BIOS 7720: Applied Functional Data Analysis

Lecture 7: Scalar on Function Regression (SoFR)

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Roadmap

1. Final Project Proposal
2. Introduction to SoFR
3. Methods for Estimation
4. In Class Exercises

Project Proposal

1. Due next Thursday (4/8), **ungraded**
2. Very short (1-2 paragraphs) text document:
    1. Description of the dataset you're using
        - Source (e.g. web scraping, data repository, etc.)
        - Data generating mechanism (e.g. clinical trial, observational, etc.)
        - Size of the data (number of observational units, covariates, etc.)
    2. Explicit description of the functional data in your dataset
        - What
    3. Scientific question and the role of your functional data in answering that question
        - What is the relevant scientific question?
        - Functional data as outcome or predictor?

Set Up

- Notation
  - Sampling unit (e.g. participant) by $i = 1, \ldots, N$
  - Scalar outcome y_i
  - Scalar predictor x_i
  - Functional predictor $z_i(s)$ observed on regular grid $s_1, \ldots, s_J$
- Observed data are then of the form

$$[\{y_i, x_i, Z_i(s_j)\}, 1 \leq j \leq J, 1 \leq i \leq N]$$

Motivating Data Set

- NHANES physical activity data
- Outcome (y_i) is 5-year all cause mortality
- Functional predictor is participants log transformed activity profile:

$$z_i(s) = M_i^{-1} \sum_{m=1}^{M_i} \log(1 + \mathrm{AC}_{im}(s))$$

where $s = 1, \ldots, 1440$, $m = 1, \ldots, M_i$ denotes "good" days of data, and $\mathrm{AC}_{im}(s)$ denotes the activity count for subject i on day m at minute s

Motivating Data Set

SoFR
Estimation
In-Class Exercises

- These data are noisy, may want to smooth the data using fPCA via *refund::fpca.face()*

$$\tilde{z}_i(s) = \sum_{k=1}^{K_z} \xi_{ik}^z \phi_k^z(s)$$

- Where ξ_{ik}^z and ϕ_k^z are the scores and PCs estimated from fPCA
- Functional predictor then $\tilde{z}_i(s)$

Motivating Data Set

```
library("here"); library("readr"); library("dplyr")
data <- read_rds(here("data","data_processed","NHANES_AC_processed.rds"))
## create the functional predictor
data <-
  data %>%
  ## only consider good days of data and indiviudals age 50 or over
  filter(good_day %in% 1, Age > 50)
## get mortality data from the rnhanesdata package
library("rnhanesdata")
data_mort <- bind_rows(Mortality_2015_C, Mortality_2015_D)
str(data_mort)

## 'data.frame': 20470 obs. of  8 variables:
##  $ SEQN         : int  21005 21006 21007 21008 21009 21010 21011 21012 21013 21
##  $ eligstat     : int  1 2 2 2 1 1 2 1 2 2 ...
##  $ mortstat     : int  0 NA NA NA 0 0 NA 1 NA NA ...
##  $ permth_exm   : int  150 NA NA NA 135 149 NA 127 NA NA ...
##  $ permth_int   : int  150 NA NA NA 135 149 NA 128 NA NA ...
##  $ ucod_leading : chr  NA NA NA NA ...
##  $ diabetes_mcod: int  NA NA NA NA NA NA NA 0 NA NA ...
##  $ hyperten_mcod: int  NA NA NA NA NA NA NA 0 NA NA ...
```

Motivating Data Set

SoFR

Estimation

In-Class Exercises

```r
## merge with our data and derive 5-year mortality indicator
data      <-
  left_join(data, data_mort, by="SEQN") %>%
  mutate(mort_5yr = as.numeric(permth_exm/12 <= 5 & mortstat %in% 1),
         ## replace accidental deaths within 5 years as NA
         mort_5yr = ifelse(mort_5yr == 1 & ucod_leading %in% "004", NA, mort_5yr))
  ## drop anyone missing mortality data or who had accidental deaths within 5 year
  filter(!is.na(mort_5yr))
```

