrm{H}}(X', Y') < r$ in Z. Thus, we can see every element of X and Y as elements od Z trough some isometry. Therefore, we can define the correspondance

$$R \coloneqq \{(x, y) \in X \times Y \colon d_Z(x, y) < r\}.$$

The set R is actually a correspondance because the fact that $d_{\mathrm{H}}(X', Y') < r$ implies that for every $x \in X$ and every $y \in Y$, $d_Z(x, y) < r$, so every x and every y have some correspondance. Now, let $(x, y), (x', y') \in R$. Using the triangle inequality of Z we have

$$\begin{aligned}
\operatorname{dis}(R) &\leq |d_X(x, x') - d_Y(y - y')| \\
&= |d_Z(x, x') - d_Z(y - y')| \\
&\leq |d_Z(x, y) + d_Z(y, x') - d_Z(y - y')| \\
&\leq d_Z(x, y) + d_Z(x', y) + d_Z(y - y') \\
&\leq d_Z(x, y) + d_Z(x', y') \leq 2r.
\end{aligned}$$

This shows

$$2 d_{\mathrm{GH}}(X, Y) \geq \inf_R \operatorname{dis}(R).$$

To see the reverse inequality, let R be any correspondance, and lets take $\operatorname{dis}(R) = 2r$. Lets construct a metric space (Z, d_Z) formed by the disjoint union of spaces $Z = X \cup Y$.

For every $z_1, z_2 \in Z$, we define d_Z as

$$d_Z(z_1, z_2) := \begin{cases} d_X(z_1, z_2) \text{ if } z_1, z_2 \in X, \\ d_Y(z_1, z_2) \text{ if } z_1, z_2 \in Y, \\ \inf\{d_X(z_1, x') + r + d_Y(z_2, y') \colon (x', y') \in R\} \text{ if } z_1 \in X, z_2 \in Y. \end{cases}$$

By definition, it is clear that d_Z respects isometrically both d_X and d_Y. By the same reason $d_Z(z_1, z_2) = d_Z(z_2, z_1)$ and $d_Z(z_1, z_2) \geq 0$ for every $z_1, z_2 \in Z$, where $d_Z(z_1, z_2) = 0$ only if either $z_1 = z_2$ or $r = 0$. To check the triangle inequality we take $z_1, z_2, z_3 \in Z$. If either all three are elements of X, or all three are elements of Y, the inequality is verified as it is granted in X and Y with d_X and d_Y respectively. In case $z_1, z_2 \in X$ and $z_3 \in Y$ we can take some $y \in Y$ such that $(z_2, y) \in R$. Thus, we have

$$\begin{aligned} d_Z(z_1, z_2) + d_Z(z_2, z_3) &\geq d_X(z_1, z_2) + d_X(z_2, z_2) + r + d_Y(z_3, y) \\ &\geq d_X(z_1, z_2) + r + d_Y(z_3, y) \\ &\geq d_Z(z_1, z_3). \end{aligned}$$

Analogously, the argument follows for $z_1 \in X$ and $z_2, z_3 \in Y$. Thus, all is left to prove is to check $d_{\mathrm{H}}(X, Y) \leq r$. But for this is immediate by the definition of distance d_Z. □

Definition 1.5.8 (Distortion of a map). Let $(X, d_X), (Y, d_Y)$ be two metric spaces and let $f \colon X \to Y$ an arbitrary map. The **distortion** of f is defined as

$$\operatorname{dis}(f) := \sup_{x_1, x_2 \in X} |d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)|.$$

Definition 1.5.9 (ε-isometry). Let X and Y be two metric spaces and let $\varepsilon > 0$. A **ε-isometry** between two metric spaces is a map $f \colon X \to Y$ such that $\operatorname{dis}(f) \leq \varepsilon$. The image $f(X)$ is called an **ε-net**.

Proposition 1.5.10 (Theorem 7.3.28.1, [4]). *Let X and Y be two metric spaces and let $\varepsilon > 0$. If $d_{\mathrm{GH}}(X, Y) < \varepsilon$, them there exists a $2\varepsilon - isometry$ from X to Y.*

Proof. Let R be a correspondance between X and Y. By 1.5.7 it holds that $\operatorname{dis}(R) < 2\varepsilon$. For every $x \in X$ we choose some $y \in Y$ such that $(x, y) \in R$ and define $f(x) := y$. This defines a map $f \colon X \to Y$. Then for every $y \in Y$, consider an $x \in X$ such that $(x, y) \in R$. Since both y and $f(x)$ are in correspondance with x, we have that

$$d(x, f(x)) \leq d(x, x) + \operatorname{dis}(R) \leq 2\varepsilon.$$

Hence, $d(y, f(X)) < 2\varepsilon$ and f is a 2ε-isometry. □

Up to this moment we have seen that Gromov-Hausdorff distance defines a pseudometric over the set of metric spaces. Note that if X and Y are isometric, directly of the definition ge get, $d_{\text{GH}}(X,Y)=0$. To make Gromov-Hausdorff distance an actual metric we need to ask one more thing to our metric spaces. That is, to be compact. Denote

$$\mathcal{X} \coloneqq \{(X,d_X)\colon (X,d_X) \text{ is a metric compact space}\}.$$

Theorem 1.5.11 (Theorem 7.3.30, [4])**.** *Gromov-Hausdorff distance is in fact a metric over the space of isometry clases of compact metric spaces.*

Proof. We just seen that if X and Y are isometric, directly of the definition ge get, $d_{\text{GH}}(X,Y)=0$. By definition, Gromov-Hausdorff distance is nonnegative and symmetric and, by Lemma 1.5.4, it verifies the triangle inequality. It only remains to prove that given two metric spaces $X, Y \in \mathcal{X}$, if $d_{\text{GH}}(X,Y)=0$ then X and Y are isometric.

Let $X, Y \in \mathcal{X}$ such that $d_{\text{GH}}(X,Y)=0$. By Proposition 1.5.10, there exists a sequence of maps $f_n\colon X \to Y$ such that $\operatorname{dis}(f_n) \to 0$. As X is compact, we can fix a countable dense set $S \subset X$. We can then choose a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that for every $x \in S$, the sequence $\{f_{n_k}\}$ converges in Y. Now, without loss of generality, we can assume that this holds for $\{f_n\}$ itself. Use this to define a map

$$\begin{aligned} f\colon S &\to Y \\ x &\mapsto \lim f_n(x). \end{aligned}$$

Hence, since for every $x, y \in S$,

$$|d(f_n(x), f_n(y)) - d(x,y)| \leq \operatorname{dis}(f_n) \to 0,$$

we have that

$$d(f(x), f(y)) = \lim d(f_n(x), f_n(y)) = d(x,y).$$

This map is a distance preserving map from S to Y and it is posible to extend it to the entire X (see [4][Proposition 1.5.9]). Repeating the process analogously, we can find a distance preserving map from Y to X, from what follows that X and Y are isometric making a composition of both maps. $\square$

In order to extend the scope of Gromov-Hausdorff distance we can endow our compact metric spaces with real-valued functions, which will still maintain good stability properties as we will see in Chapter 5.

Denote the collection of such spaces as

$$\mathcal{X}_1 \coloneqq \{(X,d_x,f)\colon (X,d_x) \in (X), f_X\colon X \to \mathbb{R} \text{ continuous}\}.$$

Definition 1.5.12. Let $X, Y \in \mathcal{X}_1$. We extend the **Gromov-Hausdorff distance over** X_1 as

$$d^1_{\mathrm{GH}}((X, d_X, f_X), (Y, d_Y, f_Y)) = \inf_R \max \left\{ \frac{1}{2} \operatorname{dis}(R), \|f_X - f_Y\|_{\ell^\infty} \right\}.$$

An analogous adaptation of Theorem 1.5.11 and the previous results proofs that d^1_{GH} defines a metric over the set of isomorphism clases od $\mathcal{X}_1$.

To motivate the inclusion of a continuous function in this extension of the Gromov–Hausdorff distance, observe that in many applications (e.g., shape analysis, metric geometry with additional structure), we are not only interested in comparing the metric spaces themselves, but also in comparing how additional structures—typically represented by continuous functions—behave under this comparison. The function $f_X \colon X \to \mathbb{R}$ can encode further geometric or topological information, such as height functions, scalar fields, or signal data defined over the space. Including these functions allows us to extend the classical Gromov–Hausdorff framework to a richer setting where both metric and functional information are taken into account.

In Chapter 5 we will proof that persistence diagrams from certain filtrations given by continuous functions are stable respect this generalized Gromov-Hausdorff distance.

Chapter 2

Structure Theorem

The Structure Theorem for persistence modules is referred to as the “first miracle” of persistence homology [14]. This algebraic property allows to express a persistence module of finite type as a direct sum of finitely many interval modules. This enables to express a persistence module by the bars in a barcode, which latter allows to compare bottleneck distance between the barcodes, with the interleaving distance between the modules.

The proof of the theorem requires the algebraic structure theorem for finitely generated modules over a principal domain. Section 2.1 introduces the algebraic structure theorem for finitely generated modules over a principal ideal domain. Section 2.2 then gives a proof of the Structure Theorem for persistence diagrams.

2.1 Algebraic structure theorem

Definition 2.1.1 (Graded ring)**.** Let R be a ring. It is said that R is a **graded ring** if it can be decomposed into a direct sum of additive groups

$$R = \bigoplus_{n=1}^{\infty} R_n = R_1 \oplus R_2 \oplus \dots$$

such that for all $n, m \geq 0$,

$$R_n \cdot R_m \subseteq R_{n+m}.$$

Example 2.1.2 (Example 4.4, [18])**.** Let $\mathbb{F}$ be a field, the polynomial ring over it, $\mathbb{F}[x]$, is a graded ring. It can be decomposed as

$$\mathbb{F}[x] = \bigoplus_{i=0}^{\infty} x^i \cdot \mathbb{F} = \bigoplus_{i=0}^{\infty} \{cx^i \mid c \in \mathbb{F}\}.$$

Also, polynomial multiplication verifies the degree of the product of two monomials is the sum of the degrees of the factors.

Definition 2.1.3 (Graded ideal)**.** Let R be a graded ring. A **graded ideal** is a two sided ideal $I \subseteq R$ that can be decomposed into a direct sum

$$I = \bigoplus_{n=0}^{\infty} I_n$$

where for each $n \geq 0$, $I_n = I \cap R_n$.

Example 2.1.4. Note that in Example 2.1.2, every ideal is a graded ideal. To see an example of non graded ideals take the two-dimensional polynomial ring over a field, $R = \mathbb{F}[x, y]$. There are also many graded ideals in R, take, for example, the ideal generated by the two dimensional monomials $I = \langle x^2, xy \rangle$.

Take now an ideal generated by non homogeneous elements as $I' = \langle x^2 + y \rangle$. Then it happens that $x^2 \in I'_2 = I' \cap R^2$ and $y \in I'_1 = I' \cap R^1$, but either $x^2 \in I'$ nor $y \in I'$.

Definition 2.1.5 (Graded module, Definition 4.7 [18])**.** Let M be a left module over a graded ring R. It is said that M is a **left graded module** if it can be decomposed into a direct sum

$$M = \bigoplus_{n=1}^{\infty} M_n$$

if for each $n, m \geq 0$, $R_n M_m \subseteq M_{n+m}$.

Theorem 2.1.6 (Chapter IV, Theorem 6.12, [12])**.** *Let M be a finitely generated module over a principal ideal domain R. There exist a finite sequence of proper ideals $(d_1) \supseteq (d_2) \supseteq \cdots \supseteq (d_n)$ such that*

$$M \cong \bigoplus_{i=1}^{n} R/(d_i).$$

Due to the lengthy concepts needed to prove it, we will refer to it as a well known fact. An introduction to module theory and a detailed proof of Theorem 2.1.6 can be found at [12, Chapter IV].

2.2 Structure theorem for persistence diagrams

In addition to Theorem 2.1.6, we will use the following simple algebraic statement.

Proposition 2.2.1 (Proposition 4.6, [18])**.** *An ideal $I \subseteq R$ is graded if and only if it is generated by homogeneous elements.*

Proof. First, if I is a graded ideal $I = \bigoplus_p I^p$ and is generated by $\bigcup_p I^p$. Then, each

$$I^p = I \cap R^p \subseteq R^p$$

is a subset of homogeneous elements. Therefore, I is generated by homogeneous elements.

Now, let I be generated by a set X of homogeneous elements. For sure, $I \cap R^p \subseteq I$, so we just need to prove the converse inclusion. As I is generated by X, its elements $u \in I$ are of the form

$$u = \sum_i r_i x_i s_i, \tag{2.1}$$

for $r_i, s_i \in R$ and $x_i \in X$. And as $I \subseteq R$, also,

$$u = \sum_p u_p,$$

for $u_p \in R^p$. For every term in (2.1), we have

$$r_i = \sum_j r_{i,j}, \qquad\qquad s_i = \sum_l s_{i,l},$$

with each $r_{i,j}$, $s_{i,l}$ being homogeneous. Therefore, combining all we have that

$$u = \sum_i \sum_{j,l} r_{i,j} x_i s_{i,l}. \tag{2.2}$$

Each term in (2.2) is homogeneous as is a product of homogeneous elements. Thus u_p is the sum of those terms, and u has degree p. Therefore $u_p \in I$ and $I \subseteq I \cap R^p$. □

Theorem 2.2.2 (Proposition 4.8, [18]). *Let (V, π) be a persistence module. There exist a barcode* $\mathrm{Bar}(V, \pi)$*, with* $\mu\colon \mathrm{Bar}(V, \pi) \longrightarrow \mathbb{N}$*, the multiplicity of the barcode intervals, such that there is a unique direct sum decomposition*

$$V \cong \bigoplus_{I \in \mathrm{Bar}(V)} \mathbb{F}(I)^{\mu(I)}. \tag{2.3}$$

Proof. V is of finite type, so it is a finite $\mathbb{F}[x]$-module. As $\mathbb{F}$ is a field, $\mathbb{F}[x]$ is a principal ideal domain, therefore, V is a finitely generated module over a principal ideal domain. Using Fact 2.1.6, V can be decompose in the direct sum of its free and torsion subgroups, $F \oplus T$. Thus, we have

$$F = \bigoplus_{i \geq q} x^i \cdot \mathbb{F}$$

$$T = \bigoplus_{i \geq q} R^i / I^i.$$

Each $x^i \cdot \mathbb{F}$ is isomorphic to ideals of the form (x^q). By Proposition 2.2.1, each R^i/I^i is isomorphic to some quotient of graded ideals of the form $(x^p)/(x^r)$. Note that the free subgroup can be seen as a particular case of the torsion group taking $r = 0$. Thus V can be decomposed as described in (2.3). $\square$

Chapter 3

Interleaving Stability Theorem

This Chapter gives a detailed proof of the first stability theorem for persistence homology. This theorem is referred to as the "Geometry miracle" of persistent homology, as it allows to describe an isometry between persistence modules and barcodes [14]. This shows that *small* changes in a data sets will perform *small* changes in their persistence modules, and therefore small changes in how persistent homology groups vary through time. The theorem claims that given two persistence modules, the distance between them using the interleaving distance, is the same as the distance between their barcodes using the bottleneck distance.

