REPUBLIC POLYTECHNIC

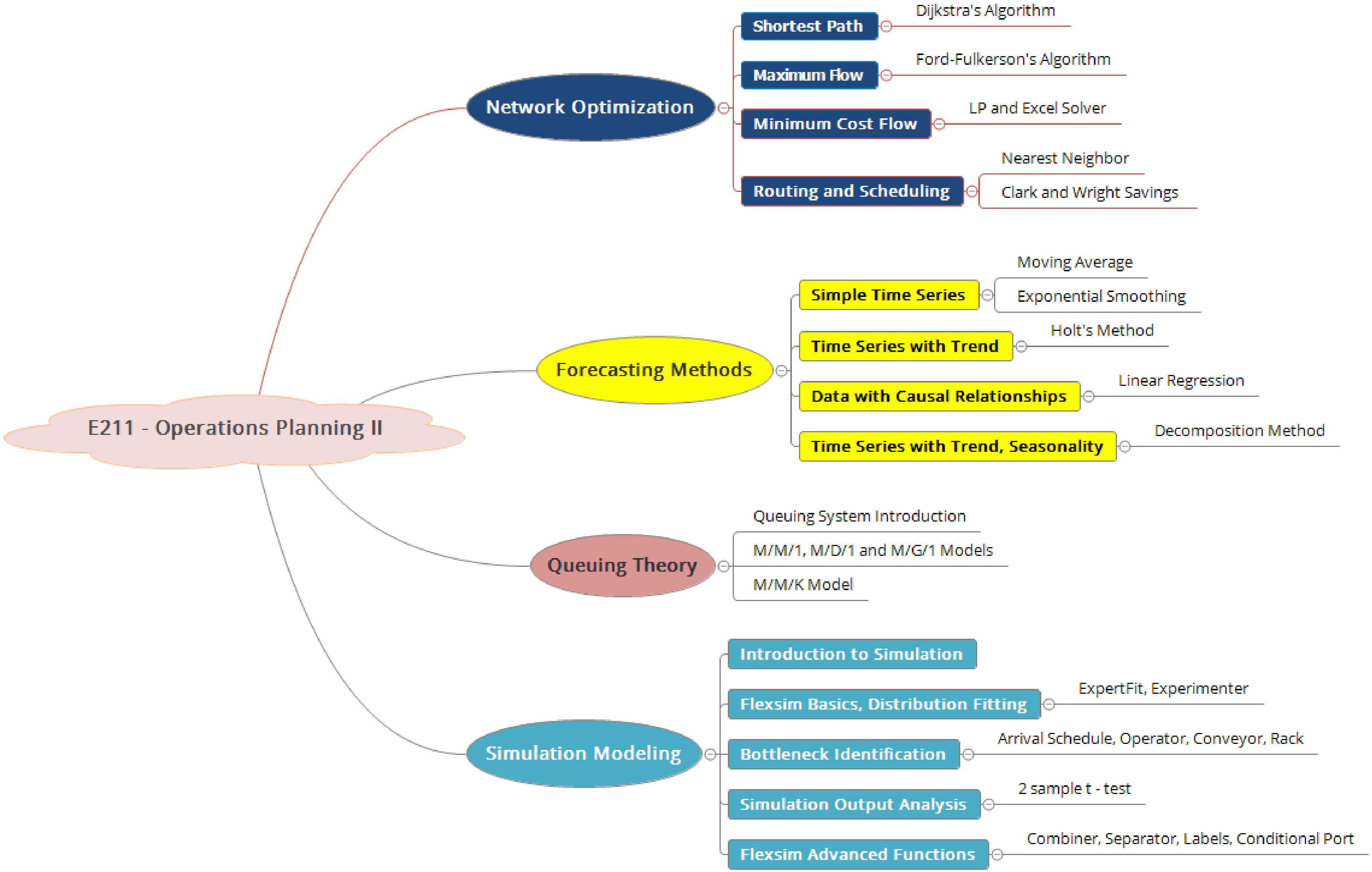
Problem 02

Be More Cost Effective

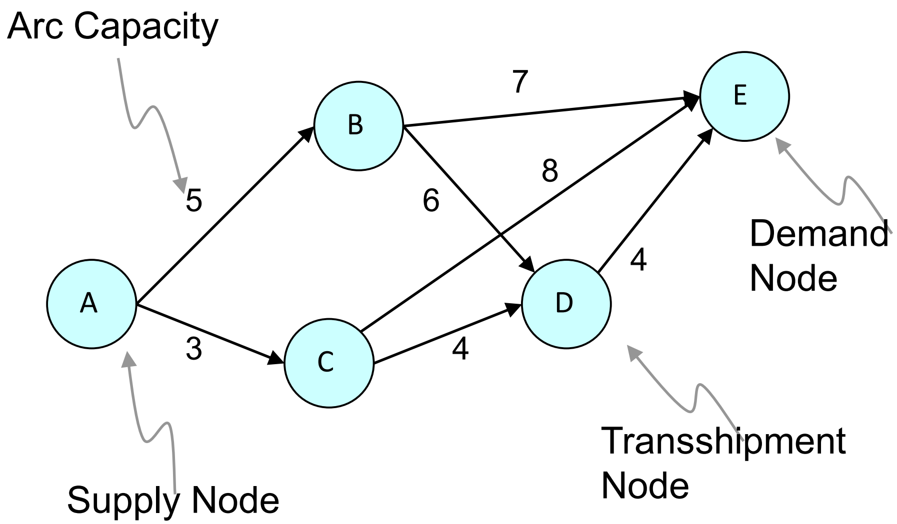
E211 – Operations Planning II

SCHOOL OF ENGINEERING

Module Coverage: E211 Topic Tree



Network Definition



- The maximum amount of flow that can be carried on a directed arc is referred to as the arc capacity.
- A supply node (or source node) has the property that the flow out of the node exceeds the flow into the node.
- The reverse case is a demand node (or sink node).
- A transshipment node (or intermediate node) satisfies the conservation of flow, so that flow in equals flow out.

