Name: Gonzalo Ortega Carpintero

Exercise 1

Let $p\colon E \to B$ be a covering map. Let X be a path connected, simply connected and locally path connected topological space, let $x \in X$ and $f\colon X \to B$ be a continuos surjective function.

a) *Proof.* By the lifting criterion we have that $\forall e \in p^{-1}(f(x))$, $\exists! \tilde{f}_e\colon X \to E$ continuos such that $p \circ \tilde{f}_e = f$ and $\tilde{f}_e(x) = e$. Suppose that f is a homeomorphism onto its image. Hence f is bijective, continuos and f^{-1} is continuos.

Take $x_1 \neq x_2 \in X$. As in particular f is injective, $f(x_1) \neq f(x_2)$ and therefore $p^{-1}(f(x_1)) \neq p^{-1}(f(x_1))$, where p^{-1} denotes the pre-image of p. As $p^{-1} \circ f = \tilde{f}_e$, we then have $\tilde{f}_e(x_1) \neq \tilde{f}_e(x_1)$. Thus $\tilde{f}_e$ is injective. Every function is surjective onto its image, so in particular, $\tilde{f}_e$ is surjective onto its image and therefore bijective.

By the lifting criterion, $\tilde{f}_e$ is continuos. Also, $\tilde{f}_e^{-1} = f^{-1} \circ p|_{\mathrm{Im}(\tilde{f}_e)}$. As p is a covering, it is continuos, and we have that, $\tilde{f}_e^{-1}$ can be formed by the composition of continuos functions. Therefore, it is continuos too and $\tilde{f}_e$ is a homeomorphism into its image. □

b) *Proof.* Let $e \neq e' \in p^{-1}(f(x))$. Suppose that there exists $h \in E$ such that $h \in \operatorname{Im} \tilde{f}_e$ and $h \in \operatorname{Im} \tilde{f}'_e$. That is, there would exist $x, x' \in X$ such that $\tilde{f}_e(x) = h$ and $\tilde{f}'_e(x') = h$. Hence we have that

$$f(x) = p(\tilde{f}_e(x)) = p(\tilde{f}'_e(x)) = f(x').$$

As f is injective, it must be $x = x'$. But by the uniqueness of lifts, as $\tilde{f}_e$ and $\tilde{f}'_e$ coincide on one point, they must be the same, and therefore $e = e'$ making a contradiction. Thus, the images of $\tilde{f}_e$ and $\tilde{f}'_e$ must be disjoint.

Let $h \in p^{-1}(f(X))$. That means that $\exists x \in X$ such that $h = p^{-1}(f(x)) = \tilde{f}_h(x) \in \mathrm{Im}(\tilde{f}_h)$. Thus,

$$p^{-1}(f(X)) \subseteq \bigsqcup_{e \in p^{-1}(f(x))} \mathrm{Im}(\tilde{f}_e).$$

If $h \in \bigsqcup_{e \in p^{-1}(f(x))} \mathrm{Im}(\tilde{f}_e)$, then $\exists e \in p^{-1}(f(x))$ such that $h \in \mathrm{Im}(\tilde{f}_e)$. That means that $\exists x \in X$ such that $h = \tilde{f}_e(x) = p^{-1}(f(x)) \in p^{-1}(f(X))$. Thus,

$$p^{-1}(f(X)) \supseteq \bigsqcup_{e \in p^{-1}(f(x))} \mathrm{Im}(\tilde{f}_e).$$

□

Exercise 5

Let $/Z_8$ act on $S^3 = (z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1$ as $[m] \cdot (z, w) \coloneqq (\xi^m z, \xi^m w)$, where ξ is a primitive 8-th root of unity, $(z, w) \in S^3$ and $[m] \in \mathbb{Z}$.

a) As $\mathbb{C}^2$ is a metric space, let d be the induced distance function over S^3. Define

$$\epsilon \coloneqq \frac{1}{2} d((z, w), \xi(z, w)).$$

Let $[m] \neq [n] \in \mathbb{Z}_8$ act over $(z, w) \in S^3$. Let $B_\epsilon((z, w))$ be the open ball of center (z, w) and radius ϵ. Then $[m] \cdot B_\epsilon((z, w)) = B_\epsilon(\xi^m(z, w))$ and $[n] \cdot B_\epsilon((z, w)) = B_\epsilon(\xi^n(z, w))$. If

$$B_\epsilon(\xi^m(z, w)) \cap B_\epsilon(\xi^n(z, w)) \neq \emptyset,$$

then exists $(x, y) \in S^3$ such that $(x, y) \in B_\epsilon(\xi^m(z, w))$ and $(x, y) \in B_\epsilon(\xi^n(z, w))$. That is

$$d((x,y), \xi^m(z,w)) < \epsilon,$$
$$d((x,y), \xi^n(z,w)) < \epsilon.$$

But then, by the triangle inequality of metric spaces,

$$d(\xi^m(z,w), \xi^n(z,w)) \leq d(\xi^m(z,w), (x,y)) + d((x,y), \xi^n(z,w)) < 2\epsilon = d((z,w), \xi(z,w)).$$