Theorem 3.0.1 (Interleaving Stability, Theorem 2.2.8, [16]). *There is an isometry between a persistence module with the interleaving distance and its barcode with the bottleneck distance. That is, given two persistence modules V and W, it holds that*

$$d_{\text{int}}(V, W) = d_{\text{bot}}(\text{Bar}(V), \text{Bar}(W)).$$

For the presented proof we have followed [16]. Hence, we will divide the proof into proving two complementary inequalities separately. This implies checking that if there exists a δ-matching between two given barcodes, then there exists a δ-interleaving morphism between their persistence modules, Proposition 3.1.2. Also, we need to check that, if there exists a δ-interleaving morphism between two persistence modules, then there exists a δ-matching between their barcodes, 3.2.8.

3.1 First inequality

The first claim can be deduced from the Structure Theorem in a rather direct way, proving first the case where our modules are just interval modules.

Lemma 3.1.1 (Exercise 2.2.7, [16]). *Let I, J be two δ-matched intervals. Then, their corresponding interval modules $(\mathbb{F}(I), \pi)$ and $(\mathbb{F}(J), \theta)$ are δ-interleaved.*

Proof. Let $I = (a, b]$, $J = (c, d]$. If ρ is the δ-matching between them, then $\rho(I) = J$ and, following Definition 1.3.4, $(a, b] \subseteq (c - \delta, d + \delta]$ and $(c, d] \subseteq (a - \delta, b + \delta]$, with $b - a > 2\delta$ and $d - c > 2\delta$. Then, the morphisms

$$\begin{aligned} \phi_\delta \colon \mathbb{F}(I) &\to \mathbb{F}(J)_\delta \\ \phi_\delta(\mathbb{F}(I)_t) &\mapsto \mathbb{F}(J)_{(t+\delta)} \end{aligned} \quad \text{and} \quad \begin{aligned} \psi_\delta \colon \mathbb{F}(J) &\to \mathbb{F}(I)_\delta \\ \psi_\delta(\mathbb{F}(J)_t) &\mapsto \mathbb{F}(I)_{(t+\delta)} \end{aligned}$$

are well defined as for any $t \in (a, b]$, $t + \delta \in (c, d]$, as $a + \delta > c$ and $b + \delta \leq d$. In the same way, for any $t \in (c, d]$, $t + \delta \in (a, b]$. Thus,

$$\psi_\delta \circ \phi_\delta(\mathbb{F}(I)_t) = \psi_\delta(\mathbb{F}(J)_{(t+\delta)}) = \mathbb{F}(I)_{(t+2\delta)} = \pi_{t \leq t+2\delta}(\mathbb{F}(I)_t).$$

and

$$\phi_\delta \circ \psi_\delta(\mathbb{F}(J)_t) = \phi_\delta(\mathbb{F}(I)_{(t+\delta)}) = \mathbb{F}(J)_{(t+2\delta)} = \theta_{t \leq t+2\delta}(\mathbb{F}(J)_t).$$

Therefore, ϕ_δ and ψ_δ are a pair of δ-interleaving morphisms. □

Once we are able to build a δ-interleaving between two δ-matched interval modules, we will use the Structure Theorem for persistence modules to generalice the construction for arbitrary persistence modules. This will be useful to prove the first inequality needed to prove Theorem 3.0.1.

Proposition 3.1.2 (Theorem 3.0.1, [16])**.** *Given two persistence modules V, W, if there is a δ-matching between their barcodes, then there is a δ-interleaving morphism between them.*

Proof. Suppose that $\rho\colon \operatorname{Bar}(V) \to \operatorname{Bar}(W)$ is a δ-matching between the barcodes of V and W. By the Structure Theorem 2.1.6, V and W decompose in a finite direct sum of interval modules

$$V \cong \bigoplus_{I \in \operatorname{Bar}(V)} \mathbb{F}(I), \qquad W \cong \bigoplus_{J \in \operatorname{Bar}(W)} \mathbb{F}(J).$$

We can express $V = V_Y \oplus V_N$, $W = W_Y \oplus W_N$ denoting

$$\begin{aligned} V_Y &\cong \bigoplus_{I \in \operatorname{coim} \rho} \mathbb{F}(I), & V_N &\cong \bigoplus_{I \in \operatorname{Bar}(V) \setminus \operatorname{coim} \rho} \mathbb{F}(I), \\ W_Y &\cong \bigoplus_{J \in \operatorname{im} \rho} \mathbb{F}(J), & W_N &\cong \bigoplus_{J \in \operatorname{Bar}(J) \setminus \operatorname{im} \rho} \mathbb{F}(J). \end{aligned}$$

The V_Y, W_Y modules separate the "yes, matched" intervals, from the V_N, W_N "not matched" intervals. For every interval $I \in \operatorname{Bar}(V_Y)$, I is δ-matched to an interval

$J \in \mathrm{Bar}(W_Y)$ by $\rho(I) = J$. Thus, by Lemma 3.1.1, for all pair I, J of matched intervals, there exist a par of δ-interleaved morphisms

$$\phi_\delta \colon \mathbb{F}(I) \to \mathbb{F}(J)_\delta \quad \text{and} \quad \psi_\delta \colon \mathbb{F}(J) \to \mathbb{F}(I)_\delta$$

which induce the pair of δ-interleaved morphisms

$$\phi_\delta \colon V_Y \to W_{Y\delta} \quad \text{and} \quad \psi_\delta \colon W_Y \to V_{Y\delta}.$$

Not matched intervals are of length smaller than 2δ, therefore both, V_N and V_Y are δ-interleaved with the empty set. We can now construct the δ-interleaving morphism $\phi \colon V \to W$ such as $\phi|_{V_Y} = \phi_Y$ and $\phi|_{V_N} = 0$. In a similar way, we also construct $\psi \colon W \to V$. □

3.2 Second inequality

With Proposition 3.1.2 we have proven the first half of Stability Theorem 3.0.1. Now we need to prove that we can build a δ-interleaving morphism from a δ-matching. To verify this claim we need several previous lemmas that will lead us to prove Proposition 3.1.2.

First, we will introduce some notation. Let (V, π), (W, θ) be two persistence modules and let $I = (b, d]$ be an interval with $d \in \mathbb{R} \cup \{+\infty\}$. Denote the set of bars of $\mathrm{Bar}(V)$ that start before b and end exactly at d as

$$\mathrm{Bar}_{I-}(V) \coloneqq \{(a, d] \in \mathrm{Bar}(V) \colon a \le b\}.$$

Analogously, denote the set of bars that start at b and end after d as

$$\mathrm{Bar}_{I+}(V) \coloneqq \{(b, c] \in \mathrm{Bar}(V) \colon c \ge d\}.$$

Lemma 3.2.1 (Proposition 3.1.1, [16])**.** *Let $I = (b, d]$ be an interval. If there exists an injective morphism $\iota \colon (V, \pi) \to (W, \theta)$, then $\#(\mathrm{Bar}_{I-}(V)) \le \#(\mathrm{Bar}_{I-}(W))$. Where $\#(\cdot)$ denotes the cardinal operator.*

Proof. For $b < s < d < r$, denote the set of elements in V_d witch come from all V_s and disappear in all V_r as

$$E_{I-} = \bigcap_{b<s<d} \mathrm{im}\,\pi_{s\le d} \cap \bigcap_{r>d} \ker \pi_{d\le r} \subseteq V_d.$$

It holds that $\dim E_{I-}(V) = \#(\mathrm{Bar}_{I-}(V))$. For every morphism $p \colon (V, \pi) \to (W, \theta)$ the following diagram conmutes

$$\begin{array}{ccc} V_s & \xrightarrow{\pi_{s\le r}} & V_r \\ {\scriptstyle p_s}\downarrow & & \downarrow{\scriptstyle p_r} \\ W_s & \xrightarrow[\theta_{s\le r}]{} & W_r \end{array}$$

This implies that

$$p_r(\operatorname{im}\pi_{s\le r}) \subseteq \operatorname{im}\theta_{s\le r}, \qquad p_s(\ker\pi_{s\le r}) \subseteq \ker\theta_{s\le r}.$$

Taking $r = d$, $b < s < d$ in the first inclusion, and $s = d$, $r > d$ in the second, we have that

$$p_d(\operatorname{im}\pi_{s\le d}) \subseteq \operatorname{im}\theta_{s\le d}, \qquad p_d(\ker\pi_{d\le r}) \subseteq \ker\theta_{d\le r},$$

and

$$p_d(E_{I-}(V)) \subseteq E_{I-}(W).$$

If we now take p as the injective morphism of the hypothesis, $p = \iota$, we get

$$\dim E_{I-}(V) \le \dim E_{I-}(W).$$

□

Lemma 3.2.2 (Exercise 3.1.3, [16]). *Let $I = (b, d]$ be an interval. Suppose there exists a surjective morphism $s\colon (V, \pi) \to (W, \theta)$. Then $\#(\mathrm{Bar}_{I+}(V)) \ge \#(\mathrm{Bar}_{I+}(W))$.*

Proof. Analogously to the proof of Lemma 3.2.1 we now define

$$E_{I+}(V) = \bigcap \operatorname{im}\pi_{d\le r}.$$

Therefore $\dim E_{I+}(V) = \#(\mathrm{Bar}_{I+}(V))$, and recalling the diagram used for the previous proof, and using the fact that is commutative, we have that

$$p_r(\operatorname{im}\pi_{s\le r}) \supseteq \operatorname{im}\theta_{s\le r}.$$

Taking $s = d$ we then have that

$$p_d(E_{I+}(V)) \supseteq E_{I+}(W).$$

And finally, taking the surjective morphism $p = s$ we have that

$$\dim E_{I-}(V) \ge \dim E_{I-}(W).$$

□

To build our δ-matching we first define two induced matchings, by an injection and by a surjection respectively. First, suppose that there exists an injection $\iota\colon V \to W$. For every $c \in \mathbb{R} \cup \{\infty\}$, sort the bars $(a_i, c] \in \mathrm{Bar}(V)$, $i \in \{1, \dots, k\}$ by decreasing length order,

$$(a_1, c] \supseteq (a_2, c] \supseteq \cdots \supseteq (a_k, c], \text{ with } a_1 \le a_2 \le \cdots \le a_k.$$

Sort in the same manner the bars $(b_j, c] \in \mathrm{Bar}(V)$, $j \in \{1, \dots, l\}$,

$$(b_1, c] \supseteq (b_2, c] \supseteq \cdots \supseteq (a_k, c], \text{ with } b_1 \leq b_2 \leq \cdots \leq b_k.$$

As there is an injection between V and W, Lemma 3.2.1 asures that the amount of bars in $\mathrm{Bar}(V)$ is lower that the amount in $\mathrm{Bar}(V)$, i.e., $k \leq l$. We define the *injective induced matching* $\mu_{inj}\colon \mathrm{Bar}(V) \to \mathrm{Bar}(W)$ matching, for each $c \in \mathbb{R} \cup \{\infty\}$, the intervals from both lists by decreasing length.

Lemma 3.2.3 (Proposition 3.1.5, [16]). *If there exists an injection* $\iota\colon (V, \pi) \in (W, \theta)$*, then the induced matching* $\mu_{inj}\colon \mathrm{Bar}(V) \to \mathrm{Bar}(W)$ *satisfies:*

1. $\operatorname{coim} \mu_{inj} = \mathrm{Bar}(V)$,
2. $\mu_{inj}(a, c] = (b, c], \ \forall b \leq a, \ \forall (a, d] \in \mathrm{Bar}(V)$.

Proof. Applying Lemma 3.2.1 with the interval $(a_k, c]$, we have that for each $c \in \mathbb{R} \cup \{\infty\}$, $\# \mathrm{Bar}_{(a_k, c]-}(V) \leq \# \mathrm{Bar}_{(a_k, c]-}(W)$, having $k \leq l$ as we note earlier. This means that every bar in $\mathrm{Bar}(V)$ is matched to some bar in $\mathrm{Bar}(W)$. Hence $\operatorname{coim} \mu_{inj} = \mathrm{Bar}(V)$. Moreover, as the matching is carried out in length descending order, for each $i \in \{1, \dots, k\}$, $\mu_{inj}(a_i, c] = (b_i, c]$, and applying Lemma 3.2.1, now with the interval $(a_i, c]$, and making the same reasoning, $a_i \leq b_i$. □

Now we suppose that there exists a surjection $\sigma\colon V \to W$. For every $a \in \mathbb{R}$, sort the intervals $(a, c_i] \in \mathrm{Bar}(V)$, $i \in \{1, \dots, k\}$ by decreasing length order as before,

$$(a, c_1] \supseteq (a, c_2] \supseteq \cdots \supseteq (a, c_k], \text{ with } c_1 \geq c_2 \geq \cdots \geq a_k,$$

and again in the same manner, sort the intervals $(a, d_j] \in \mathrm{Bar}(V)$, $j \in \{1, \dots, l\}$,

$$(a, d_1] \supseteq (a, d_2] \supseteq \cdots \supseteq (a, d_l], \text{ with } d_1 \geq d_2 \geq \cdots \geq d_l.$$

We define the *surjective induced matching* $\mu_{sur}\colon \mathrm{Bar}(V) \to \mathrm{Bar}(W)$ matching, for each $a \in \mathbb{R}$, the intervals from both lists by decreasing length.

Lemma 3.2.4 (Exercise 3.1.8, [16]). *If there exists a surjection* $s\colon (V, \pi) \to (W, \theta)$*, then the induced matching* $\mu_{sur}\colon \mathrm{Bar}(V) \to \mathrm{Bar}(W)$ *satisfies:*

1. $\operatorname{im} \mu_{sur} = \mathrm{Bar}(W)$,
2. $\mu_{sur}(a, c] = (a, d], \ \forall c \geq d, \ \forall (a, d] \in \mathrm{Bar}(V)$.

Proof. Using Lemma 3.2.2 with the interval $(b, d_k]$ for each $b \in \mathbb{R}$, we get that, as there exists a surjection between the modules, now $k \geq l$. Therefore, every bar in $\mathrm{Bar}(W)$ is matched to some bar in $\mathrm{Bar}(V)$ and $\operatorname{im}\mu_{sur} = \mathrm{Bar}(W)$. In an analogue way to the previous lemma, as the intervals in both lists are matched in a decreasing manner, for every $j \in \{1, \ldots, l\}$, $\mu_{sur}(a, c_j] = (a, d_j]$, and if we now apply Lemma 3.2.2, we get that $c_j \geq d_j$. $\square$

Hence, with the injective and the surjective induced matchings at hand, for a general morphism f, we can define the *induced matching* $\mu(f)\colon \mathrm{Bar}(V) \to \mathrm{Bar}(W)$, as the composition $\mu_{inj} \circ \mu_{sur}$, defined as $\operatorname{im}\mu_{sur} = \mathrm{Bar}(\operatorname{im} f) = \operatorname{coim}\mu_{inj}$.

The following lemma verifies that, in fact, the mapping between persistence modules with its morphisms and barcodes with induced matchings (either the injective or the surjective versions) has functorial properties between the two categories.

Lemma 3.2.5 (Claim 3.1.13, [16]). *Let U, V and W persistence diagrams and f, g, h morphisms between them defined as in the following diagram:*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend right, "h"'] & V \arrow[r, "g"] & W \end{tikzcd}.$$

If all f, g, h are all injections, or all surjections, then the corresponding diagram formed by the barcodes of the modules, and their respective matchings conmutes as well.

$$\begin{tikzcd} \mathrm{Bar}(U) \arrow[r, "\mu_*(f)"] \arrow[rr, bend right, "\mu_*(h)"'] & \mathrm{Bar}(V) \arrow[r, "\mu_*(g)"] & \mathrm{Bar}(W) \end{tikzcd}.$$

Where μ_* denotes μ_{inj} or μ_{sur} accordingly.

