Distributed constraint satisfaction

1.1 D-CSP

CSP Variables $= \{x, y, z\}$

Domains $= \{\{1..3\}, \{1..3\}, \{1..3\}\}$

Constraints $= \{$
- $x > 1$ (1)
- $y < z$ (2)
- $x \leq y \leq 2$ (3)

$\}$

(1) Node consistency

$D' = \{\{2..3\}, \{1..3\}, \{1..3\}\}$ because it prunes the x' domain without care the other variables.

(2) Arc consistency

$D'' = \{\{2..3\}, \{1..2\}, \{2..3\}\}$ because there exists all pairs (a, b) of x and y respectively meets $x < z$.

(3) Path consistency

$D''' = \{\{2\}, \{2\}, \{2\}\}$ because all pairs (a, b) in x and y meet $x \leq y$ and in addition there exists a pair (b, c) in y and z that meets $y \leq 2$.

Problem:

- Inconsistent → the problem has no solution
- Solved → Domains just have one value
- Inclusive → At lest one domain has more the one value and is no possible to prune more.
- Consistent → The problem has some solutions that accomplishs with all constraints.

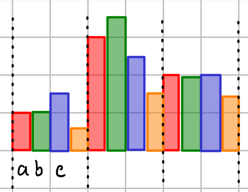
Notes about of the reading.

Ch 1: Distributed Constraint Satisfaction → CSP
- pruning
- backtraking

→ Pending distribution. (Naive)

Ch 2: Distribute Optimization.
- MDPs
- Search
- Negotiation, auctions, opt.
- Scheduling problem → integer programming
- Conventions (Laws)

Ch 3: Noncooperative games



b is Pareto dominate to a and c

a b and c are Pareto optimal

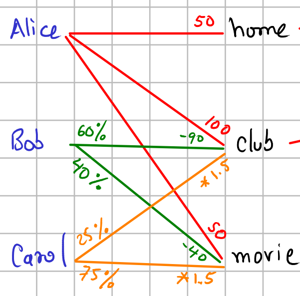
d is no pareto

Noncooperative Games ($\neg$ coalitional games)

└ individual └ group

Self-interested agents

- Utility theory → quantify preference
- utility function → mapping world to numbers
- Example:

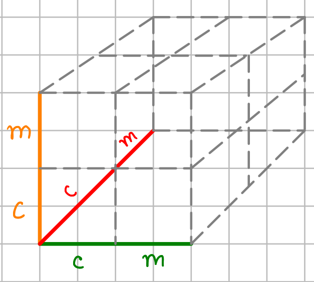


home: $+\ 50 \quad = 50$

club: $\begin{cases} +\ 0.25\big(0.6(100-90)*1.5 + 0.4(100)*1.5\big) \\ +\ 0.75\big(0.6(100-90) + 0.4(100)\big) \end{cases} = 51.75$

movie: $\begin{cases} +\ 0.25\big(0.6(50) + 0.4(50-40)\big) \\ +\ 0.75\big(0.6(50)*1.5 + 0.4(50-40)*1.5\big) \end{cases} = 46.75$

- Using Payoff Matrix



		Bob: club	Bob: movie
Carol	club	15	150
Carol	movie	10	100

Alice = club

		Bob: club	Bob: movie
Carol	club	50	10
Carol	movie	75	15

Alice = movie

Utility $\equiv$ Preference

- Outcomes $O = \{o_1 \ldots o_n\}$
- Preferences: $o_1, o_2 \in O$, let $o_1 \succeq o_2$
  let $o_1 \sim o_2$
  let $o_1 \succ o_2$
- Lottery: probability distribution over O

$$\mathcal{L} = [p_1 : o_1, \ldots, p_k : o_k]$$

where $o_i \in O$

$p_i \geq 0$ and $\sum_{i=1}^{k} p_i = 1$

Completeness

$\forall\, o_1, o_2$, $o_1 \succ o_2$ or
$o_2 \succ o_1$ or
$o_1 \sim o_2$

Transitivity

If $o_1 \succeq o_2$ and $o_2 \succeq o_3$ then
$o_1 \succeq o_3$

Substitutability

If $o_1 \sim o_2$ then

$[p : o_1, p_3 : o_3, \ldots, p_k : o_k]$
$= [p : o_2, p_3 : o_3, \ldots, p_k : o_k]$

Monotonicity

Continuity

Decomposability

If $\forall o_i \in O$, $P_{\ell_1}(o_i) = P_{\ell_2}(o_i)$ then
$\mathcal{L}_1 \sim \mathcal{L}_2$

Normal form

- TCP Dilemma

		colleague	
		correct	defective
me	correct	-1, -1	-4, 0
	defective	0, -4	-3, -3

(N, A, u)

- Set of players indexed by i till n
- $A_1 \times \dots \times A_n$
- $a = (a_1, \dots, a_n) \in A$ — Action profile
- $(u_1, \dots, u_n)$ where $u_i : A \mapsto \mathbb{R}$

why a_n? "n" means 4 or 9?

