Standard model

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Abstract

We will cover the basics of Yang-Mills theory, electroweak theory, the anomalies and tests of Standard Model. In the Yang-Mills theory, we discuss the SU(3) Lagrangian, beta function and show there is indeed asymptoptic freedom and confinement. Next, in the electroweak, we will show that it is under the symmetry of $SU(2) \times U(1)$ symmetry, and further show that the broken symmetry which gives mass to gauge bosons, leptons an quarks. Later in the anomalies, we show that the $U(1)$ charges of Standard model is further confined by the anomaly free conditions. Lastly, we give some predictions of Standard Model and show that they fit very well with the experiments.

Introduction

From the begining of 1900, quantum effects has come to our eyes. We progressively discover new particles and new interactions. However, there are too many models, too many interactions and too many composite particles. Thus, we shall find a simpler theory to combine all these effects and give an explanation to them. This is the birth of the Standard model. Standard model is a combination of all the interactions except gravity. The model is described by mainly 2 parts, first is the strong interaction, described by the Yang-Mills theory, which is an extension of QED. The second is the electroweak interaction, unifies weak interaction and electromagnetic interaction by $SU(2) \times U(1)$ symmetry. However, the symmetry broke down and give mass to gauge boson by Higgs mechanism. With this theory, it gives fairly accurate predictions to experiments, what a good theory needs.

1 Yang-Mills theory

Yang-Mills theory is a gauge theory. Our fields will go under a gauge transformation as the following equation.

$$\psi_l'(x) = \psi_l(x) + i\varepsilon^a(x)\,(T^a)_l^m\,\psi_m(x)$$

where T^a is the generators of the group. m is the index that sum over the space, and a sums over generators. If our gauge symmetry is global symmetry, there will be no gauge field. However, we impose a local symmetry on our system, which means the terms in front of the generators depend on x. In this case, we would have to write down the covariant derivative.

$$D_\mu = \partial_\mu - igA_\mu(x)$$

The use of the covariant derivative is to replace the old derivative operator, because this derivative operator transforms the same way as the fields under gauge transformation, as the following equations.

$$(D_\mu\psi)_l' = (D_\mu\psi)_l + i\varepsilon^a(x)\,(T^a)_l^m\,(D_\mu\psi)_m$$

However, to make such thing happen, the gauge field A would also have to transform under gauge transformation. If we do the calculations, we would have

$$A_\mu'^a(x) = A_\mu^a(x) + \frac{1}{g}\partial_\mu\varepsilon^a(x) + f^{abc}A_\mu^c(x)\varepsilon^b(x)$$

With all being said, we still need to relate the fields to physics, which requires the Lagrangian to be invariant under gauge transformation. To achieve this, we first define the field strength

$$F_{\mu\nu} = \frac{1}{g}[D_\mu, D_\nu]$$

And then we calculate the $\mathrm{tr}(F^{\mu\nu}F_{\mu\nu})$, which is then gauge invariant

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$$

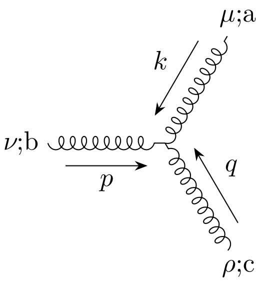
1.1 Feynman rules:

In order to calculate the feynman diagrams up to one loop, the renormalized perturbative Lagrangian is as follow:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}Z_3\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + Z_2\bar{\psi}_i\left(i\partial\!\!\!/ - Z_m m_R\right)\psi_i - Z_{3c}\bar{c}^a\Box c^a \\ &- g_R Z_{A^3} f^{abc}\left(\partial_\mu A^a_\nu\right)A^{\mu b}A^{\nu c} - \frac{1}{4}g_R^2 Z_{A^4}\left(f^{eab}A^a_\mu A^b_\nu\right)\left(f^{ecd}A^{\mu c}A^{\nu d}\right) \\ &+ g_R Z_1 A^a_\mu\bar{\psi}_i\gamma^\mu T^a_{ij}\psi_j + g_R Z_{1c} f^{abc}\left(\partial_\mu\bar{c}^a\right)A^{\mu b}c^c\end{aligned}$$

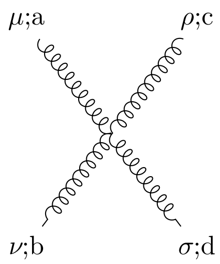
using which we have following feynman rules:

The three gluon vertex:



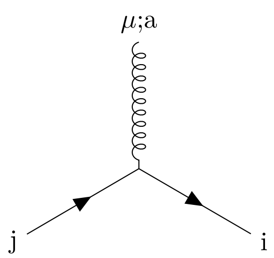
$$= gf^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right] \tag{1}$$

The four gluon vertex:



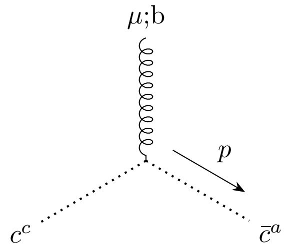
$$\begin{aligned} = \quad &-ig^2 \times \left[f^{abe}f^{cde}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right)\right. \\ &\quad + f^{ace}f^{bde}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) \\ &\quad \left. + f^{ade}f^{bce}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\right]\end{aligned} \tag{2}$$

The gluon fermion vertex:



$$= ig\gamma^\mu T^a_{ij} \tag{3}$$

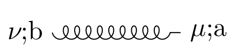
The gluon ghost vertex:



$$= -g f^{abc} p^{\mu} \tag{4}$$

And the propagators are:

gluon propagator:



$$= i \frac{-g^{\mu\nu} + (1-\xi)\frac{p^{\mu}p^{\nu}}{p^2}}{p^2 + i\varepsilon} \delta^{ab} \tag{5}$$

fermion propagator:

$$= \frac{i\delta^{ij}}{\not{p} - m + i\varepsilon} \tag{6}$$

ghost propagator:

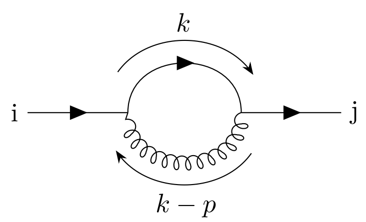
$$= \frac{i\delta^{ab}}{p^2 + i\varepsilon} \tag{7}$$

1.2 beta function and asymptotic freedom:

In order to calculate the beta function of coupling constants, we need to write the coupling in terms of the counterterms:

$$g_0 = g_R \frac{Z_1}{Z_2\sqrt{Z_3}} \mu^{\frac{4-d}{2}}$$

Therefore the relevant calculations are fermion self energy, gluon self energy, and the fermion gluon vertex one loop corrections. For the fermion self energy, we have:

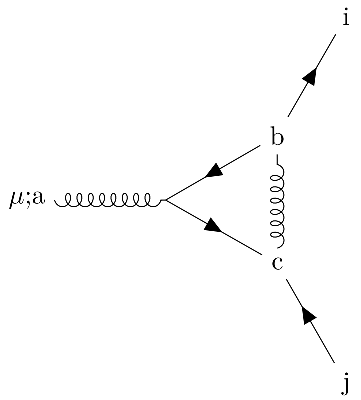


$$
\begin{aligned}
&= \sum_{a,b,k,l} T^a_{ki} T^b_{jl} \delta^{ab} \delta^{kl} (ig)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon} \gamma^\mu \frac{-i}{(k-p)^2+i\varepsilon} \\
&= \delta^{ij} C_F (ig)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon} \gamma^\mu \frac{-i}{(k-p)^2+i\varepsilon} \\
&= \delta^{ij} \left\{ -\frac{ig^2}{8\pi^2} C_F \int_0^1 dx (2m - x\not{p}) \left[\frac{2}{\varepsilon} + \ln \frac{\tilde{\mu}^2}{(1-x)(m^2-p^2x)} \right] + \delta_2 \not{p} - (\delta_m + \delta_2) m \right\} \\
&= \delta^{ij} \left\{ \frac{ig^2}{16\pi^2} C_F \left(\frac{2\not{p} - 8m}{\varepsilon} \right) + \text{ finite } + \delta_2 \not{p} - (\delta_m + \delta_2) m \right\}
\end{aligned} \tag{8}
$$

Therefore we find that the counterterm for fermion field is:

$$
\delta_2 = \frac{1}{\varepsilon} \frac{g^2}{16\pi^2} [-2C_F]
$$

For the three gluon propagator, we have:



$$
\begin{aligned}
&= ig \left(T^c T^a T^b \right)_{ij} \delta^{bc} \times \Gamma^\mu_{(2A)} \\
&= ig\delta^{bc} \times \left(C_F - \frac{1}{2} C_A \right) T^a_{ij} \times \left(F_1^{(2A)} \left(\frac{p^2}{m^2} \right) \gamma^\mu + \frac{i\sigma^{\mu\nu}}{2m} p_\nu F_2^{(2A)} \left(\frac{p^2}{m^2} \right) \right)
\end{aligned} \tag{9}
$$

$$
F_2^{(2A)}(p^2) = \frac{g^2}{4\pi^2} m^2 \int_0^1 dxdydz \delta(x+y+z-1) \frac{z(1-z)}{(1-z)^2 m^2 - xyp^2} \tag{10}
$$

