Metrika (2011) 74:297–311
DOI 10.1007/s00184-010-0304-x

Change point test for tail index for dependent data

Moosup Kim · Sangyeol Lee

Received: 6 August 2009 / Published online: 27 March 2010

Abstract To test for the constancy of tail index, Quintos et al. (Rev Econ Stud 68:633–663, 2001) proposed three types of change point tests for independent and ARCH type sequences. In this paper, we demonstrate that their tests can be successfully extended to a large class of dependent stationary sequences. Further, we designate a time-reverse version of those tests since the original tests produce very low powers in case the tail of distribution gets thinner. A simulation study is implemented for illustration.

Keywords Tail index · Regular variation · Hill's estimator · Change point test

1 Introduction

Tail index estimation has received much attention from researchers in statistics, finance, reliability, and teletraffic engineering for past decades since heavy tailed distributions with a regularly varying tail are well fitted to data in these fields. The tail index is regarded as a beacon index in extreme value theory since it determines asymptotic distributions of many statistics such as the sample maximum. In many situations, it is assumed that the tail indices are constant in the whole observed time domain. However, empirical findings suggest that it is not always true especially in the analysis of financial time series since time series frequently suffer from the changes of their underlying models due to the changes of monetary policies and critical social events. Motivated by this viewpoint, we are led to study the change point test for the tail index. For details regarding the change point test for tail index and financial time

M. Kim · S. Lee (✉)
Department of Statistics, Seoul National University,
Seoul 151-742, Korea
e-mail: sylee@stats.snu.ac.kr

series, we refer to Kim and Lee (2009) and the papers cited therein. Further, as a relevant reference, we refer to Raimondo and Tajvidi (2004), who studied the change point test to detect the discontinuities of mean functions in nonparametric regression models. Their test is based on a combination of the wavelet methods and extreme value theory and includes the extreme value index estimation in the tail of the error distribution. See also Gijbels et al. (1999).

Quintos et al. (2001) proposed three types of change point tests for the tail index for independent and ARCH type observations. We extend their method to a large class of dependent stationary sequences which includes ARMA and ARCH type processes. As a matter of fact, the three types of the change point tests are called recursive, rolling, and sequential, respectively. These tests examine the difference between Hill's estimates obtained from different time domains. It is well known that the recursive type test outperforms the others in that it is stable and produce higher powers. However, it also has a drawback that it can not well detect a change point when the tail of distribution gets thinner, in other words, the tail index gets larger. This motivates us to consider a time-reverse version change point test.

The organization of this paper as follows. In Sect. 2, we verify that the test statistics multiplied by some constant have the same asymptotic null distribution as for i.i.d. sequences. In Sect. 3, we carry out a simulation study to evaluate the performance of the tests. In Sect. 4, we provide the proofs for the results presented in Sect. 2.

2 Main results

Let $\{X_i\}$ be a sequence of positive random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Moreover, we suppose that each marginal distribution of X_i, say F_i, has regularly varying tail with the exponent $-\alpha_i$, viz., $\bar{F}_i := 1 - F_i$ satisfies

$$\bar{F}_i(x) = x^{-\alpha_i} L_i(x),$$

where L_i are slowly varying at ∞:

$$\lim_{x\to\infty} \frac{L_i(\lambda x)}{L_i(x)} = 1 \quad \text{for every } \lambda > 0.$$

Suppose that one wishes to test

$$\begin{aligned} &\mathcal{H}_0 : \text{the tail index } \alpha_i \text{ does not vary over } X_1, \ldots, X_n \qquad \text{vs.} \\ &\mathcal{H}_1 : \text{not } \mathcal{H}_0. \end{aligned}$$

Actually, under $\mathcal{H}_0$ we assume that $\{X_i\}$ is a stationary sequence. Unless, we can not go ahead. Let $k = k_n$ be positive integers satisfying that

$$k \to \infty \quad \text{and} \quad k = o(n) \quad \text{as } n \to \infty,$$

which indicates the number of upper extreme observations used in the tail index estimation. Letting $X_{(k,s,t)}$ denote the $(\lfloor k(t-s)\rfloor+1)$-th largest value of $\{X_{\lfloor ns\rfloor+1}, \ldots, X_{\lfloor nt\rfloor}\}$, we define

$$H_n(s,t) := \frac{1}{k(t-s)} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left(\log X_i - \log X_{(k,s,t)}\right)_+ ,$$

which is Hill's estimator based on $X_{\lfloor ns\rfloor+1}, \cdots, X_{\lfloor nt\rfloor}$. Let t_0 be a constant in $(0, \frac{1}{2})$. Here we consider the test statistics:

$$Q_{\mathrm{rec}} := \sup_{t_0\le t\le 1-t_0} \left\{ t\sqrt{k}\left(\frac{H_n(0,t)}{H_n(0,1)} - 1\right)\right\}^2,$$

$$Q^*_{\mathrm{rec}} := \sup_{t_0\le t\le 1-t_0} \left\{ (1-t)\sqrt{k}\left(\frac{H_n(t,1)}{H_n(0,1)} - 1\right)\right\}^2,$$

$$Q_{\mathrm{rol}} := \sup_{t_0\le t\le 1} \left\{ t_0\sqrt{k}\left(\frac{H_n(t-t_0,t)}{H_n(0,1)} - 1\right)\right\}^2$$

$$Q_{\mathrm{seq}} := \sup_{t_0\le t\le 1-t_0} \left\{ t\sqrt{k}\left(\frac{H_n(t,1)}{H_n(0,t)} - 1\right)\right\}^2,$$

$$Q^*_{\mathrm{seq}} := \sup_{t_0\le t\le 1-t_0} \left\{ (1-t)\sqrt{k}\left(\frac{H_n(0,t)}{H_n(t,1)} - 1\right)\right\}^2.$$

where Q^*_{rec} and Q^*_{seq} are the time-reverse version of Q_{rec} and Q_{seq}, respectively. All the statistics are designed to measure the difference between Hill's estimators obtained from different time domains. We reject $\mathcal{H}_0$ if test statistics provide large values.