This contradicts the fact that the minimum distance between the action of two roots of unity is the distance between the action of two consecutive roots. Hence, if $m \neq n$, then

$$B_\epsilon(\xi^m(z,w)) \cap B_\epsilon(\xi^n(z,w)) = \emptyset,$$

and the action if $\mathbb{Z}_8$ over S^3 is properly discontinuos.

b) Let $L = S^3/\mathbb{Z}_8$. As $\pi_1(S^3)$ is trivial, S^3 is simply connected. As Z_8 acts properly discontinuos on S^3, by Corollary 16, $\pi_1(L) \cong \mathbb{Z}_8$.

c) By the classification theorem for coverings, there exists a bijection between the set of coverings $p\colon E \to L$ and the conjugacy classes of groups of $\pi_1(L)$. The sub groups of $\mathbb{Z}_8$ are: $1, \mathbb{Z}_2, \mathbb{Z}_4$ and $\mathbb{Z}_8$. As all of them are normal, the conjugacy classes are the subgroups themselves. That is, up to equivalence of coverings, the four coverings of S^3 are:

$$p_1\colon S^3/\mathbb{Z}_8 \hookrightarrow_{\mathrm{id}} S^3/\mathbb{Z}_8,$$
$$p_2\colon S^3/\mathbb{Z}_4 \hookrightarrow_{/\mathbb{Z}_2} S^3/\mathbb{Z}_8,$$
$$p_3\colon S^3/\mathbb{Z}_2 \hookrightarrow_{/\mathbb{Z}_4} S^3/\mathbb{Z}_8,$$
$$p_4\colon S^3 \hookrightarrow_{/\mathbb{Z}_8} S^3/\mathbb{Z}_8,$$

where the notation $X/\mathbb{Z}_i$ with $i \in \{2, 4, 8\}$ means the quotient space of X by the action of $\mathbb{Z}_i$ given by $[m]_i \cdot (z, w) := (\xi_i^m z, \xi_i^m w)$. With $X \in \{S^3, S^3/\mathbb{Z}_2, S^3/\mathbb{Z}_4, S^3/\mathbb{Z}_8\}$, $[m]_i \in \mathbb{Z}_i$, $(z, w) \in X$ and where ξ_i is a primitive i-th root of unity.

d) *Proof.* Let $f\colon L \to T$ be any continuos map from L to $T = S^1 \times S^1$. If we consider $f_*(\pi_1(L)) \to \pi_1(T)$, we have that, since $\pi_1(L) \cong \mathbb{Z}_8$ is finite, $f_*(\pi_1(L))$ is going to be finite too. As $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$ is an infinite group, its only finite subgroup is the trivial subgroup 1. Hence, considering the universal covering of the torus, $\rho\colon \mathbb{R}^2 \to T$ we have that $f_*(\pi_1(L))$ is a subgroup of $\rho_*(pi_1(\mathbb{R}^2))$.

By the lifting property, there exists $\tilde{f}\colon L \to \mathbb{R}^2$ such that $f = \rho \circ \tilde{f}$. Fix $x_0 \in \mathbb{R}^2$, and let

$$\begin{aligned} H\colon L \times [0,1] &\to \mathbb{R}^2 \\ (l,t) &\mapsto (1-t)\tilde{f}(l) + t x_0. \end{aligned}$$

As $H(l, 0) = \tilde{f}(l)$ and $H(l, 1) = x_0$, we have that $\tilde{f}$ is nullhomotopic. If we now define

$$\begin{aligned} H'\colon L \times [0,1] &\to T \\ (l,t) &\mapsto \rho(H(l,t)), \end{aligned}$$

we have that $H'(l, 0) = \rho(\tilde{f}(l)) = f(l)$ and $H'(l, 1) = \rho(x_0)$ constant. Hence, f is nullhomotopic. □

Exercise 7

Let K be a tame knot. Let X de the 2-dimensional CW complex associated to the Wirtinger presentation of $\pi_1(X) \cong \pi_1(S^3 \setminus K)$. Let v be the 0-cell of X.

a) *Proof.* Let q be a positive prime number. If H is a normal subgroup of $\pi_1(X)$ of index q, then $|\pi_1(X)/H| = q$. The quotient is a finite group of prime order, hence it is a cyclic group. As the quotient is finite and cyclic, then it must be $\pi_1(X)/H \cong \mathbb{Z}/q\mathbb{Z}$.

