Problems set 2

1 Consider the problem

$$\min_{\boldsymbol{x}\in\mathbb{R}^2} f(\boldsymbol{x}) = -x_1$$

subject to

$$\begin{aligned} h_1(\boldsymbol{x}) &= (1-x_1)^3 - x_2 \geq 0 \\ h_2(\boldsymbol{x}) &= x_1 \geq 0, \\ h_3(\boldsymbol{x}) &= x_2 \geq 0 \end{aligned}$$

Prove that $\mathcal{Z}^1(\boldsymbol{x}^\star) \cap \mathcal{Z}^2(\boldsymbol{x}^\star) \neq \emptyset$.

2 Discuss the following optimization problem in terms of the parameter $\beta > 0$.

$$\min_{\boldsymbol{x}\in\mathbb{R}^2} f(\boldsymbol{x}) = (x_1-1)^2 + x_2^2$$

subject to

$$h(x_1, x_2) = -x_1 + \beta x_2^2 \geq 0$$

Interpret the solutions geometrically.

3 If $\left(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\mu}}\right)$ is a solution of $(\mathcal{S})$ then $\hat{x}$ is a solution of $(\mathcal{P})$. Fulfil the details from the notes in class.

4 Prove that the intersection of an arbitrary family of convex sets is also a convex set.

5 Denote by $\mathcal{C}(G)$ the convex hull of an arbitrary set $G \subset \mathbb{R}^n$. Compute $\mathcal{C}(G)$ for

- $G = \{\cup_{n=1}^{3}(x_n, y_n)\} \subset \mathbb{R}^2$.
- Set $G_1 = \{z = (x, y) \in \mathbb{R}^2 \mid |z| < 1\}$, G_2 to be the triangle whose vertices are the points $A = (-2, 0)$, $B = (2, 0)$ and $C = (0, 2)$, $G_3 = \{z = (x, y) \in \mathbb{R}^2 \mid x = -2 \text{ and } 1 \leq y \leq 2\}$. Consider $G = \cup_{n=1}^{3} G_n$.

6 Consider the two (sufficient decrease and curvature) conditions defining Wolfe's conditions (see notes in class). The conditions refer to two constants c_1 (sufficient decrease) and c_2 (curvature) with $c_1, c_2 \in (0, 1)$. We proved in class that if $0 < c_1 < c_2 < 1$ then there is an interval of step lengths (α_k's) which satisfies Wolfe's conditions. Prove that if we have $0 < c_2 < c_1 < 1$ there may be no step lengths that satisfies the Wolfe conditions.

7 Consider the Newton-like method $p_k = -B_k^{-1} \nabla f(x_k)$ where $B_k,\ k \geq 1$ is a symmetric non singular matrix given by either $Hf(x_k)$ or some approximate matrix. Assume $B_k,\ k \geq 1$ are positive definite and there exists $M > 0$ such that $||B_k||\ ||B_k^{-1}|| \leq M$ (uniform bounded condition number). Prove that $\cos(\theta_k) \geq M$ where

$$\cos(\theta_k) = -\frac{(\nabla f(x_k))^T\ p_k}{||\nabla f(x_k)||\ ||p_k||}.$$

Hint: Look for a definition of $||B||$ and its properties. Prove then that $||Bx|| \geq ||x||/||B^{-1}||$ for any non-singular matrix B.

8 Proof that for any $a \in \mathbb{R}$, the function $f(x) = \exp(ax)$ is convex in the whole domain $\mathbb{R}$.

9 Let f be a real valued function defined in $\mathbb{R}^n$. Let $\boldsymbol{x_0}, \boldsymbol{z} \in \mathbb{R}^n$ and $\theta \in \mathbb{R}$. Define $\Phi(\theta) = f(\boldsymbol{x}_0 + \theta \boldsymbol{z})$ and suppose we are looking for the minimum of Φ (that is, for the minimum of f in the direction $\boldsymbol{z}$ through the point $\boldsymbol{x}_0$). Let $\boldsymbol{x}_0 + \theta_j \boldsymbol{z}$ be three points where f is evaluated. Consider the quadratic approximation of Φ

$$\hat{\Phi}(\theta) = a + b\theta + c\theta^2,\ a, b, c \in \mathbb{R}.$$

Prove that the minimum predicted by applying the quadratic approximation method (that is substituting Φ by $\hat{\Phi}$) is achieved at the value

$$\theta^\star = \frac{(\theta_2^2 - \theta_3^2)\,\Phi(\theta_1) + (\theta_3^2 - \theta_1^2)\,\Phi(\theta_2) + (\theta_1^2 - \theta_2^2)\,\Phi(\theta_3)}{2\left[(\theta_2 - \theta_3)\,\Phi(\theta_1) + (\theta_3 - \theta_1)\,\Phi(\theta_2) + (\theta_1 - \theta_2)\,\Phi(\theta_3)\right]}.$$