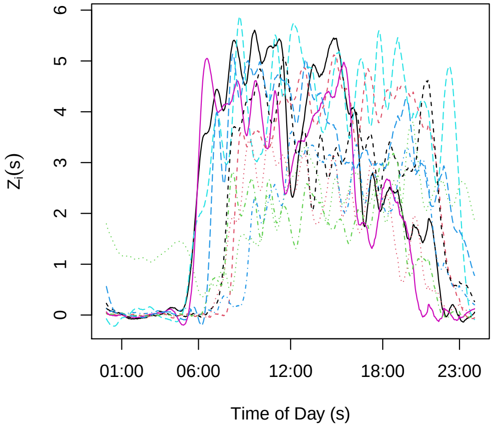
Motivating Data Set


```r
library("refund")
## extract just the activity count data
Z <- log(as.matrix(data[,paste0("MIN",1:1440)])+1)
Z[is.na(Z)] <- 0
## average across days within participants (SEQN)
uid <- unique(data$SEQN)       # unique subject identifiers
nid <- length(uid)             # number of participants
Zmat <- matrix(NA, nid, 1440) # empty container to store average profiles
inx_ls <- lapply(uid, function(x) which(data$SEQN %in% x)) # list of indices
for(i in seq_along(uid)){
  Zmat[i,] <- colMeans(Z[inx_ls[[i]],,drop=FALSE])
}
## do fpca on the log(1+AC)
fpca_Z <- fpca.face(Y=Zmat, knots=50)
Zsm    <- fpca_Z$Yhat
```

Motivating Data Set


```r
## Get a data frame for analysis which contains one row per participant
df <- data[!duplicated(data$SEQN), ]
## drop the activity count columns
df <-
  df %>%
  dplyr::select(-one_of(paste0("MIN",1:1440)))
## add in the activity count matrix using the AsIs class via I()
## note!! be careful when working with dataframes which contain matrixes
df$Zsm  <- I(Zsm)
df$Zraw <- I(Zmat)
## clean up the workspace a bit
rm(Zsm);rm(Zmat);rm(Z)
```

Motivating Data Set

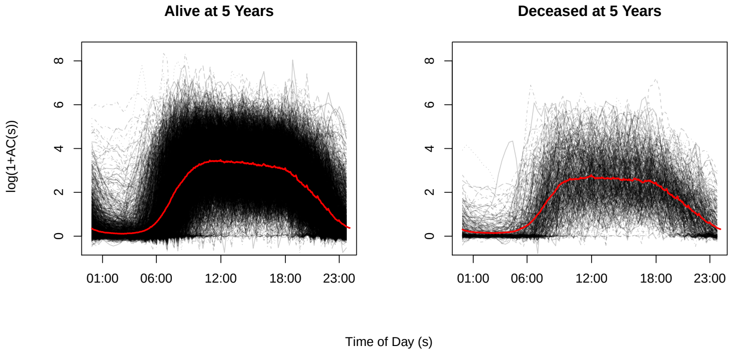


Motivating Data Set

SoFR

Estimation

In-Class Exercises



SoFR: NHANES

- We want to model the association between y and $x, z(s)$
- Naive approach:

$$g(E[y_i | x_i, \boldsymbol{z}_i]) = \alpha_0 + x_u\beta + \sum_{j=1}^{J} \gamma_j z_i(s_j)$$

or

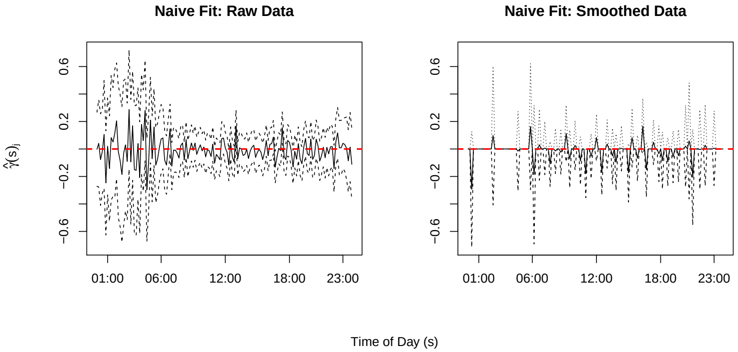
$$g(E[y_i | x_i, \boldsymbol{z}_i]) = \alpha_0 + x_u\beta + \sum_{j=1}^{J} \gamma_j \tilde{z}_i(s_j)$$

- Potential problems?