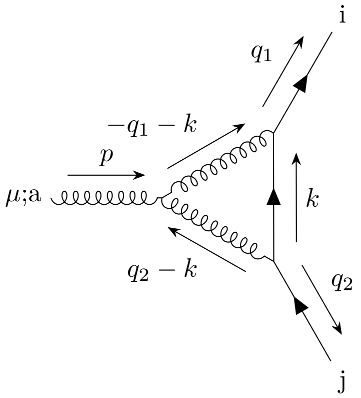
is finite, but the first term is divergent, we extract the divergent in $\bar{M}S$ scheme, we have with $\Delta = (1-z)^2 m^2 - xyp^2$:

$$
F_1^{(2A)} \left(p^2 \right) = \frac{g^2}{8\pi^2} \left(\frac{1}{\varepsilon} - \frac{1}{2} + \int_0^1 dxdydz \delta(x+y+z-1) \left[\frac{p^2(1-x)(1-y) + m^2 \left(1 - 4z + z^2 \right)}{\Delta} + \ln \frac{\tilde{\mu}^2}{\Delta} \right] \right) \tag{11}
$$

we conclude that this graph has contribution to counterterm:

$$
ig \left(2C_F - C_A \right) T^a_{ij} \gamma^\mu \left(\frac{g^2}{16\pi^2 \varepsilon} \right)
$$

The other graph is:



$$
\begin{aligned}
&= igf^{abc}\left(T^cT^b\right)_{ij}\Gamma^{\mu}_{2B} \\
&= -\frac{i}{2}C_AT^a_{ij}(ig)^2g\int\frac{d^4k}{(2\pi)^4}\gamma^{\rho}\frac{i\not{k}}{k^2}\gamma^{\nu}\frac{-i}{\left(q_1+k\right)^2+i\varepsilon}\frac{-i}{\left(q_2-k\right)^2+i\varepsilon} \\
&\quad\times\left[g^{\mu\nu}\left(2q_1+q_2+k\right)^{\rho}+g^{\nu\rho}\left(-q_1+q_2-2k\right)^{\mu}+g^{\rho\mu}\left(k-2q_2-q_1\right)^{\nu}\right]
\end{aligned}
\tag{12}
$$

To extract the divergence we can set $p^2 = 0$, and find that:

$$\Gamma^{\mu}_{(2B)}(0) = i\gamma^{\mu}\frac{g^2}{8\pi^2}\left(\frac{3}{\varepsilon}+\frac{3}{2}ln\tilde{\mu}^2+...\right)$$

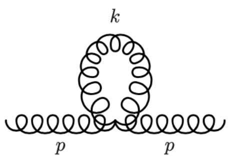
Summing all, we have:

$$\delta_1 = \frac{1}{\varepsilon}\left(\frac{g^2}{16\pi^2}[-2C_F-2C_A]\right)$$

Last but most tedious, the gluon propagator contain four main diagrams:

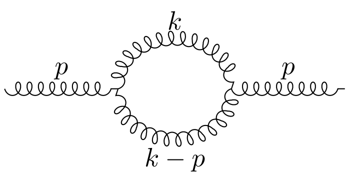
The first vanishes since it is a scaleless integral: The second diagram is:

$$i\mathcal{M}_4^{ab\mu\nu} =$$



$$\sim\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2} = 0.$$

$$i\mathcal{M}_2^{ab\mu\nu} =$$



$$= \frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{-i}{k^2}\frac{-i}{(k-p)^2}f^{ace}f^{bdf}\delta^{cf}\delta^{ed}N^{\mu\nu} \tag{13}$$

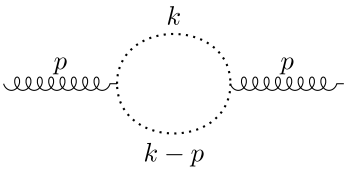
$$N^{\mu\nu} = \left[g^{\mu\alpha}(p+k)^{\rho}+g^{\alpha\rho}(p-2k)^{\mu}+g^{\rho\mu}(k-2p)^{\alpha}\right]g^{\alpha\beta}g^{\rho\sigma}\times\left[g^{\nu\beta}(p+k)^{\sigma}-g^{\beta\sigma}(2k-p)^{\nu}-g^{\sigma\nu}(2p-k)^{\beta}\right]$$

using feynman parameters and with $\Delta = x(1-x)p^2$, after tedious calculation, the final result will be:

$$
\begin{aligned}
\mathcal{M}_2^{ab\mu\nu} =& -\frac{g^2}{2}\frac{\mu^{4-d}}{(4\pi)^{d/2}}\delta^{ab}C_A\int_0^1 dx\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \times \left\{g^{\mu\nu}3(d-1)\Gamma\left(1-\frac{d}{2}\right)\Delta\right.\\
&+p^\mu p^\nu\left[6\left(x^2-x+1\right)-d(1-2x)^2\right]\Gamma\left(2-\frac{d}{2}\right)\\
&\left.+g^{\mu\nu}p^2\left[\left(-2x^2+2x-5\right)\Gamma\left(2-\frac{d}{2}\right)\right]\right\}
\end{aligned}
$$

Also the ghost bubble term:

$$i\mathcal{M}_{gh}^{ab\mu\nu} =$$



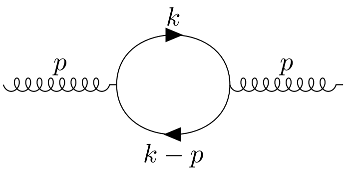
$$= -g^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{(k-p)^2}\frac{i}{k^2}f^{cad}f^{dbc}k^\mu(k-p)^\nu \tag{14}$$

(15)

$$\mathcal{M}_{gh}^{ab\mu\nu} = g^2\frac{\mu^{4-d}}{(4\pi)^{d/2}}\delta^{ab}C_A\int_0^1 dx\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\left\{g^{\mu\nu}\left[\frac{1}{2}\Gamma\left(1-\frac{d}{2}\right)\Delta\right]+p^\mu p^\nu\left[x(1-x)\Gamma\left(2-\frac{d}{2}\right)\right]\right\}$$

For the fermion loop, we have:

$$i\mathcal{M}_F^{ab\mu\nu} = \tag{16}$$



$$\mathcal{M}_F^{ab\mu\nu} = i\,\mathrm{Tr}\left[T^aT^b\right](ig)^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{(p-k)^2-m^2}\frac{i}{k^2-m^2}\,\mathrm{Tr}\left[\gamma^\mu(\not{k}-\not{p}+m)\gamma^\nu(\not{k}+m)\right] \tag{17}$$

$$= -\delta^{ab}T_F\frac{g^2}{2\pi^2}(p^2g^{\mu\nu}-p^\mu p^\nu)\int_0^1 dx x(1-x)\left[\frac{2}{\varepsilon}+ln\left(\frac{\tilde{\mu}^2}{m^2-p^2x(1-x)}\right)\right] \tag{18}$$

The sum of gluon and ghost propagator will be:

$$
\begin{aligned}
\mathcal{M}_{\text{glue}}^{ab\mu\nu} &= \mathcal{M}_{2}^{ab\mu\nu} + \mathcal{M}_{gh}^{ab\mu\nu} \\
&= \delta^{ab} C_A g^2 \frac{\mu^{4-d}}{(4\pi)^{d/2}} \int_0^1 dx \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \left\{ g^{\mu\nu}\Delta \left[\left(\frac{3-3d}{d} + (d-1) + \frac{1}{d}\right) \frac{d}{2}\Gamma\left(1-\frac{d}{2}\right)\right]\right. \\
&+ p^\mu p^\nu \left[-3\left(x^2-x+1\right) + \frac{d}{2}(1-2x)^2 + x(1-x)\right]\Gamma\left(2-\frac{d}{2}\right) \\
&+ g^{\mu\nu} p^2 \left[\left(x^2 - x + \frac{5}{2} - (1-x)^2(d-1)\right)\Gamma\left(2-\frac{d}{2}\right)\right] \} \\
&= \delta^{ab} C_A g^2 \frac{\mu^{4-d}}{(4\pi)^{d/2}} \int_0^1 dx \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \Gamma\left(2-\frac{d}{2}\right) \\
&\times \left\{ g^{\mu\nu} p^2 \left[\left(-2x^2+3x-1\right)d + x(4x-5) + \frac{7}{2}\right]\right. \\
&\left. + p^\mu p^\nu \left[\frac{d}{2}(1-2x)^2 - 4x^2 + 4x - 3\right]\right\} \\
&= C_A \delta^{ab} \frac{g^2}{16\pi^2} \left(g^{\mu\nu}p^2 - p^\mu p^\nu\right) \left[\frac{10}{3\varepsilon} + \frac{31}{9} + \frac{5}{3}\ln\frac{\tilde{\mu}^2}{-p^2} + \mathcal{O}(\varepsilon)\right]
\end{aligned} \tag{19}
$$

Adding fermion loop, we have final result:

$$
\mathcal{M}^{ab\mu\nu} = \delta^{ab}\frac{g^2}{16\pi^2}\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\left[C_A\left(\frac{10}{3\varepsilon} + \frac{5}{3}\ln\frac{\tilde{\mu}^2}{-p^2}\right) - n_f T_F\left(\frac{8}{3}\frac{1}{\varepsilon} + \frac{4}{3}\ln\frac{\tilde{\mu}^2}{-p^2}\right)\right]
$$

So the counterterm will be(in Feynman gauge):

$$
\delta_3 = \frac{1}{\varepsilon}\left(\frac{g^2}{16\pi^2}\right)\left[\frac{10}{3}C_A - \frac{8}{3}n_f T_F\right]
$$

Using renormalization group method, by:

$$
0 = \mu\frac{d}{d\mu}g_0 = \mu\frac{d}{d\mu}\left[g_R\frac{Z_1}{Z_2\sqrt{Z_3}}\mu^{\frac{4-d}{2}}\right]
$$

Calculating the beta function we have:

$$
\begin{aligned}
\beta(g_R) &= \mu\frac{d}{d\mu}g_R = g_R\left[\left(-\frac{\varepsilon}{2}\right) - \mu\frac{d}{d\mu}\left(\delta_1 - \delta_2 - \frac{1}{2}\delta_3\right)\right] + ... \\
&= -\frac{\varepsilon}{2}g_R + \frac{\varepsilon}{2}g_R^2\frac{\partial}{\partial g_R}\left(\delta_1 - \delta_2 - \frac{1}{2}\delta_3\right) \\
&= -\frac{\varepsilon}{2}g_R - \frac{g_R^3}{16\pi^2}\left[\frac{11}{3}C_A - \frac{4}{3}n_f T_F\right]
\end{aligned} \tag{20}
$$

The beta function is smaller than zero when there are less than 17 fermions, whereas $C_A = 3$, $T_F = \frac{1}{2}$. That is the interaction decreases when the energy scale is increasing, therefore the QCD interactions are asymptotically free at high energy.