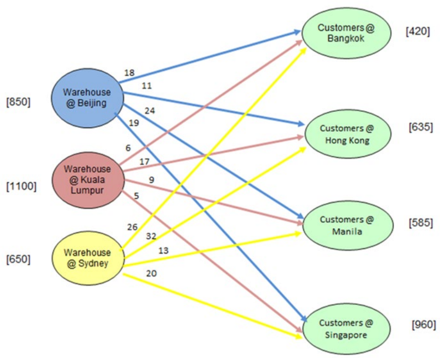
The Distribution Network Problem

- The distribution network problem
  - Goods are shipped from known supply points (factories, plants) to known demand points (customers, retail outlets), possibly through intermediate points (regional and or/field warehouses) known as the transhipment points.
  - The overall objective is to find the best distribution plan, i.e., the amount to ship along each route from the supply points, through the intermediate points, to the demand points, where “best” meant a plan that minimizes the total shipping cost.
  - It is also necessary to satisfy certain constraints:
    - ✓ Not shipping more than the specified capacity from each supply point;
    - ✓ Meeting known demand at the demand points;
    - ✓ Shipping goods only along valid routes;
    - ✓ Shipment not exceeding capacity limit imposed on certain routes.

The Minimum Cost Flow Model

- One favourable feature of the distribution network problem is that it can be precisely represented and solved by the Minimum Cost Flow Model.
- The Minimum Cost Flow Model is a network optimization model having the following features:
  - ➢ Its objective is to find the **cheapest** possible way of sending a certain amount of flow **from sources to destinations** through a flow network;
  - ➢ Like the maximum flow problem: it considers flow through a network with limited arc capacities;
  - ➢ Like the shortest path problem: it considers a cost (or distance) for flow through an arc;
  - ➢ Like the transportation problem: it can consider multiple sources and multiple destinations for the flow, again with associated costs.
  - ➢ All three problems above are special cases of the Minimum Cost Flow Problem.

The Minimum Cost Flow Model



Transportation/ Assignment Problem

- No transhipment nodes
- No limitation on flow capacity
-Each arc is directed from a source node to a destination

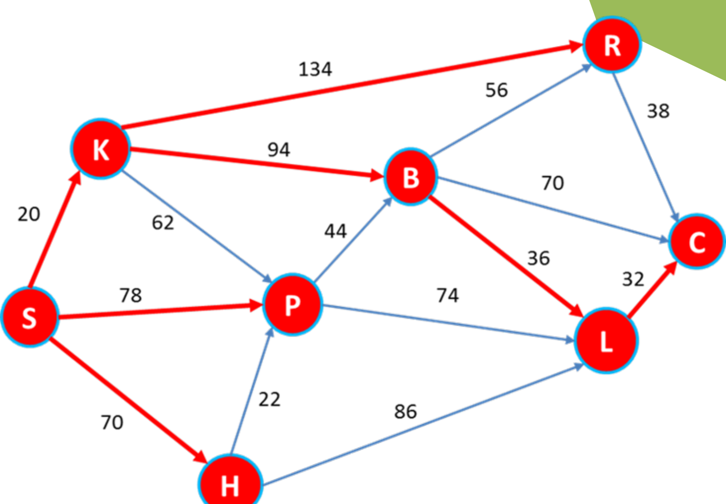
Shortest Path Problem

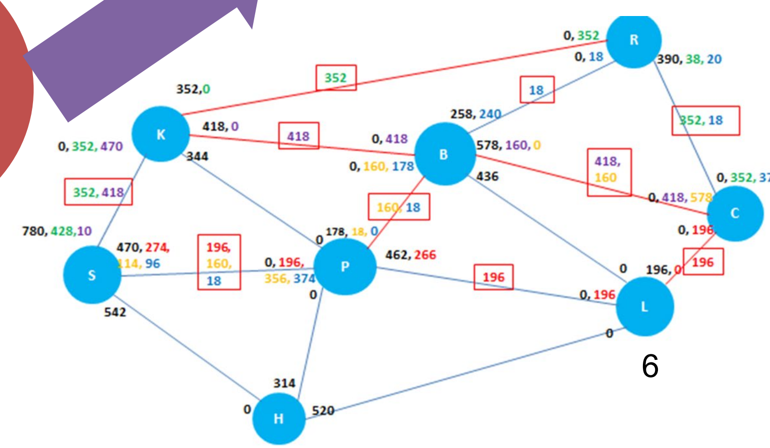
-Single source (O) having supply of '1'
-Single sink (D) having demand of '1'
-No limitation on flow capacity

Maximum Flow Problem

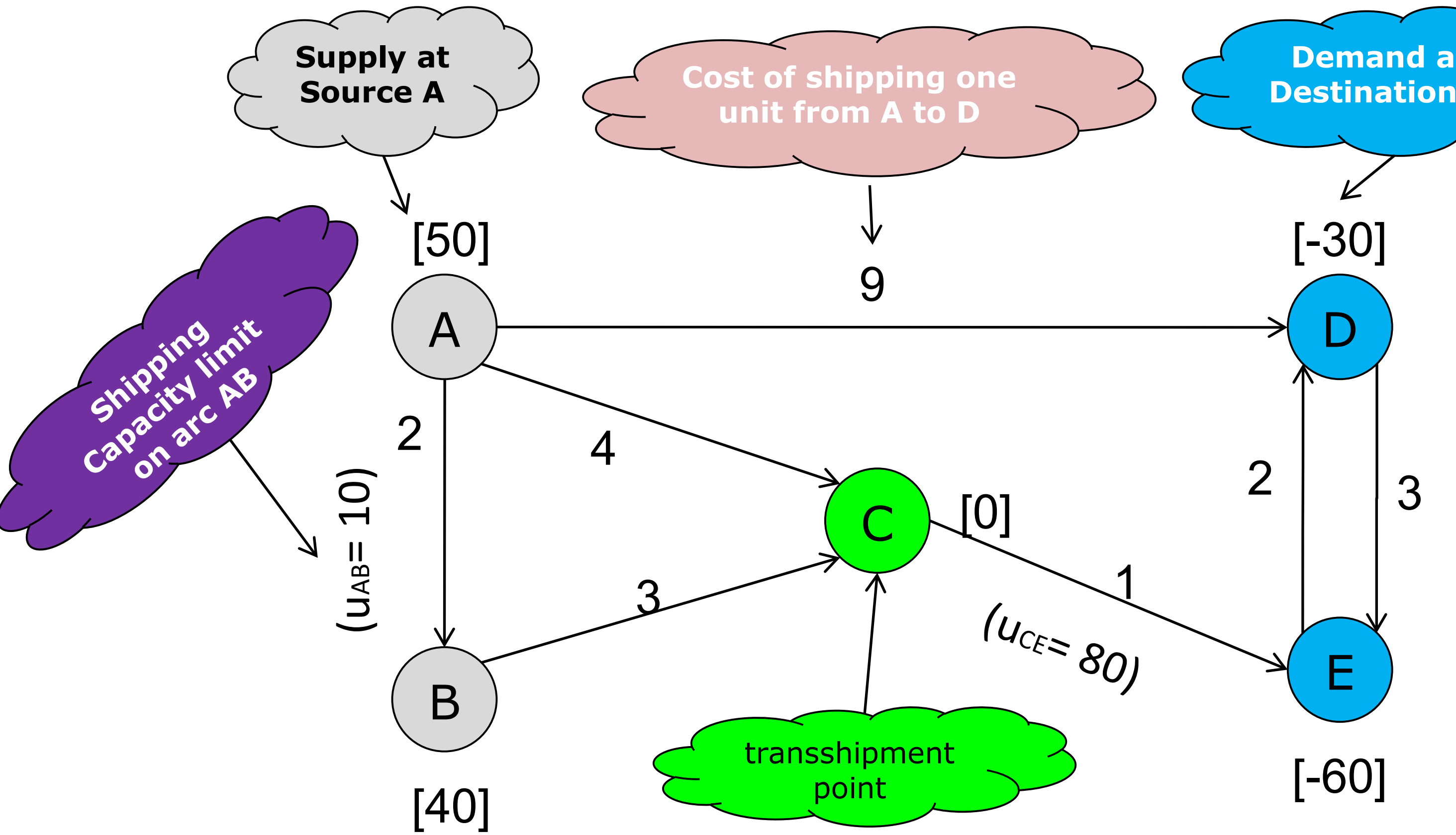
-Single source (O); single sink (D)
-With limitation on flow capacity
-No consideration on cost of flow