Common-payoff games: $\forall a \in A_1 \times \dots \times A_n$ and $i, j \in N$
$u_i(a) = u_j(a)$

	Quick	Slow
Quick	3, 3	2, 2
Slow	2, 2	1, 1

Zero-sum games: $\exists c, \forall a \in A_1 \times A_2$
$u_1(a) + u_2(a) = c$

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

No-zero-sum games ✓

Strategies (NFG)

- Pure strategy → single action
- Mixed strategy → using the probability distribution
  let (N, A, u), $S_i = \Pi(A_i)$

Not necessarily a "product"?

- Mixed strategy profile → $S_1 \times \dots \times S_n$

Doesn't matter if they are good or no?

- Support → set of actions with positive probability over S_i
  $\{a_i \mid s_i(a_i) > 0\}$
  Note: - Pure strategy → support a single action
  - Fully mixed → full support

Can I use this to obtain a probability?

Alice → Home = 50; Club = 51.75; Movie = 46.75

$s_i(a_i)$ → Probability that a_i is played under s_i

- Expected utility ?

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j=1}^{n} s_j(a_j)$$

More notes:

- $|A| = |S|$
- a_i contains utility; s_i contains probability.
- $u_i(a)$ means "utility"; $u_i(s)$ means "utility expected"

From optimality to equilibrium → strategy? output? utility? probability?

- optimal strategy → maximize the agent's expected payoff
  Note: The best strategy depends on the choices of others

Pareto → Point of view of an outside observer "utility of an strategy" is it ok?

- Pareto domination: $s \overset{PD}{\rightarrow} s'$ if $\forall i \in N$, $u_i(s) \geqslant u_i(s')$, and $\exists j \in N$, $u_j(s) > u_j(s')$
- Pareto optimality: $\nexists s' \in S$, $s' \overset{PD}{\rightarrow} s$

Equilibrium → Point of view of an individual agent the best?

- Best response: → of player i of s_{-i} is a mixed strategy $s_i^* \in S_i$
  such that $u_i(s_i^*, s_{-i}) \geqslant u_i(s_i, s_{-i})$ for all $s_i \in S$

  Note: this is not the concept of solution

- Nash equilibrium → s is a Nash equilibrium if $\forall i \in N$, s_i is the best response to s_{-i}
  → is strict if $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$
  → is weak if $u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i})$

		Husband: LW	Husband: WL
Wife	LW	2, 1	0, 0
Wife	WL	0, 0	1, 2

husband
- LW $\pi = 1/3$ $u_h(LW) = 1/3$?
- WL $\boxed{\pi = 2/3}$

wife
- LW $\boxed{\pi = 2/3}$ $u_w(LW) = 2/3$?
- WL $\pi = 1/3$

Rationalizability A strategy profile of a player i is ratinalizable if it is a best response to some beliefes that i could have about the strategies that other players have. → or will take.

		Player 2: Head	Player 2: Tail
Player 1	Head	1, -1	-1, 1
Player 1	Tail	-1, 1	1, -1

Maxmin strategy think of the worst-case

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

4 Computing

Nash equilibria two-player zero-sum games

Linear programming

Example: $G = (\{1,2\}, A_1 \times A_2, (u_1, u_2))$ Expected utility $U_1^* = -U_2^*$ Because is zero-sum games

$$\sum_{k\in A_2} \underbrace{u_1(a_1^j, a_2^k)}_{\text{constant}} \cdot s_2^k \leq U_1^*$$

$$\sum_{k\in A} s_2^k = 1, \quad s_2^k \geq 0$$

Because there are probabilities.

