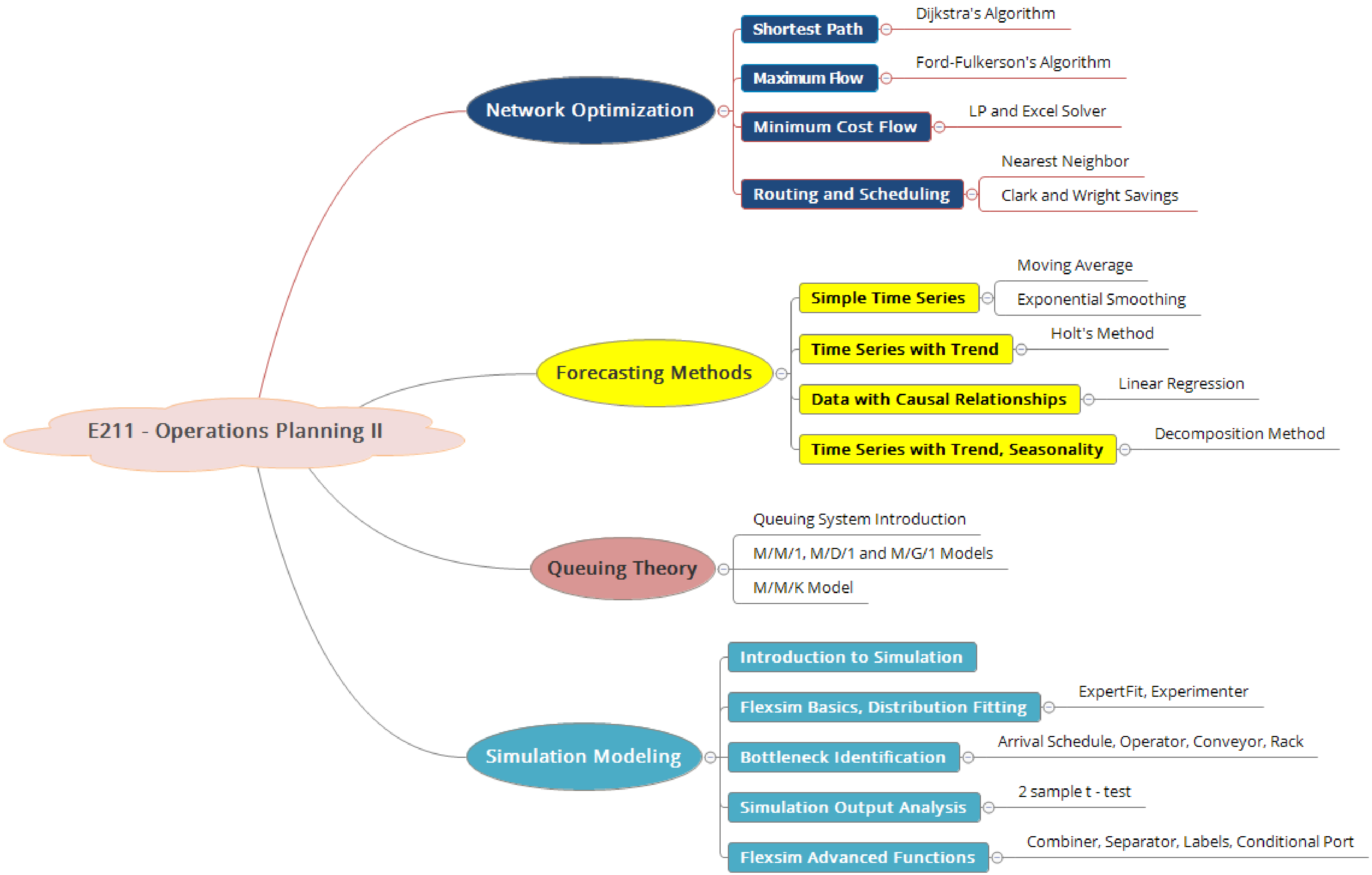
Problem 08

Single Server Queuing Models

E211 – Operations Planning II

SCHOOL OF ENGINEERING

Module Coverage: E211 Topic Tree



Purpose of Studying Queuing models

Queuing models are used to:

- describe the behaviour of queuing systems.
- determine the level of service to provide.
  - ✓ Example: waiting time in the queue
- evaluate alternate configurations for providing service.
  - ✓ Example: single line or multiple-line queue

Classification of Queuing Systems

- The **Kendall** classification of queuing systems (1953) exists in several modifications. In general, classification uses 6 symbols(parameters):

A/B/C/D/E/F

- **Parameter A -** the arrival pattern (distribution of intervals between arrivals).
  - M: Poisson (Markovian) process with exponential distribution of intervals
  - D: Deterministic(Constant) inter-arrival times
  - G: Inter-arrival time follows a general (any) distribution
- **Parameter B** – the service pattern (distribution of service duration).
  - M: Poisson (Markovian) process with exponential distribution of service duration.
  - G: Service duration follows a general (any) distribution
  - D: Deterministic(Constant) service duration
- **Parameter C** – the number of servers

Classification of Queuing Systems

A/B/C/D/E/F

- **Parameter D -** the queuing discipline (FIFO, LIFO, ...). Omitted for FIFO or if not specified.
- **Parameter E** – the system capacity. Omitted for unlimited queues.
- **Parameter F** – the population size (number of possible customers). Omitted for infinite (open) systems.