Assume that $\{X_i\}$ is a stationary sequence, so $\mathcal{H}_0$ holds. Put $F = F_i$ and $L = L_i$ for each $i \in \mathbb{Z}$. Let $b(x) := \inf\{y : F(y) \ge 1 - x^{-1}\}$,

$$Y_{ni} := (\log X_i - \log b(n/k))_+ \quad \text{and} \quad Z_{ni}^{(\zeta)} := I(X_i > e^{\zeta/\sqrt{k}} b(n/k)).$$

Define

$$\beta(l) := \sup_m \mathrm{E}\left\{ \sup_{A\in\mathcal{F}_{m+l+1}^{\infty}} \left| P(A|\mathcal{F}_{-\infty}^m) - P(A)\right| \right\},$$

where

$$\mathcal{F}_{-\infty}^m := \sigma\{X_i : i \le m\} \quad \text{and} \quad \mathcal{F}_m^{\infty} := \sigma\{X_i : i \ge m\}.$$

In what follows, we assume the regularity conditions below so that null distributions of test statistics are standard:

(A1) $\{X_i\}$ is a stationary β-mixing process and there exists a sequence $\{r_n\}$ of positive integers such that

$$\lim_{n\to\infty} \frac{n}{r_n}\beta\left(\lfloor \epsilon r_n \rfloor\right) = 0 \quad \text{for every } \epsilon > 0, \tag{2.1}$$

and

$$\left(k \vee r_n^3\right) e^{-\epsilon\sqrt{k}/r_n} = o(1). \tag{2.2}$$

(A2) There exist $\kappa(x) = K\int_1^x t^{\gamma-1}dt$, where $K \in \mathbb{R}$, and a positive measurable function $g \in RV_\gamma$ with $\gamma \le 0$, such that for all $\lambda > 0$,

$$\lim_{x\to\infty} \frac{\frac{L(\lambda x)}{L(x)} - 1}{g(x)} = \kappa(\lambda).$$

Further,

$$\sqrt{k}g(b(n/k)) \to 0 \quad \text{as } n \to \infty. \tag{2.3}$$

(A3) There exist nonnegative constants χ, ψ, and ω such that for every $0 < \epsilon < 1$,

$$\chi = \lim_{n\to\infty} \frac{2\alpha^2 n}{\lfloor \epsilon r_n \rfloor k} \sum_{1\le i<j\le\lfloor \epsilon r_n\rfloor} \operatorname{Cov}\left\{Y_{ni}, Y_{nj}\right\}, \tag{2.4}$$

$$\psi = \lim_{n\to\infty} \frac{\alpha n}{\lfloor \epsilon r_n \rfloor k} \sum_{1\le i<j\le\lfloor \epsilon r_n\rfloor} \left[\operatorname{Cov}\left\{Y_{ni}, Z_{nj}^{(0)}\right\} + \operatorname{Cov}\left\{Z_{ni}^{(0)}, Y_{nj}\right\}\right], \tag{2.5}$$

$$\omega = \lim_{n\to\infty} \frac{2n}{\lfloor \epsilon r_n \rfloor k} \sum_{1\le i<j\le\lfloor \epsilon r_n\rfloor} \operatorname{Cov}\left\{Z_{ni}^{(0)}, Z_{nj}^{(0)}\right\}. \tag{2.6}$$

Remark 1 Several types of mixing conditions are assumed in the literatures dealing with tail index estimation for dependent data. For example, Hill's estimator was proven to be asymptotically normal for stationary strong-mixing processes (cf. Hsing 1991). Moreover, smooth functionals of the tail empirical process were investigated for stationary β-mixing processes (cf. Drees 2000, 2003). Here, **(A1)** is assumed for a functional central limit theorem to hold for the tail sequential process.

Remark 2 **(A2)** is called the second order regularly varying condition. In many papers concerning tail index estimators, either **(A2)** or other conditions essentially the same as **(A2)** are assumed. For the details regarding **(A2)**, we refer to Bingham et al. (1987) and Goldie and Smith (1987).

Remark 3 Under (2.3), Hill's estimator is asymptotically unbiased. If k increases too fast, the bias does not vanish although its variance is reduced. Now, several tail index estimators are available. Among them, we can employ more efficient tail index estimators by choosing a fairly large k and comparing the degree of the bias reduction of the estimators. For details, we refer to Feuerverger and Hall (1999) and Gomes et al. (2004). A large class of short memory processes satisfy **(A3)**.

The following is the main theorem of this paper of which proof is given in Sect. 4. In what follows, $B(t)$ denotes a standard Brownian motion.