To prove the existence of H, note that as K is tame, Wirtinger presentation of $\pi_1(X)$ has a finite number of generators $g_1, \dots, g_n$. Fix any $[k] \in \mathbb{Z}/q\mathbb{Z}$ different from 0 and note that it must be a generator of $\mathbb{Z}/q\mathbb{Z}$ since q is prime. Map every generator of $\pi_1(X)$ to $[k]$ trough the homomorphism $\phi\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}/q\mathbb{Z}$. That is, $\phi(g_i) = [k]$ for all $i = 1 \dots n$. This is well-defined because the Wirtinger presentation relations imply that all generators are conjugate to one another, and they are sent to the same element under the given assignment. Note that ϕ is surjective because h is a generator. By the first group isometry theorem we have

$$\pi_1(X)/\ker(\phi) \cong \mathrm{Im}(\phi) \cong \mathbb{Z}/q\mathbb{Z}.$$

Now, just take $H = \ker(\phi)$.

To show that H is unique, note that $\pi_1(X)/H$ is unique up to isomorphism and H would be subgroup generating the quotient map $\phi\colon \pi_1(X) \twoheadrightarrow \pi_1(X)/H \cong \mathbb{Z}/q\mathbb{Z}$. That is, $H = \ker(\phi)$ for some ϕ defined as above. Note that there are $q-1$ ways of defining ϕ, one for each element in $\mathbb{Z}/q\mathbb{Z}$ were we can map all generators. Nevertheless, every posible ϕ has the same kernel, the set of all generators powered to q: $\{g_i \in G \mid i = 1, \dots n\}$. Hence, as all posible kernels are the same, H is unique. $\square$

b) Let $K = 4_1$. Let $p : Y \to X$ be the covering map associate to $\pi_1(X, v)$. Let $q = 2$. As seen in class, the Wirtinger of $\pi_1(S^3/4_1)$ is

$$\pi_1(S^3/4_1) \cong \langle a, b, c, d \mid ac^{-1}b^{-1}c, bdc^{-1}d^{-1}, ca^{-1}d^{-1}a \rangle.$$

Following the procedure seen in class, Y^0 are the elements of $\mathbb{Z}/2\mathbb{Z}$:

$$Y^0 = \{[0] =: x_0, [1] =: x_1\}.$$

The elements of Y^1 are in bijection with $\{[0], [1]\} \times \{a, b, c, d\}$:

$$\begin{aligned} Y^1 = \big\{ & e^1_{(x_0,a)} =: e_1, e^1_{(x_0,b)} =: e_2, e^1_{(x_0,c)} =: e_3, e^1_{(x_0,d)} =: e_4, \\ & e^1_{(x_1,a)} =: e_5, e^1_{(x_1,b)} =: e_6, e^1_{(x_1,c)} =: e_7, e^1_{(x_1,d)} =: e_8 \big\}. \end{aligned}$$

Finally, elements in Y^1 are in bijection with $\{[0], [1]\} \times \{ac^{-1}b^{-1}c, bdc^{-1}d^{-1}, ca^{-1}d^{-1}a\}$:

$$\begin{aligned} Y^2 = \big\{ & e^2_{(x_0,ac^{-1}b^{-1}c)} =: s_1, e^2_{(x_0,bdc^{-1}d^{-1})} =: s_2, e^2_{(x_0,ca^{-1}d^{-1}a)} =: s_3, \\ & e^2_{(x_1,ac^{-1}b^{-1}c)} =: s_4, e^2_{(x_1,bdc^{-1}d^{-1})} =: s_5, e^2_{(x_1,ca^{-1}d^{-1}a)} =: s_6 \big\}. \end{aligned}$$

where the gluing maps can be given by

$$\begin{aligned} s_1 = e^2_{(x_0,ac^{-1}b^{-1}c)} &\longrightarrow e_1 e_3^{-1} e_6^{-1} e_7, \\ s_2 = e^2_{(x_1,ac^{-1}b^{-1}c)} &\longrightarrow e_5 e_7^{-1} e_2^{-1} e_3, \\ s_3 = e^2_{(x_0,bdc^{-1}d^{-1})} &\longrightarrow e_2 e_8 e_7^{-1} e_4^{-1}, \\ s_4 = e^2_{(x_1,bdc^{-1}d^{-1})} &\longrightarrow e_6 e_4 e_3^{-1} e_8^{-1}, \\ s_5 = e^2_{(x_0,ca^{-1}d^{-1}a)} &\longrightarrow e_3 e_1^{-1} e_8^{-1} e_5, \\ s_6 = e^2_{(x_1,ca^{-1}d^{-1}a)} &\longrightarrow e_7 e_5^{-1} e_4^{-1} e_1. \end{aligned}$$

References

[1] Allen Hatcher, *Algebraic Topology*, Allen Hatcher 2001.

[2] *I collaborated with Saioa to think for ideas to complete Exercise 5.*