Proof. Let f, g, h injective morphisms, by the definition of the injective induced matching and Lemma 3.2.1 for any $d \in \mathbb{R} \cup \{+\infty\}$, there exist $k \leq l \leq q$ such that the barcodes of U, V, W consist on the following bars:

$$\begin{aligned} \mathrm{Bar}(U) :& (a_1, d] \supset \cdots \supset (a_k, d] \\ \mathrm{Bar}(V) :& (b_1, d] \supset \cdots \supset (b_k, d] \supset \cdots \supset (b_l, d] \\ \mathrm{Bar}(V) :& (c_1, d] \supset \cdots \supset (c_k, d] \supset \cdots \supset (c_l, d] \supset \cdots \supset (c_q, d]. \end{aligned}$$

Therefore, for any d the diagram conmutes as

$$\mu_{inj}(f)(a_i, d] = (b_i, d],\ \mu_{inj}(g)(b_i, d] = (c_i, d],\ \mu_{inj}(h)(a_i, d] = (c_i, d]$$

for $1 \leq i \leq k$. If f, g, h were surjective morphisms, an analogue reasoning using the surjective induced matching definition and Lemma 3.2.2 completes the proof. $\square$

Finally, we can claim the two main lemmas from which we will construct our desired δ-matching.

Lemma 3.2.6 (Lemma 3.2.1, [16]). *Let $(V,\pi),(W,\theta)$ be δ-interleaved persistence modules, with δ-interleaving morphisms $\phi\colon V\to W_\delta$ and $\psi\colon W\to V_\delta$. Let $\phi\colon V\to \operatorname{im}\phi$ be a surjection and $\mu_{sur}\colon \operatorname{Bar}(V)\to\operatorname{Bar}(\operatorname{im}\phi)$ the induced matching. Then*

1. $\operatorname{coim}\mu_{sur}\supseteq \operatorname{Bar}(V)_{\geq 2\delta}$,
2. $\operatorname{im}\mu_{sur}=\operatorname{Bar}(\operatorname{im}\phi)$ *and*
3. $\mu_{sur}(b,d]=(b,d'],\ \forall (b,d]\in\operatorname{coim}\mu_{sur},\ d'\in[d-2\delta,d]$.

Proof. 1. To check the first part, we observe that, in the following diagram, the three morphisms are surjective as ϕ and $\pi_{t\leq t+2\delta}$ are defined onto their images, and the diagram conmutes,

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[rr, bend right, "\pi_{t\leq t+2\delta}"'] & \operatorname{im}\phi \arrow[r, "\psi_\delta"] & \operatorname{im}\pi_{t\leq t+2\delta} \end{tikzcd}.$$

Therefore, because of Lemma 3.2.5 the barcode diagram also conmutes:

$$\begin{tikzcd} \operatorname{Bar}(V) \arrow[r, "\mu_{sur}(\phi)"] \arrow[rr, bend right, "\mu_{sur}(\pi_{t\leq t+2\delta})"'] & \operatorname{Bar}(\operatorname{im}\phi) \arrow[r, "\mu_{sur}(\psi_\delta)"] & \operatorname{Bar}(\operatorname{im}\pi_{t\leq t+2\delta}) \end{tikzcd}.$$

By the definition of the surjective induced matching,

$$\operatorname{coim}\mu_{sur}(\pi_{t\leq t+2\delta})=\operatorname{Bar}(V)_{\geq 2\delta}.$$

For each starting point $a\in\mathbb{R}$, we have that

$$\operatorname{Bar}(\operatorname{im}\pi_{t\leq t+2\delta})=\{(a,b-2\delta]\colon (a,b]\in\operatorname{Bar}(V), b-a>2\delta\}.$$

Sorting all bars of $\operatorname{Bar}(V)$ and of $\operatorname{Bar}(\operatorname{im}\pi_{t\leq t+2\delta})$ in length-not-increasing order and matching the bars though the longest-first order, each bar $(a,b]\in\operatorname{Bar}(V)$ is matched with the bar $(a,b-2\delta]\in\operatorname{Bar}(\operatorname{im}\pi_{t\leq t+2\delta})$ while $b-a>2\delta$. The smaller bars are not matched. Thus,

$$\operatorname{coim}\mu_{sur}(phi)\supseteq\operatorname{coim}\mu_{sur}(\operatorname{im}\pi_{t\leq t+2\delta})=\operatorname{Bar}(V)_{\geq 2\delta}.$$

2. The second part is just a reformulation of Lemma 3.2.1.

3. Let $(b, d] \in \operatorname{coim}$. There are two cases:

   On one hand, if $d - b \leq 2\delta$, $(b, d]$ is matched to $(b, d']$ where $d \geq d'$, by definition of μ_{sur}. Also, $d' > b$ and, as in this case we have $b \geq d - 2\delta$, we have $d' > d - 2\delta$. Therefore, $d' \in [d - 2\delta, d]$.

   On the other hand, if $d - b > 2\delta$, $(b, d]$ is matched to $(b, d']$ by $\mu_{sur}(\phi)$, with $(b, d'] \in W_{\leq 2\delta}$. We can therefore use Lemma 3.2.4 to check that $d' \geq d$. In the same manner, $(b, d']$ is matched to $(b, d'']$ by $\mu_{sur}(\psi)_\delta$ with $d'' \geq d'$. Finally, using the commutativity of the following diagram, we have that $(b, d''] = (b, d - 2\delta]$, making $d' \in [d - 2\delta, d]$.

$$\begin{tikzcd}
\mathrm{Bar}(V)_{\geq 2\delta} & \mathrm{Bar}(\operatorname{im}\phi) & \mathrm{Bar}(\operatorname{im}\pi_{t\leq t+2\delta}) \\
\Cup & \Cup & \Cup \\
(b, d] \arrow[r, mapsto, "\mu_{sur}(\phi)"] \arrow[rrd, bend right, mapsto, "\mu_{sur}(\pi_{t\leq t+2\delta})"'] & (b, d'] \arrow[r, mapsto, "\mu_{sur}(\psi_\delta)"] & (b, d''] \arrow[d, equal] \\
& & (b, d - 2\delta]
\end{tikzcd}$$

$\square$

Lemma 3.2.7 (Proposition 3.2.2, [16])**.** *Let $(V, \pi), (W, \theta)$ be δ-interleaved persistence modules, with δ-interleaving morphisms $\phi\colon V \to W_\delta$ and $\psi\colon W \to V_\delta$. Let $\phi\colon V \to \operatorname{im}\phi$ be an injection and $\mu_{inj}\colon \mathrm{Bar}(\operatorname{im}\phi) \to \mathrm{Bar}(W_\delta)$ the induced matching. Then*

1. $\operatorname{coim}\mu_{sur} = \mathrm{Bar}(\operatorname{im}\phi)$,
2. $\operatorname{im}\mu_{inj} \supseteq \mathrm{Bar}(W_\delta)_{\geq 2\delta}$ *and*
3. $\mu_{inj}(b, d'] = (b', d'],\ (b, d'] \in \operatorname{coim}\mu_{inj},\ b' \in [b - 2\delta, b]$.

Proof. 1. Immediate using Lemma 3.2.3.

2. As $\phi_\delta \circ \psi = \theta_{t\leq t+2\delta}$ the following diagram conmutes:

$$\begin{tikzcd}
W \arrow[r, "\psi"] \arrow[rr, bend right, "\theta_{t\leq t+2\delta}"'] & \operatorname{im}\psi \arrow[r, "\phi_\delta"] & W_{2\delta}
\end{tikzcd}.$$

   This implies that $\operatorname{im}\theta_{t\leq t+2\delta} \subseteq \operatorname{im}\phi_\delta \subseteq W_{2\delta}$, so there are some injections j and i which make the following diagram commute as well:

$$\begin{tikzcd}
\operatorname{im}\theta_{t\leq t+2\delta} \arrow[r, "j"] \arrow[rr, bend right, "\theta_{t\leq t+2\delta}"'] & \operatorname{im}\phi_\delta \arrow[r, "i"] & W_{2\delta}
\end{tikzcd}.$$

As all morphisms in the diagram above are injections, we can use the functorial properties of Lemma 3.2.5 having a commutative diagram of the barcodes of each of the previous persistence modules:

$$\begin{tikzcd}
\mathrm{Bar}(\mathrm{im}\,\theta_{t\le t+2\delta}) \arrow[r, "\mu_{inj}(j)"] \arrow[rr, bend right, "\mu_{inj}(\theta_{t\le t+2\delta})"'] & \mathrm{Bar}(\mathrm{im}\,\phi_\delta) \arrow[r, "\mu_{inj}(i)"] & \mathrm{Bar}(W_{2\delta})
\end{tikzcd}.$$

We have that

$$\begin{aligned}
\mathrm{Bar}(\mathrm{im}\,\theta_{t\le t+2\delta}) &= \{(b, d-2\delta)\colon (b,d] \in \mathrm{Bar}(W), b < d-2\delta\},\\
\mathrm{Bar}(W_{2\delta}) &= \{(b-2\delta, d-2\delta)\colon (b,d] \in \mathrm{Bar}(W)\} \text{ and}\\
\mu_{inj}(\theta_{t\le t+2\delta})((b, d-2\delta)) &= (b-2\delta, 2-2\delta)
\end{aligned}$$

Therefore $\mathrm{im}_\mu\, inj(i) \supseteq \mathrm{im}\,\mu_{inj}(\psi_{t\le t+2\delta}) = \mathrm{Bar}(W_{2\delta})_{2\delta}$. Thus, undoing the shifts, the proof is complete.

3. Let $(b,d] \in \mathrm{Bar}(\mathrm{im}\, f_\delta)$ such as for some b', $\mu_{inj}(b,d] = (b',d] \in \mathrm{Bar}(W)$. Because of Lemma 3.2.3, $b' \le b$. There are again two cases:

   If $d-b \le 2\delta$, then $b' \ge d-2\delta \ge b > b-2\delta$ and $b' \in [b-2\delta, b]$.

   Else, if $d-b > 2\delta$, there exists an interval $(b'+2\delta, d] \in \mathrm{Bar}(\mathrm{im}\,\theta_{t\le t+2\delta})$ such that

$$\mu_{inj}(\theta_{t\le t+2\delta})(b'+2\delta, d] = \mu_{inj}(i)(b,d] = (a,d].$$

   Thus, $b \le b'+2\delta$ and $b' \in [b-2\delta, b]$.

□

At last, we can now prove the other part of the Stability theorem. For so, we will construct a δ-matching out of a δ-interleaving morphism.

Proposition 3.2.8 (Theorem 3.0.2, [16]). *Given two persistence modules V, W, with a δ-interleaving morphism between them, then there is a δ-matching between their barcodes.*

Proof. Let $\mu(\phi) = \mu_{inj} \circ \mu_{sur}$ and let $\Phi_\delta\colon \mathrm{Bar}(W_\delta) \to \mathrm{Bar}(W)$ be the *shift map* that carries each bar $(a,b]$ into $(a+\delta, b+\delta]$. The composition $\Phi_\delta \circ \mu(\phi)$ is a matching between

$\mathrm{Bar}(V)$ and $\mathrm{Bar}(W)$. Hence, using Lemma 3.2.7 and 3.2.6, we get the following diagram:

$$\begin{array}{ccccccc}
\mathrm{Bar}(V) & & & & \mathrm{Bar}(W_\delta)_{\geq 2\delta} & & \mathrm{Bar}(W)_{\geq 2\delta} \\
\cup\!\shortmid & & & & \cap\!\shortmid & & \cap\!\shortmid \\
\mathrm{Bar}(V)_{\geq 2\delta} & \xrightarrow{\mu_{sur}} & \mathrm{Bar}(\mathrm{im}\, f) & \xrightarrow{\mu_{inj}} & \mathrm{im}\, \mu_{inj} & \xrightarrow{\Psi_\delta} & \mathrm{Bar}\, B(W) \\
\Cup & & \Cup & & \Cup & & \Cup \\
(b, d] & \longmapsto & (b, d'] & \longmapsto & (b', d'] & \longmapsto & (b' + \delta, d' + \delta]
\end{array}$$

The diagram shows that, by Lemma 3.2.6, a bar $(b, d] \in \mathrm{Bar}(V)_{\geq 2\delta}$ is sent to $\mu_{sur}(b, d] = (b, d'] \in \mathrm{Bar}(\mathrm{im}\, \phi)$ with $d' \in [d - 2\delta, d]$. Then, by Lemma 3.2.6, it is sent to $\mu_{sur}(b, d'] = (b', d']$ with $b' \in [b - 2\delta, b]$. At last, using the shift morphism Φ_δ it is carried to $(b' + \delta, d' + \delta]$.

This shows that any bar in $\mathrm{Bar}(V)_{\geq 2\delta}$ is matched. In the same manner it can be seen that any bar in $\mathrm{Bar}(W)_{\geq 2\delta}$ is matched. Thus, we have that

$$\begin{cases} d - 2\delta \leq d' \leq d \\ b - 2\delta \leq b' \leq b \end{cases} \Rightarrow \begin{cases} d - \delta \leq d' + \delta \leq d + \delta \\ b - \delta \leq b' + \delta \leq b + \delta \end{cases},$$

and therefore, $\Phi_\delta \circ \mu(\phi)$ is a δ-matching between $\mathrm{Bar}(V)$ and $\mathrm{Bar}(W)$. □

The constructions made by Proposition 3.1.2 and Proposition 3.2.8 asure that given a δ-interleaving morphism we can build a δ-matching, and conversely, given a δ-matching we can build a δ-interleaving morphism. This means that if one of the two exists, it fixes a δ. Both the interleaving distance and the bottleneck distance try to minimice this δ, so once fixed for one of them, the other needs an smaller or equal δ'. Thus, with each of the propositions we can prove one of the inequalities needed to reach the isomorphism between the space of persistence diagrams and the space of their barcodes.

Theorem 3.0.1 (Interleaving Stability, Theorem 2.2.8, [16]). *There is an isometry between a persistence module with the interleaving distance and its barcode with the bottleneck distance. That is, given two persistence modules V and W, it holds that*

$$d_{\mathrm{int}}(V, W) = d_{\mathrm{bot}}(\mathrm{Bar}(V), \mathrm{Bar}(W)).$$

Proof. Suppose $d_{int}(V, W) = \delta$. Proposition 3.2.8 asures there exist a δ-matching between $\mathrm{Bar}(V)$ and $\mathrm{Bar}(W)$. As $d_{bot}(V, W)$ is the infimum δ for witch exists a δ-matching, $d_{bot}(V, W) \leq d_{int}(V, W)$. On the other hand, Proposition 3.1.2 leads, with the same reasoning, to $d_{int}(V, W) \leq d_{bot}(V, W)$. Thus, it has to be $d_{int}(V, W) = d_{bot}(\mathrm{Bar}(V), \mathrm{Bar}(W))$. □

Chapter 4

Hausdorff Stability Theorem

Persistence diagrams help summarize the information given by the homology groups of a filtration over a certain data set. They represent the birth and death of every feature in an easy to analice format scattering points over the upper half of the plane $\mathbb{R}^2$ and its diagonal Δ. Once we have computed the diagrams given by two datasets we can measure distances between them using the bottleneck distance, enabling us to decide wether two diagrams are close to each other. However, this would be kind of useless if the bottleneck distance were not stable. If minor differences in original data would cause great changes in the bottleneck distance between the corresponding persistence diagrams, then this data summary method would be as good as any other random method.