Theorem 1 *Assume that $\mathcal{H}_0$ holds. Under* **(A1)–(A3)***,*

$$\alpha(1+\lambda)^{-1/2} t\sqrt{k}\left(H_n(0,t) - \frac{1}{\alpha}\right) \Rightarrow B(t) \qquad \text{in } D[t_0, 1],$$

where $\lambda = \chi - 2\psi + \omega$.

From the above theorem and the fact that $H_n(0,1) \xrightarrow{P} \frac{1}{\alpha}$ (cf. Hsing 1991), it can be obtained that

$$(1+\lambda)^{-1} Q_{\mathrm{rec}} \Rightarrow \sup_{t_0 \le t \le 1-t_0} \{B(t) - tB(1)\}^2 .$$

Similarly, it can be seen that

$$\begin{aligned}
(1+\lambda)^{-1} Q^*_{\mathrm{rec}} &\Rightarrow \sup_{t_0 \le t \le 1-t_0} \{B(t) - tB(1)\}^2 ,\\
(1+\lambda)^{-1} Q_{\mathrm{rol}} &\Rightarrow \sup_{t_0 \le t \le 1} \{(B(t) - B(t-t_0)) - t_0 B(1)\}^2 ,\\
(1+\lambda)^{-1} Q_{\mathrm{seq}} &\Rightarrow \sup_{t_0 \le t \le 1-t_0} \left\{B(t) - \frac{t}{1-t}(B(1) - B(t))\right\}^2 ,\\
(1+\lambda)^{-1} Q^*_{\mathrm{seq}} &\Rightarrow \sup_{t_0 \le t \le 1-t_0} \left\{B(t) - \frac{t}{1-t}(B(1) - B(t))\right\}^2 .
\end{aligned}$$

Remark 4 In this study, we have concentrated on the positive extreme value index case, which is another name of the tail index (cf. Dekkers 1989), and naturally designed a change point test based on Hill's estimator. In other cases, however, Hill's estimator is no longer available and other estimators must be considered. To cope with the negative (or zero) extreme value index case as well as the tail index case, Dekkers (1989) proposed a moment estimator. Hence, in all the cases, we may consider employing a test statistic with Hill's estimator replaced by the moment estimator. In order to demonstrate its validity, the property of such a test is worth to investigate. Since it is beyond the scope of this study, we leave this problem as a task of future study.

3 Simulation study

In this section, we implement a simulation study for investigating finite sample properties of the change point tests. To evaluate the stability of the tests, we employ the

Table 1 The sizes of tests at nominal level 0.05 ($\alpha = 1$)

Type	k										
	50	60	70	80	90	100	110	120	130	140	150
Recursive	0.036	0.049	0.049	0.047	0.050	0.031	0.053	0.055	0.038	0.052	0.041
Rolling	0.027	0.026	0.021	0.033	0.018	0.028	0.018	0.023	0.026	0.023	0.019
Sequential	0.122	0.103	0.088	0.093	0.093	0.082	0.089	0.071	0.074	0.064	0.075
Reverse recursive	0.046	0.049	0.055	0.048	0.047	0.047	0.051	0.053	0.036	0.047	0.054
Reverse sequential	0.116	0.104	0.103	0.094	0.108	0.087	0.073	0.082	0.088	0.069	0.063

Table 2 The sizes of tests at nominal level 0.05 ($\alpha = 3$)

Type	k										
	50	60	70	80	90	100	110	120	130	140	150
Recursive	0.054	0.050	0.073	0.053	0.051	0.064	0.061	0.055	0.061	0.081	0.065
Rolling	0.042	0.036	0.043	0.034	0.040	0.046	0.052	0.046	0.060	0.061	0.081
Sequential	0.077	0.068	0.070	0.054	0.053	0.059	0.057	0.068	0.057	0.064	0.061
Reverse recursive	0.053	0.056	0.044	0.050	0.057	0.066	0.058	0.065	0.061	0.069	0.069
Reverse sequential	0.088	0.076	0.060	0.054	0.062	0.061	0.059	0.063	0.059	0.066	0.060

following stationary MA(2) process:

$$X_i = \xi_i + 0.8\,\xi_{i-1} + 0.5\,\xi_{i-2},$$

where ξ_i are i.i.d. random variables following t-distribution with tail index α. Here the sample size is 1,000 and t_0 is set to be 0.2 except for the rolling type test: in this case $t_0 = 0.4$. Concerning the estimating method for $\lambda = \chi - 2\psi + \omega$, we refer to Hsing (1991) and Kim and Lee (2009). Tables 1 and 2 present the sizes of the tests at the nominal level 0.05 in the case of $\alpha = 1, 3$. The result appears to be satisfactory except for the case of the sequential type test when k is small and $\alpha = 1$.

