BIOS6643 Longitudinal

L18 Nonparametric

EJC

Department of Biostatistics & Informatics

Mixture Models

Zero-plus-continuous mixture distributions

Topics involved with these notes:

- Modeling mixture distributions
- Zero-plus-continuous (or 'clump at zero') data
- 2-part models
- PROC NLMIXED

- **Associated reading:**
  - related topics in course notes (see 'Modeling independent or correlated non-normal data' chapter).

Zero-plus-continuous mixture distributions

Some distributions are more complex and cannot be modeled well using standard methods. For example, some distributions have possibility of 0 but where positive values can be well-modeled as continuous. Some examples:

- Health care costs.
- Precipitation amounts.

Such a distribution is a discrete and continuous mixture, so both aspects need to be accounted for properly.

Some possible distributions for the continuous part would be: log-Normal, Gamma, Weibull, truncated normal.

With the cost example, some potential interesting questions are:

- What is the chance someone will incur a cost?
- What is the mean of costs for those who do?
- What is the overall mean, taking into account both sources (those who have some costs, and those who don't)?

With the precipitation example, equivalent questions are:

- What is the probability of rain in a given city?
- What is the mean rainfall on days when it did rain?
- What is the overall mean rainfall in each city?

We can use the Theorem on Total Probability to derive the complete '0 + continuous' distribution. Let R denote an indicator variable for positive values of Y and let p denote the probability of a positive value. Then we have

$F_Y(y) = P(Y \leq y \mid r = 0)P(R = 0) + P(Y \leq y \mid r > 0)P(R = 1) =$
$P(Y \leq y \mid r = 0)(1 - p) + P(Y \leq y \mid r = 1)p =$
$(1 - p)I_{y=0} + pF_{Y \mid r=1}(y \mid r = 1)$; where $F_{Y \mid r=1}(y \mid r = 1)$ is the CDF of a random variable with positive density for positive values of Y (e.g., Weibull, Gamma, log-Normal).

Although the CDF is easier to work with for mixed distributions, we need the PDF for the likelihood. The form can be defined mathematically as $f_Y(y) = (1 - p)\delta_0(y) + pf_{Y \mid r=1}(y \mid r = 1)$

where $\delta_0(y)$ is the Dirac delta function, defined to be 0 when $y \neq 0$ but integrates to 1 over all y on the real line. For practical purposes (e.g., in the likelihood function) we set this term to 1 so that
$f_Y(y) = (1 - p)I_{r=0} + pf_{Y \mid r=1}(y \mid r = 1)$

In our example with rainfall, we'll consider the Gamma distribution for positive amounts (i.e., given $R = 1$), which has density

$f_Y(y) = \frac{x^{\theta-1}e^{-x/\lambda)}}{\lambda^\theta \Gamma(\theta)},\ for\ y > 0$; where $\theta > 0$ is a shape parameter and $\lambda > 0$ is a scale parameter.

The mean of this distribution is $\theta\lambda$ and the variance is $\theta\lambda^2$.

We can define a mixed model for this mixed distribution as follows.

- Occurrence model: $logit(p_{ij}) = \alpha_0 + b_{0i}$, where $p_{ij} = P(Y_{ij} = 1 \mid b_{0i})$.
- Intensity model: $Y_{ij} \mid r = 1,\ b_{1i},\ b_{2i} \sim \mathcal{G}amma(\theta + b_{1i},\ \lambda + b_{2i})$
- Covariance structure of random effects: $\boldsymbol{b} = (b_{0i},\ b_{1i},\ b_{2i})^\top \sim \mathcal{N}(\mathbf{0},\ \boldsymbol{G})$, where $\boldsymbol{G}$ is a 3×3 unstructured matrix.

The addition of random intercepts to both shape and scale parameters for each city allows unique Gamma rainfall distribution by city.

Deriving the mean, variance and covariance

For our model,
$E[Y_{ij} \mid \boldsymbol{b}] = E[E[Y_{ij} \mid \boldsymbol{b},\ r]] = 0(1-p_{ij}) + \mu_{ij+}p_{ij} = p_{ij}\mu_{ij+}$ where μ_{ij+} is the mean of the positive Y values and p_{ij} is the probability of a positive value. Now $p_{ij} = \frac{1}{1+exp(-\alpha_0 - b_{0i}^{p})})$ and $\mu_{ij+} = (\theta + b_{0i}^{shape})(\lambda + b_{0i}^{scale})$; (where conditioning on random effects is implied). Thus the mean that puts the 0's and positive data together is $E[Y_{ij} \mid \boldsymbol{b}] = \frac{(\theta + b_{0i}^{shape})(\lambda + b_{0i}^{scale})}{1+exp(-\alpha_0 - b_{0i}^{p})}$.

