Equations

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$$
\begin{aligned}
Y_i|X_i, b_i \sim N\Big(\mu_i = X^T\beta,\ V_i = X_i^T G X_i + R_i\Big) \\
Y_i = X_i^T\beta + X_i^T b_i + \epsilon = X_i^T(\beta + b_i) + \epsilon_i = X_i^T\gamma_i + \epsilon_i
\end{aligned}
$$

$$
\begin{aligned}
& Y_i = X_i^T\beta + X_i^T b + \epsilon_i = X_i^T(\beta + b_i) + \epsilon_i = X_i^T\gamma_i + \epsilon_i \\
& \dot{y}_i = X_i^T\gamma_i \\
& \ddot{y}_i = Z_i^T\alpha + e_i
\end{aligned}
$$

we can assume ϵ is independent, but e is dependent across different time with in individuals

based on linear mixed model, the BLUP: $\dot{y} = x_*^T(X^TV^{-1}X)^{-1}X^TV^{-1}Y$

based on linear regression BLE: $\ddot{y} = Z^T(Z^TZ)^{-1}Z^T\dot{y} = H^T\dot{y}$, $H^T = Z^T(Z^TZ)^{-1}Z^T$

then $\ddot{y} = Z^T(Z^TZ)^{-1}Z^Tx_*^T(X^TV^{-1}X)^{-1}X^TV^{-1}Y = W^TY$, $W^T = Z^T(Z^TZ)^{-1}Z^Tx_*^T(X^TV^{-1}X)^{-1}X^TV^{-1}$ is a linear transformation from Y to $\ddot{y}$.

Assuming that $Y - X^T\beta \sim N(0, V = X^TGX + R)$, however there is violation on the independence, that $\ddot{y} = W^TY \sim N(W^TX^T\beta, \Sigma \neq W^TVW)$.

Also we have estimations from brokenstick model:

$$
\begin{aligned}
\hat{\beta} &= (X^TV^{-1}X)^{-1}X^TV^{-1}Y \\
V[\hat{\beta}] &= (X^TV^{-1}X)^{-1} \\
V[\hat{y}] &= x_*^T(X^TV^{-1}X)^{-1}x_* \\
\hat{b} &= \hat{G}X\hat{V}^{-1}(Y - X^T\hat{\beta}) \\
V[\hat{b}_i - b_i] &= G - \hat{G}XV^{-1}X^TG + GXV^{-1}X^T(XV^{-1}X)^{-1}X^TV^{-1}XG
\end{aligned}
$$

Hence

\hat ydot

$$
\begin{aligned}
\hat{Y}_i &= X_i^T\beta + X_i^T\hat{b}_i \\
&= X_i^T\beta + X_i^T\hat{G}_iX_i\hat{V}_i^{-1}(Y_i - X_i^T\beta) \\
&= (I - X_i^T\hat{G}_iX_i\hat{V}_i^{-1})X_i^T\beta + X_i^T\hat{G}_iX_i\hat{V}_i^{-1}Y_i \\
&= (\hat{R}_i\hat{V}_i^{-1})X_i^T\beta + (1 - \hat{R}_iV_i^{-1})Y_i
\end{aligned}
$$

the prediction of ydot is bended over the population mean related to the propotional to the R / (R + XGX)

and linear model $E[\dot{y}] = Z^T\alpha$ with plug-in:

$$\hat{\alpha} = (Z^T Z)^{-1} Z^T \dot{y}$$

$$V[\hat{\alpha}] = (Z^T Z)^{-1}$$

this is the one we are sampling from

Then we have the second estimations for $V[\alpha]$ and $V[\ddot{y}]$, both :

$$V[\tilde{\alpha}] = (Z^{-1})^T (\dot{y} - \ddot{y})^T (\dot{y} - \ddot{y}) Z^{-1}$$

$$= (Z^{-1})^T \Big(x_*^T (X^T V^{-1} X)^{-1} X^T V^{-1} Y - W^T Y \Big)^T \Big(x_*^T (X^T V^{-1} X)^{-1} X^T V^{-1} Y - W^T Y \Big) Z^{-1}$$