Now, we evaluate the power of the tests. To this end, we consider

$$V_i = \xi_i + 0.8\,\xi_{i-1} + 0.5\,\xi_{i-2},$$
$$W_i = \eta_i + 0.8\,\eta_{i-1} + 0.5\,\eta_{i-2},$$

where ξ_i and η_i are i.i.d. random variables following t-distribution with $\alpha = 3$ and 1, respectively. Put

$$X_i = \begin{cases} V_i & \text{for } i \le \lfloor n\tau \rfloor, \\ W_i & \text{for } i > \lfloor n\tau \rfloor, \end{cases}$$

where τ $(0 < \tau < 1)$ indicates the change point, namely, the tail of distribution gets thicker from $\lfloor n\tau \rfloor$. Tables 3, 4 and 5 present the powers of the tests at the nominal

Table 3 The powers of tests at nominal level 0.05 (α changes from 3 to 1 at $\tau = 0.25$)

Type	k										
	50	60	70	80	90	100	110	120	130	140	150
Recursive	0.186	0.184	0.167	0.190	0.221	0.248	0.250	0.262	0.324	0.340	0.358
Rolling	0.027	0.025	0.017	0.023	0.029	0.031	0.037	0.030	0.025	0.033	0.048
Sequential	0.332	0.322	0.324	0.292	0.303	0.261	0.261	0.243	0.224	0.260	0.216
Reverse recursive	0.021	0.024	0.017	0.020	0.016	0.025	0.023	0.028	0.021	0.025	0.023
Reverse sequential	0.085	0.064	0.048	0.043	0.036	0.050	0.047	0.073	0.100	0.115	0.137

Table 4 The powers of tests at nominal level 0.05 (α changes from 3 to 1 at $\tau = 0.5$)

Type	k										
	50	60	70	80	90	100	110	120	130	140	150
Recursive	0.797	0.861	0.888	0.911	0.925	0.935	0.947	0.954	0.951	0.950	0.956
Rolling	0.234	0.322	0.436	0.480	0.495	0.567	0.587	0.642	0.634	0.659	0.647
Sequential	0.675	0.651	0.707	0.670	0.695	0.753	0.709	0.761	0.743	0.710	0.751
Reverse recursive	0.009	0.013	0.018	0.021	0.018	0.029	0.028	0.034	0.049	0.059	0.078
Reverse sequential	0.011	0.009	0.003	0.001	0.009	0.024	0.027	0.033	0.039	0.063	0.064

Table 5 The powers of tests at nominal level 0.05 (α changes from 3 to 1 at $\tau = 0.75$)

Type	k										
	50	60	70	80	90	100	110	120	130	140	150
Recursive	0.917	0.918	0.918	0.929	0.945	0.922	0.913	0.941	0.949	0.933	0.942
Rolling	0.140	0.167	0.191	0.199	0.225	0.222	0.242	0.267	0.267	0.293	0.291
Sequential	0.792	0.808	0.830	0.831	0.861	0.884	0.849	0.908	0.886	0.886	0.905
Reverse recursive	0.015	0.025	0.039	0.065	0.104	0.140	0.192	0.230	0.270	0.323	0.355
Reverse sequential	0.002	0.000	0.001	0.001	0.001	0.000	0.001	0.002	0.001	0.002	0.001

level 0.05 in the cases of $\tau = 0.25, 0.5$, and 0.75, respectively. It can be seen that the recursive test is most powerful in the cases of $\tau = 0.5, 0.75$ while as might be anticipated, all the time-reverse tests produce very low powers.

Now, in opposition, we consider the case that the tail of distribution becomes thinner. We employ the same set-up as above except that the tail indices of $\{\xi_i\}$ and $\{\eta_i\}$ are set to be 1 and 3, respectively. Tables 6, 7 and 8 exhibit the powers. It is noteworthy that the time-reverse tests produce high powers in the cases of $\tau = 0.25, 0.5$ while the original tests perform poorly. All these results confirm that our tests perform adequately.

Table 6 The powers of tests at nominal level 0.05 (α changes from 1 to 3 at $\tau = 0.25$)

Type	k										
	50	60	70	80	90	100	110	120	130	140	150
Recursive	0.013	0.018	0.041	0.065	0.104	0.139	0.200	0.231	0.261	0.333	0.395
Rolling	0.143	0.160	0.184	0.186	0.219	0.217	0.238	0.269	0.288	0.291	0.296
Sequential	0.001	0.000	0.000	0.002	0.002	0.000	0.000	0.002	0.001	0.001	0.001
Reverse recursive	0.903	0.914	0.913	0.935	0.939	0.938	0.915	0.952	0.933	0.938	0.929
Reverse sequential	0.751	0.809	0.824	0.832	0.862	0.870	0.880	0.874	0.883	0.888	0.901

Table 7 The powers of tests at nominal level 0.05 (α changes from 1 to 3 at $\tau = 0.5$)

Type	k										
	50	60	70	80	90	100	110	120	130	140	150
Recursive	0.020	0.014	0.009	0.011	0.012	0.018	0.019	0.032	0.032	0.071	0.082
Rolling	0.228	0.321	0.402	0.448	0.507	0.586	0.597	0.635	0.627	0.660	0.653
Sequential	0.012	0.007	0.007	0.010	0.011	0.018	0.025	0.027	0.043	0.047	0.063
Reverse recursive	0.806	0.854	0.868	0.902	0.917	0.935	0.949	0.950	0.941	0.945	0.955
Reverse sequential	0.666	0.652	0.699	0.715	0.706	0.687	0.721	0.732	0.741	0.727	0.752

Table 8 The powers of tests at nominal level 0.05 (α changes from 1 to 3 at $\tau = 0.75$)