1.3 Wilson loops and confinement:

An interesting phenomenon in QCD is that the quarks are confined, meaning that they do not have a well defined asymptotic states, this is due to a phenomenon called confinement, opposite charge quarks are confined, and in order to study this topic, we must encounter strong coupling, which we cannot use perturbation theory, therefore we use a method called wilson loop to study it.
A wilson link is defined by ($\varepsilon << 1$):

$$W(x+\varepsilon) := exp[ig\varepsilon^{\mu}A_{\mu}(x)]$$

When the line is not infinitesimal, the **Wilson line** is defined as:

$$W_P(y,x) := W(y, y-\varepsilon_n)...W(x+\varepsilon_1+\varepsilon_2, x+\varepsilon_1)W(x+\varepsilon_1, x)$$

It has following properties:

1. Under gauge transformation $W'_P = U(y)W_P(y,x)U^{\dagger}(x)$
2. $W^{\dagger}_P(y,x) = W_{-P}(x,y)$

where the P denote the path deforming the line from x to y. When the end and start coincide, we define wilson loop as:

$$W_C := Tr[W_C(x,x)]$$

Now we study the Non-abelian case, we first factorize the spacetime into an elementary square with side a called **plaquette**, and define the sides to be μ and ν direction, and we call each wilson loop of the **plaguette** $W_{Plaq\mu\nu}$, we calculate as follow:

$$\begin{aligned} W_{Plaq\mu\nu} &\approx \mathrm{Tr}[e^{-igaA_{\mu}(x)}e^{ia(A_{\nu}(x)+a\partial_{\mu}A_{\nu}(x))}e^{-ia(A_{\mu}(x)+a\partial_{\nu}A_{\mu}(x))}e^{-iaA_{\nu}(x)}] \\ &= \mathrm{Tr}[e^{ia(A_{\mu}(x)+A_{\nu}(x)+a\partial_{\mu}A_{\nu}(x)+\frac{ia}{2}[A_{\mu}(x),A_{\nu}(x)])}e^{-ia(A_{\nu}(x)+A_{\mu}(x)+a\partial_{\nu}A_{\mu}(x))-\frac{ia}{2}[A_{\mu}(x),A_{\nu}(x)]}] \\ &= \mathrm{Tr}\, e^{ia^2F_{\mu\nu}(x)+...} \\ &= \mathrm{Tr}\left(1+ia^2F_{\mu\nu}-\frac{a^4}{2}F_{\mu\nu}F_{\mu\nu+.....}\right) \\ &\approx -\frac{a^4}{2}\,\mathrm{Tr}\,F_{\mu\nu}F_{\mu\nu}+.... \end{aligned} \tag{21}$$

We've used the BCH formula and there's no Einstein summation here. Next summing over all plaquette, and we have it is proportional to the yang mills lagrangian, we have following:

$$S_{wilson} = -\frac{\beta}{2N}\sum_{plaq}\left(W_{\mu\nu}+W^{\dagger}_{\mu\nu}\right) = \frac{\beta g^2}{4N}\int d^4xTrF^{\mu\nu}F_{\mu\nu}+....$$

By comparing we have $\beta = \frac{2N}{g^2}$, since we are dealing with strong coupling limit, g^{-2} is the perturbative power. Now we need to compute wilson loop using path integral formalism:

$$\langle 0|W_C|0\rangle = \frac{1}{Z}\int \mathcal{D}UW_C e^{-S}$$

where the measure of path integral is Haar measure, with following properties:

1.
$$\int dUU_{ij} = 0$$

2.
$$\int dUU_{ij}U_{kl} = 0$$

3.
$$\int dUU_{ij}U^*_{kl} = \frac{1}{N}\delta_{ik}\delta jl$$

4.
$$\mathcal{D}U := \prod_{links} dU_{link}$$

5.
$$\int dU = 1$$

Now in the path integral, we know that only the wilson links on the boundary of the loop with opposite orientation($W^\dagger$ has opposite orientation) will give contribution. Therefore when expanding out the action, only the plaqutte intersecting loop C survives after integrating out the dU_{link} of the boundaries, however if the term expanding out the action contains only terms with products of boundary plaqutte, the links of interior(not the boundary) generated by these plaquttes will make the integral vanish, therefore in order to integrate out all terms, we need to include all interior plaquettes so that their opposite direction cancell with the outside plaquettes next to them, hence we need to expand the action at least to the order of terms with all internal plaquette multiplied together, therefore the vev of wilson link $\langle 0|W_C|0\rangle \propto \frac{1}{g^2}^{\frac{A}{a^2}}$, with A the area of loop.
We write the area law in terms of $\tau := \frac{ln(g^2)+O(1)}{a^2} := \frac{c(g)}{a^2}$, in strong coupling limit, the leading term of τ is logarithm, and is called **string tension**.
Picking the wilson loop with length T, width R and $R << T$, note that the opposite direction in T is equivalent to a particle with same time direction but carries opposite charge, as seen in the definition of wilson loop, ignoring the width integral, the wilson loop is equivalent to particle and anti charged particle pair with distance R moving forward in time, by definition of path integral method we know that this is proportional to $e^{-E_{pair}T}$, comparing with the result of wilson loop area law, we conclude that:

$$E_{pair} = \tau R$$

which means that the non abelian gauge field has potential increasing with distance, therefore the charges are confined, this phenomenon is therefore called confinement.

2 Electroweak theory

2.1 Higgs mechanism

In Electroweak unified theory, the gauge symmetry is SU(2) $\times$ U(1)$_\mathrm{Y}$. From this symmetry, there should be 4 massless gauge bosons. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}(W^a_{\mu\nu})^2 - \frac{1}{4}B^2_{\mu\nu} \tag{22}$$

However, the symmetry is broken into U(1)$_\mathrm{EM}$ at low energy, the world we live in. The subscript Y means high energy U(1) symmetry, called hypercharge; the subscript EM means the low energy U(1) symmetry. The Higgs field break the original symmetry, we can see this by writing out the kinetic part of the Higgs field and the potential.

$$\mathcal{L} = (D_\mu H)(D^\mu H) + m^2 H^\dagger H - \lambda(H^\dagger H)^2 \tag{23}$$

Where the covariant derivative is

$$D_\mu = \partial_\mu - igW^a_\mu t^a - ig' B_\mu Y \tag{24}$$

and as spontaneous symmetry breaking suggests, the Higgs field should have a non 0 vacuum expected value (v.e.v). Also, we know that this theory is SU(2) $\times$ U(1)$_\mathrm{Y}$, we suspect the Higgs field transforms under gauge transformation as follow,

$$H = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} \to \exp\left(i\alpha^a \frac{\sigma^a}{2} + i\beta\frac{1}{2}\right)\begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} \tag{25}$$

We could definitly transform it into

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \tag{26}$$

Under this form, any direction other than $\alpha_3 = \beta$ will disturb the vev, which it means this is a symmetry. This corresponds to a neutral current we will discuss later. Bring this back to (23), we get the Lagrangian

$$\mathcal{L} = g^2\frac{v^2}{8}\left[(W^1_\mu)^2 + (W^2_\mu)^2 + \left(\frac{g'}{g}B_\mu - W^3_\mu\right)^2\right] \tag{27}$$

We could perform some basis change, explicitly saying that

$$B_\mu = \cos\theta_w A_\mu - \sin\theta_w Z_\mu \tag{28}$$

$$W^3_\mu = \sin\theta_w A_\mu + \cos\theta_w Z_\mu \tag{29}$$

And $\tan\theta_w = g'/g$.

Furthermore, we can find a basis that the W bosons have ± 1 charge, the basis is $W^{\pm}_{\mu} = \frac{1}{\sqrt{2}}(W^1_\mu \mp W^2_\mu)$. This is been done by calculating the coupling constant between W and A, this would give us $e = g\sin\theta_w$. To sum up, the resulting mass for each boson is

$$m_W = \frac{vg}{2} \tag{30}$$

$$m_Z = \frac{v}{2}\sqrt{g^2 + g'^2} = \frac{m_W}{\cos\theta} \tag{31}$$

$$m_A = 0 \tag{32}$$

We could already see an non-trivial result, suggesting that $m_Z > m_W$. Which is true, with each of them being 91.2 GeV and 80.4 GeV.

2.2 Lepton coupling

An important discovery is chiral symmetry does not apply in weak interaction, so is parity symmetry. We denote left-handed leptons as

$$L_i = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix} \tag{33}$$

i denotes the generation. We could look at the Lagrangian, it look really weird, pretty asymmetric, suggesting CP violation.

$$\mathcal{L} = i\bar{L}_i(\not{\partial} - ig\not{W}^a T^a - ig'Y_L\not{B})L_i + \bar{e}_R(\not{\partial} - ig'Y_e\not{B})e^i_R + \bar{\nu}_R(\not{\partial} - ig'Y_\nu\not{B})\nu^i_R \tag{34}$$

Note that only the doublet interacts with the W boson, the singets does not interact with the W boson.

