MAT224: Linear Algebra II

Lecture Notes

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1 Week 2: January 12, 2026 - January 16, 2026

1.1 January 13, 2026 - Lecture 3

Definition (Vector Space)**.** A **vector space** over a field $\mathbb{F}$ is a set V equipped with two operations: vector addition and scalar multiplication, satisfying the following axioms for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a, b \in \mathbb{F}$:

(i) (Closure under addition) $\mathbf{u} + \mathbf{v} \in V$.

(ii) (Commutativity of addition) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

(iii) (Associativity of addition) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

(iv) (Existence of additive identity) There exists an element $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.

(v) (Existence of additive inverses) For each $\mathbf{u} \in V$, there exists an element $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

(vi) (Closure under scalar multiplication) $a\mathbf{u} \in V$.

(vii) (Existence of multiplicative identity) $1\mathbf{u} = \mathbf{u}$, where 1 is the multiplicative identity in $\mathbb{F}$.

Definition (Span)**.** Let S be a subset of V. A linear combination of vectors in S is a vector of the form

$$a_1 \vec{\mathbf{s}}_1 + a_2 \vec{\mathbf{s}}_2 + \cdots + a_n \vec{\mathbf{s}}_n, \text{ where } a_i \in \mathbb{R} \text{ and } \mathbf{s}_j \in S.$$

– If $S \neq \emptyset$, the **span** of S, denoted by $\operatorname{span}(S)$, is the set of all linear combinations of vectors in S.

– If $S = \emptyset$, we define $\operatorname{span}(S) = \{\mathbf{0}\}$.

Theorem (Span is a Subspace of V)**.**

a) span(S) is a subspace of V.

b) Let W be any subspace of V containing S, then span(S) $\subseteq W$.

a) + b) span(S) is the smallest subspace of V containing S.

Proof of b). Let $\mathbf{W}$ be an n subspace of V containing S and let $\mathbf{v} \in \operatorname{span}(S)$. Then, by definition of span, we can write

$$a_1 \vec{\mathbf{s}}_1 + a_2 \vec{\mathbf{s}}_2 + \cdots + a_n \vec{\mathbf{s}}_n, \text{ where } a_i \in \mathbb{R} \text{ and } \mathbf{s}_j \in S.$$

Since $\mathbf{S} \in W$, therefore each $\mathbf{s}_j \in W$. Then because W is closed under addition and scalar multiplication, we have $\mathbf{v} = a_1 \vec{\mathbf{s}}_1 + a_2 \vec{\mathbf{s}}_2 + \cdots + a_n \vec{\mathbf{s}}_n \in W$. $\square$

Corollary. *If S is a subspace of V then span$(S) = S$.*

Proposition (1.3.8). *Let $W_1 = span(S_1)$ and $W_2 = span(S_2)$ be subspaces of V. Then, $W_1 + W_2 = span(S_1 \cup S_2)$.*

Example. Let $W_1 = \text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$ and $W_2 = \text{span}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$ be subspaces of $\mathbb{R}^2$. Then,

$$W_1 + W_2 = \text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = \mathbb{R}^2.$$

While $W_1 + W_2$ gives us all values in $\mathbb{R}^2$, $W_1 \cup W_2$ gives us the values along the two axes and thus $W_1 + W_2 \neq W_1 \cup W_2$

Proof. To prove that two sets are equal (equality of sets), we need to show that each set is a subset of the other. Thus, we will show the 2 claims that

$$\text{span}(S_1 \cup S_2) \subseteq W_1 + W_2 \text{ and } W_1 + W_2 \subseteq \text{span}(S_1 \cup S_2).$$

Proof span$(S_1 \cup S_2) \subseteq W_1 + W_2$.
From the theorem above, $b)$ states that it's enough to check that

$$S_1 \cup S_2 \subseteq W_1 + W_2 = \{\vec{V} = \mathbf{w}_1 + \mathbf{w}_2 | \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2\}.$$

then if $x \in S_1$ then, $\vec{x} = \vec{x} + \vec{0} \in W_1 + W_2$ because $\vec{x} \in W_1$ and $\vec{0} \in W_2$. If $S_1 \subseteq W_1 + W_2$, the same argument holds for S_2. Thus, $S_1 \cup S_2 = W_1 + W_2$. □

Proof $W_1 + W_2 \subseteq \text{span}(S_1 \cup S_2)$.
Let $\vec{v} \in W_1 + W_2$. We know that $\vec{v} = \mathbf{w}_1 + \mathbf{w}_2$ with $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$. By definition of span, we can write

$$\mathbf{w}_1 = a_1 \vec{\mathbf{s}}_{11} + a_2 \vec{\mathbf{s}}_{12} + \cdots + a_n \vec{\mathbf{s}}_{1n}, \text{ where } a_i \in \mathbb{R} \text{ and } \mathbf{s}_{1j} \in S_1,$$

$$\mathbf{w}_2 = b_1 \vec{\mathbf{s}}_{21} + b_2 \vec{\mathbf{s}}_{22} + \cdots + b_m \vec{\mathbf{s}}_{2m}, \text{ where } b_i \in \mathbb{R} \text{ and } \mathbf{s}_{2j} \in S_2.$$

Therefore, we can know that $W_1 + W_2 \in \text{span}(S_1 \cup S_2)$. □

Example. $W_1 = \text{span}\left\{\begin{pmatrix}1\\e^x\end{pmatrix}\right\}$ and $W_2 = \text{span}\left\{\begin{pmatrix}cos(x)\\x^2\end{pmatrix}\right\} \subseteq F(\mathbb{R})$. Then,

$$W_1 + W_2 = \text{span}\left\{\begin{pmatrix}1\\e^x\end{pmatrix}, \begin{pmatrix}cos(x)\\x^2\end{pmatrix}\right\}.$$

$$W_1 + W_2 \to \vec{v} = \vec{w_1} + \vec{w_2}, \text{ where } \vec{w_1} \in W_1, \vec{w_2} \in W_2.$$

Example. $\vec{w_1} = 2 + 5e^x$, $\vec{w_2} = 7cos(x) + \pi x^2$,

$$\vec{v} = \vec{w_1} + \vec{w_2} = (2 + 5e^x) + (7cos(x) + \pi x^2) = \text{span}(1, e^x, cos(x), x^2).$$

$\square$

Remark.

- Show that $W_1 + W_2$ is the smallest subspace containing $W_1 \cup W_2$ **Prop(1.3.11)**
- Prove **Prop(1.3.8)** for $\text{span}(S_1) + \text{span}(S_2) + \cdots + \text{span}(S_n)$.

Definition (Subset)**.** A subset $S = \vec{S_1}, \vec{S_2}, \cdots, \vec{S_n}$ of a vector space V is linearly independent if the only solution to the equation

$$a_1\vec{S_1} + a_2\vec{S_2} + \cdots + a_n\vec{S_n} = \vec{0}$$

is the trivial solution $a_1 = a_2 = \cdots = a_n = 0$. Otherwise, S is linearly dependent.

Example. consider the set $\{1, x, x^2\}$ in $P_2(\mathbb{R})$. To check if this set is linearly independent, we need to see if the equation

$$a_0(1) + a_1(x) + a_2(x^2) = 0$$

has only the trivial solution. Since the only way for this polynomial to be identically zero for all x is if $a_0 = a_1 = a_2 = 0$, the set $1, x, x^2$ is linearly independent.

> **Remark.** Note that $1, x, x^2$ are vectors in the vector space $P_2(\mathbb{R})$, the space of all polynomials of degree at most 2 with real coefficients and we aren't solving for specific values of x but rather the equivalency of both sides of the equation.