Type	k										
	50	60	70	80	90	100	110	120	130	140	150
Recursive	0.028	0.018	0.020	0.025	0.025	0.021	0.028	0.028	0.018	0.020	0.015
Rolling	0.029	0.028	0.027	0.024	0.018	0.015	0.027	0.044	0.036	0.040	0.049
Sequential	0.070	0.052	0.043	0.031	0.034	0.057	0.055	0.085	0.103	0.108	0.116
Reverse recursive	0.198	0.166	0.184	0.200	0.239	0.241	0.255	0.274	0.299	0.330	0.332
Reverse sequential	0.358	0.348	0.298	0.319	0.265	0.294	0.270	0.237	0.239	0.200	0.222

4 Proofs

Let t_0 be a fixed number in $(0, \frac{1}{2})$. Let $X_{(k,s,t)}$ be the $(\lfloor k(t-s)\rfloor + 1)$-th largest value among $X_{\lfloor ns\rfloor+1}, \ldots, X_{\lfloor nt\rfloor}$ and

$$U_n(s,t) := \sqrt{k}\left\{\log X_{(k,s,t)} - \log b(n/k)\right\} \quad (0 \le s < t \le 1, t-s \ge n^{-1}).$$

Define

$$\begin{aligned} U_n^{(1)}(t) &:= U_n(0,t), \quad \text{for } t_0 \le t \le 1, \\ U_n^{(2)}(t) &:= U_n(t-t_0,t), \quad \text{for } t_0 \le t \le 1, \\ U_n^{(3)}(t) &:= U_n(t,1), \quad \text{for } 0 \le t \le 1-t_0. \end{aligned}$$

Put $\varphi_1(x) = I(x > 0)$, $\varphi_2(x) = x_+$, and

$$M_n(t,\zeta;\varphi_j) := \frac{1}{\sqrt{k}} \sum_{i=1}^{[nt]} \left\{ \varphi_j\left(\log X_i - \log b(n/k) + \frac{\zeta}{\sqrt{k}}\right) \right.$$
$$\left. - \mathrm{E}\varphi_j\left(\log X_i - \log b(n/k) + \frac{\zeta}{\sqrt{k}}\right)\right\}$$

($j = 1, 2$), which plays a prominent role in this section. Let $D[a, b](0 \le a < b \le 1)$ be the space of the cadlag functions on $[a, b]$ with the Skorohod metric endowed (cf. Billingsley 1999). The following results can be found in Kim and Lee (2009).

Lemma 1 *Suppose that Conditions* **(A1)–(A3)** *hold. Then, for every* $K > 0$ *and* $j \in \{1, 2\}$,

$$\sup_{\zeta\in[-K,K]} \sup_{0\le t\le 1} \left| M_n(t,\zeta;\varphi_j) - M_n(t,0;\varphi_j) \right| = o_P(1).$$

Lemma 2 *Suppose that conditions* **(A1)–(A3)** *hold. Then,*

$$\frac{1}{\sqrt{1+\omega}} M_n(t,0;\varphi_1) \Rightarrow B(t) \quad \textit{in } D[0,1],$$

and

$$\frac{\alpha}{\sqrt{2+\chi}} M_n(t,0;\varphi_2) \Rightarrow B(t) \quad \textit{in } D[0,1].$$

For $f \in D[a, b]$, let

$$w(f,\delta) = \sup\{|f(t) - f(s)| : |t - s| < \delta, \;\; t, s \in [a, b]\},$$

which is called the modulus of continuity of f on $[a, b]$ (cf. Billingsley 1999). We prove that $U_n^{(1)}$, $U_n^{(2)}$, and $U_n^{(3)}$ are asymptotically uniformly equicontinuous, and thus tight.

Lemma 3 *Suppose that* **(A1)–(A3)** *hold. Then we have*

$$\sup\{|U_n(s,t)| : 0 \le s < t \le 1, t - s \ge t_0\} = O_P(1), \tag{4.7}$$

and for every $\epsilon > 0$ *and* $l \in \{1, 2, 3\}$,

$$\lim_{\delta\to 0} \limsup_n P\left\{ w(U_n^{(l)}, \delta) > \epsilon \right\} = 0, \tag{4.8}$$

Proof We first prove (4.7). For each $\zeta \in \mathbb{R}$, we have

$$\sqrt{k}\left\{\log X_{(k,s,t)} - \log b(n/k)\right\} \le \zeta \quad \text{if and only if} \quad \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} Z_{ni}^{(\zeta)} \le \lfloor k(t-s)\rfloor,$$

which is rewritten as

$$M_n(t,\zeta;\varphi_1) - M_n(s,\zeta;\varphi_1) \le \sqrt{k}\left(\frac{\lfloor k(t-s)\rfloor}{k} - \frac{\lfloor nt\rfloor - \lfloor ns\rfloor}{k}\bar{F}(e^{\zeta/\sqrt{k}}b(n/k))\right)$$

On the other hand, we obtain from Lemmas 1 and 2 that

$$\sup_{t\in[0,1]}|M_n(t,\zeta;\varphi_1) - M_n(t,0;\varphi_1)| = o_P(1) \quad \text{and} \quad \sup_{t\in[0,1]}|M(t,0;\varphi_1)| = O_P(1),$$

and

$$\sqrt{k}\left(\frac{\lfloor k(t-s)\rfloor}{k} - \frac{\lfloor nt\rfloor - \lfloor ns\rfloor}{k}\bar{F}(e^{\zeta/\sqrt{k}}b(n/k))\right) = \alpha(t-s)\zeta + o(1)$$

uniformly on $0 \le s < t \le 1$ and $t - s \ge t_0$. Hence, (4.7) holds.