2.2.1 Quark hypercharge

Next, we would like to look at the conserved current here. In previous section, we've mentioned there is a neutral current for charge. Let's do a basis change in covariant derivative and constraining the direction only in W^3, that is

$$D_\mu = \partial_\mu - ieA_\mu(t^3 + Y\mathbb{1}) - ieZ_\mu(\cot\theta_w t^3 - \tan\theta_w Y\mathbb{1}) \tag{35}$$

We constrain the direction because it would not change the vev of Higgs field. We could easily see that the coupling with E & M is $Q = t^3 + Y\mathbb{1}$, so that should give the charge of field we want. For example, the e_L

$$Q\begin{pmatrix} 0 \\ e_L \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} - \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 \\ e_L \end{pmatrix} = -\begin{pmatrix} 0 \\ e_L \end{pmatrix} \tag{36}$$

if we have Q act on ν and $\bar{e}$, we will have the charge 0 and $-e$.

The interaction term therefore has the form:

$$\mathcal{L}_{int} = \frac{1}{\sqrt{2}} g_2 W_\mu^+ J^{-\mu} + \frac{1}{\sqrt{2}} g_2 W_\mu^- J^{+\mu} + \frac{e}{sin\theta_W cos\theta_W} Z_\mu J_Z^\mu + eA_\mu J_{EM}^\mu$$

And the neutral currents are

$$\begin{aligned}
J^{+\mu} &:= \bar{\varepsilon}_L \gamma^\mu N_L \\
J^{-\mu} &:= \bar{N}_L \gamma^\mu \varepsilon_L \\
J_Z^\mu &:= J_3^\mu - sin^2\theta_W J_{EM}^\mu \\
J_3^\mu &:= \frac{1}{2} \bar{N}_L \gamma^\mu N_L - \frac{1}{2} \bar{\varepsilon}_L \gamma^\mu \varepsilon_L \\
J_{EM}^\mu &:= -\bar{\varepsilon} \gamma^\mu \varepsilon
\end{aligned} \tag{37}$$

$$N_L := \begin{pmatrix} \nu \\ 0 \end{pmatrix} \tag{38}$$

$$\varepsilon := \begin{pmatrix} e_L \\ e_R \end{pmatrix}$$

2.2.2 Mass Term

In the above Lagrangian, there isn't a mass term for the leptons or quarks, so now we will introduce its mass by Higgs Mechanism. The term we add is what we called Yukawa term,

$$\mathcal{L}_{\text{Yuk}} = -y\bar{L}He_R + h.c. \tag{39}$$

h.c. means to add Hermitian term.

As the Higgs field got the vev., the electrons will get a mass term of $-m_e(\bar{e}_L e_R + \bar{e}_R e_L), m_e = \frac{y\sqrt{2}}{v}$. If we get the L changed into Q, we will also give down-type quarks mass. So this Lagrangian gives leptons and down-type quarks mass.

2.3 Quark section

In order to construct the full standard model, which is the gauge group $SU(3) \times SU(2) \times U(1)_Y$, the covariant derivatives are as follows:

$$\begin{aligned}
(D_\mu q_I)_{\alpha i} &= \partial_\mu q_{I\alpha i} - ig_3 A_{3\mu}^a (T_3^a)_\alpha^\beta q_{I\beta i} - ig_2 A_{2\mu}^a (T_2^a)_i^j q_{I\alpha j} - ig_1 Y_q B_\mu q_{I\alpha i} \\
(D_\mu u_{IR})^\alpha &= \partial_\mu {u_{IR}}^\alpha - ig_3 {A_3}_\mu^a (T_{\bar{3}}^a)_\beta^\alpha {u_{IR}}^\beta - ig_1 Y_{u_R} B_\mu {u_{IR}}^\alpha \\
(D_\mu d_{IR})^\alpha &= \partial_\mu {d_{IR}}^\alpha - ig_3 {A_3}_\mu^a (T_{\bar{3}}^a)_\beta^\alpha {d_{IR}}^\beta - ig_1 Y_{d_R} B_\mu {d_{IR}}^\alpha
\end{aligned} \tag{40}$$

where i are flavor indices of up down quarks, α are color indices and I are generation indices(we will ignore it for a while), indicating the following types of quarks. The theory is also chiral since different chirality couples to different gauge representation, and we write the quark doublets q as:

$$q_I = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix} \tag{41}$$

In order to construct and find the $U(1)_Y$ generator of each quark types, using the phemenological value that upper component of the quark type has charge $+\frac{2}{3}$ and lower has $-\frac{1}{3}$, using $Q = T_2^3 + Y$, we conclude that:

$$Y_q = \frac{1}{6}, Y_{u_R} = \frac{2}{3}, Y_{d_R} = -\frac{1}{3} \tag{42}$$

The kinetic term is simply:

$$\mathcal{L}_{kin} = iq^{\dagger\alpha}\bar{\sigma}^{\mu}(D_\mu q)_{\alpha i} + iu^{\dagger}_{R\alpha}\sigma^{\mu}(D_\mu u_R)^{\alpha} + id^{\dagger}_{R\alpha}\sigma^{\mu}(D_\mu d_R)^{\alpha}$$

And in order to include a $SU(2) \times U(1)_Y$ invariant mass term, we need to use the higgs mechanism, therefore the yukawa couplings are:

$$\mathcal{L}_{Yuk} = -G_d\left(q^{\dagger\alpha i}\varphi d_R^{\alpha} + d_R^{\dagger\alpha}\varphi^{\dagger}q_{\alpha}\right) - G_u\left(q_{\alpha}^{\dagger}\tilde{\varphi}u_R^{\alpha} + u_R^{\dagger\alpha}\tilde{\varphi}^{\dagger}q_{\alpha}\right)$$

with $\varphi = i\sigma^2\varphi^*$. The reason we choose σ^2 is because we need to swap the components of Higgs field, but σ^1 do not have the nice property as σ^2 that $\sigma_k^{\dagger}\sigma_2 + \sigma_2\sigma_k^* = 0$ to preserve the $SU(2)$ transformation. And the conjugation of Higgs field is to make sure $U(1)$ invariance.

We can write the Yukawa coupling in terms of Dirac fields:

$$\mathcal{D}_\alpha = \begin{pmatrix} d_\alpha \\ d_{R\alpha} \end{pmatrix}, \mathcal{U}_\alpha = \begin{pmatrix} u_\alpha \\ u_{R\alpha} \end{pmatrix} \tag{43}$$

$$\mathcal{L}_{Yuk} = -\frac{G_d}{\sqrt{2}}(v + H(x))\bar{\mathcal{D}}_\alpha\mathcal{D}_\alpha - \frac{G_u}{\sqrt{2}}(v + H(x))\bar{\mathcal{U}_\alpha}\mathcal{U}_\alpha$$

where the bar denote dirac conjugate as normal. we have mass terms $m_d = \frac{G_d v}{\sqrt{2}}$ and $m_u = \frac{G_u v}{\sqrt{2}}$

2.4 Neutral Current

Consider the generation indices, and we denote the mass term matrix as G^d_{IJ}, and G^u_{IJ}, so the Yukawa coupling became:

$$\mathcal{L}_{Yuk.} = -G^d_{IJ}\left(q_I^{\dagger\alpha}\phi d^{\alpha}_{JR}\right) - G^u_{IJ}\left(q^{\dagger}_{\alpha I}\tilde{\phi}u^{\alpha}_{JR}\right) + h.c.$$

Using the fact that any matrix A, $AA^\dagger$ is hermitian and write the mass matrix $G_d G_d^\dagger = U_d^\dagger M_d^2 U_d$, and $G_u G_u^\dagger = U_u M_u^2 U_u^\dagger$, so we can generically write $G_d = U_d M_d K_d^\dagger$, and $G_u = U_u M_u K_u^\dagger$, M_u and M_d are diagonal matrices. And the mass term became:

$$\mathcal{L}_{mass} = -\frac{v}{\sqrt{2}}\left[d_L^\dagger U_d M_d K_d^\dagger d_R + u_L^\dagger U_u M_u K_u^\dagger u_R + h.c.\right]$$

