L7 Sept. 14, 2020 HW: page 119: #'s 67, 70,

**The Binomial Probability Distribution** (section 3.4)

$$p(y) = \binom{n}{y} p^y (1-p)^{n-y} \qquad \text{mean} \quad \mu = np \qquad \text{variance} \quad \sigma^2 = npq$$

Last lecture we went through the proofs, Now we only have to use the results.

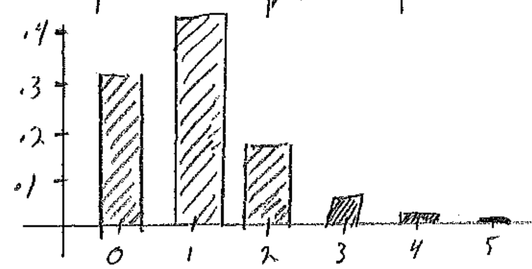
1. (part c is hard) Smith and Jones each take the same multiple choice. The test has 5 questions, and each question has 5 answers with exactly one correct answer.

   a. Assuming that your answers are random guesses, give the probability distribution, table and graph for the number of correct answers for the test.

   i) $p(y) = \binom{5}{y}(.2)^y(.8)^{5-y}$ y is the number correct

   ii)

| $y$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $p(y)$ | $(.8)^5 = .32768$ | $\binom{5}{1}.8^4 .2$ .4096 | $\binom{5}{2}(.2)^2(.8^3)$ .2048 | .0512 | .0064 | .00032 |




   b. What is the expected value and variance for this distribution?

   $E(X) = np = 5(.2) = 1$

   $Var(X) = npq = 5(.2)(.8) = .8$

This is the hard part

   c. Find the probability that Smith and Jones get the same number of answers correct.

$P\begin{pmatrix}\text{Smith and Jones} \\ \text{Get same \#} \\ \text{correct}\end{pmatrix} = P(S=0)\cdot P(J=0) + P(S=1)\cdot P(J=1) + \ldots + P(S=5)P(J=5)$

$= (.32768)^2 + (.4096)^2 + (.2048)^2 + (.0512)^2 + (.0064)^2 + (.00032)^2$

$= .3198$

## Section 3.5; The Geometric Probability Distribution

**Review of series:**

Suppose that $0 < r < 1$, what is the sum of the series $X = 1 + r + r^2 + r^3 + \cdots$ ?

This is a geometric sequence so $\frac{1}{1-r}$

To show: Let $x = 1 + r + r^2 + r^3 + \cdots$

$$\underline{-(rx = \quad r + r^2 + r^3 + \cdots)}$$

$$x(1-r) = 1$$

$$x = \frac{1}{1-r}$$

What is: $= 1 + 2r + 3r^2 + 4r^3 + 5r^4 + \cdots$ ?

Let $y = 1 + 2r + 3r^2 + 4r^3 + \cdots$

$$\underline{-(ry = \quad r + 2r^2 + 3r^3 + \cdots)}$$

$$y(1-r) = 1 + r + r^2 + r^3 + \cdots$$

$$y(1-r) = \frac{1}{1-r}$$

$$y = \frac{1}{(1-r)^2}$$

1. Let $X$ = the number of tosses of a fair die until the first "1" appears. What is $E(X)$?

$$E(X) = \underbrace{1\left(\tfrac{1}{6}\right)}_{1\text{ toss}} + \underbrace{2\left(\tfrac{5}{6}\right)\tfrac{1}{6}}_{2\text{ toss}} + \underbrace{3\left(\tfrac{5}{6}\right)^2\tfrac{1}{6}}_{3\text{ toss}} + \underbrace{4\left(\tfrac{5}{6}\right)^3\left(\tfrac{1}{6}\right)}_{4\text{ toss}} + \cdots$$

$$= \frac{1}{6}\left[1 + 2\left(\frac{5}{6}\right) + 3\left(\frac{5}{6}\right)^2 + 4\left(\frac{5}{6}\right)^3 + \cdots\right]$$

$$= \frac{1}{6} \cdot \frac{1}{\left(1-\frac{5}{6}\right)^2}$$

$$= \frac{1}{6} \cdot \frac{1}{\left(\frac{1}{6}\right)^2}$$

$$= \frac{1}{6}\,\frac{6^2}{1}$$

$$= 6$$

**Def.** A random variable $Y$ is said to have a geometric probability distribution if and only if

$$p(y) = (1-p)^{y-1} p, \qquad y = 1, 2, 3, \ldots \quad 0 \le p \le 1$$

Mean $\mu = E(Y) = \frac{1}{p}$ Variance $\sigma^2 = V(Y) = \frac{1-p}{p^2}$

The proof is at the end.

Use the die example

2. What is the probability that you will not get a "1" in the first 4 tosses?

i.e. you fail 4 times

$$q \cdot q \cdot q \cdot q = (q)^4 = \left(\frac{5}{6}\right)^4 = .4823 \quad \left(= \frac{625}{1296}\right)$$

Proof for finding the mean and variance for the Geometric Distribution.

**Mean:** $\mu = E(y) = \frac{1}{p}$

We have done this.

$$E(y) = 1\cdot p + 2\cdot qp + 3\cdot q^2 p + 4q^3 p + \dots)$$
$$= p(1 + 2q + 3q^2 + 4q^3 + \dots)$$
$$= p\,\frac{1}{(1-q)^2}$$
$$= p\left(\frac{1}{p^2}\right)$$
$$= \frac{1}{p}$$

**Variance** $= \frac{1-p}{p^2}$

Look at something interesting

$$\frac{d}{dq}(q^y) = y\cdot q^{y-1} \text{ and } \frac{d^2}{dq}(q^y) = y(y-1)q^{y-2}$$

So

$$E(Y(Y-1)) = \sum_{y=1}^{\infty} y(y-1)\,p\,q^{y-1}$$

since at $y=1$ the 1st term is zero.

$$= pq\sum_{y=2}^{\infty} y(y-1)\,q^{y-2}$$
$$= pq\,\frac{d^2}{dq^2}\left(\sum_{y=0}^{\infty} q^y\right)$$
$$= pq\left(\frac{d^2}{dq^2}\left(\frac{1}{1-q}\right)\right)$$
$$= pq\cdot\frac{2}{(1-q)^3}$$

The derivative of $(1-q)^{-1}$

$$\frac{d}{dq} = (-1)(1-q)^{-2}(-1)$$
$$\frac{d^2}{dq^2} = -2(1-q)^{-3}(-1)$$
$$= \frac{2}{(1-q)^3}$$

$$= pq\,\frac{2}{p^3}$$
$$= \frac{2q}{p^2} = \frac{2(1-p)}{p^2}$$

So: $E(Y(Y-1)) = E(Y^2 - Y) = E(Y^2) - E(Y)$

(ie) $E(Y^2) = E(Y(Y-1)) + E(Y)$

$$Var(Y) = \sigma^2 = E(Y^2) - E(Y)^2$$
$$= \left[\frac{2(1-p)}{p^2} + \frac{1}{p}\right] - \frac{1}{p^2}$$
$$= \frac{2 - 2p + p - 1}{p^2} = \frac{1-p}{p^2}$$