Fortunately for us, David Cohen-Steiner, Herbert Edelsbrunner and John Harer, proved in their 2005 paper that the bottleneck distance over persistence diagrams is indeed stable [8]. This means that, when comparing the bottleneck distance between the persistence diagrams formed by the pre images of two tame functions, see Definition 1.1.14, the first will be always lower or equal that the Lebesgue L_∞ norm between the two functions.

Along this chapter we will follow Edelsbrunner et al. paper [8] to prove Theorem 4.0.2. Recall from Definition 1.1.13 that homological critical values are the real numbers at where homology changes. Recall too the notation given in Definition 1.1.18. For a triangulable topological space X, let $f\colon X \to \mathbb{R}$ and fix some $k \in \mathbb{Z}$, we denoted

$$F_x := H_k(f^{-1}(-\infty, x]),$$

and for every $x \leq y$, we denoted the induced inclusion map from X_x to X_y as

$$f_x^y\colon F_x \to F_y.$$

The persistent homology module associated to x and y was the subspace of X_y given by

$$F_x^y := \operatorname{im} f_x^y.$$

The persistent Betti numbers were defined as

$$\beta_x^y := \dim(F_x^y).$$

We can define the multiplicity of each critical point as follows.

Definition 4.0.1 (Multiplicity)**.** Let $f\colon X \to \mathbb{R}$ be tame, and $(a_i)_{i=1,\ldots,n}$ be its homological critical values. Take $(b_i)_{i=1,\ldots,n}$ be an interleaved sequence of non critical values such that $b_{i-1} < a_i < b_i$ for all $i = 1, \ldots, n$. Define $b_{-1} = a_0 = -\infty$, $b_{n+1} = a_{n+1} = \infty$. The **multiplicity** of $(a_i, a_j) \in D(f)$, denoted μ_i^j is

$$\mu_i^j \coloneqq \beta_{b_{i-1}}^{b_j} - \beta_{b_i}^{b_j} + \beta_{b_i}^{b_{j-1}} - \beta_{b_{i-1}}^{b_{j-1}}.$$

The **total multiplicity** of a multiset A, denoted $\#(A)$ is the sum of the multiplicities of every element in A.

Note that the total multiplicity of a multiset is the the generalized concept of cardinality of a normal set. While the cardinality of a set counts the number of elements in the set, the multiplicity of a multiset counts how many elements, different or not, are there in the multiset.

When a function f is tame, we can form a finite multiset of barcodes taking each pair of critical points (a_i, a_j) with their corresponding multiplicity μ_i^j, for each $0 \leq i < j \leq n+1$. Hence we can form a persistence diagram $D(f)$. The main theorem of this chapter states that the distance of two of those persistence diagrams is never grater that the L^∞ norm between the functions that form them.

Theorem 4.0.2 (Main Theorem, [8])**.** *Let X be a triangulable space, and $f, g\colon X \to \mathbb{R}$ continuous tame functions. Then,*

$$d_{\text{bot}}(D(f), D(g)) \leq \|f - g\|_\infty$$

4.1 Hausdorff Stability

Before approaching the proof of Theorem 4.0.2, this section shows how the claim of the theorem is true when the bottleneck distance is replaced by the Hausdorff distance. On Section 4.2 this assertion is used to give an upper limit of the bottleneck distance by Hausdorff distance, proving the main theorem.

We will denote the closed upper left quadrant of a point $(x, y) \in \mathbb{R}^2$ as

$$Q_x^y \coloneqq [-\infty, x] \times [y, \infty].$$

Lemma 4.1.1 (k-Triangle Lemma, [8])**.** *Let $f\colon X \to \mathbb{R}$ be a tame function, $x < y \in \mathbb{R}$ be non critical values of f. Then the multiplicity μ of the persistence diagram of f in the closed upper left quadrant is*

$$\mu = \#(D(f) \cap Q_x^y) = \beta_x^y.$$

Proof. Let $x = b_i$, $y = b_{j-1}$.

$$\mu = \sum_{k \le i \le j \le l} \mu_k^l = \sum_{k \le i \le j \le l} \beta_{b_{k-1}}^{b_l} - \beta_{b_k}^{b_l} + \beta_{b_k}^{b_{l-1}} - \beta_{b_{k-1}}^{b_{l-1}} \tag{4.1}$$

$$= \beta_{b_{-1}}^{b_{n+1}} - \beta_{b_i}^{b_{n+1}} + \beta_{b_i}^{b_{j-1}} - \beta_{b_{j-1}}^{b-1} = \beta_{b_k}^{b_{l-1}} = \beta_x^y. \tag{4.2}$$

The fist two equalities in (4.1) are just the definition of total multiplicity. In (4.2), note that every other term in the sum cancels. Then note that $\beta_{b_{-1}}^{b_{n+1}} = \dim F_{-\infty}^{\infty}$, $\beta_{b_i}^{b_{n+1}} = \dim F_x^{\infty}$ and $\beta_{b_{j-1}}^{b-1} = \dim F_{-\infty}^{y}$. All of them are the dimension of the trivial module, therefore, equal to 0. This leaves only one remaining term and completes the proof. □

Denote the **upper left quadrants** $Q := Q_b^c = [-\infty, b] \times [c, \infty]$, $Q_\varepsilon := Q_{b-\varepsilon}^{c+\varepsilon} = [-\infty, b-\varepsilon] \times [c+\varepsilon, \infty]$.

Lemma 4.1.2 (Quadrant Lemma, [8])**.** *Let $f, g \colon X \to \mathbb{R}$ be two tame functions. With the notation abobe, the following inequality holds,*

$$\#(D(f), \cap Q_\varepsilon) \le \#(D(g) \cap Q).$$

Proof. Let $\varepsilon := \|f - g\|_\infty$. Hence, considering the pre-image of the functions, we have the following inclusions

$$f^{-1}((-\infty, x]) \subseteq g^{-1}((-\infty, x+\varepsilon)), \tag{4.3}$$

$$g^{-1}((-\infty, x]) \subseteq f^{-1}((-\infty, x+\varepsilon)). \tag{4.4}$$

Name $\varphi_x \colon F_x \to G_{x+\varepsilon}$ to the inclusion map induced by (4.3) and $\psi_x \colon G_x \to F_{x+\varepsilon}$ to the inclusion map induced by (4.4). Let $b < c \in \mathbb{R}$. With the described maps, we can form commutative diagram (4.5) where we observe that

$$\mathrm{im}(f_{c-\varepsilon}^{c+\varepsilon}) = F_{c-\varepsilon}^{c+\varepsilon} \subseteq \psi_c \circ g_b^c(G_b) = \psi_c(G_b^c).$$

$$\begin{tikzcd} F_{b-\varepsilon} \arrow[r, "f_{b-\varepsilon}^{c+\varepsilon}"] \arrow[d, "\varphi_{b-\varepsilon}"'] & F_{c+\varepsilon} \\ G_b \arrow[r, "g_b^c"] & G_c \arrow[u, "\psi_c"'] \end{tikzcd} \tag{4.5}$$

Last inclusion is enough for the requirements of this proof, nevertheless, we will make one more note that will be useful later on, through the proof of Lemma 4.1.3. Fit the maps so that they describe commutative diagram (4.6), showing that

$$\psi_c(G_b^c) = \psi_c \circ g_b^c(G_b) = f_{b+\varepsilon}^{c+\varepsilon} \circ \psi_b(G_b) \subseteq F_{b+\varepsilon}^{c+\varepsilon}.$$

$$\begin{tikzcd} F_{b+\varepsilon} \arrow[r, "f_{b+\varepsilon}^{c+\varepsilon}"] & F_{c+\varepsilon} \\ G_b \arrow[u, "\varphi_b"] \arrow[r, "g_b^c"] & G_c \arrow[u, "\psi_c"'] \end{tikzcd} \tag{4.6}$$

From both diagrams we finally obtain the inclusion chain

$$F_{c-\varepsilon}^{c+\varepsilon} \subseteq \psi_c(G_b^c) \subseteq F_{b+\varepsilon}^{c+\varepsilon}. \tag{4.7}$$

By Lemma 4.1.1, we are able to count the elements in the intersection of the diagrams with the upper left quadrants. Hence

$$\#(D(f) \cap Q_\varepsilon) = \beta_{b-\varepsilon}^{c+\varepsilon} = \dim F_{b-\varepsilon}^{c+\varepsilon},$$
$$\#(D(g) \cap Q) = \beta_b^c = \dim G_b^c.$$

As if one homology module is contain into an other, the dimension of the first must be lower or equal to the one of the second. Also, the dimension is invariant under inclusion maps. Thus, the first inclusion of (4.7) asserts that $F_{c-\varepsilon}^{c+\varepsilon} \subseteq \psi_c(G_b^c)$ and therefore we have proven that $\dim F_{c-\varepsilon}^{c+\varepsilon} \leq \dim G_b^c$. □

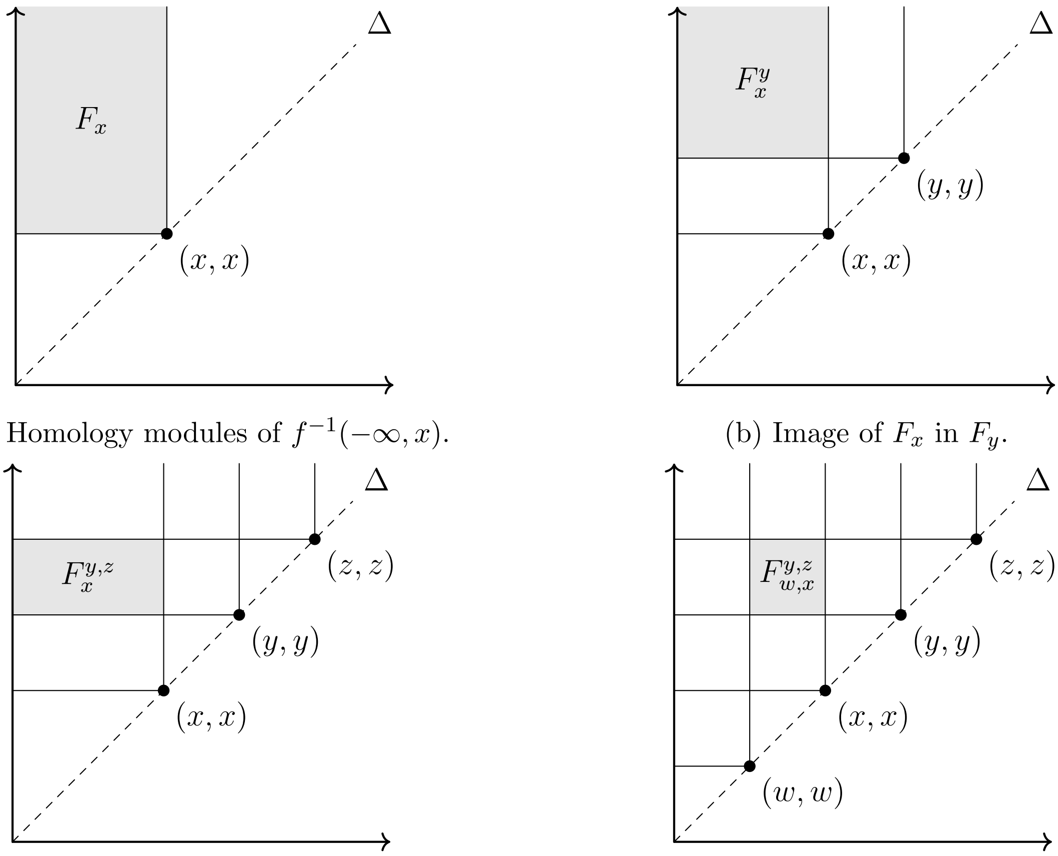
We will introduce some useful notation. Let $f\colon X \to \mathbb{R}$ be a tame function. Let $w < x < y < z \in \mathbb{R}$ be numbers different from critical values of f. Recall that $F_x = H_k(f^{-1}(-\infty, x])$, $f_x^y\colon F_x \to F_y$ and $F_x^y = \dim f_x^y$. We denote

$$f_x^{y,z} := f_y^z|_{F_x^y}, \qquad F_x^{y,z} := \dim f_x^{y,z}.$$

Note, from linear algebra, that $\dim F_x^{y,z} = \dim F_x^y - \dim F_x^z$. Also note that $F_w^y \subseteq F_x^y$. Therefore, $\ker F_w^y \subseteq \ker F_x^y$ and we can define the quotient

$$F_{w,x}^{y,z} := F_x^{y,z}/F_w^{y,z}.$$

Figure 4.1 depicts a visual interpretation and recap of this notation, based on Figure 3 from [8].



(a) Homology modules of $f^{-1}(-\infty, x)$.

(b) Image of F_x in F_y.

(c) Kernel of the surjection $F_x^y \to F_x^z$.

(d) Quotient of $F_x^{y,z}$ and $F_w^{y,z}$.

Figure 4.1: Persistent homology notation.

Let $a < b < c < d \in \mathbb{R}$. Denote the **rectangles** $R := [a, b] \times [c, d]$, $R_\varepsilon := [a + \varepsilon, b - \varepsilon] \times [c + \varepsilon, d - \varepsilon]$.