This means that we can choose a basis called **mass basis** such that $d_L = U_d d'_L$, $u_L = U_u u'_L$, $d_R = K_d d'_R$ and $u_R = K_u u'_R$, while the original **flavor basis**, therefore in terms of mass basis we can write the full Lagrangian as:

$$\begin{aligned}
&\mathcal{L}_{\text{flavor-basis}} \\
&= \begin{pmatrix} u_L^{I\dagger} & d_L^{I\dagger} \end{pmatrix} \left[i\partial\!\!\!/ + \bar{\sigma}^\mu \begin{pmatrix} \frac{g_1}{6}B_\mu + \frac{g_2}{2}W_\mu^3 & \frac{g_2}{\sqrt{2}}W_\mu^+ \\ \frac{g_2}{\sqrt{2}}W_\mu^- & \frac{g_1}{6}B_\mu - \frac{g_2}{2}W_\mu^3 \end{pmatrix} \right] \begin{pmatrix} u_L^I \\ d_L^I \end{pmatrix} \\
&+ u_R^{I\dagger}\left(i\partial\!\!\!/ + g_1\frac{2}{3}B\!\!\!/\right)u_R^I + d_R^{I\dagger}\left(i\partial\!\!\!/ - g_1\frac{1}{3}B\!\!\!/\right)u_R^I - \frac{v}{\sqrt{2}}\left[d_L^{I\dagger}(U_d M_d K_d^\dagger)_{IJ} d_R^J + u^{I\dagger}(U_u M_u K_u^\dagger)_{IJ} u_R^J + h.c.\right] \\
&= \frac{e}{\sin\theta_W} Z_\mu J_Z^\mu + eA_\mu J_{EM}^\mu - \sum_I m_I^d \left(d^{I\dagger} d_R^I + d_R^{I\dagger} d_L^I\right) - \sum_I m_I^u \left(u^{I\dagger} u_R^I + u_R^{I\dagger} u_L^I\right) \\
&+ \frac{e}{\sqrt{2}\sin\theta_W}\left[W_\mu^+ u_L^{I\dagger}\bar{\sigma}^\mu V^{IJ} d_{JL} + W^-\mu d_L^{I\dagger}\bar{\sigma}^\mu (V)^{IJ} u_{JL}\right]
\end{aligned} \tag{44}$$

where we have $\partial\!\!\!/ = \sigma^\mu \partial_\mu$, and $V := U_u^\dagger U_d$. The V matrix is called Cabibbo-Kobayashi-Maskawa(CKM) matrix. The currents are defined as follows:

$$\begin{aligned}
J_Z^\mu &:= J_3^\mu - sin^2\theta_w J_{EM}^\mu \\
J_3^\mu &:= \frac{1}{2}\bar{\mathcal{U}}_L\gamma^\mu\mathcal{U}_L - \frac{1}{2}\bar{\mathcal{D}}_L\gamma^\mu\mathcal{D}_L \\
J_{EM}^\mu &:= \frac{2}{3}\bar{\mathcal{U}}\gamma^\mu\mathcal{U} - \frac{1}{3}\bar{\mathcal{D}}\gamma^\mu\mathcal{D} \\
J^{+\mu} &:= \bar{\mathcal{D}}_{LI}(V^\dagger)_{IJ}\gamma^\mu\mathcal{U}_{LJ} \\
J^{-\mu} &:= \bar{\mathcal{U}}_{LI}V_{IJ}\gamma^\mu\mathcal{D}_{LJ}
\end{aligned}$$

2.5 CKM matrix

The CKM matrix can be explicitly written as

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}s_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{45}$$

We can count the DOFs of the CKM matrix. We can now count the DOF in V. Since it is Hermitian, there are 9 DOFs. If the matrix is all real, there will be 3 DOFs. Now there is complex involved, thus adding 6 phases to match 9 DOFs. However, we could eliminate 5 DOFs by the U(1) symmetry, so the CKM matrix only have 4 DOFs, including 3 angles and 1 phase.

Moreover, the CKM matrix is set to be unitary. So if there are more than 3 generations, the CKM matrix now, which is 3 by 3, will not be unitary. We can check this by experiments. We know that a unitary matrix should have orthogonal rows and columns, thus

$$\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} + \frac{V_{ud}V_{ub}^*}{V_{td}V_{tb}^*} + 1 = 0 \tag{46}$$

This constructs a triangle in the complex plane. We could see it in the following figure

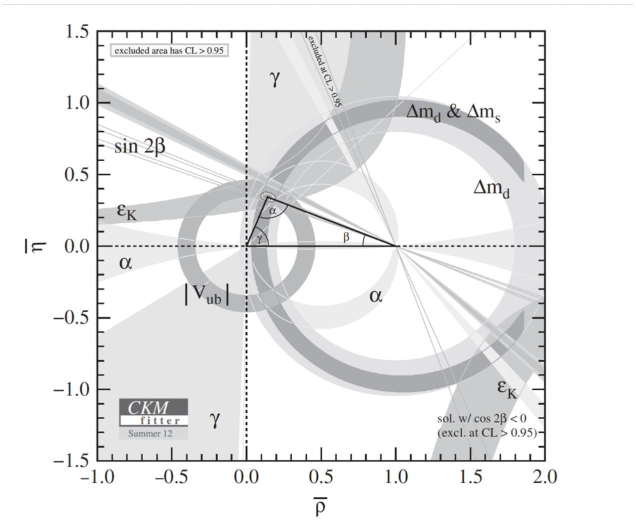


Figure 1: Measurement of quarks' mass

2.6 Neutrino

In the above analysis, we didn't account the mass for the neutrinos. To consider it, we could do the similar thing we did before

$$\mathcal{L} = -Y_{ij}^e \bar{L}^i H e_R^j - Y_{ij}^\nu \bar{L}^i \tilde{H} \nu_R^i - iM_{ij}(\nu_R^i)^c \nu_R^i + h.c. \tag{47}$$

The second term is what we call Dirac mass, the thrid term is new, Majorana mass. We might think the Majorana mass break the electroweak symmetry. However, as long as neutrinos don't bring quantum number the symmetry still holds. If we change our basis as

$$\psi_R = \begin{pmatrix} i\sigma_2 \nu_R^* \\ \nu_R \end{pmatrix}, \psi_L = \begin{pmatrix} \nu_L \\ i\sigma_2 \nu_L^* \end{pmatrix}, \tag{48}$$

and then we can write the Lagrangian as

$$\mathcal{L}_{\nu,\text{mass}} = -m\bar{\psi}_L \psi_R - \frac{M}{2}\bar{\psi}_R \psi_R \tag{49}$$

We could obtain its mass eigenstates and eigen mass, which is

$$m^* = \sqrt{m^2 + \frac{1}{4}M^2} \pm \frac{1}{2}M \tag{50}$$

So with another mass being considered, we could explain the uniquely low mass of neutrino. Consider m has the electroweak scale, about 100 GeV. Then for neutrino's mass, 10^{-6} eV, the Majorana mass has to be about 10^{19} GeV.

Moreover, there is also a similar mass-basis to flavor-basis in neutrino. To make such thing happen, there should be different flavors' interaction, which happens at the W boson interaction term.

$$\mathcal{L} = -\frac{g}{\sqrt{2}}(\bar{e}_L \not{W} \nu_{Le} + \bar{\mu}_L \not{W} \nu_{L\mu} + \bar{\tau}_L \not{W} \nu_{L\tau} + h.c.) \tag{51}$$

Now, we could write it in mass eigenstate(ν_{Li})

$$\mathcal{L} = -\frac{g}{\sqrt{2}} U^{ij} (\bar{e}_{Li} \not{W} \nu_{Lj} + h.c.) \tag{52}$$

where

$$\nu_{Le} = U^{ei} \nu_{Li} \tag{53}$$

This U is called the PMNS matrix, which has 2 more phases because of the Majarona mass. So the PMNS matrix is

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}s_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{i\alpha_{12}}{2}} & 0 \\ 0 & 0 & e^{\frac{i\alpha_{31}}{2}} \end{pmatrix} \tag{54}$$

3 Anomalies and constraint on U(1) charges:

We've seen that the U(1) charges are used in previous sections, but now we are going to show that they are constrained by anomaly free conditions.

3.1 Chiral anomaly:

3.1.1 Non perturbative:

Using Fujikawa method we can calculate the anomaly from path integral formulation. The reason this phenomenon arises is due to that the measure of path integral is not invariant under some symmetries, in this case the axial symmetry $\Psi'(x) = \int d^4y J(x,y)\Psi(y) := \int d^4y e^{-i\alpha(x)\gamma_5}\Psi(y)$.

$$Z(A) := \int \mathcal{D}\Psi\mathcal{D}\bar{\Psi} e^{iS(A)} = \int \mathcal{D}\Psi\mathcal{D}(det(J))^{-2}\bar{\Psi} e^{iS(A) - i\int d^4x \alpha(x)\partial_\mu j^\mu_A(x)} \tag{55}$$

regulating the measure we have

$$\begin{aligned}
\delta^4(x-y) &\to e^{\frac{(i\not{D}_x)^2}{M^2}}\delta^4(x-y) \\
&= \int \frac{d^4k}{(2\pi)^4} e^{\frac{(i\not{D}_x)^2}{M^2}} e^{ik(x-y)} \\
&= \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} e^{(i\not{D}-\not{k})^2/M^2} \\
&= \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} e^{(-k^2+2ik\cdot D+D^2+gS^{\mu\nu}F_{\mu\nu})/M^2} \quad S^{\mu\nu} \text{ is lorentz generator} \\
&= M^4 \int \frac{d^4k}{(2\pi)^4} e^{iMk(x-y)} e^{(-k^2+2ik\cdot D/M+D^2/M^2+gS^{\mu\nu}F_{\mu\nu}/M^2)} \\
\therefore \operatorname{Tr}\delta^4(x-y)\gamma_5 &\to M^4 \int \frac{d^4k}{(2\pi)^4} \mathrm{e}^{2\mathrm{ik}\cdot\mathrm{D}/\mathrm{M}+\mathrm{D}^2/\mathrm{M}^2+\mathrm{g}\mathrm{S}^{\mu\nu}\mathrm{F}_{\mu\nu}}\gamma_5 \\
&= \frac{g^2}{2} \int \frac{d^4k}{(2\pi)^4} e^{-k^2} (\operatorname{Tr} F_{\mu\nu}F_{\rho\sigma})(\operatorname{Tr} S^{\mu\nu}S^{\rho\sigma}\gamma_5) \\
&= \frac{g^2}{2} \int \frac{d^4k}{(2\pi)^4} e^{-k^2} (\operatorname{Tr} F_{\mu\nu}F_{\rho\sigma})(i\varepsilon^{\mu\nu\rho\sigma}) \\
&= -\frac{g^2}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}\operatorname{Tr} F_{\mu\nu}F_{\rho\sigma} \quad \text{Using formula} detJ = e^{\operatorname{Tr} lnJ} \\
\text{we have } (detJ)^{-2} &= exp\left[-\frac{ig^2}{16\pi^2}\int d^4x\alpha(x)\varepsilon^{\mu\nu\rho\sigma}\operatorname{Tr} F_{\mu\nu}(x)F_{\rho\sigma}\right] \\
\text{the path integral then became } Z(A) &\to \int \mathcal{D}\Psi\mathcal{D}\bar{\Psi} e^{iS(A)} e^{-i\int d^4x\alpha(x)\left[\frac{g^2}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}\operatorname{Tr} F_{\mu\nu}(x)F_{\rho\sigma}+\partial_\mu j_A^\mu(x)\right]}
\end{aligned}$$