SoFR: NHANES


```
library("mgcv")
## fit on a subset of minutes (could do all 1440, just long computation time)
cols_regress <- seq(1,1440,by=10)
fit_naive_raw <- gam(mort_5yr ~ df$Zraw[,cols_regress], family=binomial, data=df)
fit_naive_sm  <- gam(mort_5yr ~ df$Zsm[,cols_regress], family=binomial, data=df)
```

SoFR: NHANES


Naive Fit: Raw Data
Naive Fit: Smoothed Data
$\hat{\gamma}(s)_j$
0.6
0.2
−0.2
−0.6
01:00
06:00
12:00
18:00
23:00
Time of Day (s)

Generalized Functional Linear Model (GFLM)

$$g(E[y_i|x_i, \boldsymbol{Z}_i]) = \alpha_0 + x_i\beta + \int_{\mathcal{S}} z_i(s)\gamma(s)ds$$

- $g(\cdot)$ is a link function
- α_0 is the intercept
- β is the linear association between x_i and y_i
- $\gamma(s)$ is the functional coefficient
    - "Linear" effect over the functional domain $\mathcal{S}$
    - Can be thought of as a weight function

GFLM

- Approximation of the integral term

$$
\begin{aligned}
g(E[y_i|x_i, \boldsymbol{Z}_i]) &= \alpha_0 + x_i\beta + \int_{\mathcal{S}} z_i(s)\gamma(s)ds \\
&= \alpha_0 + x_i\beta + \int_{\mathcal{S}} \left[\sum_{k=1}^{K_z} \xi_k^z \phi_k^z(s)\right] \left[\sum_{k=1}^{K_\gamma} \xi_k^\gamma \phi_k^\gamma(s)\right] ds \\
&\approx \alpha_0 + x_i\beta + \sum_{j=1}^{J} l(s_j) \left[\sum_{k=1}^{K_z} \xi_k^z \phi_k^z(s_j)\right] \left[\sum_{k=1}^{K_\gamma} \xi_k^\gamma \phi_k^\gamma(s_j)\right]
\end{aligned}
$$

- Where $l(s_j)$ is the quadrature weight associated with the numeric approximation method

GFLM

- Basis Expansion(s)

$$
\begin{aligned}
g(E[y_i|x_i, \boldsymbol{Z}_i]) &= \alpha_0 + x_i\beta + \int_{\mathcal{S}} z_i(s)\gamma(s)ds \\
&= \alpha_0 + x_i\beta + \int_{\mathcal{S}} \left[\sum_{k=1}^{K_z} \xi_k^z \phi_k^z(s)\right] \left[\sum_{k=1}^{K_\gamma} \xi_k^\gamma \phi_k^\gamma(s)\right] ds
\end{aligned}
$$

GFLM: fPCA Basis

- One option: use the same basis for z and γ
- Convenient: fPC basis

$$g(E[y_i|x_i, \boldsymbol{Z}_i]) = \alpha_0 + x_i\beta + \int_{\mathcal{S}} \left[\sum_{k=1}^{K_z} \xi_k^z \phi_k^z(s) \right] \left[\sum_{k=1}^{K_\gamma} \xi_k^\gamma \phi_k^\gamma(s) \right] ds$$

$$= \alpha_0 + x_i\beta + \sum_{k=1}^{K_z} \xi_{ik}^z \xi_k^\gamma$$

- (generalized) linear regression on the PC scores!
- Because $\int \phi_k^z(s)\phi_l^z(s) = 0$ if $k \neq l$ and 1 if $k = l$
- Choice of K_z becomes a tuning parameter

GFLM: fPCA Basis

SoFR

Estimation

In-Class Exercises

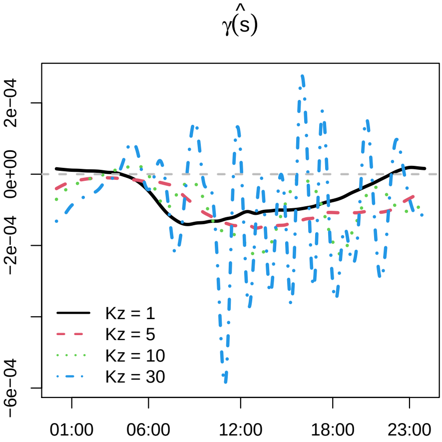
```
df$xi_z <- I(fpca_Z$scores)
Kz <- c(1,5,10,30)
coef_mat <- matrix(NA, length(Kz), 1440)
for(k in seq_along(Kz)){
  K_k    <- Kz[k]
  efuncs_k <- fpca_Z$efunctions[,1:K_k,drop=F]
  fit_k    <- gam(mort_5yr ~ df$xi_z[,1:K_k,drop=F], data=df)
  coef_mat[k,] <- efuncs_k %*% coef(fit_k)[-1]
}
```

