Stat210B: Theoretical Statistics	**Lecture Date: April 03, 2007**

Lecture 22

Lecturer: Michael I. Jordan *Scribe: Sriram Sankararaman*

1 Convergence to a Normal experiment

Definition 1. Given an observation X, a randomized statistic $T(X, U)$ is a measurable map that depends on X and also on a $Uni(0, 1)$ random variable U.

Theorem 2. *(van der Vaart, 1998, Theorem 7.10) Assume that the experiment P_θ is q.m.d at a point θ with non-singular Fisher information I_θ. Let T_n be statistics in the experiments $(P^n_{\theta+\frac{h}{\sqrt{n}}} : h \in \mathbf{R})$ such that T_n converges in distribution under every h. Then there exists a randomized statistic T in the experiment $\mathcal{N}(h, I_\theta^{-1})$ such that $T_n \overset{h}{\rightsquigarrow} T$ for every h.*

Proof of Theorem 2 reposes on the following lemma.

Lemma 3. *(van der Vaart, 1998, Lemma 7.11, p. 99) Given a RV (S, Δ) and an independent uniform RV $U \sim Unif(0, 1)$, $\exists$ a RV $T = T(\Delta, U)$ such that $(T(\Delta, U), \Delta) \sim (S, \Delta)$.*

Proof. Use quantile transform. □

Theorem 2. Define

$$P_{n,h} = P^n_{\theta+\frac{h}{\sqrt{n}}}, \quad \Delta_n = \frac{1}{\sqrt{n}} \sum_i \dot{l}_\theta(X_i), J = I_\theta$$

By assumption, the marginals of (T_n, Δ_n) converge in distribution under $h = 0$. So

Prohorov	$\Rightarrow$	uniform tightness
Marginal tightness	$\Rightarrow$	joint tightness of (T_n, Δ_n)

Applying Prohorov in the converse, there exists a subsequence $\{n\}$ along which $(T_n, \Delta_n) \overset{h=0}{\rightsquigarrow} (S, \Delta)$ jointly. Since Δ_n converges marginally to Δ, we have $\Delta \sim \mathcal{N}(0, J^{-1})$.

The assumption that P_θ is QMD

$$\begin{aligned}
&\Rightarrow \quad (T_n, \log \frac{dP_{n,h}}{dP_{n,0}}) \xrightarrow{0} (S, h^T\Delta - \frac{1}{2}h^T Jh), \text{ (van der Vaart, 1998, Theorem 7.2)} \\
&\Rightarrow \quad \log \frac{dP_{n,h}}{dP_{n,0}} \xrightarrow{0} \mathcal{N}(-\frac{1}{2}h^T Jh, h^T Jh) \\
&\Rightarrow \quad P_{n,h} \triangleleft P_{n,0}
\end{aligned}$$

Using Le Cam's third lemma(van der Vaart (1998, Example 6.7)), the limit law of T_n under h is L_h where

$$L_h(B) = E1_B(S)e^{h^T\Delta - \frac{1}{2}h^T Jh}$$

Consider an observation X in the limiting experiment

$$\begin{aligned} X &\sim \mathcal{N}(h, J^{-1}) \\ JX &\sim \mathcal{N}(Jh, J) \\ &= \mathcal{N}(0, J) \text{ under } h = 0 \end{aligned}$$

Thus, JX is equal in law to Δ.

From Lemma 3, we can find $T(JX, U)$ such that

$$(T(JX, U), JX) \sim (S, \Delta), \text{ under } h = 0 \tag{1}$$

$$\begin{aligned} P_h(T(JX, U) \in B) &= \int dx P(T(Jx, U) \in B) e^{-\frac{1}{2}(x-h)^T J(x-h)} \sqrt{\frac{|J|}{(2\pi)^k}} \\ &= \int dx P(T(Jx, U) \in B) e^{-\frac{1}{2}h^T Jh + h^T Jx} \sqrt{\frac{|J|}{(2\pi)^k}} e^{-\frac{1}{2}x^T Jx} \\ &= E_0 1_{T(JX,U)\in B} e^{h^T JX - \frac{1}{2}h^T Jh} \end{aligned}$$

□

The RHS is equal to $L_h(B)$ from Equation 1. It then follows that the statistic $T_n \overset{h}{\rightsquigarrow} T(JX, U)$ under h.

2 Application to Testing

The power function is a function of the distribution of a statistic so the theory described so far applies. In an experiment $(\mathcal{X}, \mathcal{A}, P_h : h \in H)$, a test ϕ is a map $\phi : \mathcal{X} \to [0, 1]$ *i.e.*, it is the probability of rejecting the null hypothesis based on an observation $x \in \mathcal{X}$. The power function $\pi(h) \equiv E_h\phi(X)$ is the probability of rejection under hypothesis h. The test is of level α is $sup_{h\in H_0} \pi(h) \leq \alpha$.

Theorem 4. *van der Vaart (1998, Theorem 15.1) Consider a sequence of experiments $P_{n,h}$ converging to a limit experiment P_h (For us, this just means LAN). Suppose that the sequence of power functions $\pi_n(h)$ converges to $\pi(h), \forall h$. Then $\pi(h)$ is a power function in the limiting experiment* i.e., $\exists \phi \ni \pi(h) = E_h\phi(X)$.

References

van der Vaart, A. W. (1998). *Asymptotic Statistics.* Cambridge University Press, Cambridge.