Example. consider the set $\{1, x+1, 2x+3\}$ in $P_2(\mathbb{R})$. Because $2x+3 = 2(x+1)+1$, we can see that the set is linearly dependent.

Example. consider the set $\{cos(x), sin(x), e^x\}$ in $F(\mathbb{R})$. To check if this set is linearly independent we can plug in 3 values for x such as $x = 0, \frac{\pi}{2}, \pi$ and form a system of equations:

$$\begin{cases} a_1(1) + a_2(0) + a_3(1) = 0 \\ a_1(0) + a_2(1) + a_3(e^{\frac{\pi}{2}}) = 0 \\ a_1(-1) + a_2(0) + a_3(e^{\pi}) = 0 \end{cases}$$

Solving this system, we find that the only solution is the trivial solution $a_1 = a_2 = a_3 = 0$. Thus, the set $\{cos(x), sin(x), e^x\}$ is linearly independent.

1.2 January 5, 2026 - Lecture 4

Cancelled due to snow day.

2 Week 3: January 19, 2026 - January 23, 2026

2.1 January 20, 2026 - Lecture 5

Theorem (Basis)**.** *Let V be a vector space that has a finite generating set. (i.e., $V = span(S)$, with $|S| < \infty$).*

1. *If $R \leq V$ is a linearly independent subset, then V has a basis B such that $R \leq B$ (B extends R).*
2. *If S is a finite generating set for V, then V has a basis B such that $B \leq S$ (B is contained in S).*

Proof of 2). Suppose $V = \text{span}(S)$ and $S = \{\vec{s_1}, \vec{s_2}, \cdots, \vec{s_n}\}$.

1. Check if S is linearly independent. If it is, then we are done.
2. Since S is dependent, then we can find a vector $\vec{s_i} \in S$ such that $\vec{s_i} \in \text{span}(S \setminus \{\vec{s_i}\})$. Thus, we can remove $\vec{s_i}$ from S and still have $V = \text{span}(S \setminus \{\vec{s_i}\})$.
3. Repeat steps 1 and 2 with new set $V = \text{span}(S \setminus \{\vec{s_i}\})$ until we get a linearly independent set. Since S is finite, this process will stop after a finite number of steps. Let B be the resulting set. Then, B is linearly independent and $V = \text{span}(B)$, so B is a basis for V contained in S.

$\square$

Example. Find a basis for $\text{span}(\{x^2 + 2x + 1,\ x^2 + 4x + 3,\ x^2 + x + 1\})$ in $P_2(\mathbb{R})$.

1. Method 1: Start with the set $S = \{x^2 + 2x + 1,\ x^2 + 4x + 3,\ x^2 + x + 1\}$ and shrink it down to a basis.
2. Method 2: Start with the $x^2 + 2x + 1$ and build up to a basis.

Proposition (1.6.10)**.** *Let $V = span(\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n})$. Then any subset $S = \{\vec{s_1}, \vec{s_2}, \cdots, \vec{s_m}\}$ of V with $m > n$ is linearly dependent.*

Corollary. *If S & S' are two bases of V with m & m' elements*

Proof. S spans $V +$ S' is linearly independent therefore $m' \leq m$. Additionally, S' spans $V + S$ is linearly independent therefore $m \leq m'$. Thus, $m = m'$. $\square$

Proof of Proposition 1.6.10. Let $V = \text{span}(\vec{v_1}, \vec{v_2})$ and let $S = \{\vec{s_1}, \vec{s_2}, \vec{s_3}\}$ be a subset of V. We want to consider $a_1\vec{s_1} + a_2\vec{s_2} + a_3\vec{s_3} = \vec{0}$. Lets suppose $s = 4v_1 + (-5)v_2$, $s_2 = 6v_1 + \pi v_2$, and $s_3 = ev_1 + \frac{7}{2}v_2$. Then, we can plug in these values to get

$$(4a_1 + 6a_2 + ea_3)v_1 + (-5a_1 + \pi a_2 + \frac{7}{2}a_3)v_2 = \vec{0}.$$

If $4x_1 + 6x_2 + ex_3 = 0$ and $-5x_1 + \pi x_2 + \frac{7}{2}x_3 = 0$, then $a_1 = x_1$, $a_2 = x_2$, and $a_3 = x_3$ is a non-trivial solution to the equation $a_1\vec{s_1} + a_2\vec{s_2} + a_3\vec{s_3} = \vec{0}$. Thus, S is linearly dependent. $\square$

Definition (Dimension)**.** The dimension of V is equal to the number of elements of a basis of V and is denoted by $\dim(V)$.

Remark. Revvisiting the idea of sums, If $W = \text{span}(S_1) + \text{span}(S_2)$, then $W = \text{span}(S_1 \cup S_2)$. The problem we have encountered is that $\dim(W)$ is not always equal to the $\dim(W_1) + \dim(W_2)$.

Example. $W_1 = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}, \left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$ and $W_2 = \text{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ in $\mathbb{R}^3$. $W_1 + W_2 = W_2$

$$W_1 + W_2 = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

This creates the generating set but is not linearly independent.

Theorem (1.6.8). *dim($W_1 + W_2$) = dim(W_1) + dim(W_2) - dim($W_1 \cap W_2$)*
Let W_1 and W_2 be subspaces of a vector space V, then we say the sum $W_1 + W_2$ is a direct sum, denoted by $W_1 \oplus W_2$, if $W_1 \cap W_2 = \{\vec{0}\}$. In this case, dim($W_1 \oplus W_2$) = dim($W_1$) + dim($W_2$).

Remark. In the homework we were asked to show that $F(\mathbb{R}) = F_{even} \oplus F_{odd}$, where $F_{even} = \{f \in F(\mathbb{R}) | f(x) = f(-x) \text{ for all } x \in \mathbb{R}\}$ and $F_{odd} = \{f \in F(\mathbb{R}) | f(x) = -f(-x) \text{ for all } x \in \mathbb{R}\}$. To show this, we need to prove two things:

1. $F(\mathbb{R}) = F_{even} + F_{odd}$
2. $F_{even} \cap F_{odd} = \{\vec{0}\}$

We can also take the direct sum of finitely many subspaces. Let $W_1, W_2, \cdots, W_n$ be subspaces of a vector space V. Then the sum is direct if $W_i \cap \left(\sum_{j=1} W_j\right) = \{\vec{0}\}$ for each $i = 1, 2, \cdots, n$. In this case, we denote the direct sum by $W_1 \oplus W_2 \oplus \cdots \oplus W_n$ and we have $\dim(W_1 \oplus W_2 \oplus \cdots \oplus W_n) = \dim(W_1) + \dim(W_2) + \cdots + \dim(W_n)$.

Definition (Properties of Direct Sum)**.**

1. $\dim(W_1 \oplus W_2 \oplus \cdots \oplus W_r) = \dim(W_1) + \dim(W_2) + \cdots + \dim(W_r)$
2. If B_i is a basis for W_i, then $B_1 \cup B_2 \cup \cdots \cup B_r$ is a basis for $W_1 \oplus W_2 \oplus \cdots \oplus W_r$.
3. If $V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$, then every vector $\vec{v} \in V$ can be written uniquely as $\vec{v} = \vec{w_1} + \vec{w_2} + \cdots + \vec{w_r}$, where $\vec{w_i} \in W_i$ for each $i = 1, 2, \cdots, r$.

Example.