Moreover the point $\theta^\star$ is indeed a minimum if

$$\frac{(\theta_2 - \theta_3)\,\Phi(\theta_1) + (\theta_3 - \theta_1)\,\Phi(\theta_2) + (\theta_1 - \theta_2)\,\Phi(\theta_3)}{2\left[(\theta_2 - \theta_3) + (\theta_3 - \theta_1) + (\theta_1 - \theta_2)\right]} < 0$$

10 Let f be a convex function. Prove that the set of global minimizers of f forms a convex set.

11 Consider the function $f(x_1, x_2) = (x_1 + x_2^2)^2$. At the point $\boldsymbol{x}_0^T = (1, 0)$ we consider the direction $\boldsymbol{p}^T = (-1, 1)$. Show that p^T is a descent direction and find all minimizers of the problem

$$\min_{\alpha \in \mathbb{R}^+} f(\boldsymbol{x} + \alpha \boldsymbol{p}).$$

12 Let p be a polynomial of degree d (in one variable $x \in \mathbb{R}$). Consider $N_p(x)$ the Newton method applied to the polynomial p

$$N_p(x) = x - \frac{p(x)}{p'(x)}.$$

(a) Prove that α is a zero of p if and only if it is a fixed point of N_p. *Hint: Notice that p can be written as*

$$p(x) = q(x) \prod_{j=1}^{k} (x - \alpha_j)^{m_j}$$

where $q(x) \neq 0,\ x \in \mathbb{R},\ \alpha_j \in \mathbb{R},\ m_j \in \mathbb{N}$ and $k \leq d$.

(b) Prove that if α is a simple root of p then $N_p'(\alpha) = 0$. Use this fact to prove that for any simple root α of p the Newton method always converges in a sufficiently small neighbourhood of α.

(c) Prove that if α is a simple root of p then the convergence is (precisely) quadratic.

13 Consider the polynomial

$$p(x) = -18.8496 - 143.588x + 128.148x^2 + 113.355x^3 - 144.125x^4 + 24.7208x^5 + 19.634x^6 - 8.68x^7 + x^8$$

We know it has all roots belonging to $\mathbb{R}$. Find intervals $[a_j, b_j],\ j = 1, \dots 8$ in which there is one and only one root. Program a Newton mehtod to compute all roots with an error less than 10^{-6}.

14 (This is about rate of convergence) We say that a sequence $\{\boldsymbol{x}_k\}_{k\geq 0}, \boldsymbol{x}_k \in \mathbb{R}^n$ converges *quasilinearly* to $\boldsymbol{x}^\star$ if

$$\lim_{k\to\infty} \frac{||\boldsymbol{x}_{k+1} - \boldsymbol{x}^\star||}{||\boldsymbol{x}_k - \boldsymbol{x}^\star||} = 0.$$

(a) Is the sequence (of real numbers) $\{x_k = \frac{1}{k!}\}$ converging superlinearly to $x^\star = 0$?

(b) Prove that the sequence $x_k = 1 + \left(\frac{1}{2}\right)^{2^k}$ converge quadratically to 1.

15 Let $f(x) = x^2 + \exp(x) - 3$.

(a) Prove that $f(x) = 0$ has two (and only two) solutions.

(b) Find appropriate fixed point methods (different from Newton method) to solve the equation (show that the methods you have found satisfy the assumptions of the fixed point theorem (see notes in class).

(c) Predict the number of iterates you should use to get the solutions (for each case) with an error less than 10^{-6}.

(d) Do a program to compute the solutions and discuss the previous item.

16 Let $f(x) = x^2 + \exp(x) - 3$.

(a) Prove that $f(x) = 0$ has two (and only two) solutions.

(b) Find intervals $[a_j, b_j],\ j = 1, 2$ where there is just one zero of f.

(c) Use Newton method to compute the solutions above and show that the sequence of iterates converges quadratically.