Wednesday: Sept 4, 2019, (L3)

HW: pg. 54: #'s 71, 74, 75, 81
pg. 59: #'s 84, 88, 94, 96

Review

Last time we started with counting methods in the use of finding probability of an event.

Def. 2.6 Suppose S is a sample space associated with an experiment. To every event A in S, we assign a number, $P(A)$, called the probability of A, so that the following hold:

i. $P(A) \geq 0$

ii. $P(S) = 1$

iii. If $A_1, A_2, \cdots$ form a sequence of pairwise mutually exclusive events in S

$\left(A_i \cap A_j = \phi \text{ if } i \neq j\right)$ then $P(A_1 \cup A_2 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)$

Then using your great ability of reasoning, prove the following intuitive (silly) theorems:

Thm: For each event A, $P(A) = 1 - P(\bar{A})$

$S = A \cup \bar{A}$

$P(S) = P(A \cup \bar{A})$

$1 = P(A) + P(\bar{A})$

$1 - P(\bar{A}) = P(A)$

Thm: $P(\phi) = 0$

$S = S \cup \phi$

$P(S) = P(S \cup \phi)$

$P(S) = P(S) + P(\phi)$

$0 = P(\phi)$

Thm: Given the events A and B where $A \subseteq B$, then $P(A) \leq P(B)$.

Note: $B = (A \cap B) \cup (\bar{A} \cap B)$

$P(B) = P(A \cap B) + P(\bar{A} \cap B)$

since $A \subseteq B$

$P(B) = P(A) + \underbrace{P(\bar{A} \cap B)}_{\geq 0}$

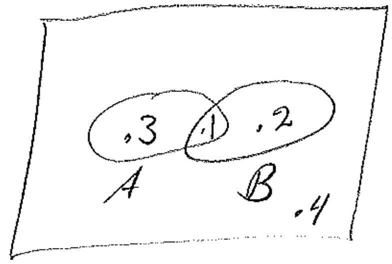
$P(B) \geq P(A)$

Let us start section 2.7 with a couple If examples.

Section 2.7: Conditional Probability and the Independence of Events

Ex. Given $P(A) = 0.4$, $P(B) = 0.3$, $P(A \cap B) = 0.1$

Draw the Venn diagram and find:



$P(\overline{A}) = .6$

$P(A \cup B) = .6$

$P(\overline{A \cup B}) = .4$

conditional prob

$P(A \mid B) = \dfrac{.1}{.3} = \dfrac{1}{3}$ P(A given B is true)

The Universial set changes

$P(B \mid A) = \dfrac{.1}{.4} = \dfrac{1}{4}$

Ex. A package contains 20 tulip bulbs, which will bloom early (E) or late (L) summer either red (R) or yellow (Y) blooms:

	Early (E)	Late (L)	total
Red (R)	5	8	13
Yellow (Y)	3	4	7
total	8	12	20

Find:

$P(E) = \dfrac{8}{20} = .40$

$P(R) = \dfrac{13}{20} = .65$

$P(R \cap E) = \dfrac{5}{20} = \dfrac{1}{4} = .25$

note

→ $P(R \mid E) = \dfrac{5}{8} = .625$ ← $P(R \cap E)$

↖ $P(E)$

Definition 2.9

The *conditional probability of an event A*, given that an event B has occurred, is equal to

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \qquad P(B) > 0$$

Definition 2.10***(know this definition)

Two events A and B are said to be *independent* if any one of the following holds:

$$P(A \mid B) = P(A)$$
$$P(B \mid A) = P(B)$$
$$P(A \cap B) = P(A) \cdot P(B)$$

Otherwise, the events are said to be dependent.

Section 2.8: Two Laws of Probability

Theorem 2.5 **The Multiplication Law of Probability**

The probability of the intersection of two events A and B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B \mid A) \\ &= P(B) \cdot P(A \mid B) \end{aligned}$$

If A and B are independent then

$$P(A \cap B) = P(A) \cdot P(B)$$

The proof follows from the definitions of *conditional probabilities* and *independent* events.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A|B) \cdot P(B) = P(A \cap B)$$

independent

$$P(A) \cdot P(B) = P(A \cap B)$$

Theorem 2.6 **The Addition Law of Probability**

The probability of the union of two events A and B is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are mutually exclusive events, then $P(A \cap B) = 0$ and

$$P(A \cup B) = P(A) + P(B)$$

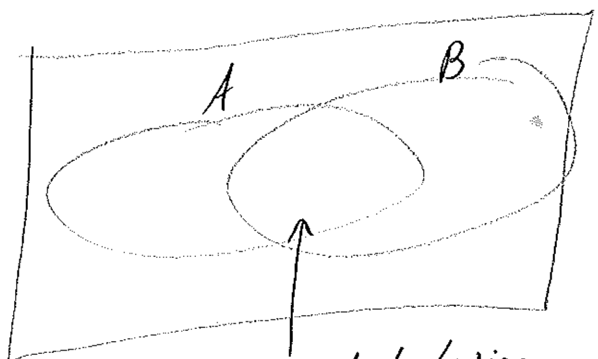
Proof

Notice the following: $A \cup B = A \cup (\overline{A} \cap B)$ and $B = (\overline{A} \cap B) \cup (A \cap B)$