It may be just as meaningful to jointly report p_{ij} and μ_{ij+}, which represent the probability of rain or snow on day j for city i and the average precipitation over days when it did rain/snow.

We can also derive $Var[Y_{ij} \mid \boldsymbol{b}] = p_{ij}(\sigma^2_{ij+} + \mu^2_{ij+}(1 - p_{ij}))$, where σ^2_{ij+} is the variance of the positive Y values (show for homework).

Covariance and correlation:

$$Cov(Y_{ij},\ Y_{ik}) = E[Cov[Y_{ij},\ Y_ik \mid \boldsymbol{b}]] + Cov[E[Y_{ij} \mid b],\ E[Y_{ik} \mid \boldsymbol{b}]]$$

- The second term is straightforward to determine since we have already defined the model in terms of mean responses given the random effects.
- The first term is more difficult since $Cov[Y_{ij},\ Y_{ik} \mid \boldsymbol{b}]$ does not come directly from the defined model. Specifically, no error term is defined for the model. (For an LMM, it would be the $(i,\ j)$th element of the error covariance matrix, $\boldsymbol{R}_i$.) We could employ residuals to estimate the quantity. Check.

For the (straightforward) term on the right side,

$$Cov[E[Y_{ij} \mid \boldsymbol{b}),\ E[Y_{ik} \mid \boldsymbol{b}]] = Cov[\mu_{ij+}p_{ij},\ \mu_{ik+}p_{ik}] =$$
$$Cov\left[\frac{(\theta+b_{1i})(\lambda+b_{2i})}{1+exp(-\alpha_0-b_{0i})},\ \frac{(\theta+b_{1i})(\lambda+b_{2i})}{1+exp(-\alpha_0-b_{0i})}\right]$$

where $\boldsymbol{b} = (b_{0i},\ b_{1i},\ b_{2i})^{\top} \sim \mathcal{N}(\boldsymbol{0},\ \boldsymbol{G})$,

$$\boldsymbol{G} = \begin{pmatrix} \sigma^2_{b_0^p} & & \\ \phi_{21}\sigma_{b_0^p}\sigma_{b_0^{shape}} & \sigma^2_{b_0^{shape}} & \\ \phi_{31}\sigma_{b_0^p}\sigma_{b_0^{scale}} & \phi_{32}\sigma_{b_0^{shape}}\sigma_{b_0^{scale}} & \sigma^2_{b_0^{scale}} \end{pmatrix}$$

Here we use an unstructured G matrix and formulate the model so that the correlation parameters are directly estimated. Note that covariances depend on city i but not day j since we only have random intercepts in the model (but we keep both subscripts on parameters for potential generalizations). To make a time-dependent structure, we could add fixed and random effects for day in the model (in the intensity and/or occurrence parts). For models I tried it did not seem to help.

Here is the analysis of rainfall data in 6 cities selected from across the U.S. Note that this is more of a demonstration of methods; cities were not randomly selected and more advanced time-series models might be used for actual analysis. But it is real data and the modeled distribution appears to fit the data well.

The cities: Atlanta, Aurora, Chicago, Houston, New York, Phoenix, Sacramento, Seattle. These are the 'subjects'.

Data collection time frame: first 100 days of 2017.

Outcome variable: precipitation, measured in inches.

SAS code and output follow. Note that the names given in the SAS code are consistent with the quantities shown above, just written out instead of in Greek symbols.

```
PROC NLMIXED DATA=precip_data2 qpoints=5 absfconv=0.0000001;

PARMS ALPHA0=-0.8 SHAPE_MEAN=1 SCALE_MEAN=0.58
 VARB0_P=0.5 VARB0_SHAPE=0.05 VARB0_SCALE=0.05 PHI21=0.1 PHI31=-0.1 PHI32=-0.4;
BOUNDS VARB0_P VARB0_SHAPE VARB0_SCALE >=0;

SHAPE=SHAPE_MEAN+B0_SHAPE;
SCALE=SCALE_MEAN+B0_SCALE;
MULOGIT=ALPHA0+B0_P;
P=1/(1+EXP(-MULOGIT));

IF PRECIP=0 THEN LOGLIKE=LOG((1-P));
ELSE LOGLIKE=LOG(P)+(SHAPE-1)*LOG(PRECIP)-SHAPE*LOG(SCALE)
         -LOG(GAMMA(SHAPE))-(PRECIP/SCALE);

MODEL precip~GENERAL(LOGLIKE);
RANDOM B0_P B0_SHAPE B0_SCALE ~ NORMAL([0,0,0],
 [VARB0_P, PHI21*(VARB0_P*VARB0_SHAPE)**.5, VARB0_SHAPE,
  PHI31*(VARB0_P*VARB0_SC)**.5, PHI32*(VARB0_SHAPE*VARB0_SC)**.5, VARB0_SC])
 SUBJECT=city out=randout;

predict p out=p;
predict SHAPE out=SHAPE;
predict SCALE out=SCALE;run;
```