1. If $B = \{\vec{b_1}, \vec{b_2}, \cdots \vec{b_n}\}$ is a basis of V, then $V = \text{span}(\vec{b_1}) \oplus \text{span}(\vec{b_2}) \oplus \cdots \oplus \text{span}(\vec{b_n})$.

2. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a diagonalizable linear transformation and let $\lambda_1, \lambda_2, \cdots, \lambda_r$ be the set of distinct eigenvalues of T. Then, $\mathbb{R}^n = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_r}$, where E_{λ_i} is the eigenspace corresponding to eigenvalue λ_i.

Theorem (Jordan-Cannonical Form)**.**

Let $T : V \to V$ be a linear transformation. Then $\exists\ V_1, V_2, \cdots, V_r$ subspaces of V such that

1. $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$

2. $T(V_i) \leq V_i$ *for each* $i = 1, 2, \cdots, r$

3. T *as a transformation of* $V_i \to V_i$ *has a matrix representation in Jordan form.*

Definition (Linear Transformation)**.** A function $T : V \to W$ between two vector spaces V and W is linear if for all $\vec{u}, \vec{v} \in V$ and all scalars $c \in \mathbb{R}$, the following properties hold:

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ (Additivity)

2. $T(c\vec{u}) = cT(\vec{u})$ (Homogeneity)

3. $T(\vec{0}) = \vec{0}$ (Preservation of Zero Vector)

Remark. We need to be able to add in both V $(\vec{u} + \vec{v})$ and W $(T(\vec{u}) + T(\vec{v}))$ for T to be a linear transformation. Additionally, scalar multiplication must be defined in both V $(T(c\vec{u}))$ and W $(cT(\vec{u}))$.

Definition (Kernel)**.** Same as in MAT223, the **kernel** of a linear transformation $T : V \to W$ is the set of all vectors in V that map to the zero vector in W. Denoted by $\ker(T) = \{\vec{v} \in V | T(\vec{v}) = \vec{0}\}$.

Definition (Image)**.** Same as in MAT223, the **image** of a linear transformation $T : V \to W$ is the set of all vectors in W that are the images of vectors in V. Denoted by $\text{im}(T) = \{T(\vec{v}) | \vec{v} \in V\}$.

Example.

1. Let $V = P_2(\mathbb{R})$ and $W = \mathbb{R}$. Let $p(x) \& q(x) \in P_2(\mathbb{R})$. Is $(p+q)(s) = p(s) + q(s)$?

   Answer. Yes, because $(p+q)(s) = p(s) + q(s)$ for all $s \in \mathbb{R}$. $p(x) = ax^2 + bx + c$ and $q(x) = dx^2 + ex + f$, then $(p+q)(s) = (a+d)s^2 + (b+e)s + (c+f) = p(s) + q(s)$.

   Remark. What about the kernel and image of this transformation? $\ker(T) = \{p(x) \in P_2(\mathbb{R}) | p(s) = 0\}$. This means that the kernel is the set of all polynomials that have s as a root. The image of this transformation is $\text{im}(T) = \mathbb{R}$ because for any real number r, we can find a polynomial $p(x)$ such that $p(s) = r$ (e.g., $p(x) = r$).

2. $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $f(x) \to f'(x)$. The claim is that this is linear. Prove this claim.

   Answer. Let $f(x), g(x) \in C^\infty(\mathbb{R})$ and $c \in \mathbb{R}$. We need to show that the following properties hold:

   (a) $T(f+g)(x) = T(f)(x) + T(g)(x)$
   (b) $T(cf)(x) = cT(f)(x)$

   For property 1, we have

   $$T(f+g)(x) = (f+g)'(x) = f'(x) + g'(x) = T(f)(x) + T(g)(x).$$

   For property 2, we have

   $$T(cf)(x) = (cf)'(x) = cf'(x) = cT(f)(x).$$

   Since both properties hold, we conclude that the differentiation operator $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by $T(f) = f'$ is indeed a linear transformation.

3. Show that the integration of 2) is also linear (see exercise 2.1.5)

   Answer. Let $f(x), g(x) \in C^\infty(\mathbb{R})$ and $c \in \mathbb{R}$. We need to show that the following properties hold:

   (a) $T(f+g)(x) = T(f)(x) + T(g)(x)$
   (b) $T(cf)(x) = cT(f)(x)$

   For property 1, we have

   $$T(f+g)(x) = \int (f+g)(x)dx = \int f(x)dx + \int g(x)dx = T(f)(x) + T(g)(x).$$

   For property 2, we have

   $$T(cf)(x) = \int (cf)(x)dx = c\int f(x)dx = cT(f)(x).$$

   Since both properties hold, we conclude that the integration operator $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by $T(f) = \int f(x)dx$ is indeed a linear transformation.

4. $\mathbb{R} \to \mathbb{R}^{20}$, $x + y = xy$ and $cx = x^c$. Some x will map to e^x under this vector space structure of positive real numbers. Prove that this is linear.

Answer. Let $x, y \in \mathbb{R}^+$ and $c \in \mathbb{R}$. We need to show that the following properties hold:

(a) $T(x+y) = T(x) + T(y)$

(b) $T(cx) = cT(x)$

For property 1, we have

$$T(x+y) = e^{x+y} = e^x \cdot e^y = T(x) + T(y).$$

For property 2, we have

$$T(cx) = e^{cx} = (e^x)^c = cT(x).$$

Since both properties hold, we conclude that the transformation $T : \mathbb{R}^+ \to \mathbb{R}^{20}$ defined by $T(x) = e^x$ is indeed a linear transformation.

2.2 January 22, 2026 - Lecture 6

Remark (More Examples)**.**

– Let $\vec{v}, \vec{w} \in \mathbb{R}^n$, $< \vec{v}, \vec{w} > = \vec{v} \cdot \vec{w} \in \mathbb{R}$. Fix some $\vec{v_n} \in \mathbb{R}^n$ and define $T_{\vec{v}_o} : \mathbb{R}^n \to \mathbb{R}$ Prove the claim that $T_{\vec{v}_o}$ is a linear transformation.

Answer. Let $\vec{w_1}, \vec{w_2} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. We need to show that the following properties hold:

1. $T_{\vec{v}_o}(\vec{w_1} + \vec{w_2}) = T_{\vec{v}_o}(\vec{w_1}) + T_{\vec{v}_o}(\vec{w_2})$
2. $T_{\vec{v}_o}(c\vec{w_1}) = cT_{\vec{v}_o}(\vec{w_1})$

For property 1, we have

$$T_{\vec{v}_o}(\vec{w_1} + \vec{w_2}) =< \vec{v_o}, \vec{w_1} + \vec{w_2} >=< \vec{v_o}, \vec{w_1} > + < \vec{v_o}, \vec{w_2} >= T_{\vec{v}_o}(\vec{w_1}) + T_{\vec{v}_o}(\vec{w_2}).$$

For property 2, we have

$$T_{\vec{v}_o}(c\vec{w_1}) =< \vec{v_o}, c\vec{w_1} >= c < \vec{v_o}, \vec{w_1} >= cT_{\vec{v}_o}(\vec{w_1}).$$

Since both properties hold, we conclude that the transformation $T_{\vec{v}_o} : \mathbb{R}^n \to \mathbb{R}$ defined by $T_{\vec{v}_o}(\vec{w}) =< \vec{v_o}, \vec{w} >$ is indeed a linear transformation.

- Observe that if $T : V \to W$ is linear, $B_v = \{\vec{v_1}, \vec{v_2}, \cdot, \vec{v_n}\}$ is a basis of V. Then T is completely determined by $T(\vec{v_i})$ for $i = 1, 2, \cdots, n$. Let $v \in V$ and we want to compute $T(\vec{v})$.