Then $P(A \cup B) = P(A) + P(\overline{A} \cap B)$ and $P(B) = P(\overline{A} \cap B) + P(A \cap B)$ $\Rightarrow P(\overline{A} \cap B) = P(B) - P(A \cap B)$

Rewriting the second equation $P(B) - P(A \cap B) = P(\overline{A} \cap B)$ followed by the substitution of the first give the desired results: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

The proof can easily be shown by drawing the Venn diagram.



counted twice in $P(A) + P(B)$

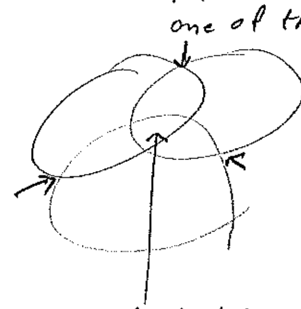
take out one of the $P(A \cap B)$'s

Helpful hints:

- When dealing with unions and intersections it is often quite helpful to draw the Venn diagram. Most problems give enough information to complete the diagram.
- From the Addition Law of Probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  it would also make sense that $P(A \cap B) = P(A) + P(B) - P(A \cup B)$.
- Extending the Addition Law of Probability using three events:

$$\begin{aligned} P(A \cup B \cup C) = P(A) &+ P(B) + P(C) \\ &- P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &+ P(A \cap B \cap C) \end{aligned}$$

Take out one of the duplicates



we took this out totally put it back

Example 2.16

Three brand of coffee, X, Y, and Z are to be ranked according to taste by a judge. Define the following events:

A: Brand X is preferred to Y.

B: Brand X is ranked the best.

C: Brand X is ranked second best.

D: Brand X is ranked third best

$E_1 = XYZ$ ✓
$E_2 = XZY$ ✓
$E_3 = YXZ$
$E_4 = YZX$
$E_5 = ZXY$ ✓
$E_6 = ZYX$

$P(A) = \frac{3}{6} = \frac{1}{2}$

If the judge actually has no taste preference and randomly assigns ranks to the brands, is event A independent of events B, C, and D?

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{6}}{\frac{2}{6}} = 1$ No

$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{\frac{1}{6}}{\frac{2}{6}} = \frac{1}{2}$ Yes

$P(A|D) \quad \frac{P(A \cap D)}{P(D)} = \frac{0}{\frac{2}{6}} = 0$ No

Example: Suppose that a fair six-sided die is tossed. What is the probability that the number tossed is ≤ 3 given that an even number was tossed?

$P(\text{even}) = P(2, 4 \text{ or } 6) = \frac{1}{2}$

$P(x \leq 3) = P(1, 2, 3) = \frac{1}{2}$

$P(\text{even} \cap x \leq 3) = \frac{1}{6}$

$P(x \leq 3 \mid \text{even}) = \frac{P(\text{even} \cap x \leq 3)}{P(\text{even})} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$

Example: Suppose you are dealt 6 cards from a standard deck of playing cards. What is the probability that the 6th card is your third spade? $= \underbrace{\_\ \_\ \_\ \_\ \_}_{2 \text{ spades}}\ \underset{P(B)}{\frac{5}{\_}}$

$A = P(2 \text{ spades out of 5 cards}) = \frac{\binom{13}{2}\binom{39}{3}}{\binom{52}{5}} = .274$

$P(A \cap B) = (.274)\frac{11}{47} = .064$

Example: Suppose that $P(A) = \frac{1}{6}$, $P(B) = \frac{5}{12}$ and $P(A \mid B) + P(B \mid A) = \frac{7}{10}$. Find $P(A \cap B)$.

$P(A|B) + P(B|A) = \frac{7}{10}$

$\frac{P(A \cap B)}{P(B)} + \frac{P(A \cap B)}{P(A)} = \frac{7}{10}$

$P(A \cap B)\left(\frac{12}{5} + \frac{6}{1}\right) = \frac{7}{10}$

$P(A \cap B) = \frac{7}{10} \cdot \frac{5}{42} = \frac{1}{12}$