Minimum Cost Flow





Network Representation of the Minimum Cost Flow Model

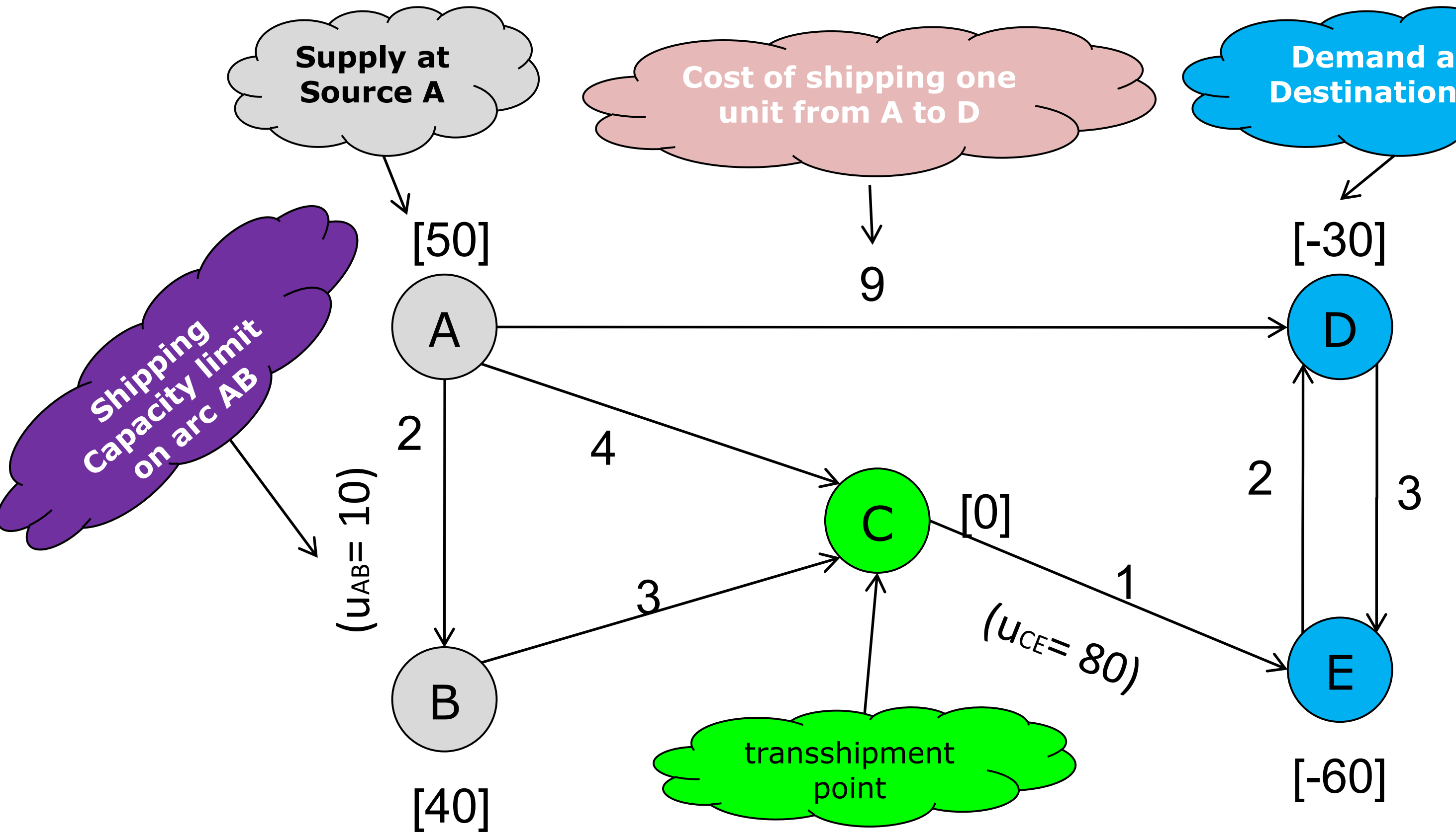


The objective is to minimize the total cost of sending the available supply through the network to satisfy the given demand.

U_{AB} =10 represents upper bound of flow capacity from A to B is 10.

- **Assumptions to simplify the problem solving:**
  - Homogenous goods
  - Total supply equals to total demands
  - Transportation costs are linear

Network Representation of the Minimum Cost Flow Model



The objective is to minimize the total cost of sending the available supply through the network to satisfy the given demand.