Lemma 4.1.3 (Box Lemma, [8])**.** *With the notation above, the following inequality holds,*

$$\#(D(f), \cap R_\varepsilon) \leq \#(D(g) \cap R).$$

Proof. Note that we can asume that $a + \varepsilon < b - \varepsilon$ and $c + \varepsilon < d - \varepsilon$. Otherwise the rectangle R_ε would not exist. Also note that

$$\#(D(f) \cap R_\varepsilon) = \dim F_{a+\varepsilon, b-\varepsilon}^{c+\varepsilon, d-\varepsilon}$$
$$\#(D(g) \cap R) = \dim G_{a,b}^{c,d}$$

To make our proof we draw diagram 4.1. Lets analice every element of the diagram.

$$\begin{tikzcd}
G_a^d \arrow[rrr, "r_1"] & & & G_b^d \\
& F_{a+\varepsilon}^{d-\varepsilon} \arrow[r, "r_2"] & F_{b-\varepsilon}^{d-\varepsilon} \arrow[ur, "s_1"] & \\
& F_{a+\varepsilon}^{c+\varepsilon} \arrow[u, "u_2"] \arrow[r, "r_3"] & F_{b-\varepsilon}^{c+\varepsilon} \arrow[u, "u_3"'] & \\
G_a^c \supseteq E_a^c \arrow[uuu, "u_1"] \arrow[ur, "s_2"] \arrow[rrr, "r_4"] & & & E_b^c \subseteq G_b^c \arrow[uuu, "u_4"'] \arrow[ul, "s_3"']
\end{tikzcd}$$

First of all, the middle upside arrows, are

$$u_2 = f_{a+\varepsilon}^{c+\varepsilon,d-\varepsilon}, \qquad u_3 = f_{b-\varepsilon}^{c+\varepsilon,d-\varepsilon}.$$

Right arrows r_1, r_2, r_3, r_4 represent the inclusions from its respective vector spaces to their destination. The objetive is to define the respective quotients to define $G_{a,b}^{c,d}$. Recall the inclusion maps defined in the proof of Lemma 4.1.2, $\varphi_x \colon F_x \to G_{x+\varepsilon}$ and $\psi_x \colon G_x \to F_{x+\varepsilon}$. We define

$$E_b^c := \psi_c^{-1}(F_{b-\varepsilon}^{c+\varepsilon,d-\varepsilon}) \cap G_b^c, \qquad E_a^c := G_a^c \cap E_b^c.$$

We have then that the outer upside arrows are the respective restrictions

$$u_1 = g_a^{c,d}|_{E_a^c}, \qquad u_4 = g_b^{c,d}|_{E_b^d}.$$

We denote

$$s_1 := \varphi_{d-\varepsilon}|_{F_{b-\varepsilon}^{d-\varepsilon}}, \qquad s_2 := \psi_c|_{E_a^c}, \qquad s_3 := \psi_c(G_b^c)|_{E_b^c}.$$

By the inclusions (4.7), we have that

$$\varphi_{d-\varepsilon}(F_{b-\varepsilon}^{d-\varepsilon}) \subseteq G_b^d, \qquad \psi_c(G_a^c), \subseteq F_{a+\varepsilon}^{c+\varepsilon} \qquad F_{b-\varepsilon}^{c+\varepsilon} \subseteq \psi_c(G_b^c).$$

Note that by the manner we have define the diagram, it need to be that

$$\operatorname{im}(s_3) = \ker(u_3), \qquad \operatorname{im}(s_1) \subseteq G_b^d.$$

Also, we can observe that $u_4 = s_1 \circ u_3 \circ s_3$. As $u_3 \circ s_3 = 0$, then $E_b^c = \ker(u_4)$. Also, as $r_1 \circ u_1 = u_4 \circ r_4 = 0$ and r_1 is the inclusion, then $E_a^c = \ker(u_1)$. Hence, we can write

$$E_b^c = E_b^{c,d} \subseteq G_b^{c,d}, \qquad E_a^c = E_a^{c,d} \subseteq G_a^{c,d}.$$

As $E_a^{c,d} = E_b^{c,d} \cap G_a^{c,d}$, the following quotient inclusion holds

$$E_{a,b}^{c,d} = E_b^{c,d}/E_a^{c,d} \subseteq G_b^{c,d}/G_a^{c,d} = G_{a,b}^{c,d}.$$

Therefore, we have

$$\dim(E_{a,b}^{c,d}) \le \dim(G_{a,b}^{c,d}).$$

Now note that

$$E_{a,b}^{c,d} = \ker(u_4)/\ker(u_1), \qquad F_{a+\varepsilon,b+\varepsilon}^{c+\varepsilon,d+\varepsilon} = \ker(u_3/\ker(u_2)).$$

By construction $s_3(\ker(u_4)) = \ker(u_3)$. As for every $x \in \ker(u_1)$, $r_2 \circ u_2 \circ s_2(x) = u_3 \circ s_3 \circ r_4(x) = 0$, and r_2 is an injection, then $s_3(\ker(u_1)) = s_2(\ker(u_1) \subseteq \ker(u_2))$, we get

$$\dim(F_{a+\varepsilon,b+\varepsilon}^{c+\varepsilon,d+\varepsilon}) \le \dim(E_{a,b}^{c,d}).$$

Hence, the desired inequality is hold as we have seen that

$$\#(D(f) \cap R_\varepsilon) = \dim F_{a+\varepsilon,b-\varepsilon}^{c+\varepsilon,d-\varepsilon} \le \dim(E_{a,b}^{c,d}) \le \dim(G_{a,b}^{c,d}) = \#(D(g) \cap R).$$

□

Theorem 4.1.4 (Hausdorff Stability, [8]). *Let X be a triangulable space, and $f, g\colon X \to \mathbb{R}$ continuous tame functions. Then,*

$$d_{\mathrm{H}}(D(f), D(g)) \le \|f - g\|_\infty.$$

Proof. As a direct consequence of Lemma 4.1.3 if $(x, y) \in D(f)$ then there must exist some point in $D(g)$ at distance less than or equal to $\varepsilon = \|f - g\|_\infty$ from (x, y) since the total multiplicity of $D(g) \cap R_\varepsilon$ is at least one. □

4.2 Bottleneck Stability

In this section we are going to prove Theorem 4.0.2.

Theorem 4.0.2 (Main Theorem, [8]). *Let X be a triangulable space, and $f, g\colon X \to \mathbb{R}$ continuous tame functions. Then,*

$$d_{\mathrm{bot}}(D(f), D(g)) \le \|f - g\|_\infty$$

Definition 4.2.1 (Very close tame functions)**.** Let $f, g\colon X \to \mathbb{R}$ be tame functions. We define

$$\delta_f = \min\{\|p - q\|_\infty : p \in D(f) \setminus \Delta,\ q \in D(f),\ p \neq q\}.$$

We say that g is **very close** to f if $\|f - g\|_\infty < \delta_f/2$.

Lemma 4.2.2 (Easy Bijection Lemma, [8])**.** *Let $f, g\colon X \to \mathbb{R}$ be tame functions, where g is very close to f. Then, following holds,*

$$d_{\mathrm{bot}}(D(f), D(g)) \leq \|f - g\|_\infty.$$

Proof. Let $p := (a_i, a_j) \in D(f) - \Delta$ be a point in the diagram of f that is not in the diagonal, and let $\mu := \beta_i^j$ denote its multiplicity. Let S_ε be the square of center p and radius $\varepsilon = \|f - g\|_\infty$. That is, the square of side 2ε. By definition of the square S_ε we have that the number of points of its intersection with the diagram of g must be grater or equal than the multiplicity at p. Hence, by the Box Lemma 4.1.3 we have

$$\mu \leq \#(D(g) \cap S_\varepsilon) \leq \#(D(f) \cap S_2\varepsilon).$$

As g is very close to f, we have $2\varepsilon \leq \delta_f$. Hence p is the only point of $D(f)$ in S_ε, and therefore the previous inequality is in fact an equivalence. If there was a point in the intersection which was not in μ then it would be inside the square S_ε and meaning the distance $\|f - g\|$ would be smaller. That is

$$\mu = \#(D(g) \cap S_\varepsilon).$$

Hence we can map every point in $D(g) \cap S_\varepsilon$ with p. We can then repeat this process for every other $p \in D(f) \setminus \Delta$. After this process, every point of $D(g)$ which have not been matched yet, must be at distance greater than ε from $D(f) \setminus \Delta$. By Theorem 4.1.4, every unmatched point must be at distance at most ε from the diagonal Δ. Hence, if we map each of this points to Δ, we have built a bijection between $D(f)$ and $D(g)$ that moves each point at most ε. □

Definition 4.2.3. Let $\hat{f}, \hat{g}$ be two piecewise linear functions over a simplicial complex K. Let $\lambda \in [0, 1]$. A **convex combination** of $\hat{f}$ and $\hat{g}$ is a function of the form

$$h_\lambda := (1 - \lambda)\hat{f} + \lambda\hat{g}.$$

Lemma 4.2.4 (Interpolation Lemma, [8])**.** *Let K be a simplicial complex. Take two piecewise linear functions $\hat{f}, \hat{g}\colon K \to \mathbb{R}$. Then, the following holds,*

$$d_{\mathrm{bot}}(D(\hat{f}), D(\hat{g})) \leq \|\hat{f} - \hat{g}\|_\infty.$$

Proof. Let $c := \|\hat{f} - \hat{g}\|_\infty$. For every $\lambda \in [0,1]$, define $\delta(\lambda) := \delta_{h_\lambda} > 0$. Let J_λ denote open intervals around each δ as follows, and consider the set C of all J_λ be

$$C := \left\{ J_\lambda := \left(\lambda - \frac{\delta(\lambda)}{4c}, \lambda + \frac{\delta(\lambda)}{4c} \right) \right\}.$$

The set C is an open cover of the interval $[0,1]$. Let C' be the minimal subcover of C. As $[0,1]$ is compact, the subcover C' must be finite. Consider then $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ the midpoints of the intervals in C'. As C' is minimal, then each intersection $J_{\lambda_i} \cap J_{\lambda_{i+1}}$ is not empty. Hence

$$\lambda_i + \lambda_{i+1} \leq \frac{\delta(\lambda_i) + \delta(\lambda_{i+1})}{4c} \leq \frac{\max\{\delta(\lambda_i), \delta(\lambda_{i+1})\}}{2c}.$$

Therefore, by definition of c and each h_{δ_i}, it holds

$$\|h_{\delta_i} - h_{\delta_{i+1}}\|_\infty = c(\lambda_{i+1} - \lambda_i) \leq \frac{\max\{\delta(\lambda_i), \delta(\lambda_{i+1})\}}{2}.$$

This implies that h_{δ_i} is very close to $h_{\delta_{i+1}}$ or viceversa. Then, by Lemma 4.2.2, for every $1 \leq i \leq n-1$,

$$d_{\mathrm{bot}}(D(h_{\lambda_i}), D(h_{\lambda_{i+1}})) \leq \|h_{\lambda_i} - h_{\lambda_{i+1}}\|_\infty. \tag{4.8}$$

Let $\lambda_0 = 0$ and $\lambda_{n+1} = 1$. Then $h_{\lambda_0} = \hat{f}$ is very close to h_{λ_1} and $h_{\lambda_1} = \hat{g}$ is very close to h_{λ_n} and therefore (4.8) also holds for $i = 0$ and $i = n+1$. Finally, using the triangle inequality we have

$$d_{\mathrm{bot}}(D(\hat{f}), D(\hat{g})) \leq \sum_{i=0}^{n} d_{\mathrm{bot}}(D(h_{\lambda_i}), D(h_{\lambda_{i+1}})) \leq \sum_{i=0}^{n} \|h_{\lambda_i} - h_{\lambda_{i+1}}\|_\infty = \|\hat{f} - \hat{g}\|_\infty.$$

□

For the final proof lets recall what the star of a simplicial complex is.

Definition 4.2.5 (Star of a simplicial complex)**.** Let σ be a simplex in a simplicial complex L. The **star** $\mathrm{St}(\sigma)$ of σ is the set of simplices in L which contain σ as a face. The **star of a subset** K of L, denoted $\mathrm{St}(K)$, is the union of the stars of each simplex of K.

Theorem 4.0.2 (Main Theorem, [8])**.** *Let X be a triangulable space, and $f, g \colon X \to \mathbb{R}$ continuous tame functions. Then,*

$$d_{\mathrm{bot}}(D(f), D(g)) \leq \|f - g\|_\infty$$

Proof. As X is triangulable, there exists a finite simplicial complex L and a homeomorphism $\Phi\colon L \to X$. Hence a persistence diagram is invariant under this change of variables. That is, $f \circ \Phi\colon L \to \mathbb{R}$ is tame and $D(f \circ \Phi) = D(f)$. Since f and g are continuous and L is compact, for every $\delta < 0$ there exists a subdivision K of L such that for every u, v points of a common simplex $\sigma \in K$,

$$|f \circ \Phi(u) - f \circ \Phi(v)| \leq \delta,$$
$$|g \circ \Phi(u) - g \circ \Phi(v)| \leq \delta.$$

Let $\hat{f}, \hat{g}\colon \mathrm{St}(K) \to \mathbb{R}$ be the piecewise linear interpolations of $f \circ \Phi$ and $g \circ \Phi$ on K. By construction of K and the definition of the L_∞-norm, these interpolations satisfy

$$\|\hat{f} - f \circ \Phi\|_\infty \leq \delta,$$
$$\|\hat{g} - g \circ \Phi\|_\infty \leq \delta.$$

Hence, by Lemma 4.2.4 and the triangle inequality

$$d_{\mathrm{bot}}(D(\hat{f}), D(\hat{g})) \leq \|\hat{f} - \hat{g}\|_\infty \leq \|f \circ \Phi - g \circ \Phi\|_\infty + 2\delta \leq \|f - g\|_\infty + 2\delta.$$

Now, we can take some δ such that $\delta \leq \delta_f/2$ so $\hat{f}$ is very close to f. This allows us to use Lemma 4.2.2 to make a bijection that satisfies

$$d_{\mathrm{bot}}(D(f), D(\hat{f})) \leq d_{\mathrm{bot}}(D(f \circ \Phi), D(\hat{f})) \leq \delta.$$

Analogously, also assuring $\delta < \delta_g$ we also have

$$d_{\mathrm{bot}}(D(g), D(\hat{g})) \leq d_{\mathrm{bot}}(D(g \circ \Phi), D(\hat{g})) \leq \delta,$$

and therefore, by triangle inequality again,

$$d_{\mathrm{bot}}(D(f), D(g)) \leq d_{\mathrm{bot}}(D(f), D(\hat{f})) + d_{\mathrm{bot}}(D(\hat{f}), D(\hat{g})) + d_{\mathrm{bot}}(D(\hat{g}), D(g)) \leq 4\delta.$$

As this holds for any δ smaller that δ_f and δ_g, taking the limit when δ tends to 0, we complete the proof. $\square$

Chapter 5

Gromov-Hausdorff's Stability Theorem

Chapter 4 established the stability of persistence diagrams under small-amplitude perturbations of filtering functions. Specifically, for two tame functions defined on the same topological space, the bottleneck distance between their persistence diagrams is bounded by the uniform norm of the function difference.

In this chapter, we extend this stability framework to distinct metric spaces, enabling quantitative comparisons of real-world datasets (e.g., point clouds representing different shapes). We achieve this through two classic filtrations from computational geometry, the Vietoris-Rips and the Čech filtrations, to obtain the corresponding modules from finite metric spaces and then prove how the bottleneck distance between their persistence diagrams is stable respect the Gromov-Hausdorff (see Definition 1.5.3) distance between the metric spaces. This bound facilitates efficient clustering and differentiation of metric spaces with distinct topological features (e.g., non-isometric shapes), as validated by experiments on 3D models.

Section 5.1 details Vietoris-Rips and the Čech filtrations and sets the necessary lemmas to prove the stability Theorem 5.1.8. Section 5.2 generalices the result comparing the distance between filtrations given by some functions with the generalized Gromov-Hausdorff distance from Definition 1.5.12.

Along all the chapter we follow the 2009 paper from Frédéric Chazal and David Cohen-Steiner, [5].

5.1 Gromov-Hausdorff stability

Definition 5.1.1 (Vietoris-Rips filtration)**.** Let (X, d) be a finite metric space and let $\alpha > 0$. The **Vietoris-Rips complex** associated with X of radius α, $\mathcal{R}_\alpha(X, d)$, is the simplicial complex whose 0-simplices are the elements of X and, for $k \geq 1$, its k-dimensional simplices are formed by every subset $\{x_0, x_1, \dots, x_k\} \subseteq X$ such that $d(x_i, x_j) \leq \alpha$ for every $i, j = 1, \dots, k$.

The family $\mathcal{R}(X, d) \coloneqq \{\mathcal{R}_\alpha(X, d)\}_{\alpha>0}$ is named **Vietoris-Rips filtration**.

Given a real function $f\colon X \to R$, let $X_\alpha \coloneqq f^{-1}((-\infty, \alpha]) \subseteq X$ be the **pre-image of f delimited by** α. We define the **Vietoris-Rips filtration associated with f** as $\mathcal{R}(X, d, f) \coloneqq \{\mathcal{R}_\alpha(X_\alpha, d)\}_{\alpha>0}$.