$$\sum_{k\in A_2} u_1(a_1^j, a_2^k) \cdot s_2^k + r_1^j = U_1^*$$

$$\sum_{k\in A_2} s_2^k = 1, \quad s_2^k \geq 0, \quad r_1^j \geq 0$$

Rock-Paper-Scissors

Sep 12/2023

	Rock	Mix (25%)	Paper	Mix (25%)	Scissors	Mix (50%)
Rock	0, 0	0.25	-1, 1	-0.25	1, -1	0
Mix (50%)	0					
Paper	1, -1		0, 0		-1, 1	
Mix (25%)	-0.25					
Scissors	-1, 1		1, -1		0, 0	
Mix (25%)	0.25					

50%(0) + 25%(1) + 25%(-1) = 0

50%(-1) + 25%(0) + 25%(1) = -0.25

50%(1) + 25%(-1) + 25%(0) = 0.25

25%(0) + 25%(-1) + 50%(1) = 0.25

25%(1) + 25%(0) + 50%(-1) = -0.25

25%(1) + 25%(-1) + 50%(0) = 0

Heads and tails

which value usually is posted when it is written just one?

	Heads	Tails	p, 1-p
Heads	1, -1	-1, 1	2p - 1
Tails	-1, 1	1, -1	1 - 2p
q, 1-q	1 - 2q	2q - 1	

$p(1) + (1-p)(-1) = p - 1 + p = 2p - 1$

$p(-1) + (1-p)(1) = -p + 1 - p = 1 - 2p$

$q(1) + (1-q)(-1) = q - 1 + q = 2q - 1$

$q(-1) + (1-q)(1) = -q + 1 - q = 1 - 2q$

Is per column or per row?

what values refers to? (oranges or greens) (columns or rows)

$P = 0.7$

$2(0.7) - 1 = 1.4 - 1 = 0.4$

$1 - 2(0.7) = 1 - 1.4 = -0.6$

	Heads 20%	Tails 80%	Expected Payoff
Heads 70%	-1	1	0.6
Tails 30%	1	-1	-0.6
Expected Payoff	-0.4	0.4	

20% (-1) + 80% (1) = -0.2 + 0.8 = 0.6

20% (1) + 80% (-1) = 0.2 + -0.8 = -0.6

70% (-1) + 30% (1) = -0.7 + 0.3 = -0.4

70% (1) + 30% (-1) = 0.7 + -0.3 = 0.4

Orange wins when =

Green wins when ≠

Orange should play Heads ! isn't it?
However, it is opposit to the logic!

Sep 13/2023

Other example:

		P_1	P_2	P_3
x_1	0.5	-2	1	2
x_2	0.5	2	-1	0
x_3	0.0	1	0	-2
		0	0	1

-2 (0.5) + 2 (0.5) + 1 (0) = 0

1 (0.5) + -1 (0.5) + 0 (0) = 0

2 (0.5) + 0 (0.5) + -2 (0) = 1

$$P_1 = -2x_1 + 2x_2 + x_3$$
$$P_2 = x_1 - x_2$$
$$P_3 = 2x_1 - 2x_3$$
$$x_1 + x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geqslant 0$$
$$z = \min\{P_1, P_2, P_3\}$$

$$z \leqslant -2x_1 + 2x_2 + x_3$$
$$z \leqslant x_1 - x_2$$
$$z \leqslant 2x_1 - 2x_3$$
$$x_1 + x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geqslant 0$$

Best is <

		25% R	50% S	25% P	
50%	R	0	1	-1	0.25
25%	S	-1	0	1	0
25%	P	1	-1	0	-0.25
		0	0.25	-0.25	0.0625

0.25(0) + 0.5(1) + 0.25(-1) = 0.25

0.25(-1) + 0.5(0) + 0.25(1) = 0

0.25(1) + 0.5(-1) + 0.25(0) = -0.25

0.25(0) + 0.5(0.25) + 0.25(-0.25) = 0.0625

0.5(0) + 0.25(-1) + 0.25(1) = ∅

0.5(1) + 0.25(0) + 0.25(-1) = 0.25

0.5(-1) + 0.25(1) + 0.25(0) = -0.25

Best is >

	P_1	P_2	P_3
q_1	-2	1	2
q_2	2	-1	0
q_3	1	0	-2

$$P_1 = -2q_1 + 2q_2 + 1q_3 = q_3 - 2(q_1 - q_2)$$
$$P_2 = 1q_1 - 1q_2 + 0q_3 = q_1 - q_2$$
$$P_3 = 2q_1 + 0q_2 - 2q_3 = 2(q_1 - q_3)$$
$$q_1 + q_2 + q_3 = 1$$
$$q_1, q_2, q_3 \geqslant 0$$
$$z = \min\{P_1, P_2, P_3\}$$

$$z \leqslant q_3 - 2(q_1 - q_2)$$
$$z \leqslant q_1 - q_2$$
$$z \leqslant 2(q_1 - q_3)$$
$$q_1 + q_2 + q_3 = 1$$
$$q_1, q_2, q_3 \geqslant 0$$