U_{AB} =10 represents upper bound of flow capacity from A to B is 10.

A and B are supply nodes: the flow out of the node exceeds the flow into the node. **Net flow generated > 0**

D and E are demand nodes: the flow into the node exceeds the flow out of the node. **Net flow generated < 0**

C is a transshipment node: flow in equals flow out. **Net flow generated = 0**

LP Formulation of the Minimum Cost Flow Problem

- **The decision variables**: x_{ij} represents the amount of flow through arc $i \rightarrow j$ where i, j represents nodes in the network: $x_{AB}, x_{AC}, x_{AD}, x_{BC}, x_{CE}, x_{DE}, x_{ED}$
- **The objective function:**

$$\text{Minimize } Z = 2x_{AB} + 4x_{AC} + 9x_{AD} + 3x_{BC} + x_{CE} + 3x_{DE} + 2x_{ED}$$

- **The constraints:**
  - **Net flow constraints** (one for each node)

Amount shipped out – Amount shipped in = Net flow generated at the node

$$\begin{aligned}
&\text{Node A: } x_{AB} + x_{AC} + x_{AD} = 50 \qquad &&\text{Node D: } x_{DE} - x_{AD} - x_{ED} = -30\\
&\text{Node B: } -x_{AB} + x_{BC} = 40 &&\text{Node E: } -x_{CE} - x_{DE} + x_{ED} = -60\\
&\text{Node C: } -x_{AC} - x_{BC} + x_{CE} = 0
\end{aligned}$$

  - **Upper-bound constraints**

Flow through an arc not exceeding the limited arc capacity

$$x_{AB} \le 10; \quad x_{CE} \le 80.$$

  - **Non-negativity constraints**: All $x_{ij} \ge 0$

eLearning Video link:
https://drive.google.com/file/d/0B7JB4jTZ9eQMVE03ODlmVE9nWGM/view?usp=sharing

The Integer Solution Property of the Minimum Cost Flow Problem

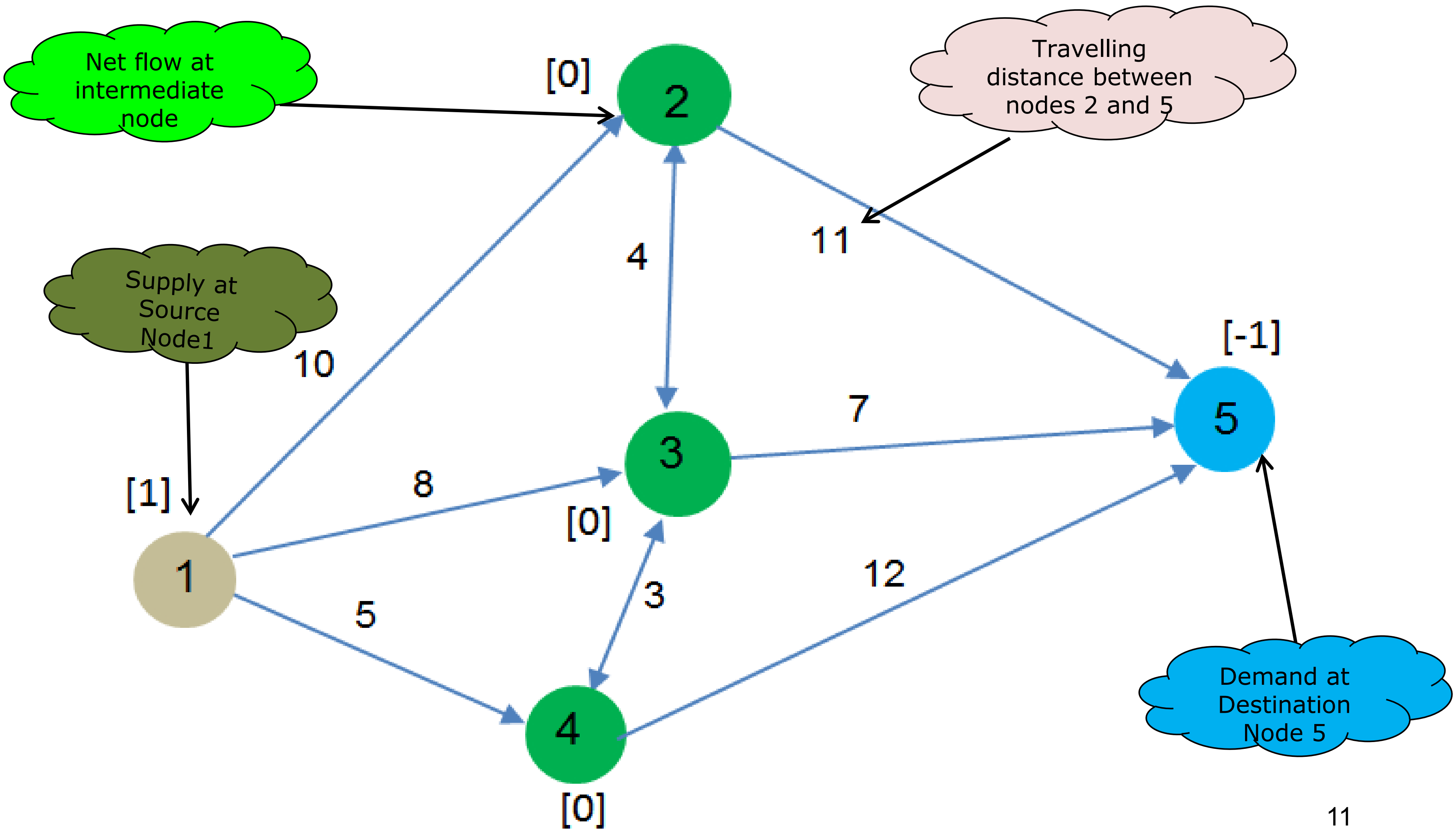
- Like the transportation problem, the minimum cost flow problem also has the integer solution property.

- As long as every demand, supply and arc capacity have integer values, a minimum cost flow problem with feasible solutions is guaranteed to have an optimal solution with integer values for all the decision variables.

- Therefore, it is not necessary to add constraints to the model to restrict the decision variables to only have integer values.

Applying Minimum Cost Flow Problem to the Shortest Path Problem

Finding the shortest path from Supply Node 1 to Demand Node 5.