GFLM: fPCA Basis

SoFR

Estimation

In-Class Exercises



GFLM: Penalized Splines

- "Fix" $z_i(s)$

$$
\begin{aligned}
g(E[y_i|x_i, \boldsymbol{Z}_i]) &= \alpha_0 + x_i\beta + \int_{\mathcal{S}} z_i(s)\gamma(s)ds \\
&= \alpha_0 + x_i\beta + \int_{\mathcal{S}} z_i(s)\left[\sum_{k=1}^{K_\gamma} \xi_k^\gamma \phi_k^\gamma(s)\right] ds \\
&\approx \alpha_0 + x_i\beta + \sum_{j=1}^{J} l(s_j) z_i(s_j)\left[\sum_{k=1}^{K_\gamma} \xi_k^\gamma \phi_k^\gamma(s_j)\right] \\
&= \alpha_0 + x_i\beta + \sum_{k=1}^{K_\gamma} \xi_k^\gamma \left[\sum_{j=1}^{J} l(s_j) z_i(s_j)\phi_k^\gamma(s_j)\right]
\end{aligned}
$$

- Where $l(s_j)$ is the quadrature weight associated with the numeric approximation method

GFLM: Penalized Splines


```
## set up the functional domain matrix
## mgcv will use this to construct the basis \phi_k^\gamma(s)
sind  <- seq(0,1,len=1440)
smat  <- matrix(sind, nrow(df), 1440, byrow=TRUE)
df$smat <- I(smat)
## set up the matrix of integration weights
df$lmat <- I(matrix(1/1440, nrow(df), 1440))
## multiply integration weights by the functional predictor
df$zlmat <- I(df$lmat*df$Zsm)
fit_fglm_ps  <- gam(mort_5yr ~ s(smat, by=zlmat, bs="cc",k=30), data=df,
                    method="REML", family=binomial)
```

GFLM: Penalized Splines

SoFR

Estimation

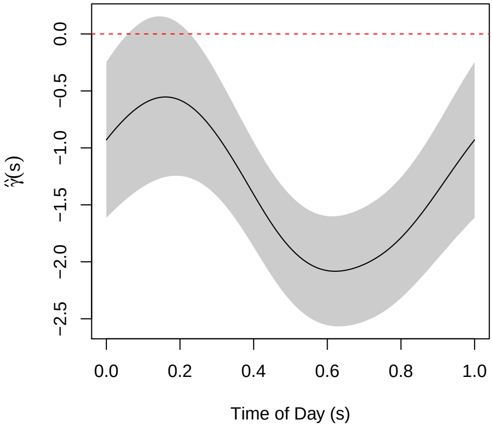
In-Class Exercises

```
summary(fit_fglm_ps)

##
## Family: binomial
## Link function: logit
##
## Formula:
## mort_5yr ~ s(smat, by = zlmat, bs = "cc", k = 30)
##
## Parametric coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept)   0.8465     0.1971   4.295 1.75e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Approximate significance of smooth terms:
##                 edf Ref.df Chi.sq p-value
## s(smat):zlmat 3.022  3.462  204.5  <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## R-sq.(adj) =  0.0804   Deviance explained = 9.64%
## -REML =   1089  Scale est. = 1         n = 3425
```

GFLM: Penalized Splines

SoFR
Estimation
In-Class Exercises



GFLM: Penalized Splines

SoFR

Estimation

In-Class Exercises

```
library("pROC")
(roc_in_sample <- roc(df$mort_5yr, fit_fglm_ps$fitted.values))

##
## Call:
## roc.default(response = df$mort_5yr, predictor = fit_fglm_ps$fitted.values)
##
## Data: fit_fglm_ps$fitted.values in 3042 controls (df$mort_5yr 0) < 383 cases (d
## Area under the curve: 0.7252

auc(roc_in_sample)

## Area under the curve: 0.7252
```

In-Class Exercises

1. Fit the unadjusted model using non-cyclic splines.
    - Do you see any differences?
    - Which model predicts better in terms of AUC?
2. Compare the fPC approach for $K_z = 1, 5, 10$ to the penalized regression approach using
    - In-sample AUC
    - 5-fold cross validation

References I