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Exercise 1

Let $p\colon E \to B$ be a covering map. Let X be a path connected, simply connected and locally path connected topological space, let $x \in X$ and $f\colon X \to B$ be a continuos surjective function.

a) *Proof.* By the lifting criterion we have that $\forall e \in p^{-1}(f(x))$, $\exists! \tilde{f}_e\colon X \to E$ continuos such that $p \circ \tilde{f}_e = f$ and $\tilde{f}_e(x) = e$. Suppose that f is a homeomorphism onto its image. Hence f is bijective, continuos and f^{-1} is continuos.

   Take $x_1 \neq x_2 \in X$. As in particular f is injective, $f(x_1) \neq f(x_2)$ and therefore $p^{-1}(f(x_1)) \neq p^{-1}(f(x_1))$. As $p^{-1} \circ f = \tilde{f}_e$, we then have $\tilde{f}_e(x_1) \neq \tilde{f}_e(x_1)$. Thus $\tilde{f}_e$ is injective. Every function is surjective onto its image, so in particular, $\tilde{f}_e$ is surjective onto its image and therefore bijective.

   By the lifting criterion, $\tilde{f}_e$ is continuos. Also, $\tilde{f}_e^{-1} = f^{-1} \circ p$. As p is a covering, it is continuos, and we have that, $\tilde{f}_e^{-1}$ can be formed by the composition of continuos functions. Therefore, it is continuos too and $\tilde{f}_e$ is a homeomorphism into its image. $\square$

b) *Proof.* Let $e \neq e' \in p^{-1}(f(x))$. Suppose that there exists $h \in E$ such that $h \in \operatorname{Im} \tilde{f}_e$ and $h \in \operatorname{Im} \tilde{f}'_e$. That is, there would exist $x, x' \in X$ such that $\tilde{f}_e(x) = h$ and $\tilde{f}'_e(x') = h$. Hence we have that

$$f(x) = p(\tilde{f}_e(x)) = p(\tilde{f}'_e(x)) = f(x').$$

   As f is injective, it must be $x = x'$. But by the uniqueness of lifts, as $\tilde{f}_e$ and $\tilde{f}'_e$ coincide on one point, they must be the same, and therefore $e = e'$ making a contradiction. Thus, the images of $\tilde{f}_e$ and $\tilde{f}'_e$ must be disjoint.

   Let $h \in p^{-1}(f(X))$. That means that $\exists x \in X$ such that $h = p^{-1}(f(x)) = \tilde{f}_h(x) \in \operatorname{Im}(\tilde{f}_h)$. Thus,

$$p^{-1}(f(X)) \subseteq \bigsqcup_{e \in p^{-1}(f(x))} \operatorname{Im}(\tilde{f}_e).$$

   If $h \in \bigsqcup_{e \in p^{-1}(f(x))} \operatorname{Im}(\tilde{f}_e)$, then $\exists e \in p^{-1}(f(x))$ such that $h \in \operatorname{Im}(\tilde{f}_e)$. That means that $\exists x \in X$ such that $h = \tilde{f}_e(x) = p^{-1}(f(x)) \in p^{-1}(f(X))$. Thus,

$$p^{-1}(f(X)) \supseteq \bigsqcup_{e \in p^{-1}(f(x))} \operatorname{Im}(\tilde{f}_e).$$

$\square$

Exercise 5

Exercise 7

References

[1] Allen Hatcher, *Algebraic Topology*, Allen Hatcher 2001.