LP Formulation for the Shortest Path Problem (with mandatory pass of each node)

Decision variables

Let X_{ij} be the route from node *i* to node *j*. X_{ij} is a binary variable:

$$X_{ij} = \begin{cases} 1 & \text{arc } i \to j \text{ is on the shortest path} \\ 0 & \text{otherwise} \end{cases}$$

In total, there are 10 decision variables: X_{12}, X_{13}, X_{14}, X_{23}, X_{25}, X_{32}, X_{34}, X_{35}, X_{43}, X_{45}

Objective function

Minimize distance traveled = $10X_{12} + 8X_{13} + 5X_{14} + 4X_{23} + 11X_{25} + 4X_{32}$
$+ 3X_{34} + 7X_{35} + 3X_{43} + 12X_{45}$

We ignored those decision variables going into source node and coming out of destination node.

LP Formulation for the Shortest Path Problem (with mandatory pass of each node)

Constraints

Subject to: -

Net flow contraints

Node 1: $x_{12} + x_{13} + x_{14} = 1$

Node 2: $-x_{12} - x_{32} + x_{23} + x_{25} = 0$

Node 3: $x_{32} + x_{34} + x_{35} - x_{13} - x_{23} - x_{43} = 0$

Node 4: $x_{43} + x_{45} - x_{14} - x_{34} = 0$

Node 5: $-x_{25} - x_{35} - x_{45} = -1$

Mandatory pass constraints

Node 2: $x_{12} + x_{32} >= 1$

Node 3: $x_{13} + x_{23} + x_{43} >= 1$

Node 4: $x_{14} + x_{34} >= 1$

Non-negativity constraints

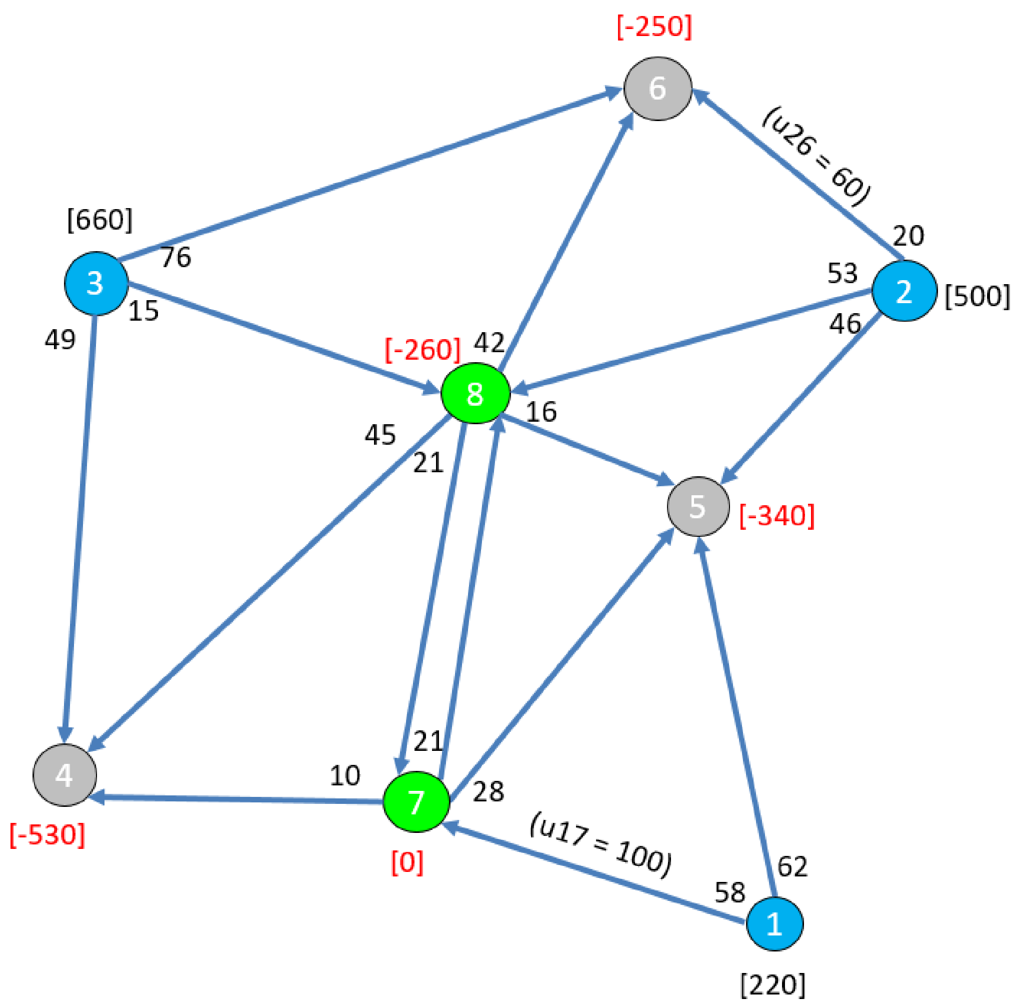
All $X_{ij} >= 0$

- Net flow constraints already imposed mandatory pass on source node 1 and destination node 5.
- Mandatory pass constraints are required only for intermediate (transhipment) nodes 2, 3 and 4.
- Here we use incoming arcs to formulate the mandatory pass constraints. We can also use outgoing arcs to formulate them.
- Due to the 'integer solution' property of the minimum cost flow problem, we do not need to specify the decision variables to be binary.

For the shortest path problem with no mandatory pass requirement, we can ignore this set of constraints.