The NLMIXED Procedure

Specifications	Dimensions
Random Effects B0_P B0_SHAPE B0_SC	Observations Used 800
Distribution for Random Effects Normal	Subjects 8
Subject Variable city	Max Obs Per Subject 100
Optimization Technique Dual Quasi-Newton	Parameters 9
Integration Method Adaptive Gaussian Quadrature	Quadrature Points 5

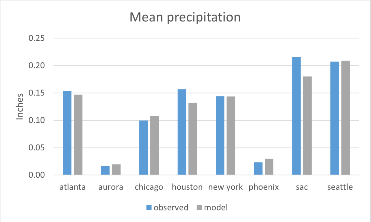
Parameter Estimates

Parameter	Estimate	SE	DF	t Value	Pr>|t|	Lower	Upper	Gradient
ALPHA0	-0.8370	0.2810	5	-2.98	0.0308	-1.5593	-0.1147	-0.00115
SHAPE_MEAN	0.7085	0.07268	5	9.75	0.0002	0.5216	0.8953	0.007636
SCALE_MEAN	0.5458	0.1008	5	5.41	0.0029	0.2866	0.8050	0.002785
VARB0_P	0.5749	0.3250	5	1.77	0.1371	-0.2605	1.4103	-0.0006
VARB0_SHAPE	0.008	0.01025	5	0.78	0.4682	-0.01832	0.03441	0.03419
VARB0_SCALE	0.0474	0.03798	5	1.25	0.2676	-0.05027	0.1450	0.000624
PHI21	0.1917	0.9727	5	0.20	0.8515	-2.3086	2.6920	0.001991
PHI31	0.2855	0.4463	5	0.64	0.5506	-0.8618	1.4327	0.001384
PHI32	-0.8628	0.3944	5	-2.19	0.0803	-1.8767	0.1511	-0.00495

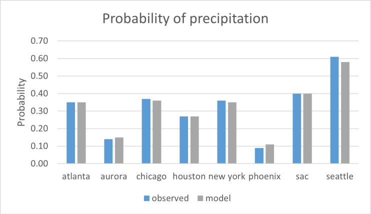
We could also add a fixed or fixed and random effect for day in either the Occurrence or Intensity models (or both), which would induce a slightly more time-sensitive correlation structure. However for the data at hand, such additions did not improve the model fit.

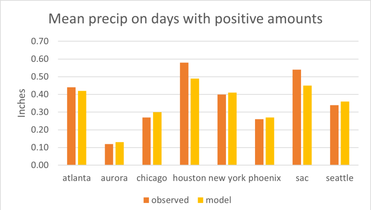
Predicted values that include random effect variations are obtained by the 'predict' statements given at the end of the SAS code. Here is some additional code that gets quantities of interest (p_{ij}, μ_{ij+}, and μ_{ij})

The graph below shows overall means for each city using both descriptive statistics and the model-based approach. They are generally in agreement.



The graphs below show how modeled values of $\mu_{ij} \mid b_{1i}, b_{2i}$ and $p_{ij} \mid b_{0i}$ versus descriptive quantities. These graphs demonstrate how information is lost when we only consider mean precipitation as in the last graph. For example, Seattle has greater likelihood of rain on any given day, but when we restrict to days where it did rain, Sacramento and Houston had higher mean daily precipitation amounts. Note that modeled amounts by city tend to exhibit shrinkage towards to overall mean (a bit higher for drier cities; less for wetter cities) which is expected for empirical Bayes estimates.





Important note: historical data might yield somewhat different results. Here we only used the early part of 2017 to build the model, so inference should be restricted to 'around this time or times that are similar in climate' and during winter.

One might wonder why the model approach is of any use, since we can estimate these quantities directly. Remember that we additionally can address correlation in our model. Also, the modeling approaches allows for the addition of other covariates and random effects, if we so wish.

Given that probabilities and means were not time sensitive in our given model, the correlation between responses should be somewhat like the compound symmetric structure.

Other mixture distributions

The previous example involved a mixture of a discrete and continuous distributions. The same principle can be used when mixing discrete distributions or mixing continuous distributions. In fact, mixing like-type distributions is probably easier, particularly when mixing discrete distributions.

A zero-inflated Poisson (ZIP) takes a standard Poisson distribution, and, as the name implies, adds to the probability of 0 occurring. This is obtained by mixing a degenerate distribution with point mass of one at zero, with a standard Poisson.

The random variable Y with ZIP distribution (Lambert, 1992) can be summarized as follows.