(1) $z \le q_3 - 2(q_1 - q_2)$
(2) $z \le q_1 - q_2$
(3) $z \le 2(q_1 - q_3)$
(4) $q_1 + q_2 + q_3 = 1$
(5) $q_1, q_2, q_3 \geq 0$

(4) $q_1 = 1 - q_2 - q_3$

(4)(1) $z \le q_3 - 2(1 - q_2 - q_3 - q_2)$
$z \le q_3 - 2 + 4q_2 + 2q_3$
(6) $z \le 3q_3 + 4q_2 - 2$

(4)(2) $z \le 1 - q_2 - q_3 - q_2$
(7) $z \le 1 - q_3 - 2q_2$

(4)(3) $z \le 2(1 - q_2 - q_3 - q_3)$
(8) $z \le 2 - 4q_3 - 2q_2$

(7)(6) $1 - q_3 - 2q_2 \le 3q_3 + 4q_2 - 2$
$-q_3 - 3q_3 \le 4q_2 + 2q_2 - 2 - 1$
$-4q_3 \le 6q_2 - 3$
(9) $q_3 \le -6/4\, q_2 + 3/4$

(7)(8) $1 - q_3 - 2q_2 \le 2 - 4q_3 - 2q_2$
$4q_3 - q_3 \le 2q_2 - 2q_2 + 2 - 1$
$3q_3 \le 1$
(10) $q_3 \le 1/3$

(9)(10) $-6/4\, q_2 + 3/4 \le 1/3$
$-6/4\, q_2 \le 1/3 - 3/4$
$-6/4\, q_2 \le -5/12$
$q_2 \le 5/12 \cdot 4/6$
$q_2 \le 20/72$
(11) $q_2 \le 5/18$

(4)(11)(10) $q_1 = 1 - 5/18 - 1/3$
(12) $q_1 = 7/18$

(2)(11)(12) $z \le 7/18 - 5/18$
$\boxed{z \le 2/18}$

(12) $q_1 = 7/18$
(11) $q_2 = 5/18$
(10) $q_3 = 6/18$

Two players general-sum

One player is not traying to minimize the other's utility.
NP-complete

PPAD Polinomial Parity argument, Directed version

Sep 14/2023

feasibility program rather than optimization problem

LCP Linear Complementary Problem

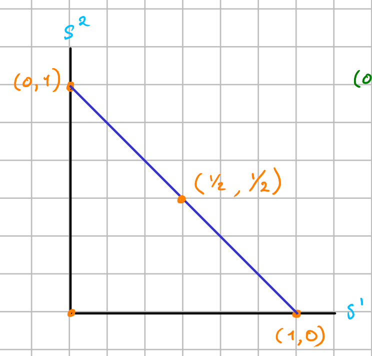
$$\sum_{k\in A_2} u_1(a_1^j, a_2^k)\cdot s_2^k + r_1^j = U_1^*$$

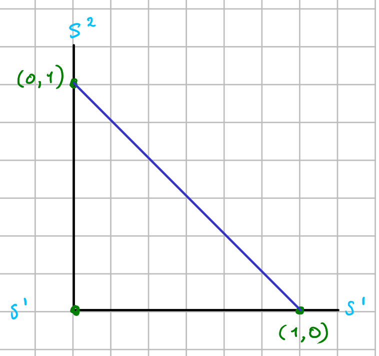
$$\sum_{k\in A_1} u(a_1^j, a_2^k)\cdot s_1^j + r_2^k = U_2^*$$

$$\sum_{k\in A_2} s_2^k = 1, \quad s_2^k \geq 0, \quad r_1^j \geq 0$$

$$\sum_{j\in A_1} s_1^j = 1, \quad s_1^j \geq 0, \quad r_2^k \geq 0$$

$$r_1^j \cdot s_1^j = 0, \quad r_2^k \cdot s_2^k = 0$$





Lemke-Howson algorithm

Example

0, 1	6, 0	0
2, 0	5, 2	1/3
3, 4	3, 3	2/3
1/3	2/3	