  Answer. We can write $\vec{v}$ as a linear combination of the basis vectors:

  $$\vec{v} = a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n}, \text{ where } a_i \in \mathbb{R}.$$

  Then, by linearity of T, we have

  $$T(\vec{v}) = T(a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n}) = a_1T(\vec{v_1}) + a_2T(\vec{v_2}) + \cdots + a_nT(\vec{v_n}).$$

  Thus, knowing $T(\vec{v_i})$ for each basis vector $\vec{v_i}$ allows us to compute $T(\vec{v})$ for any vector $\vec{v} \in V$. We can then turn this observation around. Let $B_w = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_m}\}$ be a basis of V. We may define a linear transformation $T : V \to W$ by the following:

  1. Find $\vec{w_1}, \vec{w_2}, \cdots, \vec{w_n} \in W$ and set $T(\vec{v_i}) = \vec{w_i}$ for each $i = 1, 2, \cdots, n$.
  2. Extend linearly for any $\vec{v} \in V$. We can write $\vec{v} = a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n}$, where $a_i \in \mathbb{R}$. Then, define the following and check for linearity:

     $$T(\vec{v}) = a_1T(\vec{v_1}) + a_2T(\vec{v_2}) + \cdots + a_nT(\vec{v_n}).$$

- $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$. Note $B_3 = \{1, x, x^2, x^3\}$ is a basis for $P_3(\mathbb{R})$. Given the linear transformation defined by $T(1) = 0$, $T(x) = 1$, $T(x^2) = 2x$, and $T(x^3) = 3x^2$. Let $p(x)$ be a general polynomial in $P_3(\mathbb{R})$. $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Find $T(p(x))$.

  Answer. We can use the linearity of T to find $T(p(x))$:

  $$T(p(x)) = T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_0T(1) + a_1T(x) + a_2T(x^2) + a_3T(x^3).$$

  Plugging in the values of T on the basis elements, we get:

  $$T(p(x)) = a_0(0) + a_1(1) + a_2(2x) + a_3(3x^2) = a_1 + 2a_2x + 3a_3x^2.$$

  Thus, $T(p(x)) = a_1 + 2a_2x + 3a_3x^2$. If you notice, this is the derivative of $p(x)$. Therefore, in this case, T acts as the differentiation operator on polynomials in $P_3(\mathbb{R})$ and differentiation is a linear transformation.

Definition (Matrices)**.** From MAT223, we know that any transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with a basis $B = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ of $\mathbb{R}^n$ can be represented as a matrix $[T]_B$ such that

$$[T]_B = \begin{pmatrix} | & | & & | \\ T(\vec{v_1}) & T(\vec{v_2}) & \cdots & T(\vec{v_n}) \\ | & | & & | \end{pmatrix}$$

This matrix will have properties:

1. For any $\vec{x} \in \mathbb{R}^n$, $[T]_B[\vec{x}]_B = [T(\vec{x})]_B$.
2. If $S : \mathbb{R}^m \to \mathbb{R}^p$ is another linear transformation, then $[S \circ T]_B = [S]_B[T]_B$.
3. If $I : \mathbb{R}^n \to \mathbb{R}^n$ is the identity transformation, then $[I]_B = I_n$, the $n \times n$ identity matrix.

Definition (Defining Matrix)**.** The defining matrix of a linear transformation $T : V \to W$ with respect to bases $B_V = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ of V and $B_W = \{\vec{w_1}, \vec{w_2}, \cdots, \vec{w_m}\}$ of W is the matrix $[T]_{B_V}^{B_W}$ represented by:

$$[T]_{B_V}^{B_W} = \begin{pmatrix} | & | & & | \\ [T(\vec{v_1})]_{B_W} & [T(\vec{v_2})]_{B_W} & \cdots & [T(\vec{v_n})]_{B_W} \\ | & | & & | \end{pmatrix}$$

This unique matrix has the defining property such that $[T]_{B_V}^{B_W}[\vec{v}]_{B_V} = [T(\vec{v})]_{B_W}$ for all $\vec{v} \in V$.

Example. Let $D : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be the differentiation operator. Given a $B_{P_3(\mathbb{R})} = \{2, 2x, x^2 + x, x^3\}$ basis for $P_3(\mathbb{R})$ and $B_{P_2} = \{1, x, x^2\}$ basis for $P_2(\mathbb{R})$. Find the defining matrix $[D]_{B_{P_3(\mathbb{R})}}^{B_{P_2(\mathbb{R})}}$.

Answer. We can begin by writing $[D]_{B_{P_3(\mathbb{R})}}^{B_{P_2(\mathbb{R})}}$ as follows:

$$[D]_{B_{P_3(\mathbb{R})}}^{B_{P_2(\mathbb{R})}} = \begin{pmatrix} | & | & | & | \\ [D(2)]_{B_{P_2(\mathbb{R})}} & [D(2x)]_{B_{P_2(\mathbb{R})}} & [D(x^2+x)]_{B_{P_2(\mathbb{R})}} & [D(x^3)]_{B_{P_2(\mathbb{R})}} \\ | & | & | & | \end{pmatrix}$$

There exist 4 columns because there are 4 basis vectors in $B_{P_3(\mathbb{R})}$ and 3 rows because there are 3 basis vectors in $B_{P_2(\mathbb{R})}$. We can now compute each of the columns:

- $D(2) = 0$. Thus, $[D(2)]_{B_{P_2(\mathbb{R})}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.
- $D(2x) = 2$. Thus, $[D(2x)]_{B_{P_2(\mathbb{R})}} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$.
- $D(x^2 + x) = 2x + 1$. Thus, $[D(x^2 + x)]_{B_{P_2(\mathbb{R})}} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.
- $D(x^3) = 3x^2$. Thus, $[D(x^3)]_{B_{P_2(\mathbb{R})}} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$.

Putting this all together, we have:

$$[D]_{B_{P_3(\mathbb{R})}}^{B_{P_2(\mathbb{R})}} = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

3 Week 4: January 26, 2026 - January 30, 2026

3.1 January 27, 2026 - Lecture 7

Proposition (2.2.15)**.** *Proving the defining matrix:*
Let $v \in V$. Since $V = \operatorname{span}(B_V)$, we can write v as a linear combination of the basis vectors:

$$\vec{v} = a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n}, \text{ where } a_i \in \mathbb{R}.$$

Then we can rewrite $[v]_{B_V}$ as follows:

$$[\vec{v}]_{B_V} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

We can then multiply $[T]_{B_V}^{B_W}$ by $[\vec{v}]_{B_V}$:

$$[T]_{B_V}^{B_W}[\vec{v}]_{B_V} = \begin{pmatrix} | & | & & | \\ [T(\vec{v_1})]_{B_W} & [T(\vec{v_2})]_{B_W} & \cdots & [T(\vec{v_n})]_{B_W} \\ | & | & & | \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

By matrix multiplication, we have:

$$[T]_{B_V}^{B_W}[\vec{v}]_{B_V} = a_1[T(\vec{v_1})]_{B_W} + a_2[T(\vec{v_2})]_{B_W} + \cdots + a_n[T(\vec{v_n})]_{B_W}$$

By linearity of T, we have:

$$[T]_{B_V}^{B_W}[\vec{v}]_{B_V} = [T(a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n})]_{B_W} = [T(\vec{v})]_{B_W}$$

Thus, we have shown that $[T]_{B_V}^{B_W}[\vec{v}]_{B_V} = [T(\vec{v})]_{B_W}$ for all $\vec{v} \in V$. $\square$

Remark (Different Representation of Defining Matrix)**.** We can also represent the defining matrix $[T]_{B_V}^{B_W}$ as follows where $n = \dim(V)$ and $m = \dim(W)$:

$$\begin{tikzcd} \vec{v} \in V \arrow[r, "T"] \arrow[d, "{[\ ]_{B_V}}"'] & W \arrow[d, "{[\ ]_{B_W}}"] \\ {[\vec{v}]_{B_V} \in \mathbb{R}^n} \arrow[r, "{[T]_{B_V}^{B_W}}"'] & \mathbb{R}^m \end{tikzcd}$$

Remark. If $B = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ is a basis of a vector subspace, then the map $[\ \ ]_B : V \to \mathbb{R}^n$ is a bijective linear map.

