# BIOS 6612 Homework 1: Likelihood theory

1. (**35 points**) Consider the Poisson distribution with mass function

$$f(y) = \frac{\lambda^y \exp(-\lambda)}{y!}, \lambda > 0, y \in \{0, 1, 2, \ldots\}$$

You have a sample of size $n = 50$ from this distribution with the following summary statistics:

- the sample mean of $Y_1, \ldots, Y_n$, i.e.,

$$\frac{1}{n}\sum_{i=1}^{n} Y_i = 1.3$$

- the sample mean of $Y_1^2, \ldots, Y_n^2$, i.e.,

$$\frac{1}{n}\sum_{i=1}^{n} Y_i^2 = 2.62$$

- the sample mean of $\log(Y_1!), \log(Y_2!), \ldots, \log(Y_n!)$, i.e.,

$$\frac{1}{n}\sum_{i=1}^{n} \log(Y_i!) = 0.4148$$

Assume we are interested in making inferences about $\lambda$.

(a) Derive the log-likelihood and score functions for this problem. (**5 points**)

(b) Compute $\hat{\lambda}$, the maximum likelihood estimator of $\lambda$. (**5 points**)

(c) Derive the expected information about $\lambda$ for this problem. (**5 points**)

(d) Construct a 95% Wald confidence interval for $\lambda$. (**2 points**)

(e) Test $H_0 : \lambda = 1$ versus a two-sided $H_1$ using the score test. Give the test statistic, null distribution, and $p$ value. (**6 points**)

(f) Repeat question 1e using the likelihood ratio test. (**6 points**)

(g) Repeat question 1e using the Wald test. (**6 points**)

2. (**20 points**) Suppose now we only are able to observe $Y_i^* = \mathbb{1}(Y_i > 0)$, i.e., a value of 1 if $Y_i > 0$ and 0 if $Y_i = 0$. As before, assume that $Y_i$ has the Poisson distribution with mean $\lambda$.

   (a) Give the likelihood function required to estimate $\lambda$ based on a sample of $Y_1^*, Y_2^*, \ldots, Y_n^*$. (**4 points**)

   (b) Find the MLE of $\lambda$ based on this likelihood function. *Hint: Your answer should be a function of the sample mean of the $Y_i^*$'s.* (**6 points**)

   (c) Derive the information about $\lambda$ based on this sample and hence find the variance of the MLE. (**6 points**)

   (d) Using the invariance property of MLEs, explain an easier method for finding the MLE of $\lambda$ in this problem. Recall that this property states that the MLE of a function $g(\theta)$ is the function applied to the MLE of $\theta$, i.e., $g(\hat{\theta})$. (**4 points**)

3. (**20 points**) Now consider estimation of $\zeta = P(Y_1 = 0) = \exp(-\lambda)$ based on $Y_1, \ldots, Y_n$.

   (a) One possible estimator is the proportion of 0's in the sample, i.e.,

$$\tilde{\zeta} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(Y_i = 0).$$

   Find the variance of this estimator. (**10 points**)

   (b) Another possible estimator is a transformation of the MLE, i.e.,

$$\hat{\zeta} = \exp(-\hat{\lambda}).$$

   Find the approximate variance of this estimator. *Hint: Use the delta method.* (**10 points**)