Definition 5.1.2 (Čech filtration)**.** Let (X, d) be a finite metric space and let $\alpha > 0$. The **Čech complex** associated with X of radius α, $\check{\mathcal{C}}_\alpha(Z, d)$, is the simplicial complex whose 0-simplices are the elements of X and, for $k \geq 1$, its k-dimensional simplices are formed by every subset $\{x_0, x_1, \dots, x_k\} \subseteq X$ such that there exists some $x \in X$ such that $d(x, x_i) \leq \alpha$ for all $i = 1, \dots, k$.

The family $\check{\mathcal{C}}(Z, d) \coloneqq \{\check{\mathcal{C}}_\alpha(Z, d)\}_{\alpha>0}$ is named **Čech filtration**.

Given a real function $f\colon X \to R$, let $X_\alpha \coloneqq f^{-1}((-\infty, \alpha]) \subseteq X$. We define the **Čech filtration associated with f** as $\check{\mathcal{C}}(X, d, f) \coloneqq \{\check{\mathcal{C}}_\alpha(X_\alpha, d)\}_{\alpha>0}$.

To prove Theorem 5.1.8 we first give tree auxiliary lemmas that we will need. The first, just shows how to embed a finite metric space into the Euclidean space with the uniform norm.

Lemma 5.1.3 (Exercise 3.5.4, [4])**.** *Any finite metric space of cardinality n can be isometrically embedded into $(\mathbb{R}^n, \ell^\infty)$.*

Proof. Let X be a compact metric space and let $C(X)$ be the space of all continuous functions from X to $\mathbb{R}$. Let $f, g \in C(X)$. Recall the uniform distance given by

$$d_\infty(f, g)\colon \sup |f(x) - g(x)|.$$

First, we will check that the pair $(C(X), d_\infty)$ is a metric space. Naturally nonnegativity holds and

$$d_\infty(f, f) = \sup |f(x) - f(x)| = 0.$$

Commutativity also holds as

$$d_\infty(f, g) = \sup |f(x) - g(x)| = \sup |g(x) - f(x)| = d_\infty(g, f).$$

Finally, if $h \in C(X)$, triangle inequality holds because

$$\begin{aligned} d_\infty(f,h) &= \sup |f(x) - h(x)| = \sup |f(x) + g(x) - g(x) - h(x)| \\ &\le \sup |f(x) - g(x)| + \sup |g(x) - h(x)| = d_\infty(f,h) + d_\infty(h,g). \end{aligned}$$

Now we are going to verify that the map $E \colon X \to C(X)$ defined by $E(x) = d(x,\cdot)$ is an isometric embedding onto its image. Note that

$$d_\infty(d(x,\cdot), d(y,\cdot)) = \sup_z |d(x,z) - d(y,z)| \le \sup_z |d(x,y)| = d(x,y).$$

On the other hand if we take $z = y$ we then have

$$|d(x,y) - d(y,y)| = d(x,y),$$

and therefore

$$\sup_z |d(x,z) - d(y,z)| \ge= d(x,y).$$

The proof of the lemma is just an analogous case taking $C_n(X)$ as the set of continuous functions $f \colon X \to \mathbb{R}^n$, and, for every $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $\ell^\infty(x,y) = \max_i |x_i - y_i|$. □

Lemma 5.1.4 (Lemma VII, [9]). *Let $X \subset \mathbb{R}^n$ and $\alpha > 0$. Then the α-Čech and the α-Rips complexes coincide when using the ℓ^∞-norm. That is*

$$\check{C}_\alpha(X, \ell^\infty) = R_\alpha(X, \ell^\infty).$$

Proof. For $n = 1$ the result is immediate. For $n > 1$ note that balls with the ℓ^∞ norm are actually cubes, and the entire problem decomposes into cross-products of the $d = 1$ case. □

We will give the third lemma without proof as it involves going deep into homotopy theory, and presenting it here will require a long detour. nonetheless, we give a couple of definitions to make the lemma understandable. The lemma is proven in [6][Lemma 3.3] and it is an adaptation for persistence theory of the Nerve Theorem (see [11][Corollary 4G.3.]).

Definition 5.1.5 (Paracompact space). Let X be a topological space. It is said to be **paracompact** if for all covering $\mathcal{U}$ of X, there exists $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V}$ is a finite covering.

Definition 5.1.6 (Good cover). Let S be a topological space and I a set of indexes. A **good cover** of S is a family $\mathcal{U} = \{U_i\}_{i \in I}$ of open subsets covering S such that for every finite subset $J \subset I$, the common intersection

$$\bigcap_{j \in J} U_j$$

is either empty or contractible.

Lemma 5.1.7 (Lemma 3.3, [6]). *Let $S \subset S'$ be two paracompact spaces. Let $\mathcal{U} = \{U_x\}_{x \in A}$, $\mathcal{U}' = \{U'_x\}_{x \in A'}$ be two good covers of S and S' respectively, based on finite parameter sets $A \subset A'$ such that $U_x \subset U'_x$ for all $x \in A$. Then the homotopy equivalences $\mathcal{N}\mathcal{U} \to S$ and $\mathcal{N}U' \to S'$ commute with the canonical inclusions $S \to S'$ and $\mathcal{N}\mathcal{U} \to \mathcal{N}\mathcal{U}'$ at homology level.*

Theorem 5.1.8 (Theorem 3.1, [5]). *Let (X, d_X), (Y, d_Y) be finite metric spaces. Then, for any $k \in \mathbb{N}$,*

$$d_{\text{bot}}(D_k(\mathcal{R}(X, d_X)), D_k(\mathcal{R}(Y, d_Y))) \leq d_{\text{GH}}((X, d_X), (Y, d_Y)).$$

Proof. Let $\varepsilon = d_{\text{GH}}((X, d_X), (Y, d_Y))$. As X and Y are finite, they are compact, and therefore the infimum when computing Gromov-Hausdorff distance using Definition 1.5.3 is in fact a minimum. That is, there exists a metric space (Z, d_Z) and two isometric embeddings $\gamma_X \colon X \to Z$ and $\gamma_Y \colon Y \to Z$ such that

$$d_{\text{H}}^Z(\gamma_X(X), \gamma_Y(Y)) = \varepsilon,$$

where d_{H}^Z denotes the Hausdorff distance respect the distance d_Z. Consider the subspace $\gamma_X(X) \cup \gamma_Y(Y) \subseteq Z$ with the induced metric from Z. As both X and Y are finite, let

$$n \coloneqq \#(X) + \#(Y).$$

Hence, by Lemma 5.1.3, there exists an isometric embedding

$$\gamma \colon (\gamma_X(X) \cup \gamma_Y(Y), d_Z) \to (\mathbb{R}^n, \ell^\infty).$$

Let d_{H}^∞ denote the Hausdorff distance respect the distance d_∞. We then have

$$d_{\text{H}}^\infty(\gamma \circ \gamma_X(X), \gamma \circ \gamma_Y(Y)) = d_{\text{H}}^Z(\gamma_X(X), \gamma_Y(Y)) = \varepsilon.$$

Let δ_X be the distance function from a point in $\mathbb{R}^n$ to X, and analogously, let δ_X be the distance function to Y. In ℓ^∞ norm, by hoy we defined ε, we have

$$\|\delta_X - \delta_Y\|_\infty = \max_{i=1,\dots,n} |\delta_{x_i} - \delta_{y_i}| \leq \varepsilon.$$

As distance functions are linear, both δ_X and δ_Y are lower envelopes of piecewise linear functions and therefore they are piecewise linear too. Hence, both δ_X and δ_Y are tame and continuous so by Theorem 4.0.2 we have

$$d_{\text{bot}}(D(\delta_X), D(\delta_Y)) \leq \|\delta_X - \delta_Y\|_\infty \leq \varepsilon.$$

Let $\alpha \in \mathbb{R}$. Define an off-set of radius α around the image of the embedding of X into $\mathbb{R}^n$ as

$$\gamma \circ \gamma_X(X)^\alpha := \bigcup_{x \in \gamma \circ \gamma_X(X)} B_\alpha^{\ell^\infty}(x),$$

where $B_\alpha^{\ell^\infty}(x)$ denotes de ball of radius α and center x using distance d_∞. As balls in ℓ^∞ are hypercubes, they are convex, and therefore their intersection is either empty or contractible. By Lemma 5.1.7 know that δ_X has the same persistence diagram as the Čech complex $\check{\mathcal{C}}(\gamma \circ \gamma_X, \ell^\infty)$. By Lemma 5.1.4, when using the ℓ^∞-norm, Čech and Rips complexes coincide and so do their filtrations. As $\gamma \circ \gamma_X$ is an isometric embedding, we then have

$$\check{\mathcal{C}}(\gamma \circ \gamma_X, \ell^\infty) = \mathcal{R}(\gamma \circ \gamma_X, \ell^\infty) = \mathcal{R}(X, d_X).$$

Hence, the persistence diagram of $\mathcal{R}(X, \ell^\infty)$ is the same as the persistence diagram of γ_X. The same is true taking Y and therefore we have

$$d_{\text{bot}}(D(\mathcal{R}(X, d_X)), D(\mathcal{R}(Y, d_Y))) = d_{\text{bot}}(D(\gamma_X), D(\gamma_Y)) \leq \varepsilon.$$

$\square$

Proposition 5.1.9. *Let (X, d_X), (Y, d_Y) be finite metric spaces. Then, for any $k \in \mathbb{N}$, the bottleneck distance*

$$d_{\text{bot}}(D_k(\mathcal{R}(X, d_X)), D_k(\mathcal{R}(Y, d_Y))),$$

is a tight lower bound of

$$d_{\text{GH}}((X, d_X), (Y, d_Y)).$$

That is, it is the largest possible lower bound.

Proof. Its enough to find an example were both distances are equal. For so, take the two point spaces $X = \{a, b\}$ with distance $d_X(a, b) = 2$, and $Y = \{c, d\}$ with $d_Y(c, d) = 2 + 2\varepsilon$. Both spaces can be isometrically mapped into the real line $\mathbb{R}$, with X mapped to $\{0, 2\}$ and Y mapped to $\{-\varepsilon, 2 + \varepsilon\}$. Hence $d_{\text{GH}}(X, Y) \leq \epsilon$.

On the other hand, the 0-dimensional persistence diagram of the Rips filtration of (X, d_X) and (Y, d_Y) are

$$D_0(\mathcal{R}(X, d_X)) = \{(0, \infty), (0, 1)\},$$
$$D_0(\mathcal{R}(Y, d_Y)) = \{(0, \infty), (0, 1 = \varepsilon)\},$$

and therefore, $d_{\text{bot}}(D_k(\mathcal{R}(X, d_X)), D_k(\mathcal{R}(Y, d_Y))) = \varepsilon$. $\square$

5.2 A generalization with endowed functions

The following theorem generalizes Theorem 5.1.8.

Theorem 5.2.1 (Theorem 3.2, [5]). *Let (X, d_X), (Y, d_Y) be finite metric spaces endowed with the functions $f\colon X \to \mathbb{R}$ and $g\colon Y \to \mathbb{R}$. Then*

$$d_{\text{bot}}(D_k(\mathcal{R}(X, d_X, f)), D_k(\mathcal{R}(Y, d_Y, g))) \leq d^1_{\text{GH}}((X, d_X, f), (Y, d_Y, g)).$$

Proof. We follow a similar procedure to the proof of Theorem 5.1.8. Start setting

$$\varepsilon := d^1_{\text{GH}}((X, d_X, f), (Y, d_Y, g)).$$

For every $\alpha \in \mathbb{R}$, recall the notation for the pre-images of f and g by α,

$$X_\alpha := f^{-1}((-\infty, \alpha]) \subseteq X,$$
$$Y_\alpha := g^{-1}((-\infty, \alpha]) \subseteq Y.$$

As before, as X and Y are finite, the infimum in d^1_{GH} of Definition 1.5.12 is actually a minimum realized by some correspondance $R \in (X \times Y)$. Also, the disjoint union $Z = X \cup Y$, can be endowed with a metric d_Z and a pair of inclusions $\gamma_X\colon X \to Z$, $\gamma_Y\colon Y \to Z$ such that for every $(x, y) \in R$,

$$d_Z(\gamma_X(X), \gamma_Y(Y)) \leq \frac{1}{2}\operatorname{dis}(R) \leq \varepsilon, \text{ and}$$
$$|f(x) - g(y)| \leq \|f - g\|_{\ell^\infty} \leq \varepsilon.$$

By Lemma 5.1.3, $(\gamma_X(X) \cup \gamma_Y(Y), d_Z)$ can be isometrically embedded by some γ into $(\mathbb{R}^n, \ell^\infty)$, where

$$n := \#(X) + \#(Y).$$

Hence, for every $(x, y) \in R$ we have

$$\|\gamma \circ \gamma_X(X) - \gamma \circ \gamma_Y(Y)\|_{\ell^\infty}.$$

Note that the filtrations given by the off-set defined in the proof of Theorem 5.1.8, can be seen as persistence modules $V := \{\gamma \circ \gamma_X(X_\alpha)^\alpha\}_{\alpha>0}$ and $W := \{\gamma \circ \gamma_Y(Y_\alpha)^\alpha\}_{\alpha>0}$ are ε-interleaved. That is, for all $\alpha > 0$,

$$\gamma \circ \gamma_X(X_\alpha)^\alpha \subseteq \gamma \circ \gamma_Y(Y_\alpha)^{\alpha+\varepsilon} \subseteq \gamma \circ \gamma_X(X_\alpha)^{\alpha+2\varepsilon}.$$

This is because for every element $p \in \gamma \circ \gamma_X(X_\alpha)^\alpha$, there exists some $x \in X$ such that

$$\|p - \gamma \circ \gamma_X(x)\|_{\ell^\infty} \leq \alpha.$$

Hence, taking some $y \in Y$ such that $(x, y) \in R$ we have that

$$\|\gamma \circ \gamma_X(x) - \gamma \circ \gamma_Y(y)\|_{\ell^\infty} \leq \varepsilon,$$

and as

$$g(y) \leq f(x) + \varepsilon \leq \alpha + \varepsilon,$$

we have that $y \in Y_{\alpha+\varepsilon}$ and therefore

$$\|p - \gamma \circ \gamma_Y(y)\|_{\ell^\infty} \leq \alpha + \varepsilon,$$

and so $p \in \gamma \circ \gamma_Y(Y_\alpha)^{\alpha+\varepsilon}$. The second inclusion follows analogously.

Note that as we have two ε-interleaved modules we can use 3.0.1 to expresses the bottleneck distance as the interleaving distance. Thus

$$d_{\text{bot}}(D_k(V), D_k(W)) = d_{\text{int}}(V, W) \leq \epsilon.$$

Analogously to the previous proof, Lemma 5.1.7 tells this inequality is also valid taking the Čech filtrations, and Lemma 5.1.4 let us take the Rips filtrations, completing the proof. □

Chapter 6

Vectorizations' Stability Theorems

The study of stability in topological data analysis is crucial for ensuring that small perturbations in the input data lead to controlled changes in its topological representations. This chapter explores various vectorization methods—persistence landscapes, persistence images, and Euler curves—and their associated stability theorems. By establishing bounds on how these vectorizations change with respect to distances between persistence modules or diagrams, we gain a deeper understanding of their robustness in practical applications.

