Master in Foundations of Data Science — 2020-2021

NUMERICAL LINEAR ALGEBRA

Exercises from previous exams on *linear equation solving*, the *least square problem* and the *singular value decomposition.*

1. Let ε be a small positive real number and consider the matrix and the vector

$$A = \begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(1) Compute the condition number with respect to the ∞-norm of A and of the matrices in its LU factorization.
(2) Suppose that ε is smaller than the machine precision, so that $1 \oplus \varepsilon = 1$. Show that Gaussian elimination without pivoting is a numerically unstable algorithm for solving the linear equation $A\,x = b$.
(3) Show that solving $A\,x = b$ using Gaussian elimination with partial pivoting is numerically stable.

2. Given a symmetric and positive definite matrix $A \in \mathbb{R}^{n\times n}$, *Cholesky's algorithm* computes a factorization

$$A = L \cdot L^T$$

where L is a lower triangular matrix with positive diagonal entries.
(1) Write down Cholesky's algorithm in pseudocode notation and describe how it works.
(2) Using this pseudocode, derive a bound for the complexity of this algorithm in terms of floating point operations *(flops)*.
(3) Describe how you proceed to solve $Ax = b$, once computed the factorization $A = L \cdot L^T$.
(4) Compute the Cholesky factorization of the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 10 & 3 \\ 0 & 3 & 5 \end{pmatrix}$$

3. Let $A \in \mathbb{R}^{m\times n}$ be a matrix of rank n and $b \in \mathbb{R}^m$.
(1) Explain how to use the QR factorization of A to solve the least square problem (LSP) that asks to find the vector $x_{\min} \in \mathbb{R}^n$ minimizing the quantity $\|A\,x - b\|_2$ for $x \in \mathbb{R}^n$, and give the expression for the residual error $\|A\,x_{\min} - b\|_2$.
(2) Find the affine function $\ell(x) = \alpha x + \beta$ whose graph fits better the points $(-1, 1)$, $(0, 0)$ and $(1, 1)$, in the sense that the Euclidean norm of the vector

$$\big(\ell(-1) - 1, \ell(0) + 1, \ell(1) - 0\big) \in \mathbb{R}^3$$

is minimal among all possible choices of $\alpha, \beta \in \mathbb{R}$.

4. Consider the matrix

$$A = \begin{pmatrix} 4 & 1 \\ -2 & -1 \end{pmatrix}$$

(1) Compute its QR factorization using Householder reflexions.
(2) Compute the same factorization, but this time using Givens rotations instead of reflexions.

5. Consider the *singular value decomposition (SVD)*

$$A = \begin{pmatrix} 1 & 1 & 0.41 \\ -1 & 0 & 0.41 \\ 0 & 1 & -0.41 \end{pmatrix} = \begin{pmatrix} -0.82 & 0 & -0.58 \\ 0.41 & -0.71 & -0.58 \\ -0.41 & -0.71 & 0.58 \end{pmatrix} \cdot \begin{pmatrix} 1.73 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.71 \end{pmatrix} \cdot \begin{pmatrix} -0.71 & -0.71 & 0 \\ 0.71 & -0.71 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(1) Compute the condition number of the matrix A with respect to the 2-norm.
(2) Compute the best rank 1 and rank 2 approximations of A with respect to the same norm, and determine the distance to A of these approximations.

6. Let $A \in \mathbb{R}^{2\times 2}$ such that the eigenvalues of $A \cdot A^T$ are 9 and $\frac{1}{4}$ with respective eigenvectors

$$\begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix} \approx \begin{pmatrix} 0.71 \\ -0.71 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \approx \begin{pmatrix} 0.71 \\ 0.71 \end{pmatrix},$$

and the eigenvalues of $A^T \cdot A$ are 9 and $\frac{1}{4}$ with respective eigenvectors

$$\begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \end{pmatrix} \approx \begin{pmatrix} 0.50 \\ -0.87 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \approx \begin{pmatrix} 0.87 \\ 0.50 \end{pmatrix}.$$

(1) Compute a singular value decomposition (SVD) of A.
(2) Using this SVD, compute condition number of A with respect to the 2-norm.
(3) Determine the image of the unit disk of $\mathbb{R}^2$ under the linear map defined by A.