Therefore

$$\langle\partial_\mu J_\mu^5\mathcal{O}(x_1,...,x_n)\rangle = -\frac{g^2}{16\pi^2}\langle\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\mathcal{O}(x_1,...,x_n)\rangle$$

3.1.2 perturbative:

We can show that perturbative result is equivalent to the all loop result, . Consider the triangle diagram in QED defined as below:

$$\begin{aligned}
&iM_5^{\alpha\mu\nu}(p,q_1,q_2)(2\pi)^4\delta^4(p-q_1-q_2) \\
&= \int d^4xd^4yd^4z e^{-ipx}e^{iq_1y}e^{iq_2z}\left\langle J^{\alpha5}(x)J^\mu(y)J^\nu(z)\right\rangle \\
&= \int d^4xd^4yd^4z e^{-ipx}e^{iq_1y}e^{iq_2z}\left\langle\left[\bar\psi(x)\gamma^\alpha\gamma^5\psi(x)\right]\left[\bar\psi(y)\gamma^\mu\psi(y)\right]\left[\bar\psi(z)\gamma^\nu\psi(z)\right]\right\rangle
\end{aligned}$$

we have:

$$iM_5^{\alpha\mu\nu} = -\int\frac{d^4k}{(2\pi)^4}\operatorname{Tr}\left[\gamma^\mu\frac{i}{\not{k}}\gamma^\nu\frac{i}{\not{k}+\not{q}_2}\gamma^\alpha\gamma^5\frac{i}{\not{k}-\not{q}_1}+\gamma^\nu\frac{i}{\not{k}}\gamma^\mu\frac{i}{\not{k}+\not{q}_1}\gamma^\alpha\gamma^5\frac{i}{\not{k}-\not{q}_2}\right]$$

The partial derivative acting on α will be:

$$\begin{aligned} p_\alpha M_5^{\alpha\mu\nu} &= \int \frac{d^4k}{(2\pi)^4}\left[\frac{\mathrm{Tr}\left[\gamma^\mu \not{k}\gamma^\nu(\not{k}+q/2)\not{p}\gamma^5(\not{k}-\not{q}/1)\right]}{k^2\left(k+q_2\right)^2\left(k-q_1\right)^2} + \begin{pmatrix}\mu\leftrightarrow\nu\\ 1\leftrightarrow 2\end{pmatrix}\right] \\ &= 4i\varepsilon^{\mu\nu\rho\sigma}\int\frac{d^4k}{(2\pi)^4}\left[\frac{k^\rho q_2^\sigma}{k^2\left(k+q_2\right)^2}+\frac{k^\rho q_1^\sigma}{k^2\left(k-q_1\right)^2}\right] + \begin{pmatrix}\mu\leftrightarrow\nu\\ 1\leftrightarrow 2\end{pmatrix} \end{aligned}$$

by using $\not{p}\gamma^5 = (\not{q}_1+\not{q}_2\gamma^5) = \gamma^5(\not{k}-\not{q}_1)+(\not{k}+\not{q}_2)\gamma^5$

Also evaluating:

$$\begin{aligned} q_\mu^1 M_5^{\alpha\mu\nu} &= \int\frac{d^4k}{(2\pi)^4}\left[\frac{\mathrm{Tr}\left[\not{q}_1\not{k}\gamma^\nu\left(\not{k}+\not{q}_2\right)\gamma^\alpha\gamma^5(\not{k}-\not{q}_1)\right]}{k^2\left(k+q_2\right)^2\left(k-q_1\right)^2}+\frac{\mathrm{Tr}\left[\gamma^\nu\not{k}\not{q}_1\left(\not{k}+\not{q}_1\right)\gamma^\alpha\gamma^5(\not{k}-\not{q}_2)\right]}{k^2\left(k+q_1\right)^2\left(k-q_2\right)^2}\right] \\ &= \int\frac{d^4k}{(2\pi)^4}\left[\frac{\mathrm{Tr}\left[\gamma^\nu(\not{k}+\not{q}_2)\gamma^\alpha\gamma^5(\not{k}-\not{q}_1)\right]}{\left(k-q_1\right)^2\left(k+q_2\right)^2}-\frac{\mathrm{Tr}\left[\not{k}\gamma^\nu(\not{k}+\not{q}_2)\gamma^\alpha\gamma^5\right]}{k^2\left(k+q_2\right)^2}\right. \\ &\quad\left.+\frac{\mathrm{Tr}\left[\gamma^\nu\not{k}\gamma^\alpha\gamma^5(\not{k}-\not{q}_2)\right]}{k^2\left(k-q_2\right)^2}-\frac{\mathrm{Tr}\left[\gamma^\nu(\not{k}+\not{q}_1)\gamma^\alpha\gamma^5(\not{k}-\not{q}_2)\right]}{\left(k+q_1\right)^2\left(k-q_2\right)^2}\right] \end{aligned} \tag{56}$$

Since above is a linearly divergent integral, we need to use a different method to regulate them, the result is (see appendix) by changing $k^\mu \to k^\mu + b_2 q_1^\mu + b_1 q_2^\mu$ in the first contraction and changing $k^\mu \to k^\mu + b_2 q_1^\mu + b_1 q_2^\mu$, in order to maintain Bose symmetry for photons. The result became:

$$q_\mu^1 M_5^{\alpha\mu\nu} = \frac{1}{4\pi^2}\varepsilon^{\alpha\nu\rho\sigma}q_\rho^1 q_\sigma^2(1-b_1-b_2)$$

and

$$p_\alpha M_5^{\alpha\mu\nu} = \frac{1}{4\pi^2}\varepsilon^{\mu\nu\rho\sigma}q_\rho^1 q_\sigma^2(b_1-b_2)$$

In order to preserve the symmetry in massive limit, i.e. only break the chiral current, we have to choose $b_1 - b_2 = 1$. Therefore the final result is:

$$p_\alpha M_5^{\alpha]\mu\nu} = \frac{1}{4\pi^2}\varepsilon^{\mu\nu\rho\sigma}q_\rho^1 q_\sigma^2, \quad q_\mu^1 M_5^{\alpha\mu\nu} = 0$$

This result is consistent with our path integral calculation. As a final comment, we can see that for chiral theories(different coupling for different chirality), with $\langle J_L^\alpha J_L^\mu J_L^\nu\rangle$ denoted as $M_L^{\mu\nu\alpha}$

$$M_L^{\alpha\mu\nu} = \int\frac{d^4k}{(2\pi)^4}\left[\frac{\mathrm{Tr}\left[\gamma^\mu P_L\ \not{k}\gamma^\nu P_L(\not{k}+q/2)\gamma^\alpha P_L(\not{k}-q/1)\right]}{k^2\left(k+q_2\right)^2\left(k-q_1\right)^2}+\begin{pmatrix}\mu\leftrightarrow\nu\\ 1\leftrightarrow 2\end{pmatrix}\right]$$

Obviously we see by moving projection operators,

$$M_L^{\alpha\mu\nu} = \frac{1}{2}\left(M_V^{\alpha\mu\nu}-M_5^{\alpha\mu\nu}\right)$$

$$M_R^{\alpha\mu\nu} = \frac{1}{2}\left(M_V^{\alpha\mu\nu} + M_5^{\alpha\mu\nu}\right)$$

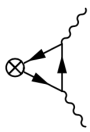
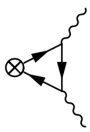
Therefore for different chiralities to combine, the triangle diagram will be the product of left chirality charge minus the one with right chirality charges.