Section 6.1 reviews the results of the 2015 paper from Peter Bubenik [2]. Section 6.2 follows the 2020 paper Henry Adams et al. [1] and Section 6.3 presents the results from Paweł Dłotko and Davide Gurnari 2023 paper, [7].

6.1 Persistence landscapes

Persistence landscapes provide a functional summary of persistence diagrams, converting them into a sequence of real-valued functions that capture topological features in a more tractable form. This section introduces the definition of persistence landscapes, their key properties, and stability results that relate the landscape distance to the bottleneck and Wasserstein distances between persistence diagrams.

Recall from Definition 1.1.18 that given some persistence module (V, π) and a fixed k the k-th persistent Betti number associated to $x \leq y \in \mathbb{R}$ is given by

$$\beta_x^y(V) = \dim(\operatorname{im} \pi_{x \leq y}).$$

Lemma 6.1.1 (Lemma 1, [2]). *Let (V, π) be a persistence module and let $a \leq b \leq c \leq d \in \mathbb{R}$. Then $\beta_b^c \geq \beta_a^d$.*

Proof. Since $\pi_{a\leq d} = \pi_{c\leq d} \circ \pi_{b\leq c} \circ \pi_{a\leq b}$, the dimension of the image of $\pi_{a\leq d}$ must be smaller than the one of just $\pi_{b\leq c}$. □

Definition 6.1.2 (Rank function)**.** The **rank function** of a persistence module V is the function $\delta\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$\lambda(b,d) = \begin{cases} \beta_b^d & \text{if } b \leq d \\ 0 & \text{otherwise.} \end{cases}$$

We can change coords to define an analogous function but defined on the upper half plane. Let

$$m = \frac{b+d}{2}, \qquad h = \frac{d-b}{2}.$$

The **rescaled rank function** is the function $\delta\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$\lambda(m,h) = \begin{cases} \beta^{m-h,m+h} & \text{if } \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 6.1.3 (Lemma 2, [2])**.** *Let $0 \leq h_1 \leq h_2$ be real numbers. Then $\beta_{t-h_1}^{t+h_1} \geq \beta_{t-h_2}^{t+h_2}$.*

Proof. As $t - h_2 \leq t - h_1 \leq t + h_1 \leq t + h_2$, the inequality comes from Lemma 6.1.1. □

Let $\overline{\mathbb{R}} \coloneqq \mathbb{R} \cup \{-\infty, \infty\}$ denote the extended real numbers.

Definition 6.1.4 (Persistence landscape)**.** A **persistence landscape** is a function $\lambda\colon \mathbb{N} \times \mathbb{R} \to \overline{\mathbb{R}}$, defined as

$$\lambda(k,t) \coloneqq \sup\{m \geq 0 \mid \beta^{t-m,t+m} \geq k\}.$$

Note that this function can also be seen as a sequence of function $\lambda_k\colon \mathbb{R} \to \overline{\mathbb{R}}$, where $\lambda_k(t) = \lambda(k,t)$.

Definition 6.1.5 (K-Lipschitz)**.** Let (X, d_X), (Y, d_Y) be two metric spaces and let $K > 0$. A **K-Lipschitz** map is a map $f\colon (X, d_X) \to (Y, d_Y)$, such that for every $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2).$$

Lemma 6.1.6 (Lemma 3, [2])**.** *Let $\lambda_k\colon \mathbb{R} \to \overline{\mathbb{R}}$ be an element of a persistence landscape. The following properties are verified.*

1. $\lambda_k(t) \geq 0$,

2. $\lambda_k(t) \geq \lambda_{k+1}(t)$,

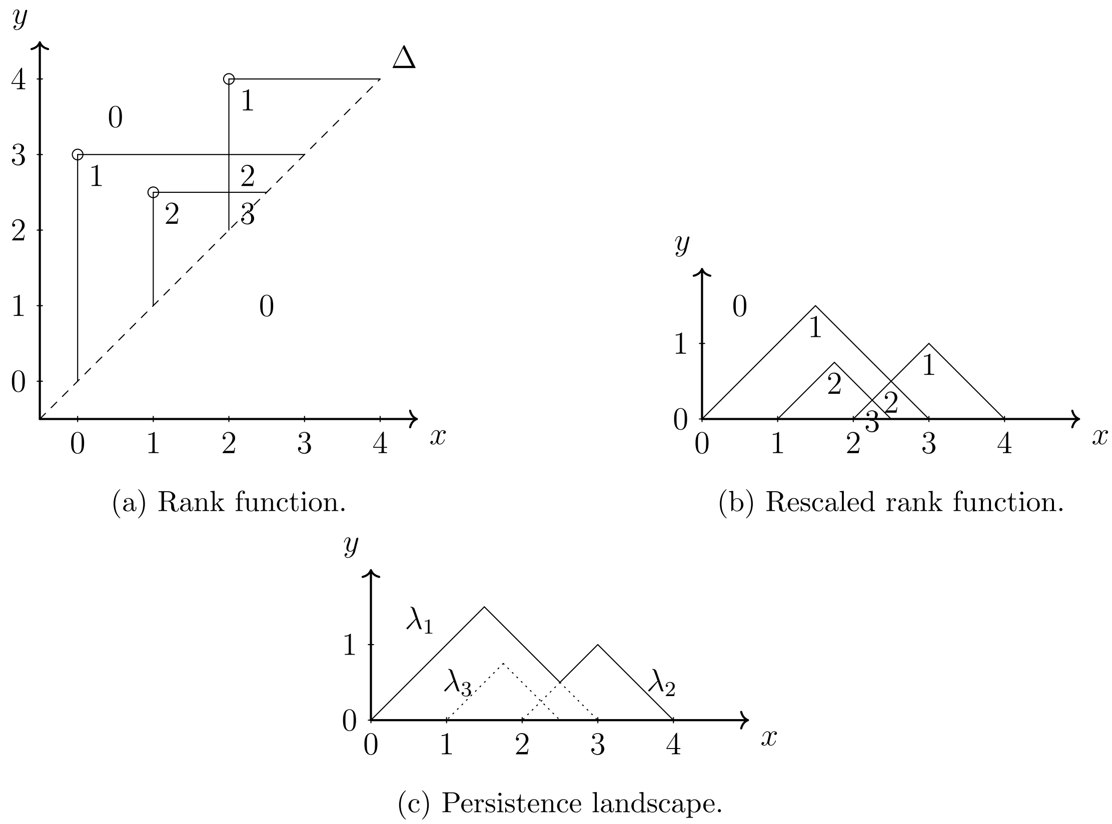
3. λ_k *is* 1*-Lipschitz, that is, for* $t, s \in \mathbb{R}$*,* $|\lambda_k(t) - \lambda_k(s)| \leq |t - s|$*.*

Proof. Properties 1. and 2. came directly from the definition. For 3., suppose $\lambda_k(s) \leq \lambda_k(t)$. If $\lambda_k(t) \leq |t-s|$ then, of course, $\lambda_k(t) - \lambda_k(s) \leq \lambda_k(t) \leq |t-s|$. Else, if $\lambda_k(t) > |t-s|$, we can take some $h \in (0, \lambda_k(t) - |t-s|)$ verifying

$$t - \lambda_k(t) < s - h < s + h < t + \lambda_k(t).$$

Hence, by Lemma 6.1.1 we have $\beta_{s-h}^{s+h} \geq k$ and $\lambda_k(s) \geq \lambda_k(t) - |t-s|$. Therefore, $\lambda_k(t) - \lambda_k(s) \leq |t-s|$. ☐

Example 6.1.7 (Persistence landscape)**.** Figure 6.1 shows the rank function, the rescaled rank function and the persistence diagram of one of the persistence diagrams of Example 1.4.13.



(a) Rank function.

(b) Rescaled rank function.

(c) Persistence landscape.

Figure 6.1: Persistence landscape of a persistence diagram.

Example 6.1.8. One of the main advantages of persistence landscapes in comparison with persistence diagrams is that they count with a nice geometry to work with. For example, in Figure 6.2 we can see how meanwhile persistence diagrams might not have a unique mean, persistence landscapes do have.

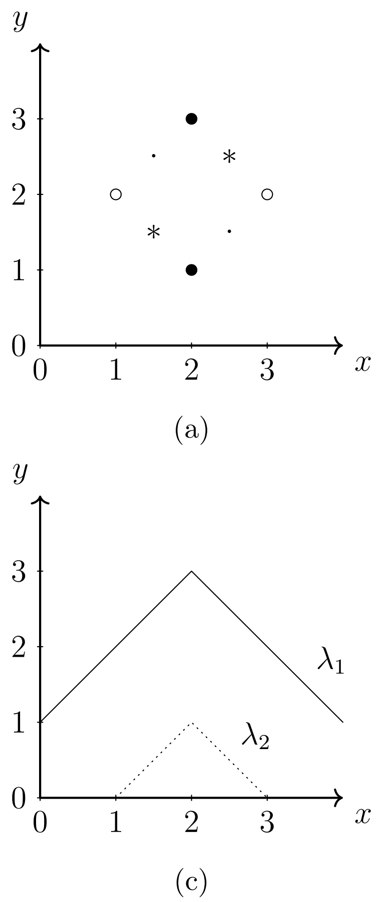


Figure 6.2: The rescaled persistence diagrams of 6.2a, $\{(1,2),(3,2)\}$ and $\{(2,1),(2,3)\}$ have two (Fréchet) means: $\{(1.5,2.5),(2.5,1.5)\}$ and $\{(1.5,1.5),(2.5,2.5)\}$. In contrast, their corresponding persistence landscapes 6.2b and 6.2c have a unique mean 6.2d.

Definition 6.1.9 (p-landscape distance)**.** Let W and W be two persistence modules, and let λ and λ' its corresponding persistence landscapes. Let $1 \leq p \leq \infty$. The **p-landscape distance between persistence modules** V and W is defined as

$$\Lambda_p(V,W) \coloneqq \|\lambda - \lambda'\|_p.$$

Similarly, if D and D' are two persistence diagrams, and are λ and λ' its corresponding persistence landscapes, The **p-landscape distance between persistence diagrams** V and W is defined as

$$\Lambda_p(D,D') \coloneqq \|\lambda - \lambda'\|_p.$$

Theorem 6.1.10 (Theorem 17, [2])**.** *Let (V,π) and (W,θ) two persistence modules. Then their ∞-landscape distance between them is a lower bound of their interleaving distance.*

That is

$$\Lambda_\infty(V, W) \leq d_{\text{int}}(V, W)$$

Proof. Suppose V and W are δ-interleaved, recall definition 1.2.6. Then, considering $\pi_{t-m \leq t+m}$ and $\pi_{t-m+\delta \leq t+m-\delta}$, we have that $t-m \leq t-m+\delta \leq t+m-\delta \leq t+m$, hence by Lemma 6.1.1 we have

$$\beta_{t-m+\delta}^{t+m-\delta}(W) \geq \beta_{t-m}^{t+m}(V).$$

Now if δ and δ' are the corresponding persistence landscapes of V and W, by Definition 6.1.4, for every $k \leq 1$, $\lambda'(k,t) \geq \lambda(k,t) - \delta$. Hence, it follows

$$\|\lambda - \lambda'\|_\infty \leq \delta.$$

□

Theorem 6.1.11 (Theorem 12, [2])**.** *Consider the persistence modules $V = F_x$, $W = G_x$ given by the maps $f, g \colon X \to \mathbb{R}$. Then*

$$\Lambda_\infty(V, W) \leq \|f - g\|_\infty$$

Proof. The proofs follows from applying Theorem 6.1.10, Theorem 3 and Theorem 4.0.2, in such order. Hence

$$\Lambda_\infty(V, W) \leq d_{\text{int}}(V, W) = d_{\text{bot}}(D(f), D(g)) \leq \|f - g\|_\infty.$$

□

Theorem 6.1.12 (Theorem 13, [2])**.** *Let D and D' be two persistence diagrams, then*

$$\Lambda_\infty(D, D') \leq d_{\text{bot}}(D, D').$$

Proof. To prof this theorem is enough to consider the persistence modules given by the sum of every interval module given by each point in the diagrams, following Theorem 2.1.6. Then the result follows by previous Theorem 6.1.11. □

This stability results can be extended for $p \neq \infty$ obtaining some less nice, jet still interesting, upper bounds (see [2]).

6.2 Persistence images

Persistence images offer an alternative vectorization by transforming persistence diagrams into finite-dimensional representations through kernel density estimation. On practice, the use of persistence images with vector-based machine learning tools can be useful to identify features of data containing discriminating topological information (see [1]). Here, we define persistence surfaces and images, analyze their stability under the 1-Wasserstein distance, and derive bounds that ensure their reliability in machine learning and statistical applications.

Definition 6.2.1 (Persistence surface)**.** Let D be a persistence diagram. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(x, y) = (x, y - x)$. Fix a nonnegative weighting function $f\colon \mathbb{R}^2 \to R$ that is zero along the horizontal axis, continuous and picewise differentiable. Fix a differentiable probability distribution $\phi_u\colon \mathbb{R}^2 \to \mathbb{R}$, with mean $u \in \mathbb{R}^2$. The **persistence surface** associated to D, by f and ϕ_u is a function $\rho_D\colon \mathbb{R}^2 \to \mathbb{R}$ defined as

$$\rho_D(z) \coloneqq \sum_{u \in T(D)} f(u)\phi_u(z).$$

Definition 6.2.2 (Persistence image)**.** Let D be a persistence diagram with an associated persistence surface ρ_D. The **persistence image** of D by ρ_D is the collection ρ of **pixels**

$$I(\rho_D)_p \coloneqq \iint_p \rho_B dydx.$$

Let $h\colon \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. We will denote the maximal norm of the gradient vector of h as

$$|\nabla h| = \sup_{z \in \mathbb{R}^2} \|\nabla h(z)\|_2.$$

Hence, by the fundamental theorem of calculus for line integrals, for every $u, v \in \mathbb{R}^2$, we have

$$|h(u) - h(v)| \leq |\nabla h| \|u - v\|_2.$$

Now, for a differentiable probability distribution $\phi_u\colon \mathbb{R}^2 \to \mathbb{R}$, with mean $u \in \mathbb{R}^2$ we can denote $|\nabla \phi_u|$ as $|\nabla \phi|$ and $\|\phi_u\|$ as $\|\phi\|$ because both the maximal directional derivative and the uniform norm of a fixed differentiable probability function are invariable under translation.

Example 6.2.3 (Persistence images)**.** Figure 6.3 (taken from [1]) depicts the pipeline proposed by persistence images. It starts with a dataset from were we compute, for a desired $k \geq 0$, the k-th persistence diagram B, given a filtration of the data. After applying the linear transformation $T(B)$ we compute the corresponding persistence surface. Finally, we can compute a persistence image with a chosen resolution, which will serve as a proper data resume.

