Gamma and Bessel functions

The variance gamma distribution

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1. **Scott Nestler & Andrew Hall (2019)**

2. **Bessel function**

The usual approach to investing

Harry Markowitz's "mean–variance" portfolio optimisation

- To maximise return on investment
- Also to constrain risk at the same time
- With risk viewed synonymously with variability and measured by variance or standard deviation
- The *return* is defined as $r_t = \ln(s_t/s_{t-1})$, where s_t represents the price of a stock at time t.
- Markowitz's model assumes that log returns of stocks and index funds are independent and identically distributed Gaussian random variables.
- However, the data are actually skewed, with higher kurtosis than would be expected: heavier tails and a more peaked center than a normal distribution.

- A few such distributions meet early mentioned criteria
  - The normal inverse Gaussian
  - The generalised hyperbolic,
  - The variance gamma (VG)
- The VG, also known as the generalised Laplace or Bessel function distribution:

$$f_X(x|\mu,\ \sigma,\ \theta,\ \nu) = \frac{2\exp(\theta(x-\mu)/\sigma^2)}{\sigma\sqrt{2\pi}\nu^{1/\nu}\Gamma(1/\nu)}\left(\frac{|x-\mu|}{\sqrt{2\theta^2/\nu+\sigma^2}}\right)^{1/\nu-1/2} K_{1/\nu-1/2}\left(\frac{|x-c|\sqrt{2\sigma^2/\nu+\theta^2}}{\sigma^2}\right)$$

$\Gamma(.)$ is the gamma function
$K_\eta(.)$ is a modified Bessel function of the third kind, of order η
$\mu \in (-\infty,\ \infty)$ is the location parameter
$\sigma \in [0,\ \infty)$ is the scale/spread parameter
$\theta \in (-\infty, \infty)$ is the asymmetry parameter
$\nu \in [0, \infty)$ is the shape parameter

What does it look like?

- A Unimodal, with a single peak, and heavy tails
- Decreases algebraically rather than decreasing exponentially
- The VG distribution is quite flexible

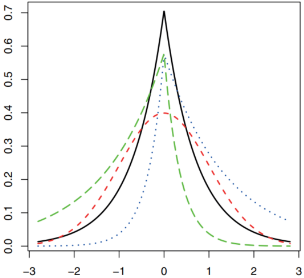


FIGURE 1 Effect of θ (asymmetry). Black curve (with $\theta = 0$) is symmetric or not skewed; blue dotted curve (with positive θ) is skewed right; green dashed curve (with negative θ) is skewed left; red dashed curve is a standard normal, for comparison.

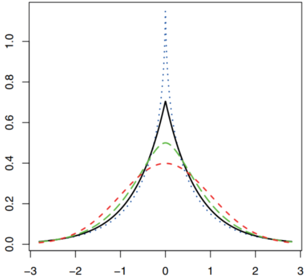


FIGURE 2 Effect of v (shape). Black curve has value of $v = 1$. Blue dotted curve shows how decreasing v adds to the peakedness, while green dashed curve increases v and is closer to the red dashed curve, which is a standard normal, for comparison.

Moments

Cannot be completely described by the first two moments, mean and variance The shape parameter ν for kurtosis to the normal distribution with a kurtosis of 3 With an assumed small skewness (θ)

The first four moments of the VG are:

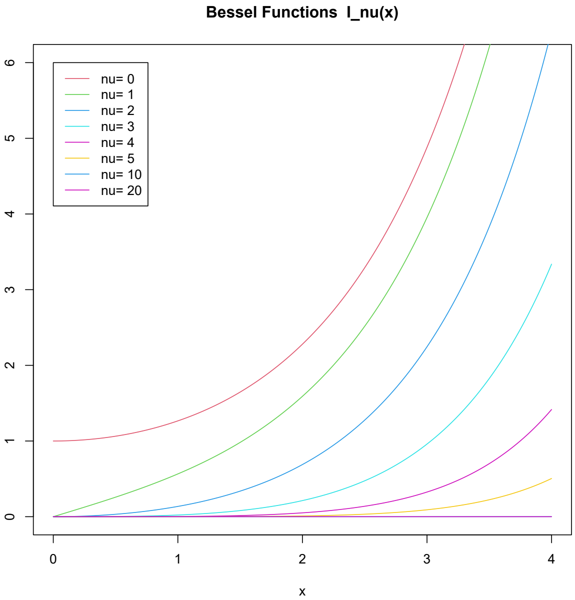
1. mean: $E[X_t] = \mu + \theta$
2. variance: $Var[X_t] = \sigma^2$
3. skewness: $\gamma[X_t] = 3\theta\nu/\sigma$
4. kurtosis: $\kappa[X_t] = 3(1 + \nu)$

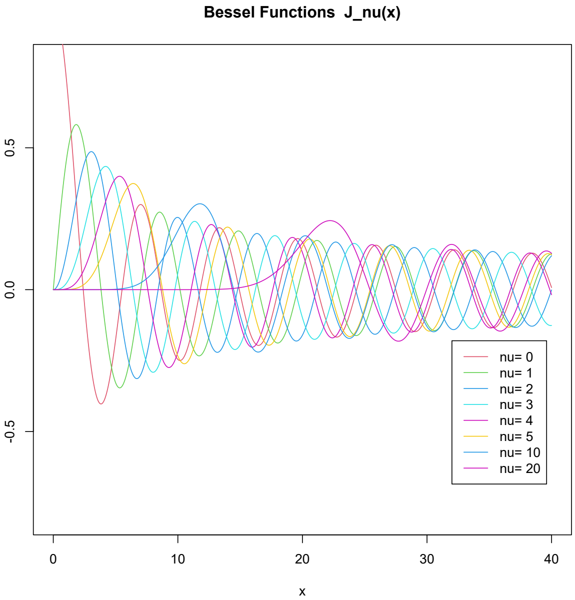
Summary

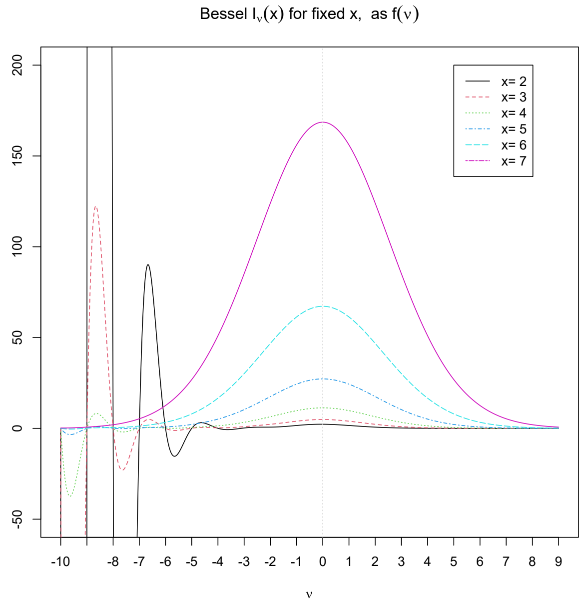
Three common ways to simulate from the VG distribution are:

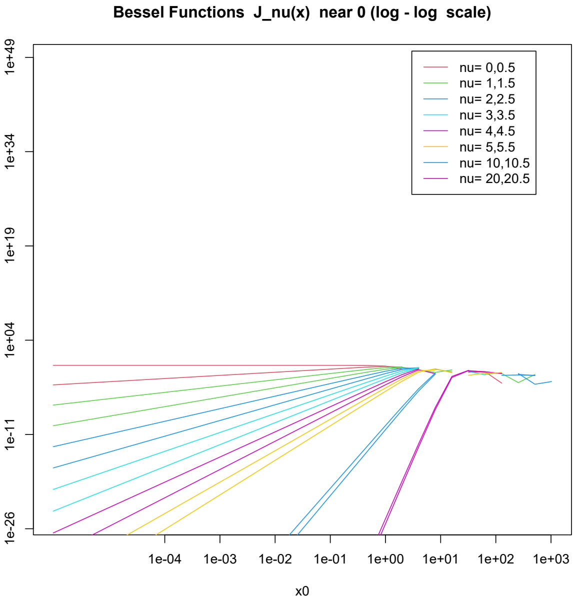
1. As a time-changed Brownian motion with a gamma
2. The difference of two non-decreasing (Gamma) processes
3. Using a compound Poisson process, for an approximation

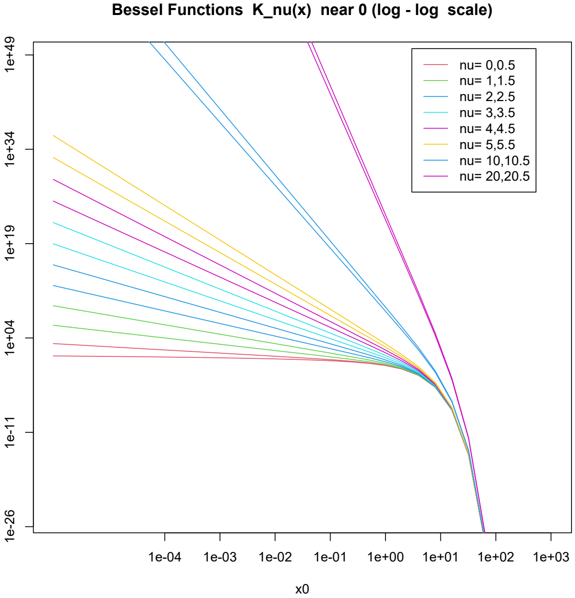
An R package, VarianceGamma (bit.ly/2NZZnJ2), includes the ability to estimate parameters from data and also simulate random variates.

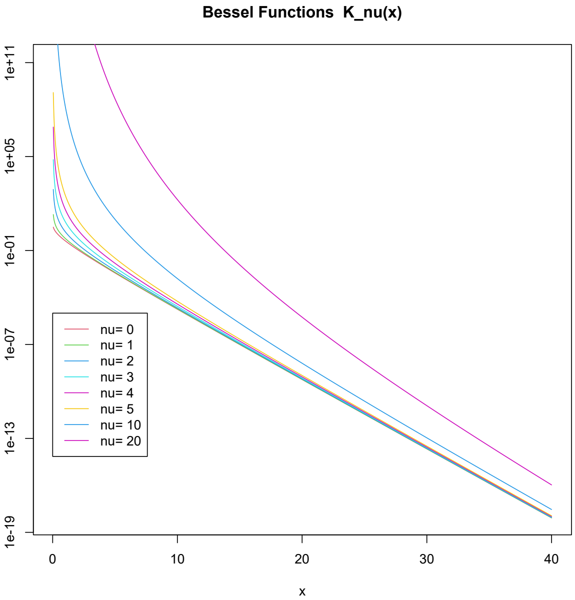

Bessel Functions I_nu(x)
nu= 0
nu= 1
nu= 2
nu= 3
nu= 4
nu= 5
nu= 10
nu= 20
0
1
2
3
4
5
6
0
1
2
3
4
x