P02 Suggested Solution

Today's problem: Formulation as a minimum cost flow Problem



1 – Guangzhou (Supply)
2 – Qingdao (Supply)
3 – Xining (Supply)
4 – Lijiang (Demand)
5 – Wuhan (Demand)
6 – Beijing (Demand)
7 – Guiyang (Transshipment)
8 – Xi'an (Transshipment / Demand)

Today's Problem: LP Formulation

<u>Decision Variables:</u>

Let X_{ij} be the **number of units of chandeliers** to be shipped from node *i* to node *j* . In total , we have 15 **decision variables (same as the number of arcs)**:

$X_{15}, X_{17}, X_{25}, X_{26}, X_{28}, X_{34}, X_{36}, X_{38}, X_{74}, X_{75}, X_{78}, X_{84}, X_{85}, X_{86}, X_{87}$

<u>Objective function:</u>

Minimize cost =

$62X_{15}+58X_{17}+46X_{25}+20X_{26}+53X_{28}+49X_{34}+76X_{36}$
$+15X_{38}+10X_{74}+28X_{75}+21X_{78}+45X_{84}+16X_{85}+42X_{86}+21X_{87}$

Today’s Problem: LP Formulation

Subject to: - Constraints

Net flow constraints

		Sign	RHS
Supply Node 1:	$x_{15} + x_{17}$	=	220
Supply Node 2:	$x_{25} + x_{26} + x_{28}$	=	500
Supply Node 3:	$x_{34} + x_{36} + x_{38}$	=	660
Demand Node 4:	$-x_{34} - x_{74} - x_{84}$	=	-530
Demand Node 5:	$-X_{15} - X_{25} - X_{75} - X_{85}$	=	-340
Demand Node 6:	$- X_{26} - X_{36} - X_{86}$	=	-250
Transhipment Node 7	$x_{74} + x_{75} + x_{78} - x_{17} - x_{87}$	=	0
Transhipment /Demand Node 8	$x_{84} + x_{85} + x_{86} + x_{87} - x_{28} - x_{38} - x_{78}$	=	-260

Arc capacity constraints

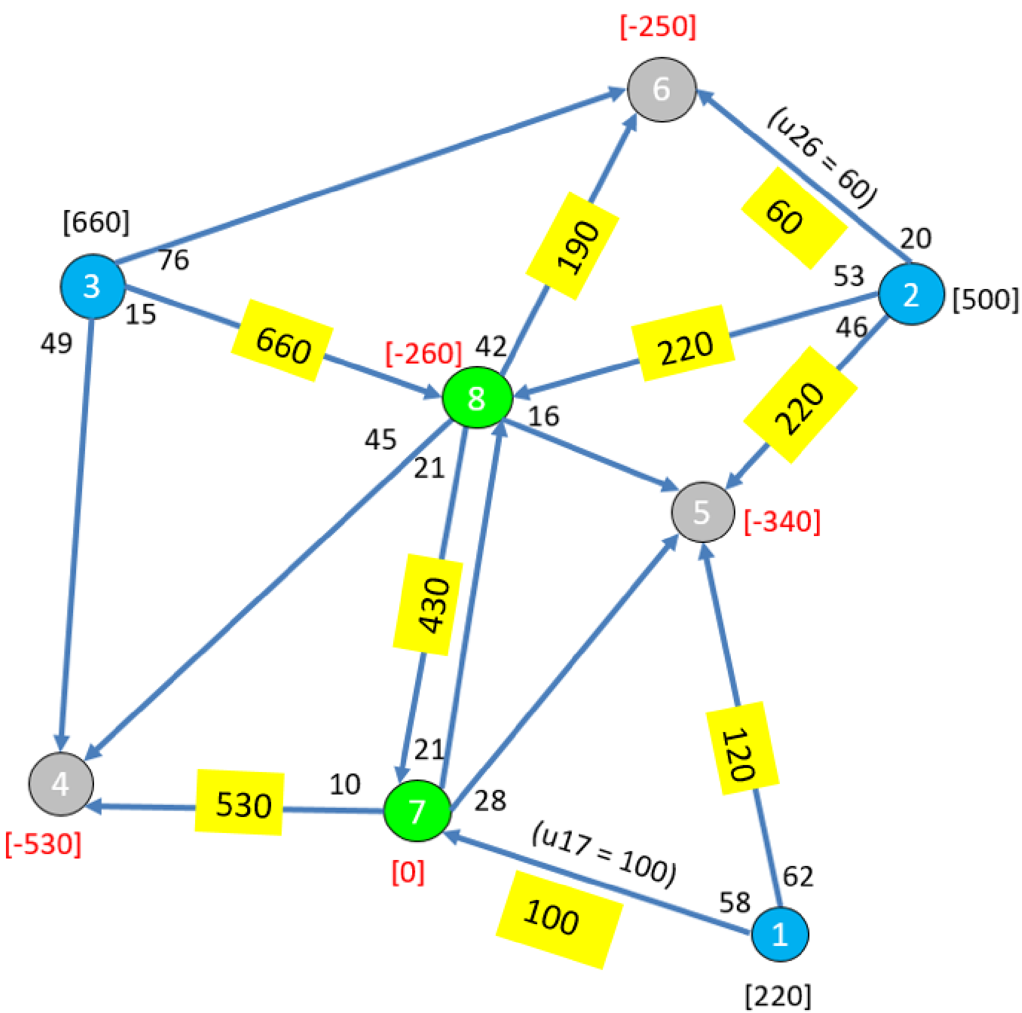
Arc 1-7	X_{17}	<=	100
Arc 2-6	X_{26}	<=	60

Nonnegativity constraints

All $X_{ij} >= 0$

Today's problem: Excel Solver solution

Total shipment cost from LP solution = $68,430



Decision Variables	Amount of Flow
X15	120
X17	100
X25	220
X26	60
X28	220
X34	0
X36	0
X38	660
X74	530
X75	0
X78	0
X84	0
X85	0
X86	190
X87	430