Examples

- **D/M/1/LIFO/10/50**: Deterministic (known) arrivals, one server with exponentially distributed service time, queue is limited by a maximum size of 10 with a last-in-first-out queueing discipline, and the total number of customers is 50.
- **D/G/3**: Multiple-server (3 servers) queuing model with deterministic (known) arrivals, service time following a general distribution (e.g.. Normal), FIFO queuing discipline, infinite number of waiting positions, and unlimited customer population.
- **M/M/1**: Single server queuing model with Poisson arrivals, exponentially distributed service time, FIFO queuing discipline, unlimited waiting positions, and unlimited customer population.

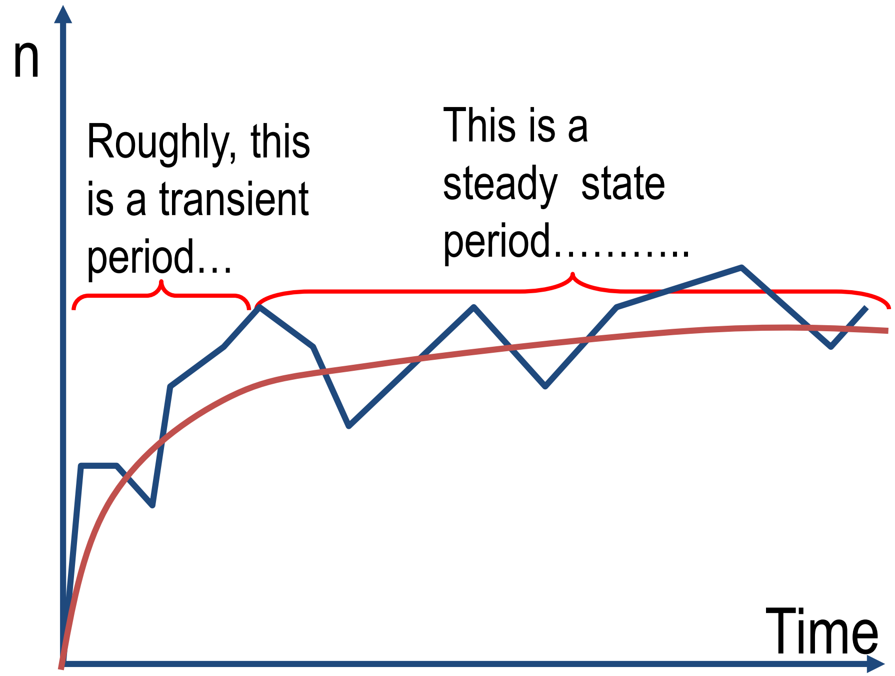
Queuing System Performance Measures

- System utilization ρ (Rho)
- Average queue time, Wq
- Average queue length, Lq
- Average time in system, Ws
- Average number in system, Ls
- Probability of idle service facility, P_0
- Probability of n units in system, P_n

Performance Measures

- Queue Performance is measured for steady state

Initial (transient) behaviour not representative of long run performance



EQUILIBRIUM CONDITION:

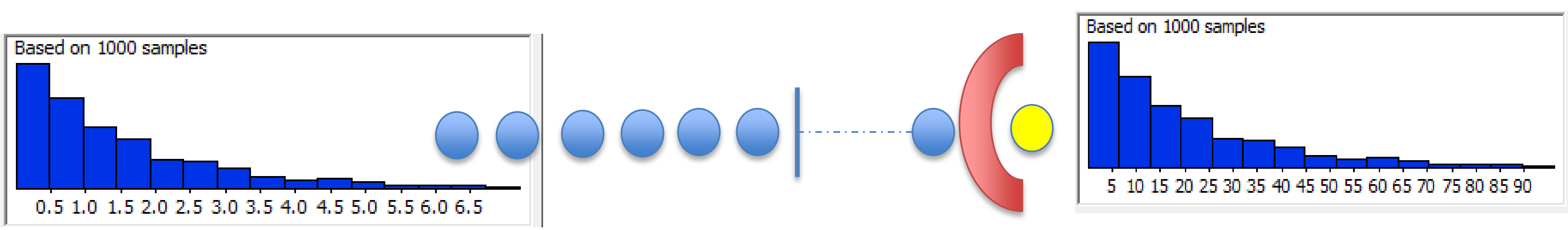
In order to achieve steady state, effective arrival rate (λ) must be less than sum of effective service rates (μ).