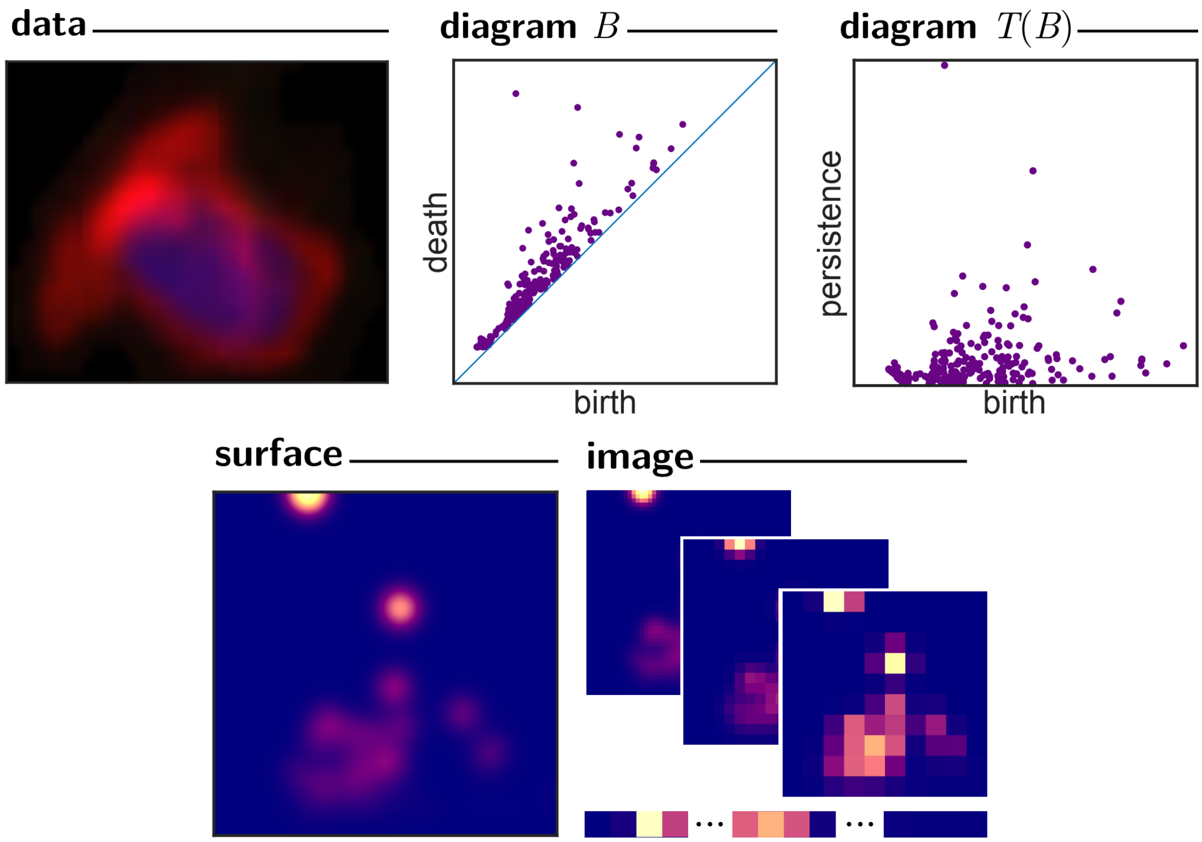


Figure 6.3: (From [1][Figure 1]) Algorithm pipeline to transform data into a persistence image.

Lemma 6.2.4 (Lemma 3, [1]). *Let $u, v \in \mathbb{R}^2$. Fix a nonnegative weighting function $f\colon \mathbb{R}^2 \to R$ that is zero along the horizontal axis, continuous and picewise differentiable. Fix a differentiable probability distribution $\phi_u\colon \mathbb{R}^2 \to \mathbb{R}$, with mean $u \in \mathbb{R}^2$. The following inequality asserts.*

$$\|f(u)\phi_u - f(v)\phi_v\|_\infty \leq (\|f\|_\infty |\nabla\phi| + \|\phi\|_\infty |\nabla f|)\|u - v\|_2.$$

Proof. For any $z \in \mathbb{R}^2$ we have

$$|\phi_u(z) - \phi_v(z)| = |\phi_u(z) - \phi_u(z + u - v)| \leq |\nabla\phi|\|u - v\|_2.$$

Hence,

$$\|\phi_u - \phi_v\| \leq |\nabla\phi|\|u - v\|_2.$$

Applying this, we can now develop

$$\begin{aligned}|f(u)\phi_u(z) - f(v)\phi_v(z)| &= |f(u)(\phi_u(z) - \phi_v(z)) + (f(u) - f(v))\phi_v(z)| \\ &\leq \|f\|_\infty |\phi_u(z) - \phi_v(z)| + \|\phi\|_\infty |f(u) - f(v)| \\ &\leq \|f\|_\infty \|\nabla\phi\| \|u - v\|_2 + \|\phi\|_\infty \|\nabla f\| \|u - v\|_2 \\ &= (\|f\|_\infty \|\nabla\phi\| + \|\phi\|_\infty \|\nabla f\|) \|u - v\|_2.\end{aligned}$$

□

Persistence surfaces are stable with respect to the 1-Wasserstein distance (see 1.4.8).

Theorem 6.2.5 (Theorem 4, [1])**.** *Let D, D' be two persistent diagrams and $\rho_D, \rho_{D'}$ two persistence surfaces associated to each diagram respectively. Then*

$$\|\rho_B - \rho_{B'}\|_\infty \leq \sqrt{10}(\|f\|_\infty |\nabla\phi| + \|\phi\|_\infty |\nabla f|)\omega_1(D, D').$$

Proof. Since D and D' consist of finitely many points, there exists a matching γ that achieves the infimum in the Wasserstein distance. Then

$$\begin{aligned}\|\rho_B - \rho_{B'}\|_\infty &= \left\| \sum_{u \in T(B)} f(u)\phi_u - \sum_{u \in T(B)} f(\gamma(u))\phi_{\gamma(u)} \right\|_\infty \\ &\leq \sum_{u \in T(B)} \left\| f(u)\phi_u - f(\gamma(u))\phi_{\gamma(u)} \right\|_\infty ,\end{aligned}$$

We now apply Lemma 6.2.4 and then note that $\| \cdot \|_2 \leq \sqrt{2} \| \cdot \| in \mathbb{R}^2$. and that $\|T(\cdot)\|_2 \leq \sqrt{5}\| \cdot \|$,

$$\begin{aligned}\sum_{u \in T(B)} \left\| f(u)\phi_u - f(\gamma(u))\phi_{\gamma(u)} \right\|_\infty &\leq \sqrt{2}(\|f\|_\infty \|\nabla\phi\| + \|\phi\|_\infty \|\nabla f\|) \sum_{u \in T(B)} \|u - \gamma(u)\|_\infty \\ &\leq \sqrt{10}(\|f\|_\infty \|\nabla\phi\| + \|\phi\|_\infty \|\nabla f\|) \sum_{u \in B} \|u - \gamma(u)\|_\infty \\ &= \sqrt{10}(\|f\|_\infty \|\nabla\phi\| + \|\phi\|_\infty \|\nabla f\|) W_1(B, B').\end{aligned}$$

□

Persistence images are stable with respect to the 1-Wasserstein distance.

Theorem 6.2.6 (Theorem 5, [1])**.** *Let A be be the maximum area of any pixel in the image, A' the total area of the image, and n the number of pixels in the image. Then*

$$\begin{aligned}\|I(\rho_B) - I(\rho_{B'})\|_\infty &\leq \sqrt{10}A(\|f\|_\infty |\nabla\phi| + \|\phi\|_\infty |\nabla f|)\omega_1(D, D'), \\ \|I(\rho_B) - I(\rho_{B'})\|_1 &\leq \sqrt{10}A'(\|f\|_\infty |\nabla\phi| + \|\phi\|_\infty |\nabla f|)\omega_1(D, D'), \\ \|I(\rho_B) - I(\rho_{B'})\|_2 &\leq \sqrt{10n}A(\|f\|_\infty |\nabla\phi| + \|\phi\|_\infty |\nabla f|)\omega_1(D, D').\end{aligned}$$

Proof. Note for any pixel p with area $A(p)$ we have

$$\begin{aligned}
|I(\rho_B)_p - I(\rho_{B'})_p| &= \left|\iint_p \rho_B \, dydz - \iint_p \rho_{B'} \, dydx\right| \\
&= \left|\iint_p \rho_B - \rho_{B'} \, dydx\right| \\
&\leq A(p)\|\rho_B - \rho_{B'}\|_\infty \\
&\leq \sqrt{10}A(p)\big(\|f\|_\infty|\nabla\phi| + \|\phi\|_\infty|\nabla f|\big)\omega_1(B, B'),
\end{aligned}$$

where the last inequality comes by applying Theorem 6.2.5. Hence we have

$$\begin{aligned}
\|I(\rho_B) - I(\rho_{B'})\|_\infty &\leq \sqrt{10}A(\|f\|_\infty|\nabla\phi| + \|\phi\|_\infty|\nabla f|)\omega_1(B, B') \\
\|I(\rho_B) - I(\rho_{B'})\|_1 &\leq \sqrt{10}A'\big(\|f\|_\infty|\nabla\phi| + \|\phi\|_\infty|\nabla f|\big)\omega_1(B, B') \\
\|I(\rho_B) - I(\rho_{B'})\|_2 &\leq \sqrt{n}\|I(\rho_B) - I(\rho_{B'})\|_\infty \\
&\leq \sqrt{10n}A\big(\|f\|_\infty|\nabla\phi| + \|\phi\|_\infty|\nabla f|\big)\omega_1(B, B').
\end{aligned}$$

□

6.3 Euler curves

The Euler characteristic curve provides a concise topological signature by tracking the evolution of the Euler characteristic across a filtration. This section examines its definition, connection to persistence diagrams, and stability properties, demonstrating how it serves as a stable summary for comparing filtered cell complexes.

Definition 6.3.1 (Euler characteristic)**.** Let K be a simplicial complex, and let K^p be its p-skeleton. The **Euler characteristic** of K is the alternating sum of the number of cells in its dimension

$$\chi(K) := \sum_d (-1)^d \#(K^d).$$

Definition 6.3.2. Let K be a simplicial complex. Let $f\colon K \to \mathbb{R}$ be a filtration function. The **Euler characteristic curve** is a function that assign an Euler characteristic χ for each filtration level $t \in \mathbb{R}$.

$$\mathrm{ECC}(K, t) := \chi(K_t),$$

where $K_t = f^{-1}(-\infty, t]$.

Using the L_1 norm, we can measure distances between Euler curves of different filtered simplicial complexes in a manner that results to be stable.

Definition 6.3.3 (k-th Betti curve)**.** Let K be a simplicial complex with filtration function $f\colon K \to \mathbb{R}$. Its **k-th Betti curve** is a function that assigns a Betti number for each filtration level $t \in \mathbb{R}$.

$$\beta_k(K,t) \coloneqq \beta_k(K_t).$$

Definition 6.3.4 (**k-th Betti curve for a persistence diagram**)**.** Let D be a persistence diagram. Given a point $(b,d) \in D$, let $I_{[b,d)}$ be the indicator function defined as

$$I_{[b,d)}(t) \coloneqq \begin{cases} 1 & \text{if } t \in [b,d), \\ 0 & \text{otherwise.} \end{cases}$$

The **k-th Betti curve for a persistence diagram** D with finitely many off-diagonal points is defined as

$$\beta_k(D,t) = \sum_{(b,d)\in D} I_{[b,d)}(t).$$

Proposition 6.3.5 (Proposition 1, [7])**.** *Let D, D' be two k-dimensional persistence diagrams. Their Betti curves are stable respect the 1-Wasserstein distance,*

$$\|\beta_k(D,t) - \beta_k(D',t)\|_1 \leq 2\omega_1(D,D').$$

Proof. Order the points in each diagram $(b_i,d_i) \in D$, and $(b_i',d_i') \in D'$ such that points with the same index i are paired by the optimal matching given by the 1-Wasserstein distance. We then can write the difference between the two Betti curves as

$$\|\beta_k(D,t) - \beta_k(D',t)\|_1 = \| \sum_i \big(I_{[b_i,d_i)}(t) - I_{[b_i',d_i')}(t)\big) \|_1 \leq \sum_i \| \big(I_{[b_i,d_i)}(t) - I_{[b_i',d_i')}(t)\big) \|_1.$$

We can now focus on each term of the sum and differentiate three cases. Case 1 comes up when $b_i \leq b_i' \leq d_i \leq d_i'$. Unfolding L_1 norm we get

$$\begin{aligned} \| \big(I_{[b_i,d_i)}(t) - I_{[b_i',d_i')}(t)\big) \|_1 =& \int_{b_i}^{b_i'} \left|I_{[b_i,d_i)(t)}\right| dt + \int_{b_i'}^{d_i} \left|I_{[b_i,d_i)(t)} - I_{[b_i',d_i')(t)}\right| dt + \int_{d_i}^{d_i'} \left|I_{[b_i',d_i')(t)}\right| dt \\ =& \int_{b_i}^{b_i'} |1|\,dt + \int_{b_i'}^{d_i} |1-1|\,dt + \int_{d_i}^{d_i'} |1|\,dt \\ =& |b_i' - b_i| + |d_i' - d_i| \leq 2\max(|b_i' - b_i|, |d_i' - d_i|). \end{aligned}$$

Case 2 comes up when $b_i \leq b_i' \leq d_i' \leq d_i$, getting

$$\begin{aligned} \| \big(I_{[b_i,d_i)}(t) - I_{[b_i',d_i')}(t)\big) \|_1 &= \int_{b_i}^{b_i'} |1|\,dt + \int_{b_i'}^{d_i'} |1-1|\,dt + \int_{d_i'}^{d_i} |1|\,dt \\ &= |b_i' - b_i| + |d_i - d_i'| \leq 2\max(|b_i' - b_i|, |d_i' - d_i|). \end{aligned}$$

A degenerate case 2 comes when some point $(b_i, d_i) \in C$ is matched to a point in the diagonal of D, and thus $b_i \leq b'_i = d'_i \leq d_i$. The final posible case comes when $b_i \leq d_i \leq b'_i \leq d'_i$, but this will never happen as if so, a better matching between the diagrams can be found matching both points to the diagonal.

Summing up both posible cases we complete the proof as

$$\begin{aligned} \|\beta_k(D,t) - \beta_k(D',t)\|_1 &= \| \sum_i \big(I_{[b_i,d_i)}(t) - I_{[b'_i,d'_i)}(t) \big) \|_1 \\ &\leq \sum_i 2\max(|b'_i - b_i|, |d'_i - d_i|) = 2\omega_1(D, D'). \end{aligned}$$

□

Theorem 6.3.6 (Proposition 2, [7]). *Let X, Y be two filtered cell complexes and let $D(X), D(Y)$ be its respective persistence diagrams. Then,*

$$\| \operatorname{ECC}(X,t) - \operatorname{ECC}(Y,t)\|_1 \leq \sum_k 2\omega_1(D(X), D(Y)).$$

Proof. The Euler-Poincaré formula implies that we can express the Euler characteristic curve of a filtered simplicial complex K as the alternating sum of its Betti curves. That is

$$\operatorname{ECC}(K,t) = \sum_k (-1)^k \beta_k(K,t).$$

Hence, applying Proposition 6.3.5 and the triangular inequality we get

$$\begin{aligned} \| \operatorname{ECC}(X,t) - \operatorname{ECC}(Y,t)\|_1 &= \| \sum_{k=0}^{n} \big((-1)^k (\beta_k(D(X),t) - \beta_k(D(Y),t)) \big) \|_1 \\ &\leq \sum_{k=0}^{n} \|(\beta_k(D(X),t) - \beta_k(D(Y),t))\|_1 \\ &\leq \sum_{k=0}^{n} 2\omega_1(D(X), D(Y)). \end{aligned}$$

□

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