Master in Foundations of Data Science — 2016-2017

NUMERICAL LINEAR ALGEBRA

Final exam, January 21th, 2019, from 15:00h till 19:00h, at room B1

1. Given a symmetric and positive definite matrix $A \in \mathbb{R}^{n\times n}$, *Cholesky's algorithm* computes a factorization

$$A = L \cdot L^T$$

where L is a lower triangular matrix with positive diagonal entries.

(1) Write down Cholesky's algorithm in pseudocode notation and describe how it works.
(2) Using this pseudocode, derive a bound for the complexity of this algorithm in terms of floating point operations *(flops)*.
(3) Compute the Cholesky factorization of the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 10 & 3 \\ 0 & 3 & 5 \end{pmatrix}$$

2. Let $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ of rank n and $b \in \mathbb{R}^m$. The *least square problem (LSP)* asks to find the vector $x_{\min} \in \mathbb{R}^n$ minimizing the quantity $\|Ax - b\|_2$ among all $x \in \mathbb{R}^n$.

(1) Explain how to use the QR factorization of A to solve the LSP and give the expression of the residual error.
(2) Find the affine function $\ell(x) = \alpha x + \beta$ whose graph fits better the points $(-1, 1)$, $(0, 0)$ and $(1, 1)$, in the sense that the Euclidean norm of the vector

$$\big(\ell(-1) - 1, \ell(0) + 1, \ell(1) - 0\big) \in \mathbb{R}^3$$

is minimal among all possible choices of $\alpha, \beta \in \mathbb{R}$.

3. Consider the *singular value decomposition (SVD)*

$$A = \begin{pmatrix} 1 & 1 & 0.41 \\ -1 & 0 & 0.41 \\ 0 & 1 & -0.41 \end{pmatrix} = \begin{pmatrix} -0.82 & 0 & -0.58 \\ 0.41 & -0.71 & -0.58 \\ -0.41 & -0.71 & 0.58 \end{pmatrix} \cdot \begin{pmatrix} 1.73 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.71 \end{pmatrix} \cdot \begin{pmatrix} -0.71 & -0.71 & 0 \\ 0.71 & -0.71 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(1) Compute the condition number of the matrix A with respect to the 2-norm.
(2) Compute the best rank 1 and rank 2 approximations of A with respect to the same norm, and determine the distance to A of these approximations.

4. A matrix $H = (h_{i,j})_{i,j} \in \mathbb{R}^{n\times n}$ is *upper Hessenberg* if all its coefficients below the lower secondary diagonal are zero, that is, if $h_{i,j} = 0$ whenever $i \geq j + 2$.

Show that the Hessenberg form is preserved under the QR iteration algorithm.

5. Consider the matrix and the vector

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(1) Compute the corresponding *iterative scheme* for the Jacobi method and for the Gauss-Seidel method.
(2) Show that for any choice of starting point $x_0 \in \mathbb{R}^2$, both iterative schemes converge to the solution of the equation $A\,x = b$.
(3) Choosing the starting point $x_0 = (0, 0)$, how many iterations of each of these schemes are necessary to compute 10 decimal digits of the coordinates of this solution? And how many if we want to compute 30 decimal digits?