Proof. Injective: Let $\vec{v} \in \ker([\ \ ]_B)$. Then $[\vec{v}]_B = \vec{0}$. This means that $\vec{v}$ can be written as a linear combination of the basis vectors with all coefficients equal to zero. Thus, $\vec{v} = \vec{0}$. Therefore, the kernel of the map is $\{\vec{0}\}$, which implies that the map is injective. Surjectove: Let $\vec{y} \in \mathbb{R}^n$. We need to find a vector $\vec{v} \in V$ such that $[\vec{v}]_B = \vec{y}$. Since B is a basis of V, we can write $\vec{v}$ as a linear combination of the basis vectors:

$$\vec{v} = y_1\vec{v_1} + y_2\vec{v_2} + \cdots + y_n\vec{v_n},$$

where y_i are the components of $\vec{y}$. Then, by definition of the map, we have:

$$[\vec{v}]_B = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \vec{y}.$$

Thus, for every $\vec{y} \in \mathbb{R}^n$, there exists a $\vec{v} \in V$ such that $[\vec{v}]_B = \vec{y}$. Therefore, the map is surjective. Since the map is both injective and surjective, it is bijective. □

Remark (Kernel and Image via matrices)**.**

Proposition. *Let $T : V \to W$ be a linear map with B_V and B_W as bases of V and W respectively. Then,*

a) $\vec{v} \in \mathit{ker}(T)$ *if and only if* $[\vec{v}]_{B_V} \in \mathit{Null}([T]_{B_V}^{B_W})$.

b) $\vec{w} \in \mathit{im}(T)$ *if and only if* $[\vec{w}]_{B_W} \in \mathit{Col}([T]_{B_V}^{B_W})$.

Proof of a).
$\vec{v} \in \ker(T) \Leftrightarrow T(\vec{v}) = \vec{0}_W$
$\Leftrightarrow [T(\vec{v})]_{B_W} = [\vec{0}_W]_{B_W}$
$\Leftrightarrow [T]_{B_V}^{B_W}[\vec{v}]_{B_V} = [\vec{0}]_{B_W}$
$\Leftrightarrow [\vec{v}]_{B_V} \in \mathrm{Null}([T]_{B_V}^{B_W})$.

□

Corollary (Rank-Nullity Theorem)**.** *Let $T : V \to W$ be a linear map between finite-dimensional vector spaces. Then,*

$$\dim(\ker(T)) + \dim(\mathit{im}(T)) = \dim(V).$$

Proof. Pick bases B_V and B_W for V and W, respectively. By the previous proposition, we have:

$$\dim(\ker(T)) + \dim(\mathrm{im}(T)) = \dim(\mathrm{Null}([T]_{B_V}^{B_W})) + \dim(\mathrm{Col}([T]_{B_V}^{B_W})).$$

By the Rank-Nullity Theorem for matrices, we know that:

$$\dim(\mathrm{Null}([T]_{B_V}^{B_W})) + \dim(\mathrm{Col}([T]_{B_V}^{B_W})) = n,$$

where n is the number of columns of the matrix $[T]_{B_V}^{B_W}$. Since B_V is a basis for V, we have that $n = \dim(V)$. Therefore, we conclude that:

$$\dim(\ker(T)) + \dim(\mathrm{im}(T)) = \dim(V).$$

□

Example. Consider $B_1 = \{1+x, 2x, x^2\} \subseteq P_2(\mathbb{R})$ and $B_2 = \{3, x^2+2x, 2x^2\} \subseteq \mathbb{R}^2$.

1. Show B_1 & B_2 are bases of their respective vector spaces.

   Answer. To show that B_1 is a basis for $P_2(\mathbb{R})$, we need to show that it is linearly independent and spans $P_2(\mathbb{R})$.

   - Linear Independence: Suppose $a(1+x)+b(2x)+c(x^2) = 0$ for some scalars $a, b, c \in \mathbb{R}$. This gives us the equation:

     $$a + (a+2b)x + cx^2 = 0.$$

     Equating coefficients, we get:

     $$\begin{aligned} a &= 0, \\ a + 2b &= 0, \\ c &= 0. \end{aligned}$$

     From the first equation, we have $a = 0$. Substituting this into the second equation gives $2b = 0$, so $b = 0$. The third equation gives $c = 0$. Thus, the only solution is $a = b = c = 0$, which means B_1 is linearly independent.
   - Spanning: Any polynomial $p(x) \in P_2(\mathbb{R})$ can be written as $p(x) = d_0 + d_1x + d_2x^2$ for some scalars $d_0, d_1, d_2 \in \mathbb{R}$. We can express $p(x)$ as a linear combination of the basis vectors:

     $$p(x) = a(1+x) + b(2x) + c(x^2),$$

     where $a = d_0$, $b = \frac{d_1 - a}{2}$, and $c = d_2$. Thus, B_1 spans $P_2(\mathbb{R})$.

   Therefore, B_1 is a basis for $P_2(\mathbb{R})$.

2. Consider $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(1+x) = x^2 + x + 1$, $T(2x) = x^2 - 2x - 2$, and $T(x^2) = 3x + 3$. Find $T(x^2 + 3x + 2)$.

Answer. First, we need to express $x^2 + 3x + 2$ as a linear combination of the basis vectors in B_1:

$$x^2 + 3x + 2 = a(1+x) + b(2x) + c(x^2).$$

Expanding the right-hand side, we get:

$$a + (a+2b)x + cx^2.$$

Equating coefficients, we have the following system of equations:

$$\begin{aligned} a &= 2, \\ a + 2b &= 3, \\ c &= 1. \end{aligned}$$

From the first equation, we have $a = 2$. Substituting this into the second equation gives $2 + 2b = 3$, so $2b = 1$ and $b = \frac{1}{2}$. The third equation gives $c = 1$. Thus, we have:

$$x^2 + 3x + 2 = 2(1+x) + \frac{1}{2}(2x) + 1(x^2).$$

Now, we can use the linearity of T to find $T(x^2 + 3x + 2)$:

$$T(x^2 + 3x + 2) = T(2(1+x) + \frac{1}{2}(2x) + 1(x^2)).$$

By linearity, we have:

$$T(x^2 + 3x + 2) = 2T(1+x) + \frac{1}{2}T(2x) + 1T(x^2).$$

Substituting the given values of T on the basis vectors, we get:

$$T(x^2 + 3x + 2) = 2(x^2 + x + 1) + \frac{1}{2}(x^2 - 2x - 2) + 1(3x + 3).$$

Simplifying this expression, we have:

$$T(x^2 + 3x + 2) = 2x^2 + 2x + 2 + \frac{1}{2}x^2 - x - 1 + 3x + 3.$$

Combining like terms, we get:

$$T(x^2 + 3x + 2) = \left(2 + \frac{1}{2}\right)x^2 + (2 - 1 + 3)x + (2 - 1 + 3) = \frac{5}{2}x^2 + 4x + 4.$$

