Noether's theorem

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1 Noether's theorem in d dimension:

The statement of Noether's theorem of general continuum fields $\mathbf{\Phi}(x)$ (containing whether spinor or vector index, for EM field $\mathbf{\Phi}(x) = A^\mu(x)$) which vanishes at infinity is that:
For a transformation of fields and coordinates defined as under $x \to x' = f(x)$

$$\mathbf{\Phi}'(x') = \mathcal{F}(\mathbf{\Phi}(x))$$

Or in the infinitesimal form under(ω_a are constants) $x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a}$:

$$\mathbf{\Phi}'(x') = \mathbf{\Phi}(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x)$$

When this is a symmetry transformation(defined in 1.2), there exists a conserved current j^μ_a as below:

$$T^\mu_a := \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \mathbf{\Phi})} \partial_\nu \mathbf{\Phi} - \delta^\mu_\nu \mathcal{L} \right\} \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \mathbf{\Phi})} \frac{\delta \mathcal{F}}{\delta \omega_a}$$

$$j^\mu_a = T^\mu_a + F^\mu_a$$

where $\mathcal{L}$ is the lagrangian density of the system and F^μ_a is some function relating to the transformation.

1.1 The transformation of the action

Suppose now we consider local transformations(i.e. $\omega_a = \omega_a(x)$). Show that the infinitesimal transformation of action S transforms as(we haven't consider it to be a symmetry transformation):

$$S'_\Omega = S_\Omega + \int_\Omega d^d x \, \omega_a K^a - \int_\Omega d^d x \, \partial_\mu \omega_a T^\mu_a$$

$$K^a = \left[\frac{\partial \mathcal{L}}{\partial \mathbf{\Phi}} \frac{\delta \mathcal{F}}{\delta \omega_a} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \mathbf{\Phi})} \partial_\mu \left(\frac{\delta \mathcal{F}}{\delta \omega_a} \right) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \mathbf{\Phi})} \partial_\nu \mathbf{\Phi} + \mathcal{L} \delta^\mu_\nu \partial_\mu \left(\frac{\delta x^\nu}{\delta \omega_a} \right) \right]$$

and T^μ_a defined as above, Ω is some region of spacetime manifold. (Hint: use the identity $det(A + \delta A) = det(A) \times (1 + tr(\delta A))$, and do not implement the EOM throughout all problems or throw away boundary terms.)

Solution.

$$
\begin{aligned}
S'_\Omega &= \int_\Omega d^dx'\mathcal{L}\left(\mathcal{F}(\mathbf{\Phi}(x')), \partial'_\mu \mathcal{F}(\mathbf{\Phi}(x'))\right) \\
&= \int_\Omega d^dx \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}(\mathcal{F}(\mathbf{\Phi}(x'), \partial'_\mu \mathcal{F}(\mathbf{\Phi}(x'))) \\
&= \int_\Omega d^dx \left[1 + \partial_\mu\left(\omega_a \frac{\delta x^\mu}{\delta\omega_a}\right)\right]\left(\mathcal{L}(\Phi(x), \partial_\mu\Phi(x)) + \omega_a \frac{\partial\mathcal{L}}{\partial\Phi}\frac{\delta\mathcal{F}}{\delta\omega_a} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\mathbf{\Phi})}\left(\partial_\mu\left(\omega_a\frac{\delta\mathcal{F}}{\delta\omega_a}\right) - \partial_\mu\left(\omega_a\frac{\delta x^\nu}{\delta\omega_a}\right)\partial_\nu\mathbf{\Phi}\right)\right) \\
&= S + \int_\Omega d^dx\omega_a\left(\frac{\partial\mathcal{L}}{\partial\mathbf{\Phi}}\frac{\delta\mathcal{F}}{\delta\omega_a} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\mathbf{\Phi})}\partial_\mu\left(\frac{\delta\mathcal{F}}{\delta\omega_a}\right) - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\mathbf{\Phi})}\partial_\nu\mathbf{\Phi}\partial_\mu\left(\frac{\delta x^\nu}{\delta\omega_a}\right) + \mathcal{L}\delta^\mu_\nu\partial_\mu\left(\frac{\delta x^\nu}{\delta\omega_a}\right)\right) \\
&+ \int_\Omega d^dx\partial_\mu\omega_a\left[\left(\mathcal{L}\delta^\mu_\nu - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\mathbf{\Phi})}\right)\frac{\delta x^\nu}{\delta\omega_a} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\mathbf{\Phi})}\frac{\delta\mathcal{F}}{\delta\omega_a}\right] \\
&= S + \int_\Omega d^dx\omega_a K_a - \int d^dx\partial_\mu\omega_a T^\mu_a
\end{aligned}
$$