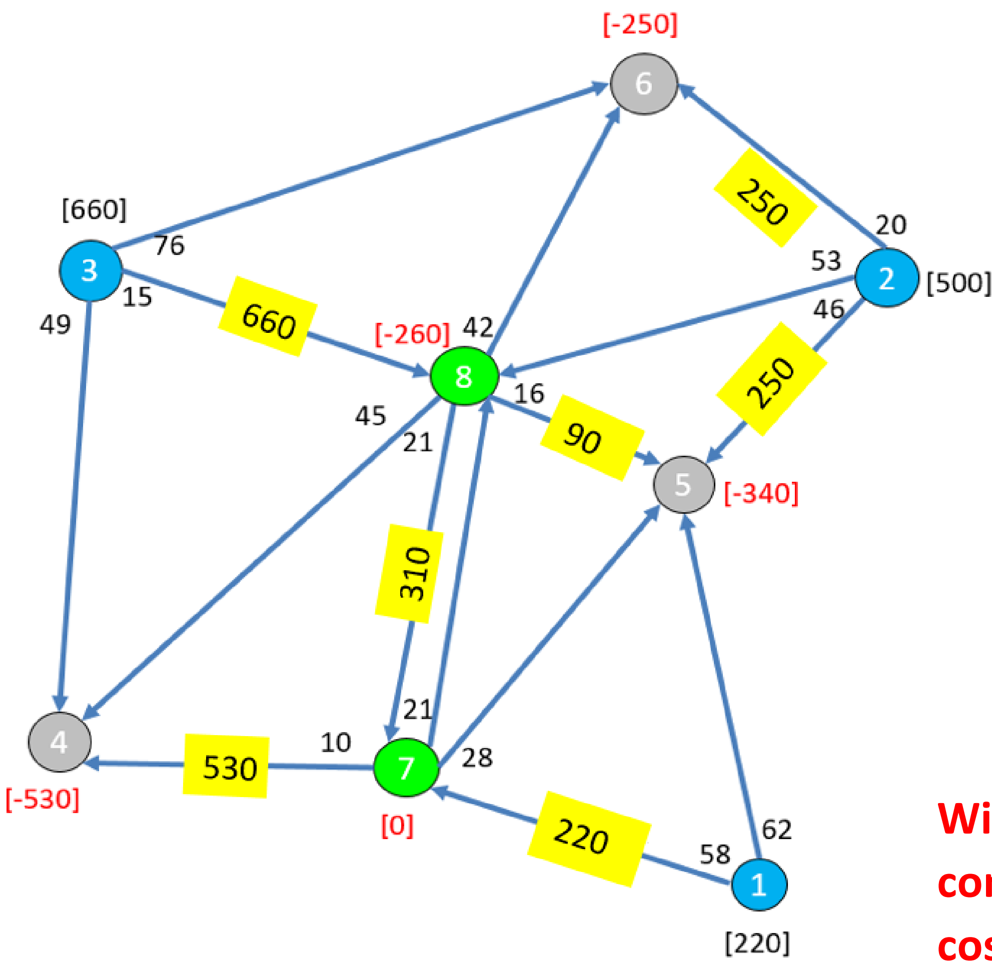
Today's Problem – A Special Case

If Amy can have unlimited amount of shipment from Guangzhou to Guiyang and from Qingdao to Beijing, what will happen to the LP model, the solution and the total shipment cost?

Arc capacity or Upper-Bound constraints are not necessary and therefore can be removed.

Today's problem: A Special Case

Revised shipment cost from LP solution = $52,410



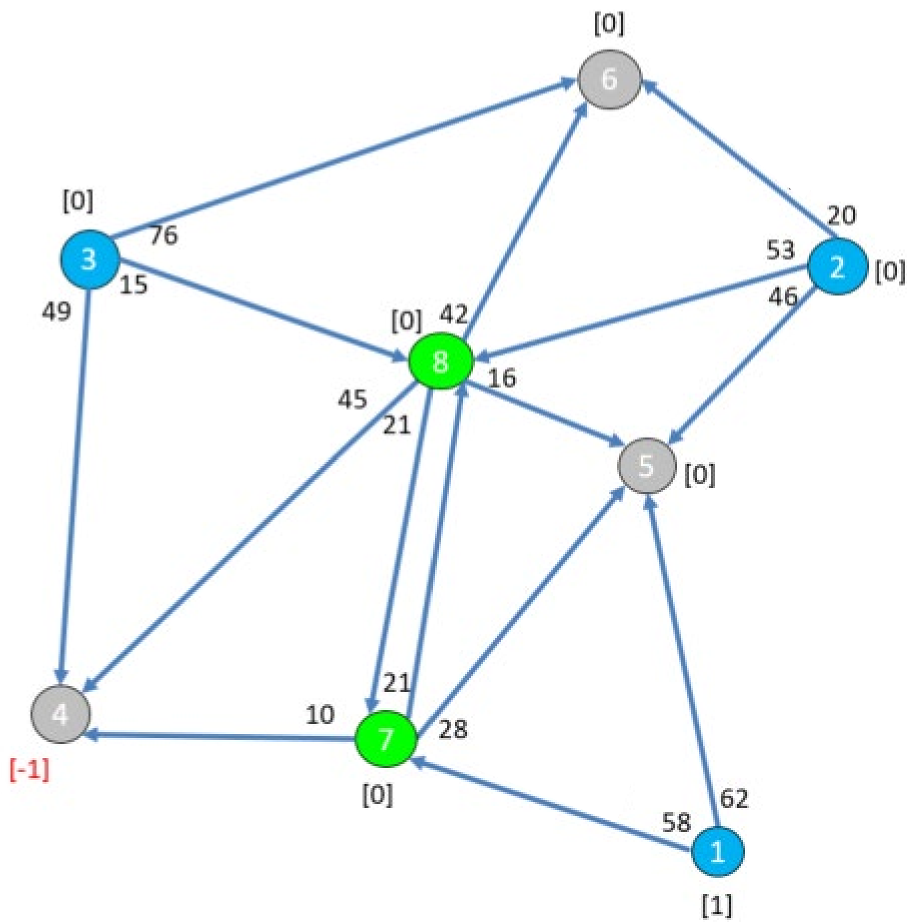
Decision Variables	Amount of Flow
X15	0
X17	220
X25	250
X26	250
X28	0
X34	0
X36	0
X38	660
X74	530
X75	0
X78	0
X84	0
X85	90
X86	0
X87	310

With the arc capacity or Upper-Bound constraints relaxed, the total shipment cost will decrease by:

$68,430 - $52,410 = $16,020

Solving the Shortest Path for Today's Problem

Assume that Amy wants to determine the shortest path (in terms of traveling cost) to ship the chandlers from Guangzhou (1) to Lijiang (4) by passing through Xi'an (8) mandatory. Numbers given below are shipment cost through the routes.



1 – Guangzhou
2 – Qingdao
3 – Xining
4 – Lijiang
5 – Wuhan
6 – Beijing
7 – Guiyang
8 – Xi'an

In the shortest path problem, origin is the supply node with 1 unit of net flow, destination is the demand node with -1 unit of net flow. All the other nodes will be transshipment nodes with net flow of 0.

