Explorations and Experiments on INLA with NHANES data

BIOS 7720

Randy

April 20, 2021

Outline

- INLA ***Integrated Nested Laplace Approximations***
- Describe the class of models INLA can be applied to
- Look at simple examples in R-INLA.

INLA

- Integrated Laplace Approximation
- Introduced by Rue, Martino and Chopin (2009).
- Posteriors are estimated using numerical approximations.
  - It is a deterministic approach to approximate
    Bayesian inference for latent Gaussian models (LGMs)
  - INLA is both faster and more accurate than MCMC
- three key components required by INLA:
  - the LGM framework
  - the Gaussian Markov random field (GMRF)
  - the Laplace approximation

Latent Gaussian Models (LGM) framework

- LGMs have a wide-ranging list of applications and most structured Bayesian models
  - Regression models, the most extensively used subset of LGMs.
  - Dynamic models, Spatial models and Spatial-temporal models
- Although the likelihood function does not have to be Gaussian,
  each latent parameter η_i must be a Gaussian given its hyperparameter in LGM.

$$\eta_i = g(\mu_i) = \beta_0 \ + \ \sum_{j=1}^{J} \beta_j x_{ij}$$

- the assumption must be hold:
  for example, if we have two parameters

$$\beta_0 \sim Normal(\mu_0,\ \sigma_0^2)$$

$$\beta_1 \sim Normal(\mu_1,\ \sigma_1^2)$$

Latent Gaussian Models (LGM) framework

- then, we have the latent effect follows Gaussian:

$$\eta_i \sim Normal(\mu_0 + \mu_1 x_{i1},\ \sigma_0^2 + \sigma_1^2 x_{i1}^2)$$

$$\boldsymbol{\eta} = (\eta_1,\ \eta_2,\ ...,\ \eta_n)^\top$$

$$\boldsymbol{\eta \sim GP\ (\ \mu,\ \Sigma\ )}$$

- extended for additive models
  - relax the assumption of linear relationship
  - introduce random effects

$$\eta_i = \beta_0\ +\ \sum_{j=1}^{J} \beta_j x_{ij}\ +\ \sum_{k=1}^{K} f_k(z_{ik})$$

Latent Gaussian Models (LGM) framework

$$\boldsymbol{y \mid \eta,\ \theta_1} \sim \prod_{i=1}^{n} p(y_i \mid \eta_i,\ \boldsymbol{\theta_1}); \quad p(\eta \mid \boldsymbol{\theta_2})\ \propto\ |\boldsymbol{Q_{\theta_2}}|_+^{1/2} exp(-\frac{1}{2}\boldsymbol{\eta}^\top \boldsymbol{Q_{\theta_2}\eta})$$

$$\pi(\boldsymbol{\eta,\ \theta \mid y}) \propto\ \pi(\boldsymbol{\theta})\ \pi(\boldsymbol{\eta \mid \theta_2}) \prod_i p(y_i \mid \boldsymbol{\eta_i},\ \boldsymbol{\theta_1})$$

$$\propto\ \pi(\boldsymbol{\theta})\ |\boldsymbol{Q_{\theta_2}}|^{1/2} exp(-\frac{1}{2}\boldsymbol{\eta}^\top \boldsymbol{Q_{\theta_2}\eta}\ +\ \sum_i log\boldsymbol{\pi}(y_i \mid \eta_i,\ \boldsymbol{\theta_1}))$$

instead of using Σ, we apply the precision matrix Q_θ

Guassian Markov Random Fields (GMRFs)

- the latent field η should not only be Gaussian
  but also Guassian Markov Random Field
  - We say η is a GMRF if it has a multivariate normal density
    with additional conditional independence
    (also called the "Markov property").
  - One common thing between different GMRFs:
    they all have a sparse precision matrix.
    * Sparse matrix provides a huge computational benefit
      when making Bayesian inference.
    * "Magic" in INLA: The joint distribution of of GMRF is also a GMRF
    * Precision matrix consists of sums of
      the precision matrices of the covariates and model components.

Additional notes AR(1)

- band matrix example AR(1)

$$\boldsymbol{Q} = \sigma_\eta^{-2} \begin{pmatrix} 1 & -\rho & & & \\ -\rho & 1+\rho^2 & -\rho & & \\ & \ddots & \ddots & \ddots & \\ & & -\rho & 1+\rho^2 & -\rho \\ & & & -\rho & 1 \end{pmatrix}$$

- conditional independent for $|i-j| > step\ 1$

Additional notes AR(1)

- conditional independent for $|i-j| > step$

$$p(\eta_2,\ \eta_4 \mid \eta_1,\ \eta_3) \ = \ p(\eta_2 \mid \eta_1)\ p(\eta_4 \mid \eta_1\ ,\ \eta_2,\ \eta_3) \ = p(\eta_2\ |\eta_1)\ p(\eta_4\ |\eta_3)$$