We will prove (4.8) only for the case of $j = 1$ since all the other cases can be proven similarly. Let K and ϵ be positive real numbers. Let $\zeta_0, \zeta_1, \ldots, \zeta_l$ be the numbers satisfying $-K = \zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_l = K$ and $\epsilon/4 \le \zeta_i - \zeta_{i-1} \le \epsilon/2$ for each $i \in \{1, 2, \ldots, l\}$. Then, on $\|U_n^{(1)}\| \le K$,

$$\begin{aligned} w(U_n^{(1)},\delta) > \epsilon \Leftrightarrow & \bigcup_{0<t-s<\delta}\left\{|U_n^{(1)}(t) - U_n^{(1)}(s)| > \epsilon\right\} \\ \Leftrightarrow & \bigcup_{0<t-s<\delta}\left\{U_n^{(1)}(t) - U_n^{(1)}(s) > \epsilon \ \text{ or } \ U_n^{(1)}(s) - U_n^{(1)}(t) > \epsilon\right\}, \end{aligned}$$

which implies

$$\begin{aligned} &\bigcup_{0<t-s<\delta}\bigcup_{i=1}^{l-2}\left\{U_n^{(1)}(t) > \zeta_{i+1},\ \ U_n^{(1)}(s) \le \zeta_i\right\} \text{ or} \\ &\bigcup_{0<t-s<\delta}\bigcup_{i=1}^{l-2}\left\{U_n^{(1)}(s) > \zeta_i,\ \ U_n^{(1)}(t) \le \zeta_{i+1}\right\}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\bigcup_{0<t-s<\delta}\left\{U_n^{(1)}(t) > \zeta_{i+1} \text{ and } U_n^{(1)}(s) \le \zeta_i\right\} \\ &\Leftrightarrow \bigcup_{0<t-s<\delta}\left\{\frac{1}{k}\sum_{j=1}^{\lfloor nt\rfloor} Z_{nj}^{(\zeta_{i+1})} > \frac{\lfloor kt\rfloor}{k}, \frac{1}{k}\sum_{j=1}^{\lfloor ns\rfloor} Z_{nj}^{(\zeta_i)} \le \frac{\lfloor ks\rfloor}{k}\right\} \\ &\Rightarrow \bigcup_{0<t-s<\delta}\left\{-\frac{1}{k}\sum_{j=1}^{\lfloor ns\rfloor}\left\{Z_{nj}^{(\zeta_i)} - Z_{nj}^{(\zeta_{i+1})}\right\} + \frac{1}{k}\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} Z_{nj}^{(\zeta_{i+1})} \ge \frac{\lfloor kt\rfloor - \lfloor ks\rfloor}{k}\right\} \end{aligned}$$

$$\Rightarrow \bigcup_{0<t-s<\delta} \left\{ \frac{1}{k} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} Z_{nj}^{(\zeta_{i+1})} \geq \frac{\lfloor kt \rfloor - \lfloor ks \rfloor}{k} + \frac{1}{k} \sum_{i=1}^{\lfloor nt_0 \rfloor} \left\{ Z_{nj}^{(\zeta_i)} - Z_{nj}^{(\zeta_{i+1})} \right\} \right\}. \tag{4.9}$$

Furthermore, since

$$\begin{aligned}
&\frac{\lfloor kt \rfloor - \lfloor ks \rfloor}{k} - \frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n} \frac{n}{k} \bar{F}\left(e^{\zeta_{i+1}/\sqrt{k}} b(n/k)\right) \\
&= \frac{\lfloor kt \rfloor - \lfloor ks \rfloor}{k} - \frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n} \left(1 - \frac{\alpha \zeta_{i+1}}{\sqrt{k}} + o\left(\frac{1}{\sqrt{k}}\right)\right) \\
&= \frac{\alpha \zeta_{i+1}}{\sqrt{k}} (t-s) + o\left(\frac{1}{\sqrt{k}}\right) \quad \text{uniformly on } 0 < s < t < 1,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor nt_0 \rfloor} \left\{ Z_{nj}^{(\zeta_i)} - Z_{nj}^{(\zeta_{i+1})} \right\} \\
&= \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor nt_0 \rfloor} \left\{ Z_{nj}^{(\zeta_i)} - Z_{nj}^{(\zeta_{i+1})} - \mathrm{E}\left(Z_{nj}^{(\zeta_i)} - Z_{nj}^{(\zeta_{i+1})} \right) \right\} + \frac{\lfloor nt_0 \rfloor}{\sqrt{k}} \mathrm{E}\left(Z_{n1}^{(\zeta_i)} - Z_{n1}^{(\zeta_{i+1})} \right) \\
&\geq \frac{\alpha \epsilon t_0}{4} + o_P(1),
\end{aligned}$$