LP Formulation for the Shortest Path Problem (with mandatory pass at Xi'an)

Decision variables

Let X_{ij} be the route from node *i* to node *j* . X_{ij} is a binary variable:

$$X_{ij} = \begin{cases} 1 & \text{arc } i \to j \text{ is on the shortest path} \\ 0 & \text{otherwise} \end{cases}$$

In total, there are 15 decision variables:

X_{15}, X_{17}, X_{25}, X_{26}, X_{28}, X_{34}, X_{36}, X_{38}, X_{74}, X_{75}, X_{78}, X_{84}, X_{85}, X_{86}, X_{87}

Objective function

Minimize total traveling cost =

$62X_{15}+58X_{17}+46X_{25}+20X_{26}+53X_{28}+49X_{34}+76X_{36}$
$+15X_{38}+10X_{74}+28X_{75}+21X_{78}+45X_{84}+16X_{85}+42X_{86}+21X_{87}$

LP Formulation for the Shortest Path Problem (with mandatory pass at Xi'an)

Constraints

Subject to:-

Net flow constraints

Node 1 (Source)	:	$X_{15} + X_{17}$	$= 1$
Node 2	:	$X_{25} + X_{26} + X_{28}$	$= 0$
Node 3	:	$X_{34} + X_{36} + X_{38}$	$= 0$
Node 4 (Destination)	:	$-X_{34} - X_{74} - X_{84}$	$= -1$
Node 5	:	$-X_{15} - X_{25} - X_{75} - X_{85}$	$= 0$
Node 6	:	$-X_{26} - X_{36} - X_{86}$	$= 0$
Node 7	:	$X_{74} + X_{75} + X_{78} - X_{17} - X_{87}$	$= 0$
Node 8	:	$X_{84} + X_{85} + X_{86} + X_{87} - X_{28} - X_{38} - X_{78}$	$= 0$

Mandatory Pass Constraints

Xi'an (Node 8)

Node 8 : $X_{84} + X_{85} + X_{86} + X_{87} >= 1$

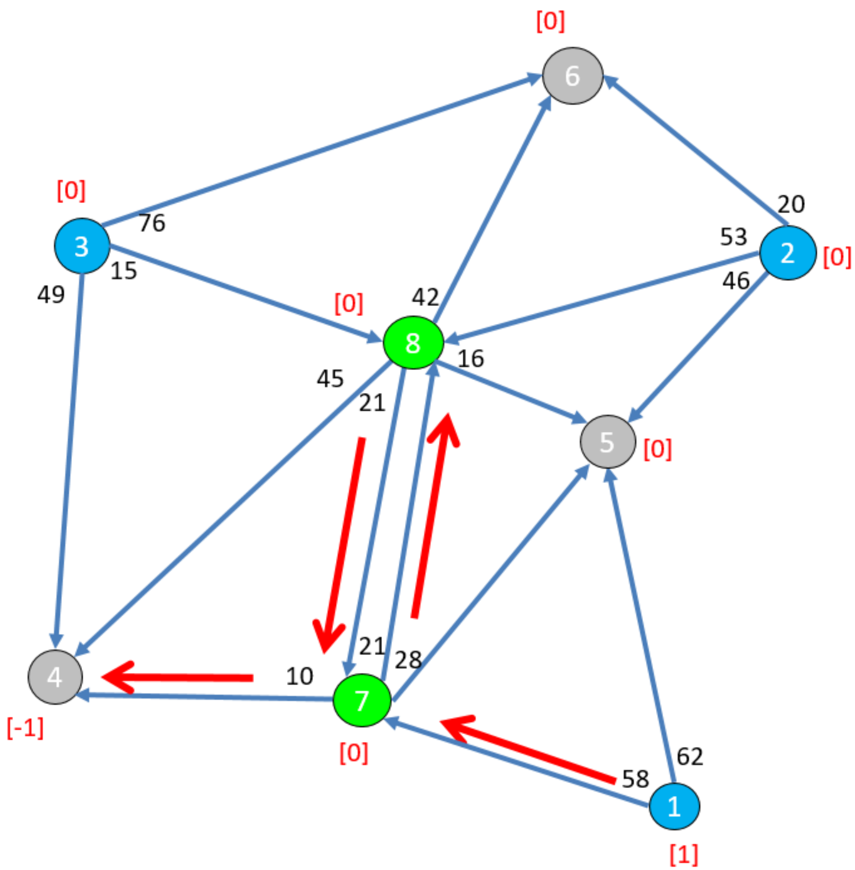
Non-negativity constraints

All $X_{ij} >= 0$

- Net flow constraints imposed mandatory pass on source node 1 and destination node 4.
- For the mandatory pass required at node 8, outgoing arcs are used to formulate this mandatory pass constraint.
- Due to the 'integer solution' property of the minimum cost flow problem, we do not need to specify the decision variables to be binary.

For the shortest path problem with no mandatory pass requirement, we can ignore this type of constraints.

EXCEL Solver Solution (Mandatory Pass at Xi'an)



Decision Variables	Amount of Flow	Cost
X15	0	62
X17	1	58
X25	0	46
X26	0	20
X28	0	53
X34	0	49
X36	0	76
X38	0	15
X74	1	10
X75	0	28
X78	1	21
X84	0	45
X85	0	16
X86	0	42
X87	1	21

Total traveling cost is $110.

Conclusion

- The minimum cost flow problem holds a central position in the network optimization models.
- The transportation problem, shortest path problem, and maximum flow problem are all special cases of the minimum cost flow problem.
- One special application of the minimum cost flow problem is the distribution network problem.
- The distribution network problem is to find the minimum cost shipment arrangements distributing products from ‘sources’ to ‘destinations’ considering both transshipment points and flow capacity in a transportation network.
- Flow networks can also be used to model liquids flowing through pipes, parts through assembly lines, current through electrical networks, information through communication networks.

Learning Objectives

At the end of the lesson, students shall be able to:

- Recognize that the distribution network problem deals with the distribution of goods from ‘sources’ to ‘destinations’ considering both transhipment points and flow capacity in a transportation network.
- Identify the sources, destinations and transhipment nodes, flow capacity, and mandatory pass of a distribution network.
- Recognize that transportation, assignment, maximum flow, and shortest path problem are special cases of the minimum cost flow problem.
- Formulate Linear Programming (LP) models for the minimum cost flow network problem & the shortest path problem.
- Use MS Excel Solver to find the optimum solution of a minimum cost flow problem & shortest path problem.

Overview of E211 Operations Planning II Module