1. Let the random variable X have the probability distribution function:

$$f(x) = \frac{(|x|+1)^2}{9} \quad for\ x = -1, 0, 1$$

Find:

$\mu = E(X)$

$E(X) = (-1)\frac{(|-1|+1)^2}{9} + 0\frac{(0+1)^2}{9} + 1\left(\frac{(1+1)^2}{9}\right)$

$= \frac{-4}{9} + 0 + \frac{4}{9}$

$= 0$

$E(X^2)$

$= (-1)^2\frac{(|-1|+1)^2}{9} + 0^2\left(\frac{(0+1)^2}{9}\right) + 1^2\left(\frac{(1+1)^2}{9}\right)$

$= \frac{4}{9} + 0 + \frac{4}{9}$

$= \frac{8}{9}$

$E(3X^2 - 2X + 4)$

$= 3E(X^2) - 2E(X) + 4$

$= 3\left(\frac{8}{9}\right) - 2(0) + 4$

$= \frac{8}{3} + 4$

$= 6\frac{2}{3}$

2. Given $E(Y+4) = 10$ and $E[(Y+4)^2] = 116$,

a. Find $Var(Y+4)$

$Var(Y+4) = E((Y+4)^2) - [E(Y+4)]^2$

$= 116 - 10^2$

$= 16$

b. Find $\mu = E(Y)$

$E(Y+4) = E(Y) + E(4) = 10$

$E(Y) + 4 = 10$

$E(Y) = 6$

$E((Y+4)^2) = E(Y^2 + 8Y + 16)$

$= E(Y^2) + 8E(Y) + 16 = 116$

$E(Y^2) = 116 - 16 - 8(6)$

$= 52$

c. Find $\sigma^2 = Var(Y)$

$Var(Y) = E(Y^2) - [E(Y)]^2$

$= 52 - 6^2$

$= 16$

Properties of Variance.

Find:

a. Var(c) $= E(c^2) - [E(c)]^2$

$= c^2 - c^2$

$= 0$

b. Var(-X) $= E[(-X)^2] - [E(-X)]^2$

$= E(X^2) - (-E(X))^2$

$= E(X^2) - E(X)^2$

$=$ Var X

c. Var (cX) $= E((cX)^2) - [E(cX)]^2$

$= c^2 E(X^2) - [cE(X)]^2$

$= c^2 [E(X^2) - (E(X))^2$

Sections 3. 4; The Binomial Probability Distribution

Consider the experiment where you flip a coin. Heads you win (succeed) tails you lose (fail).

Def.

A Bernoulli experiment is a single random experiment with two mutually exclusive outcomes, success and failure. The probability of success is *p*, *P(success) = p*, and the probability of failure is *P(fail) = 1 – p = q.*

(examples: male-female, win-lose, heads-tails, live-die)

Def 3.6.

A Binomial experiment is a sequence of Bernoulli experiments done *n* times where each Bernoulli experiment is identical and independent from each other. The set of random variables of a binomial experiment, *Y* = {0, 1, 2, … , *n*}, is the number of possible successes observed during the *n* trials.

Example:

Suppose the probability of a seed germinating is 0.8. If we plant 10 seeds and can assume that the germination of one seed is independent from another seed, what is the probability that 7 out of the 10 seeds will germinate?

$P(X=7) = \binom{10}{7}(.8)^7(.2)^3 = .2013$

success failure

mix up the order

For the Bernoulli what is i) the probability distribution function, ii) the mean or expected value and iii) the variance?