3. Find the defining matrix $[T]_{B_1}^{B_2}$.

Answer. We can begin by writing $[T]_{B_1}^{B_2}$ as follows:

$$[T]_{B_1}^{B_2} = \begin{pmatrix} | & | & | \\ [T(1+x)]_{B_2} & [T(2x)]_{B_2} & [T(x^2)]_{B_2} \\ | & | & | \end{pmatrix}$$

We can now compute each of the columns:

- $T(1+x) = x^2 + x + 1$. To express this in terms of the basis B_2, we need to find scalars a, b, c such that:

$$x^2 + x + 1 = a(3) + b(x^2 + 2x) + c(2x^2).$$

Expanding the right-hand side, we get:

$$3a + (b + 2c)x^2 + 2bx.$$

Equating coefficients, we have the following system of equations:

$$\begin{aligned} 3a &= 1, \\ b + 2c &= 1, \\ 2b &= 1. \end{aligned}$$

From the first equation, we have $a = \frac{1}{3}$. The third equation gives $b = \frac{1}{2}$. Substituting b into the second equation gives $\frac{1}{2} + 2c = 1$, so $2c = \frac{1}{2}$ and $c = \frac{1}{4}$. Thus, we have:

$$[T(1+x)]_{B_2} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}.$$

- $T(2x) = x^2 - 2x - 2$. To express this in terms of the basis B_2, we need to find scalars d, e, f such that:

$$x^2 - 2x - 2 = d(3) + e(x^2 + 2x) + f(2x^2).$$

Expanding the right-hand side, we get:

$$3d + (e + 2f)x^2 + 2ex.$$

Equating coefficients, we have the following system of equations:

$$\begin{aligned} 3d &= -2, \\ e + 2f &= 1, \\ 2e &= -2. \end{aligned}$$

From the first equation, we have $d = -\frac{2}{3}$. The third equation gives $e = -1$. Substituting e into the second equation gives $-1 + 2f = 1$, so $2f = 2$ and $f = 1$. Thus, we have:

$$[T(2x)]_{B_2} = \begin{pmatrix} -\frac{2}{3} \\ -1 \\ 1 \end{pmatrix}.$$

- $T(x^2) = 3x + 3$. To express this in terms of the basis B_2, we need to find scalars g, h, i such that:

  $$3x + 3 = g(3) + h(x^2 + 2x) + i(2x^2).$$

  Expanding the right-hand side, we get:

  $$3g + (h + 2i)x^2 + 2hx.$$

  Equating coefficients, we have the following system of equations:

  $$\begin{aligned} 3g &= 3, \\ h + 2i &= 0, \\ 2h &= 3. \end{aligned}$$

  From the first equation, we have $g = 1$. The third equation gives $h = \frac{3}{2}$. Substituting h into the second equation gives $\frac{3}{2} + 2i = 0$, so $2i = -\frac{3}{2}$ and $i = -\frac{3}{4}$. Thus, we have:

  $$[T(x^2)]_{B_2} = \begin{pmatrix} 1 \\ \frac{3}{2} \\ -\frac{3}{4} \end{pmatrix}.$$

Putting this all together, we have:

$$[T]_{B_1}^{B_2} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & 1 \\ \frac{1}{2} & -1 & \frac{3}{2} \\ \frac{1}{4} & 1 & -\frac{3}{4} \end{pmatrix}$$

4. Finding the rank-nullity of T.

   Answer.
   $\text{rank}(T) = \dim(\text{im}(T)) = \dim(\text{col}([T]_{B_1}^{B_2})) = 2$ (since there are 2 pivot columns).
   $\text{nullity}(T) = \dim(\ker(T)) = \dim(\text{null}([T]_{B_1}^{B_2})) = 1$ (since there is 1 free variable).
   Therefore, by the Rank-Nullity Theorem, we have:

$$\begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} & 1 \\ \frac{1}{2} & -1 & \frac{3}{2} \\ \frac{1}{4} & 1 & -\frac{3}{4} \end{pmatrix}$$

   This can be row reduced to:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

   Thus, we have 2 pivot columns and 1 free variable, confirming our earlier calculations

5. Finding the kernel and image of T.

   Answer.

$$[T]_{B_1}^{B_2} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & 1 \\ \frac{1}{2} & -1 & \frac{3}{2} \\ \frac{1}{4} & 1 & -\frac{3}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

   To find the kernel, we solve the equation $[T]_{B_1}^{B_2} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. This gives us the system of equations:

$$x_1 + x_3 = 0,$$
$$x_2 - x_3 = 0.$$

   From the first equation, we have $x_1 = -x_3$. From the second equation, we have $x_2 = x_3$. Letting $x_3 = t$ (a free variable), we can express the solution as:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \text{ where } t \in \mathbb{R}.$$

   Thus, the kernel of T is spanned by the vector $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

   To find the image, we look at the pivot columns of the row-reduced matrix. The pivot columns correspond to the first two columns of the original matrix:

$$\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{2} & -1 \\ \frac{1}{4} & 1 \end{pmatrix}$$

Thus, the image of T is spanned by the vectors $\begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$ and $\begin{pmatrix} -\frac{2}{3} \\ -1 \\ 1 \end{pmatrix}$.

3.2 January 29, 2026 - Lecture 8

Theorem (Rank-Nullity)**.** *Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces. Then,*

$$\dim(\ker(T)) + \dim(\mathit{im}(T)) = \dim(V).$$

Remark. The dimension of any vector space is non-negative. Thus, we have $\dim(\mathrm{im}(T)) \leq \dim(V)$

Theorem. *Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces. Then, the following are true:*

a) If T is injective, then $\dim(V) \leq \dim(W)$.

b) If T is surjective, then $\dim(V) \geq \dim(W)$.

Proof of a).
Since T is injective, we have $\ker(T) = \{\vec{0}\}$. Thus, $\dim(\ker(T)) = 0$. By the Rank-Nullity Theorem, we have:

$$0 + \dim(\mathrm{im}(T)) = \dim(V) \to \dim(\mathrm{im}(T)) = \dim(V).$$

Since $\mathrm{im}(T)$ is a subspace of W, we have $\dim(\mathrm{im}(T)) \leq \dim(W)$. Therefore, we conclude that $\dim(V) \leq \dim(W)$. □

Proof of b).
Since T is surjective, we have $\mathrm{im}(T) = W$. Thus, $\dim(\mathrm{im}(T)) = \dim(W)$. By the Rank-Nullity Theorem, we have:

$$\dim(\ker(T)) + \dim(W) = \dim(V) \to \dim(W) = \dim(V) - \dim(\ker(T)).$$

Since $\dim(\ker(T)) \geq 0$, we have $\dim(V) - \dim(\ker(T)) \leq \dim(V)$. Therefore, we conclude that $\dim(W) \leq \dim(V)$. □

As consequence, if T is bijective, then $\dim(V) = \dim(W)$.

Definition (Isomorphism)**.** A bijective linear map $T : V \to W$ between vector spaces V and W is called an **isomorphism**. If there exists an isomorphism between V and W, we say that V and W are **isomorphic**, denoted by $V \cong W$.