$Y \sim 0$ with probability p $Y \sim \mathcal{P}oisson(\lambda)$ with probability $1 - p$

The distribution has a binomial process (whether or not Y is 0) and a count process, associated with the Poisson distribution. However, there are two ways in which a '0' can be obtained; one is by sampling from the Poisson, and one is by the added 0 element. For some applications, it may be meaningful to distinguish these two types of zeroes, which is discussed in the last slides of this set.

Given the ZIP formulation above, we can combine the information to write a specific probability mass function:

$Y = 0$ with probability $p + (1-p)e^{-\lambda}$ $Y = k$ with probability $(1-p)e^{-\lambda}\lambda^k/k!,\ for\ k = 1,\ 2,\ ...$

If desired, models can specify correlation between parameters p and λ. Similar approaches can also be used to construct zero-inflated negative binomial (ZINB) and zero-inflated binomial (ZIB) models.

Mixing continuous distributions can also be performed. For example, mixing of normal distributions has been suggested to obtain more complex distributions for random effects in mixed models (see 'Heterogeneity Models' in Verbeke and Molenberghs, Linear Mixed Models for Longitudinal Data, 2000).

Additional thoughts.

Mixture distributions may be useful even if the distribution is completely discrete or continuous. For example, a zero-inflated Poisson distribution takes a standard Poisson and then adds a binomial random variable such that the probability that the mixed random variable takes on a value of 0 is increased.

We can also define mixture models based on how values in the mixture can be distinguished with respect to structure and sampling.

Hurdle models versus zero-inflated models

From McDowell (2003): A hurdle model is “a modified count model in which the two processes generating the zeros and the positives are not constrained to be the same” (Cameron and Trivedi 1998). Mullahy (1986) states, “The idea underlying the hurdle formulations is that a binomial probability model governs the binary outcome of whether a count variate has a zero or a positive realization. If the realization is positive, the “hurdle is crossed”, and the conditional distribution of the positives is governed by a truncated-at-zero count data model.

A zero-inflated model is one where the 0’s could come from 2 different types of processes (structural and sampling), and the 0’s versus nonzero’s are not governed by one overlying Bernoulli process. So, for example, we have a Poisson process, which could include 0’s and positive integers, but then is also a structural source for the 0’s.

As an example, consider a type of number of packs of cigarettes smoked in the last week. For smokers, most will likely smoke, but there is the chance that some will not; these will be 'sampling 0's'; but if the cohort also includes non-smokers, then those would be structural 0's since, by definition, they do not smoke. This would be an example of a zero-inflated model.

However, say that the time frame considered is much longer, like 3 months. In this case, it may be reasonable to assume that 0's only come from non-smokers and positive values come from smokers. We might use a hurdle model in this case.

Consider a model that needs to account for added 0's (either zero inflated, or via hurdle model). For simplicity of notation, let $p = P(Y = 0 \mid z,\ \gamma)$. Also f may represent either a pdf or pmf, depending on whether the distribution of positive values is continuous or discrete.

`For a zero-inflated model, we have $f_ZIP (x,\ z,\ \beta,\ \gamma)=p_{z,\`

`For a hurdle model, we have`

$$f_{hurdle}(x,\ z,\ \beta,\ \gamma) = \begin{cases} p_{z,\ \gamma} & y = 0 \\ (1 - p_{z,\ \gamma}) \frac{f_{count}(y \mid x,\ \beta)}{(1 - f_{count}(0 \mid x,\ \beta))} & y > 0 \end{cases}$$

The primary difference between models is that for the ZIP model, we have a standard distribution (f_count), such as a Poisson distribution and add some 0's to it, while for the hurdle model, we distinguish modeling of the 0's versus modeling of the positive values based on their structural differences. In order to model the positive values, we take a standard distribution like the Poisson and truncate it so that a value of 0 has no positive probability/mass.

Going back to the rainfall application, we combined a discrete and continuous model, the latter of which already does not have any probability mass on 0 (no need to truncate it). In this sense we intrinsically have a hurdle model. It may also make sense theoretically if there are not 'structural' and 'sampling' 0's.

However a zero-inflated model might make sense theoretically if there is some condition considered. For example, clouds must be present for rain or snow. But precipitation is not guaranteed when clouds are present. Thus, 0's could be distinguished by those on sunny (structural) and cloudy (sampling) days. One model governs rainfall when it is cloudy, and one whether it is cloudy or sunny. For the 'cloudy' model, we'd need some distribution that allows positive probability for 0 but also for positive values. A count-type model might work if we categorize the precipitation levels.

In some cases we may not need to consider the theoretical constructs of zero and nonzero values. We may use a model and be more concerned with how accurate the distribution is, and not estimate parameters based on distinguishing sampling versus structural-based zeroes.