Master in Foundations of Data Science — 2021-2022

NUMERICAL LINEAR ALGEBRA

Final exam, 24 January 2021 from 15:00h till 18:00h

Exercises should be delivered in separated pages, and all answers should be suitably justified.

R **1.** Consider the matrix

$$A = \begin{bmatrix} 2 & -4 & 0 \\ -4 & 7 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

R (1) Compute the PLU factorization of A produced by the GEPP algorithm.
R (2) Compute the LDL^T factorization of A.
R (3) Explain how you would solve the equation $Ax = b$ for the vector $b = (0, 2, -1)^T$ using each of these factorizations, and compute the solution vector x using one of them.

2. A Schur factorization of the matrix in Exercise 1 is $A = UTU^T$ with

$$U = \begin{bmatrix} -0.61 & -0.48 & 0.63 \\ -0.25 & 0.87 & 0.42 \\ 0.75 & -0.09 & 0.65 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0.34 & 0 & 0 \\ 0 & 9.30 & 0 \\ 0 & 0 & -0.64 \end{bmatrix}$$

up to two decimal digits.

(1) Compute the eigenvalues and eigenvectors of A from this factorization. Is A a positive-definite symmetric matrix?
(2) Determine the rate of convergence of the power method applied to the matrix A.

3. Let A be again the matrix in Exercise 1.

(1) Compute a singular value decomposition (SVD) of A using the data obtained in Exercise 2(1).
(2) Using this SVD, compute the 2-norm and the Frobenius norm of A.
(3) Compute the best rank 1 approximation to A with respect to the 2-norm, and determine its distance to A. Repeat this task replacing the 2-norm by the Frobenius norm.

4. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

(1) For this matrix and vector, write down the iterative scheme given by the Jacobi method, the Gauss-Seidel method, and the SOR(ω) method for a parameter $\omega \in \mathbb{R}$.
(2) Using the criterium based on the spectral radius, check if you can guarantee if the Jacobi and the Gauss-Seidel methods for A and b converge for any choice of initial vector $x_0 \in \mathbb{R}^2$.
(3) Determine for which values of $\omega \in \mathbb{R}$ we can guarantee the convergence of the SOR(ω) method for A and b, for any choice of initial vector x_0.