Example. $[\ \ ]_B : V \to \mathbb{R}^{\dim(V)} = \mathbb{R}^{\#B}$ is bijective. As such, $\vec{v} \mapsto [\vec{v}]_B$ is an isomorphism between V and $\mathbb{R}^{\dim(V)}$. Thus, any finite-dimensional vector space is isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

Theorem. *Two finite-dimensional vector spaces V and W are isomorphic if and only if* $\dim(V) = \dim(W)$*.*

Proof.
($\Rightarrow$) If V and W are isomorphic, then there exists a bijective linear map $T : V \to W$. By the previous theorem, we have $\dim(V) = \dim(W)$.
($\Leftarrow$) If $\dim(V) = \dim(W)$, let $B_V = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ and $B_W = \{\vec{w_1}, \vec{w_2}, \cdots, \vec{w_n}\}$ be bases for V and W, respectively. We can define a linear map $T : V \to W$ by defining $T(\vec{v_i}) = \vec{w_i}$ for each $i = 1, 2, \cdots, n$ and extending linearly. Since T maps a basis of V to a basis of W, it is bijective. Thus, T is an isomorphism between V and W. □

According to this theorem, every finite-dimensional vector space is isomorphic to $\mathbb{R}^n$.

Proof. Let $B_v = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ be a basis for V and $B_w = \{\vec{w_1}, \vec{w_2}, \cdots, \vec{w_n}\}$ be the basis for W. Consider a linear map $T : V \to W$ defined by $\vec{v_i} \mapsto \vec{w_i}$ for each $i = 1, 2, \cdots, n$ and extending linearly. Since T maps a basis of V to a basis of W, it is bijective. Thus, T is an isomorphism between V and W. □

Remark. If $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis of a vector space V, then $[\ \ ]_B$ agrees with the following map:

$$T : V \to \mathbb{R}^n,$$
$$\mathbf{v}_i \mapsto \mathbf{e}_i$$

for $i = 1, \ldots, n$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$.

Remark (Sum of Subspaces)**.** W_1 and W_2 are subspaces of a vector space V. We have that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$ and $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2)$.

Proof 1.
Define $W_1 \times W_2 = \{(\vec{w_1}, \vec{w_2}) \mid \vec{w_1} \in W_1, \vec{w_2} \in W_2\}$. This is a subspace of $V \times V$. Consider the linear map $T : W_1 \times W_2 \to W_1 + W_2$ defined by $T(\vec{w_1}, \vec{w_2}) = \vec{w_1} + \vec{w_2}$. We have that $\mathrm{im}(T) = W_1 + W_2$ and $\ker(T) = \{(\vec{w}, -\vec{w}) \mid \vec{w} \in W_1 \cap W_2\}$. Thus, $\dim(\ker(T)) = \dim(W_1 \cap W_2)$. By the Rank-Nullity Theorem, we have:

$$\dim(W_1 \times W_2) = \dim(\ker(T)) + \dim(\mathrm{im}(T)) \to \dim(W_1) + \dim(W_2) = \dim(W_1 \cap W_2) + \dim(W_1 + W_2).$$

□

Proof 2.

Claim 1: $W_1 \times W_2$ with $(\vec{a_1}, \vec{b_1}) + (\vec{a_2}, \vec{b_2}) = (\vec{a_1} + \vec{a_2}, \vec{b_1} + \vec{b_2})$ and $c(\vec{a}, \vec{b}) = (c\vec{a}, c\vec{b})$ is a vector space.

Claim 2: If $B_1 = \{\vec{a_1}, \vec{a_2}, \cdots, \vec{a_m}\}$ is a basis of V and $B_2 = \{\vec{b_1}, \vec{b_2}, \cdots, \vec{b_n}\}$ is a basis of W, then $\{(\vec{a_1}, \vec{0}), (\vec{a_2}, \vec{0}), \cdots, (\vec{a_m}, \vec{0}), (\vec{0}, \vec{b_1}), (\vec{0}, \vec{b_2}), \cdots, (\vec{0}, \vec{b_n})\}$ is a basis of $W_1 + W_2$
$\Rightarrow \dim(W_1 \times W_2) = \dim(W_1) + \dim(W_2)$.

Claim 3: Consider T defined by $W_1 \times W_2 \to V$. $(\vec{w_1}, \vec{w_2}) \mapsto \vec{w_1} + \vec{w_2}$. Then, $\text{im}(T) = W_1 + W_2$ and $\ker(T) = \{(\vec{w}, -\vec{w}) \mid \vec{w} \in W_1 \cap W_2\}$.

Thus, $\dim(\ker(T)) = \dim(W_1 \cap W_2)$. By the Rank-Nullity Theorem, we have:
$\dim(W_1 \times W_2) = \dim(\ker(T)) + \dim(\text{im}(T)) \to \dim(W_1) + \dim(W_2) = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$.

□

Theorem (Composition of Linear Maps)**.** *Let $F : V_1 \to V_2$ and $G : V_2 \to V_3$ be linear maps. B_i is a basis for V_i for $i = 1, 2, 3$. Then,*

1. *$[G \circ F] : V_1 \to V_3$ is a linear map defined by $\vec{v} \mapsto G(F(\vec{v}))$ for all $\vec{v} \in V_1$.*

   Proof of 1). For any $\vec{u}, \vec{v} \in V_1$ and $c \in \mathbb{R}$:

$$\begin{aligned} [G \circ F](\vec{u} + \vec{v}) &= G(F(\vec{u} + \vec{v})) \\ &= G(F(\vec{u}) + F(\vec{v})) \\ &= G(F(\vec{u})) + G(F(\vec{v})) \\ &= [G \circ F](\vec{u}) + [G \circ F](\vec{v}), \end{aligned}$$

$$\begin{aligned} [G \circ F](c\vec{v}) &= G(F(c\vec{v})) \\ &= G(cF(\vec{v})) \\ &= cG(F(\vec{v})) \\ &= c[G \circ F](\vec{v}). \end{aligned}$$

   Thus, $G \circ F$ is a linear map. □

2. *$[G \circ F]_{B_1}^{B_3} = [G]_{B_2}^{B_3}[F]_{B_1}^{B_2}$.*

   Proof of 2). For any $\vec{v} \in V_1$, we have:

$$\begin{aligned} [G \circ F(\vec{v})]_{B_3} &= [G(F(\vec{v}))]_{B_3} \\ &= [G]_{B_2}^{B_3}[F(\vec{v})]_{B_2} \\ &= [G]_{B_2}^{B_3}[F]_{B_1}^{B_2}[\vec{v}]_{B_1}. \end{aligned}$$

   Thus, $[G \circ F]_{B_1}^{B_3} = [G]_{B_2}^{B_3}[F]_{B_1}^{B_2}$. □

4 Week 5: February 2, 2026 - February 6, 2026

4.1 February 3, 2026 - Lecture 9

Definition (Inverse of a Linear Map)**.** A function $f : X \to Y$ is invertible if we may find a function $f^{-1} : Y \to X$ such that $f^{-1}(f(x)) = x$ for all $x \in X$ and $f(f^{-1}(y)) = y$ for all $y \in Y$. This agrees with the identity functions on X and Y, respectively.