3.2 Triangle diagram:

We need to maintain the ward takahashi identity in quantum level, since we require the theory to have gauge invariance, the anomally generated from our gauge group should cancell, this will give a constraint on hypercharges, which we claimed without reasoning why the charges should be the numerical result stated previously. For non abelian gauge theory, we need to include the group factors since the currents are:

$$J_\mu^a = \sum_\psi \bar{\psi}_i T_{ij}^a \gamma^\mu \psi_j$$

Since the one loop result is exact, we can calculate only one loop, and higher order terms will automatically satisfy ward takahashi identity. Calculating the diagram: We use the following relation:

$i\mathcal{M} = \mathrm{tr}\big[T^aT^bT^c\big] \times$  $+ \quad \mathrm{tr}\big[T^aT^cT^b\big] \times$ 

$$\mathrm{Tr}[T^aT^bT^c] = \frac{1}{2}\,\mathrm{Tr}[[T^a, T^b], T^c] + \frac{1}{2}\,\mathrm{Tr}[\{T^a, T^b\}, T^c] = \frac{i}{2} T_R f^{abc} + \frac{1}{4} d_R^{abc}$$

noting that the first term corresponds to the renormalization of $f^{abc} A_\mu^a A_\nu^b \partial_\mu A_\nu^c$ vertex, so it could be renormalized during renormalization procedure. We need only care about the second term, where

$$d_R^{abc} = 2\,\mathrm{Tr}[T_R^a, \{T_R^b, T_R^c\}] = 2A(R)\,\mathrm{Tr}[T^a\{T^b, T^c\} := A(R) d^{abc}$$

Therefore we need only modify the QED result into:

$$\partial_\alpha \langle J^{a\alpha}(x) J^{b\mu}(y) J^{c\nu}(z)\rangle = \left(\sum_l A(R_l) - \sum_r A(R_r)\right) \frac{g^2}{128\pi^2} d^{abc} \varepsilon^{\mu\nu\alpha\beta} \langle F_{\mu\nu}^b F_{\alpha\beta}^c(x) J^\mu(y) J^\nu(z)\rangle$$

There are in fact $H_3^3 = 10$ combinations to contribute to the anomally, and we have following possibilities:

1. $SU(3) \times SU(3) \times SU(3)$ since QCD is non-chiral(same coupling for different chirality), the group factor $d^{abc} = 0$
2. $SU(N) \times U(1) \times U(1)$ since $2\,\mathrm{Tr}[T^a\{1, 1\}] = 0$, it vanishes
3. $SU(m) \times SU(m) \times SU(n)$, since the trace of them is just the product between generators, and vanishes obviously

4. $SU(3) \times SU(2) \times U(1)$ vanish as above

5. $SU(3) \times SU(3) \times U(1)$: $2Y_Q - Y_u - Y_d = 0$, 2 for quark doublet

6. $SU(2) \times SU(2) \times U(1)$: $Y_L + 3Y_Q = 0$, three for additional color symmetry.

7. $SU(2) \times SU(2) \times SU(2)$: $d^{abc} = \delta^{bc}\,\mathrm{Tr}[\tau^a] = 0$

8. $U^3(1)$: $(2Y_L^3 - Y_e^3 - Y_\nu^3) + 3(2Y_Q^3 - Y_u^3 - Y_d^3) = 0$

9. $grav^2 U(1)$: $\partial_\alpha J^a_\alpha(x) \propto \mathrm{Tr}[T^a_R]\varepsilon^{\mu\nu\alpha\beta} R_{\mu\nu\gamma\delta} R^{\gamma\delta}_{\alpha\beta}$, there's only $U(1)$ anomaly, hence: $(2Y_L - Y_e - Y_\nu) + 3(2Y_Q - Y_u - Y_d) = 0$

The solution to these are:

$$Y_L = -\frac{a}{2} - b,\ Y_e = -a - b,\ Y_\nu = -b,\ Y_Q = \frac{a}{6} + \frac{b}{3},\ Y_u = \frac{2a}{3} + \frac{b}{3},\ Y_d = -\frac{a}{3} + \frac{b}{3}$$

and

$$Y_Q = Y_L = 0,\ Y_u = -Y_d = c,\ Y_e = -Y_\nu = d$$

Since we know neutrino has no charge, only first condition is possible, and the first condition has $b = 0$, the charges are uniquely determined. Therefore the standard model has a very good predicting power even on the charges.

4 Standard Model Tests

4.1 Neutrino Oscillation

When the mass basis differs from the flavor basis, there will be oscillations between different flavors. The mass basis will be labeled as ν_i, and the flavor basis will be label as ν_α. Then we shall have

$$|\nu_\alpha\rangle = U_{\alpha i}\,|\nu_i\rangle$$

where U is the PKMS matrix.

As we know, the evolution of a state in time should be computate with the mass basis, so

$$|\nu_j(t)\rangle = e^{-iE_j t}\,|\nu_j(0)\rangle \tag{57}$$

Since the particle is moving at near light speed,

$$E_j = \sqrt{p_j^2 + m_j^2} \approx E + \frac{m_j^2}{2E} \tag{58}$$

Now, we could calculate the transition probability from flavor α to flavor β.

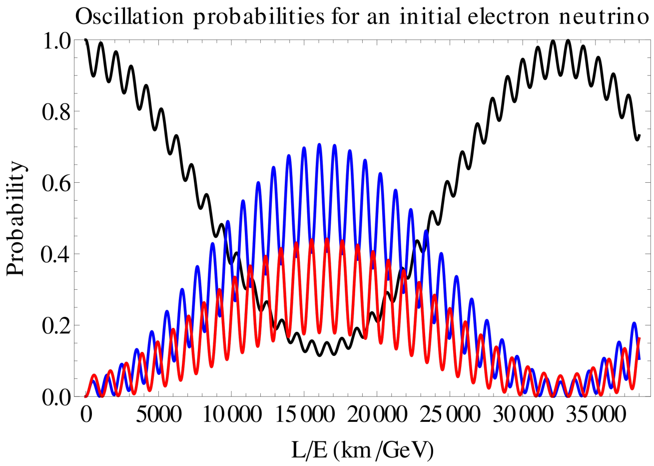
$$P_{\alpha\to\beta} = |\langle\nu_\beta\rangle\,\nu_\alpha(t)|^2 = \left|\sum_j U^*_{\alpha j}U_{\beta j}e^{-i\frac{m_j^2 t}{2E}}\right|^2 \tag{59}$$

To show the oscillation more explicitly, we could have

$$\begin{aligned} P_{\alpha\to\beta} = \delta_{\alpha\beta} - 4\sum_{j>k}\mathcal{R}_e\left\{U^*_{\alpha j}U_{\beta j}U_{\alpha k}U^*_{\beta k}\right\}\sin^2\left(\frac{\Delta_{jk}m^2 t}{4E}\right) \\ + 2\sum_{j>k}\mathcal{I}_m\left\{U^*_{\alpha j}U_{\beta j}U_{\alpha k}U^*_{\beta k}\right\}\sin\left(\frac{\Delta_{jk}m^2 t}{2E}\right) \end{aligned} \tag{60}$$

where $\Delta_{jk}m^2 = m_j^2 - m_k^2$.

The figure of the oscillation is shown as below



4.2 Electroweak Precision Test

To precision test Standard Model, we could start by Electroweak. There are some good measurement in electroweak that we can use, they are

1. Electromagnetic dipole moment g_e
2. The lifetime of muon τ_μ
3. The Z boson pole mass m_Z
4. The W boson pole mass m_W
5. The polarization asymmetry in Z boson production

$$A_e = \frac{\sigma(e_L^- e_L^+ \to Z) - \sigma(e_R^- e_R^+ \to Z)}{\sigma(e_L^- e_L^+ \to Z) + \sigma(e_R^- e_R^+ \to Z)} \tag{61}$$

4.2.1 Tree Level

In the tree level, we have follwing equation that could relate m_W and m_Z

$$m_Z = \frac{ev}{2sc}, m_W = \frac{ev}{2c} \tag{62}$$

where $s = \sin\theta_w, c = \cos\theta_w$.

The rate of τ_μ is calculate under tree level,

$$\tau_\mu^{-1} = \Gamma(\mu^- \to \nu_\mu e^- \bar{\nu}_e) = G_F^2 \frac{m_\mu^5}{192\pi^3}(1 - 8r + 8r^3 - r^4 - 12r^2 \ln r), r = \frac{m_e^2}{m_\mu^2} \tag{63}$$

This would give an accurate value of $G_F = \frac{1}{\sqrt{2}v^2}$.

The polarization asymmetry in Z boson production A_e can be given by Z boson coupling to the electrons, this part is

$$\mathcal{L}_Z = -\frac{e}{sc} Z_\mu \left[\left(\frac{1}{2} - s^2 \right) \bar{e}_L \gamma^\mu e_L - s^2 \bar{e}_R \gamma^\mu e_R \right] \tag{64}$$

so we have

$$A_e = \frac{(\frac{1}{2} - s^2) - s^4}{(\frac{1}{2} - s^2) + s^4} \tag{65}$$

Let's see how the measurement can relate to each other. We will now denote the meausurement with a circumflex above. $\hat{g}_e$ can give us $\hat{e}$, the $\hat{\tau}_\mu$ can give us G_F then we have $\hat{v}$, and then we will have $\hat{s} = \sin\theta_w$ from $\hat{m}_Z$. m_W and A_e will be the value we want to check, then we have

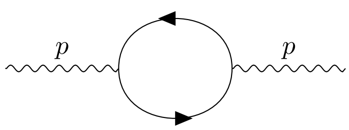
$$\hat{m}_W = \frac{\hat{e}\hat{v}}{2\hat{s}} = 79.794\,\text{GeV} \tag{66}$$

$$A_e = \frac{(\frac{1}{2} - \hat{s}^2) - \hat{s}^4}{(\frac{1}{2} - \hat{s}^2) + \hat{s}^4} = 0.1252 \tag{67}$$

This is 40 standard deviations from the actual value, so we should include the loop corrections.