(4.9) is equivalent to

$$\bigcup_{0<t-s<\delta} \left\{ M_n(t, \zeta_i; \varphi_1) - M_n(s, \zeta_i; \varphi_1) \geq \alpha \zeta_{i+1}(t-s) + \frac{\alpha \epsilon t_0}{4} + o_P(1) \right\},$$

where the remainder term $o_P(1)$ is uniformly negligible on $0 < s < t < 1$ as $n \to \infty$. Since

$$\lim_{\delta \to 0} \limsup_n P\left(w\left\{ M_n(\cdot, \zeta_{i+1}; \varphi_1), \delta \right\} \geq \frac{\alpha \epsilon t_0}{8} \right) = 0$$

holds by Lemmas 1 and 2, we have

$$\begin{aligned}
&\lim_{\delta \to 0} \limsup_n P\Bigg(\bigcup_{0<t-s<\delta} \{ M_n(t, \zeta_{i+1}; \varphi_1) - M_n(s, \zeta_{i+1}; \varphi_1) \\
&\qquad\qquad \geq \alpha \zeta_{i+1}(t-s) + \frac{\alpha \epsilon t_0}{4} + o_P(1) \Big\} \Bigg) \\
&\leq \lim_{\delta \to 0} \limsup_n P\left(\bigcup_{0<t-s<\delta} \left\{ M_n(t, \zeta_{i+1}; \varphi_1) - M_n(s, \zeta_{i+1}; \varphi_1) \geq \frac{\alpha \epsilon t_0}{8} \right\} \right) = 0,
\end{aligned}$$

which implies

$$\lim_{\delta\to 0}\limsup_n P\left(\bigcup_{0<t-s<\delta}\bigcup_{i=1}^{l-2}\left\{U_n^{(1)}(t)>\zeta_{i+1},\;\; U_n^{(1)}(s)\le\zeta_i\right\}\right)=0.$$

Similarly, we can verify

$$\lim_{\delta\to 0}\limsup_n P\left(\bigcup_{0<t-s<\delta}\bigcup_{i=1}^{l-2}\left\{U_n^{(1)}(s)>\zeta_{i+1},\;\; U_n^{(1)}(t)\le\zeta_i\right\}\right)=0.$$

Thus, we obtain

$$\lim_{\delta\to 0}\limsup_n P\left(w(U_n^{(1)},\delta)>\epsilon\ \text{ and }\ \|U_n^{(1)}\|\le K\right)=0.$$

Hence, since $\|U_n^{(1)}\|=O_P(1)$,

$$\lim_{\delta\to 0}\limsup_n P\left(w(U_n^{(1)},\delta)>\epsilon\right)=0.$$

This completes the proof. □

Lemma 4 *Suppose that* **(A2)** *holds. Then,*

$$\mathrm{E}\left(\log X_1-\log b(n/k)+\frac{\zeta}{\sqrt{k}}\right)_+=\frac{k}{n}\left(\frac{1}{\alpha}+\frac{\zeta}{\sqrt{k}}+o\left(\frac{1}{\sqrt{k}}\right)\right)$$

uniformly on any ζ-compact set.

Proof Let K be a positive real number. Note that

$$\mathrm{E}\left(\log X_1-\log b(n/k)+\frac{\zeta}{\sqrt{k}}\right)_+=\int_0^\infty \bar{F}\left(e^x e^{-\zeta/\sqrt{k}}b(n/k)\right)dx$$

$$=\bar{F}\left(e^{-\zeta/\sqrt{k}}b(n/k)\right)\int_0^\infty e^{-\alpha x}\frac{L\left(e^x e^{-\zeta/\sqrt{k}}b(n/k)\right)}{L\left(e^{-\zeta/\sqrt{k}}b(n/k)\right)}dx.$$

Putting $y = e^x$, we get

$$\int_0^\infty e^{-\alpha x} \frac{L\left(e^x e^{-\zeta/\sqrt{k}} b(n/k)\right)}{L\left(e^{-\zeta/\sqrt{k}} b(n/k)\right)} dx = \int_1^\infty y^{-\alpha-1} \frac{L\left(y e^{-\zeta/\sqrt{k}} b(n/k)\right)}{L\left(e^{-\zeta/\sqrt{k}} b(n/k)\right)} dy$$
$$= \frac{1}{\alpha} + \frac{K}{\alpha(\alpha-\gamma)} g\left(e^{-\zeta/\sqrt{k}} b(n/k)\right)(1+o(1))$$
$$= \frac{1}{\alpha} + \frac{K}{\alpha(\alpha-\gamma)} g\left(b(n/k)\right)(1+o(1))$$
$$= \frac{1}{\alpha} + o\left(\frac{1}{\sqrt{k}}\right)$$

uniformly in $\zeta \in [-K, K]$ (cf. Goldie and Smith 1987). Moreover,

$$\bar{F}(e^{-\zeta/\sqrt{k}} b(n/k)) = \frac{k}{n}\left(1 + \frac{\alpha\zeta}{\sqrt{k}} + o\left(\frac{1}{\sqrt{k}}\right)\right)$$

uniformly in $\zeta \in [-K, K]$ (cf. Goldie and Smith 1987). Hence, the proof is completed. □

Proof of Theorem 1 First, we claim that

$$\frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor nt \rfloor} \left(\log X_i - \log X_{(k,0,t)}\right)_+ \tag{4.10}$$
$$= \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor nt \rfloor} (\log X_i - \log b(n/k))_+ - t\sqrt{k}\left\{\log X_{(k,0,t)} - \log b(n/k)\right\} + o_P(1)$$

uniformly in $t_0 \le t \le 1$. Due to Lemma 1 and (4.7), we have

$$\sup_{t \in [t_0,1]} \left| M_n(t, U_n^{(1)}(t); \varphi_2) - M_n(t, 0; \varphi_2) \right| = o_P(1).$$