Corollary. *If $T : V \to W$ is an invertible linear map, then $dim(V) = dim(W)$.*

Proof.
This works because T is a bijective linear map. Thus, by the earlier theorem, we have $\dim(V) = \dim(W)$. ☐

Theorem. *Let $T : V \to W$ be an invertible linear map.*

1. *$T^{-1} : W \to V$ is a linear map.*

   Proof of 1).
   For any $\vec{u}, \vec{v} \in W$ and $c \in \mathbb{R}$:

$$\begin{aligned} T^{-1}(\vec{u} + \vec{v}) &= T^{-1}(T(T^{-1}(\vec{u})) + T(T^{-1}(\vec{v}))) \\ &= T^{-1}(T(T^{-1}(\vec{u}) + T^{-1}(\vec{v}))) \\ &= T^{-1}(\vec{u}) + T^{-1}(\vec{v}), \end{aligned}$$

$$\begin{aligned} T^{-1}(c\vec{v}) &= T^{-1}(T(cT^{-1}(\vec{v}))) \\ &= T^{-1}(T(cT^{-1}(\vec{v}))) \\ &= cT^{-1}(\vec{v}). \end{aligned}$$

   Thus, T^{-1} is a linear map. ☐

2. *If B_1 is a basis for V and B_2 is a basis for W, then $[T^{-1}]_{B_2}^{B_1} = ([T]_{B_1}^{B_2})^{-1}$.*

   Proof of 2).
   For any $\vec{w} \in W$, we have:

$$\begin{aligned} [T^{-1}(\vec{w})]_{B_1} &= [T^{-1}]([\vec{w}]_{B_2}) \\ &= [T^{-1}]_{B_2}^{B_1}[\vec{w}]_{B_2}, \end{aligned}$$

$$\begin{aligned} [T(T^{-1}(\vec{w}))]_{B_2} &= [T]_{B_1}^{B_2}[T^{-1}(\vec{w})]_{B_1} \\ &= [T]_{B_1}^{B_2}[T^{-1}]_{B_2}^{B_1}[\vec{w}]_{B_2}. \end{aligned}$$

   Since $T(T^{-1}(\vec{w})) = \vec{w}$, we have:

$$[\vec{w}]_{B_2} = [T]_{B_1}^{B_2}[T^{-1}]_{B_2}^{B_1}[\vec{w}]_{B_2}.$$

   Thus, $[T]_{B_1}^{B_2}[T^{-1}]_{B_2}^{B_1} = I$, where I is the identity matrix. Therefore, we conclude that $[T^{-1}]_{B_2}^{B_1} = ([T]_{B_1}^{B_2})^{-1}$. ☐

Example. $P_3(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$.

1. Verify that the following sets are bases for their respective vector spaces:

$$B_1 = \{1, x, x^2, x^3\} \subseteq P_3(\mathbb{R}),$$

$$B_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subseteq M_{2\times 2}(\mathbb{R}).$$

   Answer.
   B_1 is linearly independent since the only solution to

$$a_0(1) + a_1(x) + a_2(x^2) + a_3(x^3) = 0$$

   is $a_0 = a_1 = a_2 = a_3 = 0$. Furthermore, B_1 spans $P_3(\mathbb{R})$ since any polynomial of degree at most 3 can be expressed as a linear combination of the elements of B_1. Thus, B_1 is a basis for $P_3(\mathbb{R})$.
   B_2 is linearly independent since the only solution to

$$a_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

   is $a_1 = a_2 = a_3 = a_4 = 0$. Furthermore, B_2 spans $M_{2\times 2}(\mathbb{R})$ since any 2×2 matrix can be expressed as a linear combination of the elements of B_2. Thus, B_2 is a basis for $M_{2\times 2}(\mathbb{R})$

2. Define $T : P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by:

$$T(1) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad T(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$T(x^2) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad T(x^3) = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

Prove that T is invertible.

Answer (Method 1)**.**

$$\operatorname{span}\{T(1), T(x), T(x^2), T(x^3)\} = M_{2\times 2}(\mathbb{R}) \quad \text{and} \quad \ker(T) = \{0\}.$$

If T is surjective, then

$$\dim(\operatorname{im}(T)) = \dim(M_{2\times 2}(\mathbb{R})) = 4.$$

By the Rank–Nullity Theorem,

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(P_3(\mathbb{R})) = 4.$$

Thus,

$$\operatorname{nullity}(T) = 0,$$

so T is injective. Therefore, T is invertible.

Answer (Method 2)**.**

Find $[T]_{B_1}^{B_2}$, the rank and nullity of T, and the kernel and image of T. To find the matrix representation of T with respect to B_1 and B_2, compute

$$[T]_{B_1}^{B_2} = \Big([T(1)]_{B_2} \quad [T(x)]_{B_2} \quad [T(x^2)]_{B_2} \quad [T(x^3)]_{B_2}\Big).$$

We can then row reduce the given matrix as shown below to find the rank and nullity of T, as well as the kernel and image of T.

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Row reducing this, we get:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{array}\right)$$

As such, we know that:

$$[T]_{B_1}^{B_2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

Therefore, we can now find:

$$T^{-1}\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right) = x, \qquad T^{-1}\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right) = 1 - x,$$

$$T^{-1}\left(\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right) = -x^2, \qquad T^{-1}\left(\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right) = -x^2 - x^3.$$

Definition (Change of Basis Matrix)**.** Let B, C be bases for a vector space V. The **change of basis matrix** from B to C is defined as:

$$[\mathrm{Id}]_B^C = \begin{pmatrix}[\vec{v_1}]_C & [\vec{v_2}]_C & \cdots & [\vec{v_n}]_C\end{pmatrix},$$

where $B = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$.

1. $[\mathrm{Id}_V]_C^B[\vec{v}]_C = [\mathrm{Id}_V(\vec{v})]_B = [\vec{x}]_B$ for all $\vec{v} \in V$.
2. $[\mathrm{Id}_V]_B^C[\mathrm{Id}_V]_C^B = [\mathrm{Id}_V \circ \mathrm{Id}_V]_C^C = [\mathrm{Id}_V]_C^C = I$.
3. For $T : V \to V$, $[T]_C^C = [\mathrm{Id}_V]_B^C[T]_B^B[\mathrm{Id}_V]_C^B = [\mathrm{Id}_V \circ T \circ \mathrm{Id}_V]_C^C = [T]_C^C$.

$n \times n$ matricies $A \& B$ are similar if there exists an invertible matrix Q such that $B = Q^{-1}AQ$.

Remark. Let $B = \{\vec{b_1}, \vec{b_2}, \cdots, \vec{b_n}\}$ and $C = \{\vec{c_1}, \vec{c_2}, \cdots, \vec{c_n}\}$ be bases of R^n. Given $M_{B\leftarrow C}$ and $M_{C\leftarrow B}$ they have the following properties:

1. $M_{B\leftarrow C}[\vec{x}]_C = [\vec{x}]_B$ for all $\vec{x} \in R^n$
2. $M_{B\leftarrow C} = M_{C\leftarrow B}^{-1}$
3. $T : R^n \to R^n$
4. $M_{B\leftarrow C} = \begin{bmatrix}[\vec{c_1}]_B & [\vec{c_2}]_B & \cdots & [\vec{c_n}]_B\end{bmatrix}$
5. $[T]_C^C = M_{B\leftarrow C}[T]_B^B M_{B\leftarrow C}$
6. $[\mathrm{Id}]_C^B[\vec{x}]_C = [\mathrm{Id}(\vec{x})]_B$ for all $\vec{x} \in R^n$

Remark (Generalization of Properties)**.**

Theorem (2.7.5)**.** *Let $T : V \to W$ be a linear map between finite-dimensional vector spaces. Let B_1, B_1 be bases for V and C_1, C_2 be bases for W. Then,*

$$[T]_{B_2}^{C_1} = [Id_W]_{C_2}^{C_1}[T]_{B_2}^{C_2}[Id_V]_{B_1}^{B_2}$$

Proof. $[\mathrm{Id}_W]_{C_2}^{C_1}[T]_{B_2}^{C_2}[\mathrm{Id}_V]_{B_1}^{B_2} = [\mathrm{Id}_W \circ T \circ \mathrm{Id}_V]_{B_1}^{C_1} = [T]_{B_1}^{C_1}$. ∎