$\lambda < \mu$ (for single server)

$\lambda < K\mu$ (for K servers)

Random Service Time: M/M/1 Model Characteristics

- Type: Single-channel, single-phase system
- Input source: Infinite; no balks, no reneging
- Arrival process: Number of arrivals follows Poisson distribution
- Queue: Unlimited; single line
- Queue discipline: FIFO (FCFS)
- Service time: **Negative exponential** distribution
- Arrivals and service times are independent of the number of customers in system
- Service rate > arrival rate



M/M/1 Example

A cashier takes 1.5 minutes to process a transaction and one customer arrives in every 2 minutes on average.

What is the Arrival Rate (λ)?

- Lambda (λ) = 1 / 2 per minute = 0.5 per minute

What is the Service Rate (μ)?

- Mu (μ) = 1 / 1.5 per minute = 0.667 per minute

M/M/1 Equations

Arrival rate

Service rate

- System Utilization (Rho) $$\rho = \frac{\lambda}{\mu}$$
- Probability of no customer in system $$P_o = (1-\rho)$$
- Probability of n customers in system $$P_n = \rho^n(1-\rho)$$

M/M/1 Equations

- Average number of people or units waiting for service $$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\rho^2}{1 - \rho}$$
- Average time a person or unit spends in the queue $$W_q = \frac{L_q}{\lambda}$$
- Average number of people or units in the system $$\mathrm{L_s} = \mathrm{L_q} + \frac{\lambda}{\mu}$$
- Average time a unit spends in the system $$\mathrm{W_s} = \mathrm{W_q} + \frac{1}{\mu}$$

General Service Time: M/G/1 Model Characteristics

- Type: Single-channel, single-phase system
- Input source: Infinite; no balks, no reneging
- Arrival process: Number of arrivals follows Poisson distribution
- Queue: Unlimited; single line
- Queue discipline: FIFO (FCFS)
- Service time: **General distribution** (mean, standard deviation), e.g. normal, lognormal
- Arrivals and service times are independent of the number of customers in system
- Service rate > arrival rate

M/G/1 Equations

Standard deviation of service times

- Average number of people or units waiting for service $$L_q = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}$$
- Average time a person or unit spends in the queue $$W_q = L_q / \lambda$$
- Average number of people or units in the system $$L_s = L_q + \frac{\lambda}{\mu}$$
- Average time a unit spends in the system $$W_s = W_q + \frac{1}{\mu}$$

Constant Service Time: M/D/1 Model Characteristics

- Type: Single-channel, single-phase system
- Input source: Infinite; no balks, no reneging
- Arrival process: Number of arrivals follows Poisson distribution
- Queue: Unlimited; single line
- Queue discipline: FIFO (FCFS)
- Service time: **Constant**
- Arrivals and service times are independent of the number of customers in system
- Service rate > arrival rate

M/D/1 Equations

- Average number of people or units waiting for service $$L_q = \frac{\rho^2}{2(1-\rho)}$$
- Average time a person or unit spends in the queue $$W_q = L_q / \lambda$$
- Average number of people or units in the system $$L_s = L_q + \frac{\lambda}{\mu}$$
- Average time a unit spends in the system $$W_s = W_q + \frac{1}{\mu}$$

P08 Suggested Solution

Pallet Wrapping Process

- Modeling the pallet wrapping process as a queuing system
  - **Arrivals** are pallets loaded with boxes ready to be wrapped
    - ✓ Assume infinite population and Poisson distribution for pallets' arrivals.
    - ✓ Assume that no pallets arrive in batches and pallets arrive randomly throughout the day.
  - **Queue** is assumed to be infinite in length; FIFO
  - **Service** is the actual pallet wrapping process. Assume single server and single phase.

Is the Wrapping Time Exponentially Distributed?

- In exponential distribution, mean is equal to standard deviation.
  - The standard deviation is rather large.
  - It is possible for service time to assume very small values.
- The current operator is not formally trained and there is no proper standard operating procedure for him to follow. Hence, the wrapping time may vary a lot and we can assume a large standard deviation.
- With the same mean pallet wrapping time, the average number of pallets in the queue and average waiting time of each pallet in queue <u>doubles</u> when the standard deviation increases from <u>0</u> to <u>that of an exponential distribution</u>.