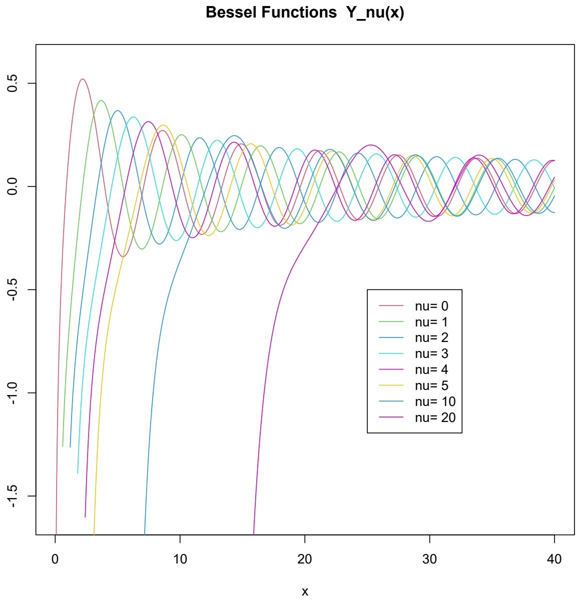

Bessel Functions J_nu(x)
0.5
0.0
-0.5
0
10
20
30
40
x
nu= 0
nu= 1
nu= 2
nu= 3
nu= 4
nu= 5
nu= 10
nu= 20

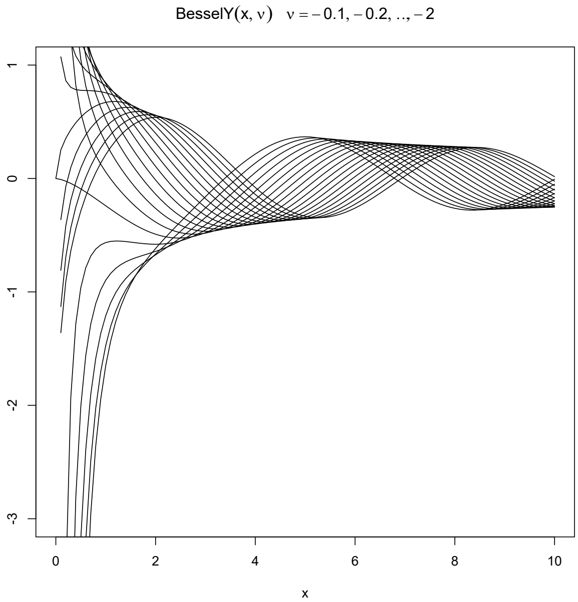

Bessel $I_\nu(x)$ for fixed x, as $f(\nu)$
x= 2
x= 3
x= 4
x= 5
x= 6
x= 7
200
150
100
50
0
-50
-10
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
ν


Bessel Functions J_nu(x) near 0 (log - log scale)
nu= 0,0.5
nu= 1,1.5
nu= 2,2.5
nu= 3,3.5
nu= 4,4.5
nu= 5,5.5
nu= 10,10.5
nu= 20,20.5
1e+49
1e+34
1e+19
1e+04
1e-11
1e-26
1e-04
1e-03
1e-02
1e-01
1e+00
1e+01
1e+02
1e+03
x0


Bessel Functions K_nu(x) near 0 (log - log scale)
nu= 0,0.5
nu= 1,1.5
nu= 2,2.5
nu= 3,3.5
nu= 4,4.5
nu= 5,5.5
nu= 10,10.5
nu= 20,20.5
1e+49
1e+34
1e+19
1e+04
1e-11
1e-26
1e-04
1e-03
1e-02
1e-01
1e+00
1e+01
1e+02
1e+03
x0


Bessel Functions K_nu(x)
1e+11
1e+05
1e-01
1e-07
1e-13
1e-19
0
10
20
30
40
x
nu= 0
nu= 1
nu= 2
nu= 3
nu= 4
nu= 5
nu= 10
nu= 20


Bessel Functions Y_nu(x)
0.5
0.0
-0.5
-1.0
-1.5
0
10
20
30
40
x
nu= 0
nu= 1
nu= 2
nu= 3
nu= 4
nu= 5
nu= 10
nu= 20



BesselY(x, ν) ν = −0.1, −0.2, .., −2
1
0
-1
-2
-3
0
2
4
6
8
10
x