Define $Y = 1$ for success and $Y = 0$ for a failure with $P(Y = 1) = p$ and $P(Y = 0) = (1-p) = q$

i) The probability function : $f(y) = p^y(1-p)^{1-y}$ $= x\begin{cases} 0 & \text{if failure} \\ 1 & \text{if success} \end{cases}$

ii) The mean: $\mu = E(Y) = 1 \cdot p + 0 \cdot (1-p) = p$

$E(Y^2) = 1^2 \cdot p + 0^2 \cdot (1-p) = p$

iii) So the variance:
$$\begin{cases} \sigma^2 = E(Y^2) - \mu^2 \\ = p - p^2 \\ = p(1-p) \quad or \quad = pq \end{cases}$$

For the **Binomial distribution**:

i) $$f(y) = \binom{n}{y} p^y (1-p)^{n-y}$$

ii) The mean: $$\mu = E(Y) = \sum_{y=0}^{n} y \cdot \binom{n}{y} p^y q^{n-y}$$ ← for $y=0$ $0 \cdot \binom{n}{0} p^0 q^n = 0$

$$= \sum_{y=1}^{n} y \cdot \binom{n}{y} p^y q^{n-y}$$

$$= \sum_{y=1}^{n} y \frac{n!}{(n-y)!\,y!} p^y q^{n-y}$$ Factor out a y

$$= \sum_{y=1}^{n} \frac{n!}{(n-y)!\,(y-1)!} p^y q^{n-y}$$ (note: each term has a np factor)

$$= np \sum_{y=1}^{n} \frac{(n-1)!}{(n-y)!\,(y-1)!} p^{y-1} q^{n-y}$$

Make a change in variable $z = y - 1$, then the equality becomes:

$$= np \sum_{z=0}^{n-1} \frac{(n-1)!}{(n-1-z)!\,(z)!} p^z q^{n-1-z}$$

$$= np \sum_{z=0}^{n-1} \binom{n-1}{z} p^z q^{n-1-z}$$ ← This is what I want you to see.

$$= np$$

To Find Variance,

Need to find $E(Y^2)$ but this is hard to find. Using the above derivation as inspiration where we factor out the y, let us factor out $y(y-1)$. Consider

$$E[Y(Y-1)] = E[Y^2 - Y] = E(Y^2) - \mu \quad \text{or}$$
$$E(Y^2) = E[Y(Y-1)] + \mu$$

So

$$E[Y(Y-1)] = \sum_{y=0}^{n} y(y-1) \cdot \binom{n}{y} p^y q^{n-y}$$ ← at y=0 and y=1, the term is 0

$$= \sum_{y=2}^{n} y(y-1) \cdot \frac{n!}{(n-y)!(y)!} p^y q^{n-y}$$

$$= n(n-1)p^2 \sum_{y=2}^{n} \frac{(n-2)!}{(n-y)!\,(y-2)!} p^{y-2} q^{n-y}$$

Make a change in variable $z = y - 2$, then the equality becomes:

$$E[Y(Y-1)] = n(n-1)p^2 \sum_{z=0}^{n-2} \frac{(n-2)!}{(n-2-z)!\,(z)!} p^z q^{n-2-z}$$
$$= n(n-1)p^2 \sum_{z=0}^{n-2} \binom{n-2}{z} p^z q^{n-2-z}$$
$$= n(n-1)p^2$$

Which means $E(Y^2) = E[Y(Y-1)] + \mu = n(n-1)p^2 + np$, and

$$\sigma^2 = E(Y^2) - \mu^2 = n(n-1)p^2 + np - (np)^2$$
$$= np[(n-1)p + 1 - np]$$
$$= np[np - p + 1 - np]$$ ← $-p+1 = 1-p = q$
$$= npq$$

Example:

1. Let Y = number of days of rain in a 30 day period. (#36)
   a. Does Y have a binomial distribution?

   yes (rains or does not rain)

   b. If so, what is n and p?

   $n = 30$, p = not given

2. The probability that a patient recovers from a stomach disease is .8. Suppose 20 people are known to have contracted this disease. Use Appendix 3 on page 839 the find the probability that :
   a. exactly 14 recover? $P(X=14) = \binom{20}{14}(.8)^{14}(.2)^{6} = .1091$
   b. at least 10 recover? $P(X \geq 10) = 1 - P(Y \leq 9) = (1 - .001) = .999$
   c. at least 14 but not more than 18 recover? $P(14 \leq X \leq 18) = P(X \leq 18) - P(X \leq 13)$

      $= .931 - .087$

      $= .844$
   d. at most 16 recover?

      $P(X \leq 16) = .589$