The Current Pallet Wrapping Process: Queuing Model (M/M/1)

- The current pallet wrapping process can be described by **M/M/1** queuing model. (One server is assumed: only one pallet is being wrapped by one operator at any time)
- Arrival: Poisson distribution with a mean of 13 pallets per hour; service time (time needed to wrap one pallet): negative exponential distribution with a mean of 4 minutes per pallet.
- Performance measures

Average system utilization	ρ	0.867	
Probability (no pallet in system)	P_0	0.133	
Average number of pallets in queue	Lq	5.633	
Average number of pallets in system	Ls	6.500	
Average waiting time in queue	Wq	26.000	minutes
Average time in system	Ws	30.000	minutes

Option 1: Pallet Wrapping Process with Standard Operating Procedure and Training: Queuing Model (M/G/1)

- With a standard operating procedure and training, the standard deviation of the wrapping time reduces. The process can be described by M/G/1 (one server is assumed: only one pallet is being wrapped by one operator at any time) queuing model.
- Arrival: Poisson distribution with a mean of 13 pallets per hour; service time (time needed to wrap one pallet): general distribution with mean of 3.6 minutes and standard deviation of 1.8 minutes.
- Performance measures

Average system utilization	ρ	0.780	
Probability (no pallet in system)	P_0	0.220	
Average number of pallets in queue	Lq	1.728	
Average number of pallets in system	Ls	2.508	
Average waiting time in queue	Wq	7.977	minutes
Average time in system	Ws	11.577	minutes

- With the SOP and training, the cycle time (total time in system) would be reduced by: (30-11.577)/30 = 61.41%.

Option 2: Automated Pallet Wrapping Process: Queuing Model (M/D/1)

- Automated pallet wrapping process: **M/D/1**
- Arrival: Poisson distribution with mean of 13 pallets per hour;
- Service time (time needed to wrap one pallet automatically): **3.6 minutes (constant)**.
- M/D/1 model is a special case of the M/G/1 model; the service time is a constant (known) value with a standard deviation equal to zero.

Average system utilization	ρ	0.780	
Probability (no pallet in system)	P_0	0.220	
Average number of pallets in queue	Lq	1.383	
Average number of pallets in system	Ls	2.163	
Average waiting time in queue	Wq	6.382	minutes
Average time in system	Ws	9.982	minutes

- With the automated pallet wrapping process, the cycle time (total time in system) would be reduced by: (30-9.982)/30 = 66.73%

Automated Wrapping Process with the Same Average Wrapping Time of the Current Process: Queuing Model (M/D/1)

- Automated pallet wrapping process: **M/D/1**
- Arrival: Poisson distribution with mean of 13 pallets per hour;
- Service time (time needed to wrap one pallet automatically): **4 minutes (constant), which is the same as the average wrapping time of the current manual process**.

Average system utilization	ρ	0.867	
Probability (no pallet in system)	P_0	0.133	
Average number of pallets in queue	Lq	2.817	
Average number of pallets in system	Ls	3.683	
Average waiting time in queue	Wq	13.000	minutes
Average time in system	Ws	17.000	minutes

- By removing the variability in the pallet wrapping process **only**, the cycle time (total time in system) would be reduced by: (30-17)/30 = 43.33%.

Comparison of Performance Measures

	Current Wrapping Process	With SOP and Training	Automated Wrapping Process	
	M/M/1	M/G/1	M/D/1	M/D/1
λ per min	13/60	13/60	13/60	13/60
μ per min	**1/4**	1/3.6	1/3.6	**1/4**
Standard Deviation	4	1.8	0	0
ρ	0.867	0.780	0.780	0.867
Lq	5.633	1.728	**1.383**	2.817
Ls	6.500	2.508	2.163	3.683
Wq (min)	26.000	7.977	**6.382**	13.000
Ws (min)	30.000	11.577	9.982	17.000

Refer to slide 18

Conclusion

- ❑ The pallet wrapping process could be modelled with **M/M/1, M/D/1** and **M/G/1** queuing models, depending on the **variation** of the wrapping (service) time. **M/M/1** and **M/D/1** are special cases of the **M/G/1** queuing model.
- ❑ To decide whether or not to invest in automatic wrapping machine:
  - ➢ Compare the cost of late shipment against cost of investment.
  - ➢ Compare the increase in income from throughput gain against cost of investment.
  - ➢ Compare the long term cost of hiring additional operators to the investment, maintenance and running cost of the automatic wrapping machine.

Learning Objectives

At the end of the lesson, students should be able to:

- Model the queue situations given using M/M/1, M/G/1 and M/D/1 models applying the Kendall’s notation.
- Explain how to characterize a queuing problem so as to optimally match available resource to meet customer expectations and service level.
- Recognize how variability in service time affects the performance measures of a queuing system.
- Apply queuing theory to analyze situations where the service time is either constant (M/D/1), exponential (M/M/1) or could be represented by some arbitrary distribution (M/G/1).
- Analyze queue performance measures for a single server queuing model.

Overview of E211 Operations Planning II Module