4.2.2 1-Loop Correction

Note that the parameters shown in this section is under $\overline{\text{MS}}$ scheme. The most important correction comes from the loop correction of boson propagators. First we determine the Z - boson propagator. By Lorentz invariance,



$$= i\Pi_{ZZ} g^{\mu\nu} + i\Pi_{ZZ}^{pp} p^\mu p^\nu \tag{68}$$

Z boson's propagator is done by summing all 1-irreducible contributions

$$iG_Z^{\mu\nu}(p) = \frac{-ig^{\mu\nu}}{p^2 - m_Z^2 - \Pi_{ZZ}(p^2)} + p^\mu p^\nu (\text{terms}) \tag{69}$$

thus the mass is given by

$$\hat{m}_Z^2 = m_Z^2 + \mathrm{Re}[\Pi_{ZZ}(p^2)] \tag{70}$$

So is the W boson's mass

$$\hat{m}_W^2 = m_W^2 + \mathrm{Re}[\Pi_{WW}(p^2)] \tag{71}$$

For the photon propagators, we also have

$$iG_\gamma^{\mu\nu}(p) = \frac{-ig^{\mu\nu}}{p^2 - \Pi_{\gamma\gamma}(p^2)} \tag{72}$$

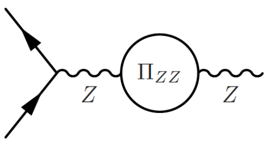
Then we want to relate the MS scheme e and $\hat{e}(m_Z)$, which is

$$\hat{e}^2(m_Z) = e^2 \frac{1}{1 - \frac{1}{m_Z^2}\Pi_{\gamma\gamma}(m_Z^2)} \tag{73}$$

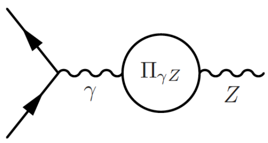
Next, we want to calculate the muon lifetime. Since it comes from W boson, we include Π_{WW}. So that

$$\frac{\hat{G}_F}{\sqrt{2}} = -\frac{g^2}{8}\frac{1}{p^2 - m_W^2 - \Pi_{WW}\left(p^2\right)}\bigg|_{p^2=0} = \frac{e^2}{8s^2c^2m_Z^2}\left(1 - \frac{\Pi_{WW}(0)}{m_W^2} + \cdots\right) \tag{74}$$

Last, we want to relate A_e. There are two loops to consider



and



.

The first loop can be accounted by replacing $\hat{m}_Z$ as m_Z, because it is from the correction of Z boson propagator. The second loop gives correction of the photon interchange, which is e. That gives a factor of $\frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2}$, and a replacement for the field A_μ to Z_μ, thus

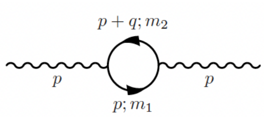
$$\begin{aligned}\mathcal{L}_Z^{\text{eff}} &= -\frac{e}{sc}Z_\mu\left[\left(\frac{1}{2} - s^2\right)\bar{e}_L\gamma^\mu e_L - s^2\bar{e}_R\gamma^\mu e_R\right] - e\frac{\Pi_{\gamma Z}\left(m_Z^2\right)}{m_Z^2}Z_\mu\left[\bar{e}_L\gamma^\mu e_L + \bar{e}_R\gamma^\mu e_R\right] \\ &= \frac{e}{sc}Z_\mu\left[\left(\frac{1}{2} - s_{\text{eff}}^2\right)\bar{e}_L\gamma^\mu e_L - s_{\text{eff}}^2\bar{e}_R\gamma^\mu e_R\right]\end{aligned} \tag{75}$$

where

$$s_{eff}^2 = s^2 - sc\frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2} \tag{76}$$

Now the measured value are related to MS scheme quantity. The problem now is to detemine those Π.

$i\Pi^{\mu\nu} =$



$$
\begin{aligned}
i\Pi_{LL}^{\mu\nu}=i\Pi_{RR}^{\mu\nu}&=(-1)e^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{\operatorname{Tr}\left[(i\gamma^\mu)P_L i(\not k+m_1)(i\gamma^\nu)P_L i(\not k+\not p+m_2)\right]}{[k^2-m_1^2]\left[(k+p)^2-m_2^2\right]}\\
&=ig^{\mu\nu}\frac{e^2}{(4\pi)^{d/2}}\mu^{4-d}\int_0^1 dx\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}}\left[2xm_2^2+2(1-x)m_1^2-4x(1-x)p^2\right]+p^\mu p^\nu\text{ terms,}
\end{aligned}
\tag{77}
$$

$$
\begin{aligned}
i\Pi_{LR}^{\mu\nu}=i\Pi_{RL}^{\mu\nu}&=(-1)e^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{\operatorname{Tr}\left[(i\gamma^\mu)P_L i(\not k+m_1)(i\gamma^\nu)P_R i(\not k+\not p+m_2)\right]}{[k^2-m_1^2]\left[(k+p)^2-m_2^2\right]}\\
&=-ig^{\mu\nu}\frac{e^2}{(4\pi)^{d/2}}\mu^{4-d}\int_0^1 dx\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}}2m_1m_2+p^\mu p^\nu\text{ terms,}
\end{aligned}
\tag{78}
$$

where $\Delta=xm_2^2+(1-x)m_1^2-x(1-x)p^2$, and L,R are the left hand and right handed.

After these, we could now plug in the charges to calculate γ,Z,W's correction.

$$
\begin{aligned}
&\Pi_{\gamma\gamma}\left(p^2\right)=N\sum_{i=t,b}Q_i^2\Pi_{VV}\left(\Delta_{ii}\right)\\
&\Pi_{\gamma Z}\left(p^2\right)=\frac{1}{sc}N\sum_{i=t,b}\left(T_i^3Q_i\frac{1}{2}\Pi_{VV}\left(\Delta_{ii}\right)-s^2Q_i^2\Pi_{VV}\left(\Delta_{ii}\right)\right)\\
&\Pi_{WW}\left(p^2\right)=\left|V_{tb}\right|^2\frac{1}{s^2}N\frac{1}{2}\Pi_{LL}\left(\Delta_{tb}\right)\\
&\Pi_{ZZ}\left(p^2\right)\\
&=\frac{1}{s^2c^2}N\sum_{i=t,b}\left(\left(T_i^3\right)^2\Pi_{LL}\left(\Delta_{ii}\right)-2s^2T_i^3Q_i\frac{1}{2}\Pi_{VV}\left(\Delta_{ii}\right)+s^4Q_i^2\Pi_{VV}\left(\Delta_{ii}\right)\right)
\end{aligned}
\tag{79}
$$

We only account top quark and bottom quark because they are the heaviest. The photon is the whole vector field, the W boson only accounts the left hand side since it only interacts with them. The Z boson couples in terms of vector and left handed amplitudes. Δ_ij means Δ with $m_1=m_i$ and $m_2=m_j$. N is the 3 colors of the quark. So in the end,

$$
\begin{aligned}
\Pi_{\gamma\gamma}\left(m_Z^2\right)&=-\frac{e^2m_Z^2}{2\pi^2\varepsilon}\left(Q_t^2+Q_b^2\right),\\
\Pi_{WW}\left(m_W^2\right)&=\left|V_{tb}\right|^2\frac{3e^2}{16\pi^2s^2\varepsilon}\left(m_b^2+m_t^2-\frac{2}{3}m_W^2\right),\\
\Pi_{\gamma Z}\left(m_Z^2\right)&=\frac{e^2m_Z^2}{8\pi^2sc\varepsilon}\left(Q_b-Q_t+4s^2\left(Q_t^2+Q_b^2\right)\right),\\
\Pi_{ZZ}\left(m_Z^2\right)&=\frac{e^2}{16s^2c^2\pi^2\varepsilon}\left[3m_b^2+3m_t^2-2m_Z^2\left(1+2\left(Q_b-Q_t\right)s^2+4\left(Q_b^2+Q_t^2\right)s^4\right)\right].
\end{aligned}
\tag{80}
$$

So we could put these back into the values we want, should eliminate ϵ for a physical observartion. This leads to $|V_{tb}|=1$. So we now can plug in the observation and we obtain

$$
m_W=80.368\,\mathrm{GeV}
\tag{81}
$$

$$
A_e=0.1491
\tag{82}
$$

which is now in the error range.

5 Conclusions

To sum up, we have established the strong interaction with Yang-Mills theory. The most important prediction it gives is the confinement and the asymptotic freedom, which results in quarks cannot be isolated. In the electroeweak part, we find that the broken symmertry is responsible for the mass of leptons and gauge bosons. We also found out neutrinos are weirder than we think. In the anomalies we've seen that since we want the theory to maintain gauge invariance, the local transformation should not only preserve the lagrangian, but also in quantum level ,the ward takahashi identity, as well, therefore this constraint gives us a unique condition as long as we know neutrino is neutral. In the predictions, we find out that neutrinos have mass and we could calculate the prediction by evaluating tree levels and 1-loops.

A Linear divergent integrals:

For integral such as the form:

$$\Delta^{\alpha}(a^{\mu}) = \int \frac{d^4k}{(2\pi)^4}(F^{\alpha}[k+a] - F^{\alpha}[k])$$

After wick rotaion and Tayler expanding around a we have:

$$\Delta^{\alpha}(a^{\mu}) = i\int \frac{d^4k_E}{(2\pi)^4}\left[a^{\mu}\frac{\partial}{\partial k_E^{\mu}}(F^{\alpha}[k_E]) + \frac{1}{2}a^{\mu}a^{\nu}\frac{\partial^2}{\partial k_E^{\mu}\partial k_E^{\nu}}(F^{\alpha}[k_E]) + ...\right]$$

For linear divergent integrals, we have:

$$\lim_{k_E\to\infty} F^{\alpha\cdots}(k_E) = A\frac{k_E^{\alpha}}{k_E^4}$$

Finally we have using:

$$\Delta^{\alpha}\left(a^{\mu}\right) = ia^{\mu}\int \frac{d^4k_E}{(2\pi)^4}\frac{\partial}{\partial k_E^{\mu}}\left(F^{\alpha}\left[k_E\right]\right) = ia^{\mu}\int \frac{d^3S_{\mu}}{(2\pi)^4}F^{\alpha}\left[k_E\right] = \frac{i}{32\pi^2}Aa^{\alpha}$$

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