where we have used $\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu\left(\omega_a\frac{\delta x^\mu}{\delta\omega_a}\right)$, whereas $\frac{\partial x^\mu}{\partial x'^\nu} = \delta^\mu_\nu - \partial_\nu\left(\omega_a\frac{\delta x^\mu}{\delta\omega_a}\right)$, also by using levi civita symbol we have the identity:

$$
\begin{aligned}
det(A+\delta A) &= det(A)det(I + A^{-1}\delta A) \\
&= (\varepsilon^{i_1 i_2 \dots i_d}(I + A^{-1}\delta A)^1_{i_1}\dots(I + A^{-1}\delta A)^d_{i_d})detA \\
&= (\varepsilon^{i_1 i_2 \dots i_d}(\delta^1_{i_1} + (A^{-1}\delta A)^1_{i_1})\dots(\delta^d_{i_d} + (A^{-1}\delta A)^d_{i_d}))detA \\
&= (1 + tr(A^{-1}\delta A))det(A)
\end{aligned}
$$

or we can use the formula $exp(tr(A)) = detA$, so that we have $exp(tr(A+\delta A)) = detA(1+tr(\delta A)) = det(A+\delta A)$ to see the identity(however this formula is proven by using the identity we wish to show XD). ■

1.2 Symmetry transformation:

Consider the global symmetry transformation($\omega_a = const.$) leaving the action invariant up to an integration of total derivatives, i.e.

$$S'_\Omega = S_\Omega + \int_\Omega d^dx\omega_a\partial_\mu F^\mu_a$$

For some F^μ_a. Now prove that when the transformation is a symmetry the local transformation version yields following:

$$S'_\Omega = S_\Omega + \int_\Omega d^dx\partial_\mu Y^\mu - \int_\Omega d^dx j^\mu_a\partial_\mu\omega_a$$

Find the explicit form of j^μ_a and Y^μ in terms of F^μ_a and ω_a and other variables defined.

Solution. obviously we have $S'_\Omega = S_\Omega + \int_\Omega d^dx\omega_a K_a$ when ω_a are constants, that is we have after comparing both sides, we have $\partial_\mu F^\mu_a = K_a$, therefore plug into original equation we have:

$$
\begin{aligned}
S'_\Omega &= S_\Omega + \int_\Omega d^dx\omega_a\partial_\mu F^\mu_a - \int_\Omega d^dx\partial_\mu\omega_a T^\mu_a \\
&= S_\Omega + \int_\Omega d^dx\partial_\mu(\omega_a F^\mu_a) - \int_\Omega d^dx\partial_\mu\omega_a(F^\mu_a + T^\mu_a)
\end{aligned}
$$

That is $Y^\mu = \omega_a F^\mu_a$, and $j^\mu_a = T^\mu_a + F^\mu_a$. ■

1.3 symmetry implies conservation law

Now since we have defined the symmetry, and assume that the field configurations vanishes at infinity boundary, argue that when the configurations are on shell, i.e. S is extremized, or the condition $\delta S = S' - S = 0$,

$$\partial_\mu (T^\mu_a + F^\mu_a) = \partial_\mu j^\mu_a = 0$$

(Hint: Now we can take the integral region Ω to be whole space)

Solution. since configurations are on shell, we have $\delta S = S' - S = 0$, therefore we need the term which is not total derivative to vanish, since the total derivative is an integration on the boundary it vanishes, so

$$\begin{aligned} S' &= S + \int d^d x \partial_\mu (\omega_a F^\mu_a) - \int d^d x \partial_\mu \omega_a (F^\mu_a + T^\mu_a) \\ &= S - \int d^d x \partial_\mu (\omega_a T^\mu_a) + \int d^d x \partial_\mu (F^\mu_a + T^\mu_a) \omega_a \end{aligned}$$

We require the last term $\int \omega_a d^d x \partial_\mu j^\mu_a = 0$, and since the integral region is not restricted to any part of the manifold, the integrand is zero for any function ω_a so we have a conserved current $j^\mu_a = T^\mu_a + F^\mu_a$! ■