Moreover, from (4.7) and Lemma 4, we have

$$M_n(t, U_n^{(1)}(t); \varphi_2)$$
$$= \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor nt \rfloor} \left(\log X_i - \log b(n/k) + \frac{U_n^{(1)}(t)}{\sqrt{k}}\right)_+$$
$$-\sqrt{k}\,\frac{\lfloor nt \rfloor}{n}\left\{\frac{1}{\alpha} + \frac{U_n^{(1)}(t)}{\sqrt{k}} + o_P\left(\frac{1}{\sqrt{k}}\right)\right\}$$
$$= \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor nt \rfloor} \left(\log X_i - \log X_{(k,0,t)}\right)_+ - \sqrt{k}\,\frac{\lfloor nt \rfloor}{\alpha n} - \frac{\lfloor nt \rfloor}{n} U_n^{(1)}(t) + o_P(1)$$

and

$$M_n(t,0;\varphi_2) = \frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}(\log X_i - \log b(n/k))_+ - \sqrt{k}\,\frac{\lfloor nt\rfloor}{\alpha n} + o_P(1)$$

uniformly in $t_0 \le t \le 1$. Thus, (4.10) is asserted.

We put

$$\Lambda_n(t) = \begin{pmatrix} \frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}(Y_{ni} - \mathrm{E}Y_{ni}) \\ t\sqrt{k}\left\{\log X_{(k,0,t)} - \log b(n/k)\right\} \end{pmatrix}.$$

It is well known that

$$\tilde{\Lambda}_n(t) := \begin{pmatrix} \frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}(Y_{ni} - \mathrm{E}Y_{ni}) \\ \frac{1}{\alpha\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\left(Z_{ni}^{(0)} - \mathrm{E}Z_{ni}^{(0)}\right) \end{pmatrix}$$

has a continuous limit process, say, Λ, and $\alpha(1+\lambda)^{-1/2}(\,1\ \ -1\,)\Lambda$ is distributed as a standard Brownian motion (cf. Hsing 1991; Kim and Lee 2009). Note that for each $t \in [t_0, 1]$ and $\zeta \in \mathbb{R}$,

$$t\sqrt{k}\left\{\log X_{(k,0,t)} - \log b(n/k)\right\} \le \zeta \quad \text{if and only if}$$
$$\frac{1}{\alpha\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\left(Z_{ni}^{(\zeta/t)} - \mathrm{E}Z_{ni}^{(\zeta/t)}\right) \le \zeta + o(1),$$

and

$$\frac{1}{\alpha\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\left(Z_{ni}^{(\zeta/t)} - \mathrm{E}Z_{ni}^{(\zeta/t)}\right) = \frac{1}{\alpha\sqrt{k}}\sum_{i=1}^{\lfloor nt\rfloor}\left(Z_{ni}^{(0)} - \mathrm{E}Z_{ni}^{(0)}\right) + o_P(1)$$

(cf. Lemma 1). Subsequently, we have that for every $\{t_1, \ldots, t_k\} \subset [t_0, 1]$,

$$(\Lambda_n(t_1), \ldots, \Lambda_n(t_k)) \Rightarrow (\Lambda(t_1), \ldots, \Lambda(t_k)).$$

Since $\Lambda_n(t)$ is tight (cf. Lemma 3), Λ_n also has Λ as its limit process. This together with (4.10) asserts the theorem.

Acknowledgments We are grateful to the two referees for their valuable comments that improve the quality of the paper. This work was supported by the National Research Foundation of Korea (NRF) granting funded by the Korea government(MEST) (2009-0079414).

References

Billingsley P (1999) Convergence of probability measures. 2nd edn. Wiley, New York

Bingham N, Goldie C, Teugels J (1987) Regular variation. Cambridge University Press, Cambridge
Dekkers ALM, Einmahl JHJ, de Haan L (1989) A moment estimator for the index of an extreme-value distribution. Ann Stat 17:1833–1855
Drees H (2000) Weighted approximations of tail processes for β-mixing random variables. Ann Appl Probab 10:1274–1301
Drees H (2003) Extreme quantile estimation for dependent data, with applications to finance. Bernoulli 9:617–657
Feuerverger A, Hall P (1999) Estimating a tail exponent by modelling departure from a Pareto distribution. Ann Stat 27:760–781
Gijbels I, Hall P, Kneip A (1999) On the estimation of jump points in smooth curves. Ann Inst Stat Math 51:231–251
Goldie CM, Smith RL (1987) Slow variation with remainder: theory and applications. Q J Mat. Oxford Ser 38(2):45–71
Gomes MI, Caeiro F, Figueiredo F (2004) Bias reduction of a tail index estimator through an external estimation of the second-order parameter. Statistics 38:497–510
Hsing T (1991) On tail index estimation using dependent data. Ann Stat 19:1547–1569
Kim M, Lee S (2009) Test for tail index change in stationary time series with Pareto-type marginal distribution. Bernoulli 15:325–356
Quintos C, Fan Z, Phillips PCB (2001) Structural change tests in tail behaviour and Asian crisis. Rev Econ Stud 68:633–663
Raimondo R, Tajvidi N (2004) A peaks over threshold model for change-point detection by wavelets. Stat Sinica 14:396–412