Unconstrained optimization

1 Do the exercises left in the regular classes.

2 Find the points satisfying necessary conditions for extrema of the function

$$f(x_1, x_2) = \frac{x_1 + x_2}{3 + x_1^2 + x_2^2 + x_1 x_2}.$$

Check the sufficient conditions to decide whether they are extrema or not.

3 Investigate whether the origin is an extremum of the function

$$f(x_1, x_2, x_3) = \alpha x_1^2 \exp(x_2) + x_2^2 \exp(x_3) + x_3^2 \exp(x_1)$$

4 Consider the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ where A is an $m \times n$ matrix and $\boldsymbol{b} \in \mathbb{R}^m$. Assume $m < n$ and that A has rank m. Find an explicit solution in terms of A and $\boldsymbol{b}$ to that solution $\boldsymbol{x}^\star$ of the linear equation for which $\boldsymbol{x}^T\boldsymbol{x}$ is minimal. You can check your result by showing that $\boldsymbol{x}^\star = (41, -9, -1)^T$ is indeed such a solution of

$$\begin{pmatrix} 2 & -1 & 5 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 86 \\ 43 \end{pmatrix}.$$

5 Find the shortest distance from the point $\boldsymbol{x} = (1, 0)$ to the curve $4x_1 - x_2^2 = 0$. Use the direct elimination method and Lagrange method. Same answer?

6 A company produce two kind of dishwashers. Denote by x and y the unities produced (and sold in the market) of each type. The revenue and total cost functions are (in millions of euros):

$$\begin{aligned} I(x, y) &= 2x + 3y \\ C(x, y) &= x^2 - 2xy + 2y^2 + 6x - 9y + 5 \end{aligned}$$

How many dishwashers of each type the company should produce to maximize the profit? Which is the profit for these pair?

7 The calculators market is a duopoly (homogeneous good). In the suitable unities the demand function is given by $Q(p) = 100 - Q$ where p is the unitary price and $Q = q_1 + q_2$ are the unities of each company (A and B) sold in the market. Suppose that the cost functions are $C(q_1) = 2q_1$ and $C(q_2) = 3q_2$, respectively.

(a) Give the profit of the companies if their production is $q_1 = 10$ and $q_2 = 20$.

(b) If company A knows *in advance* that company B is producing $q_2 = 20$ unities and it acts rationally which is the unities he will produce? Which is the profit of the two companies in this case?

(c) Find the Nash equilibrium (i.e., the pair $(q_1^\star, q_2^\star)$ such that no one of the companies wants to deviate unilaterally).

8 Solve the problem (complementary goods in a market).

$$\max_{(x,y)\in\mathbb{R}^2} \min\{x,y\} \quad \text{subject to} \quad x+2y=1.$$

9 Solve the optimization problem

$$f(x,y) = x^2+y^2 \quad \text{subject to} \quad g(x,y) = x^2+xy+y^2-3 = 0.$$

10 Consider

$$\max_{(x_1,x_2)\in\mathbb{R}^2} f(x_1,x_2) \quad \text{subject to} \quad g(x_1,x_2) = c,$$

where $c \in \mathbb{R}$ is a parameter. Suppose the problem admits a solution $(\boldsymbol{x}^\star(c), \lambda^\star(c))$ for every c (at least in some open interval for c). Define $f^\star(c) = f\left(x_1^\star(c), x_2^\star(c)\right)$. Prove that

$$\frac{df^\star(c)}{dc} = \lambda^\star(c).$$

Consider the problem above corresponds to the maximization of the utility function of an individual consumer (that is, (x,y) represents the basket of consume of two goods) with budget constrain $g(x,y) = p_x x + p_y y = m$

11 Solve

$$\max(\min) x+y+z \quad \text{subject to} \quad \begin{cases} x^2+y^2+z^2 = 1 \\ x-y-z = 1 \end{cases}$$

12 Let $\in \mathbb{R}^2$ and A a 2×2 symmetric matrix. Consider $f(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x}$. Solve the problem

$$\max f(\boldsymbol{x}) \quad \text{subject to} \quad \boldsymbol{x}^T\boldsymbol{x} = 1.$$

In particular, prove that if $(\boldsymbol{x}^\star, \lambda^\star)$ is a solution of the problem then $\boldsymbol{x}^\star$ is an eigenvector of A and $\lambda^\star$ it the corresponding eigenvalue. Prove that $\lambda^\star = \boldsymbol{x}^\star A\boldsymbol{x}^\star$. Show that the arguments extend to A being an $n\times n$ symmetric matrix.

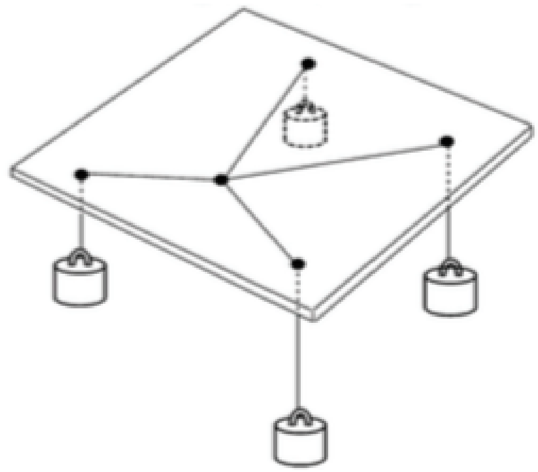
13 (The Fermat point of a set of points)

Given set of points $\boldsymbol{y}_1,\ldots,\boldsymbol{y}_m$ in the plane, we want to find a point $\boldsymbol{x}^*$ whose sum of weighted distances to the given set of points is minimized. Mathematically, the problem is

$$\min \sum_{j=1}^{m} w_j \|\boldsymbol{x}^* - \boldsymbol{y}_j\|, \quad \left[\text{subject to } \boldsymbol{x}^* \in \mathbb{R}^2\right],$$

where $w_1, \ldots, w_m$ are given positive real numbers.

(a) Show that there exists a global minimum for this problem. Argue that the problem can be viewed as the mechanical model shown in the following figure (where $x^\star$ being the *central point of the network* and each other point y_j supporting a mass w_j):



(b) Is the optimal solution always unique (convexity)?

(c) Show that an optimal solution minimizes the potential energy of the mechanical model defined as

$$\sum_{i=1}^{m} w_i h_i,$$

where h_i is the height of the i-th weight measured from some reference level.