# Chapter 1

# Ramsey Theory

"Complete disorder is impossible."

A typical statement in Ramsey Theory is that "any colouring of a set $S$ with finite number of colours contains a (structured) subset $S'$ where all its elements have the same colour". One simple example of Ramsey Theory is the pigeonholes principle. For $n \in \mathbb{N}$, denote by $[n]$ the set of $\{1, 2, \ldots, n\}$.

**Proposition 1.1** (Pigeonholes principle)**.** *Let $k \in \mathbb{N}$ and $s_1, \ldots, s_k$ be non-negative integers. Let $[n]$ be coloured with colours $c_1, \ldots, c_k$ and $n > \sum_{i\in[k]} s_i$. Then there exists $i \in [k]$ such that there are at least $s_i + 1$ numbers of colour $c_i$.*

**Proof.** Suppose the contrary. For each $i \in [k]$, there are at most $s_i$ numbers of colour $c_i$. Therefore, $n \leq s_1 + s_2 + \cdots + s_k$, a contradiction. □

## 1.1 Ramsey numbers for complete graphs

**Question.** In a party of $n$ people, does there always exists a group of $3$ people that are all friends to each other or non of them are friends (i.e. strangers)?

If $n$ is small, say $n = 2$, then it is impossible. If $n$ is large, then this is likely to be true. What is the smallest $n$ such that the question is yes? We now study this problem more formally.

We need the following basic definitions from graph theory. Let $K_n$ be the complete graph on $n$ vertices (with vertex set $V(K_n) = \{x_1, \ldots, x_n\}$ and edge set $E(K_n) = \{x_ix_j : 1 \leq i < j \leq n\}$). An *edge-colouring* of $K_n$ is an assignment of colours to each of it edges. (Sometime we say a colouring of edges of $K_n$ instead.) A red/blue-edge-colouring of $K_n$ is an edge-colouring of $K_n$ using colours red and blue. It is *monochromatic* if all of its edges are coloured the same colour. We say a *red $K_n$* to mean a monochromatic copy of $K_n$ where all its edges are coloured red.

**Definition 1.2.** For $s, t \in \mathbb{N}$, the *Ramsey number of $s$ and $t$*, denoted by $R(s, t)$, is the smallest $n \in \mathbb{N}$ such that any red/blue-edge-colouring of $K_n$ yields a red copy of $K_s$ or a blue copy of $K_t$.

We write $R(t)$ for $R(t, t)$.

In our original question, let people be the vertices of a complete graph $K_n$. We colour the edge between two people red if they are friends and blue otherwise. So a group of $3$ people that are all friends to each other corresponds to a red $K_3$. A group of $3$ people that are all strangers to each other corresponds to a blue $K_3$. Hence, our question translates to what is $R(3)$?

**Proposition 1.3.** $R(3) = R(3, 3) = 6.$

**Proof.** First, we prove the lower bound by presenting a red/blue edge-colouring of $K_5$ with no monochromatic triangles. Let $1, 2, 3, 4, 5$ denote the vertices of $K_5$. For every distinct $i, j \in [5]$, colour the edge $ij$ red if $j - i \equiv \pm 1 \pmod 5$ and blue otherwise. See Figure 1.1. It is easy to see that both the red graph and the blue graph form a cycle of length (number of edges) 5 and is thus triangle-free. We conclude that $R(3, 3) > 5$.

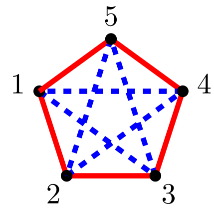


Figure 1.1: A red/blue-edge colouring of $K_5$ without a monochromatic copy of $K_3$.

Next, we prove the upper bound, that is, we will prove that in any red/blue edge-colouring of $K_6$ there is a monochromatic triangle. Consider an arbitrary red/blue edge-colouring $K_6$. Let $x$ be an arbitrary vertex of $K_6$. Note that $x$ is incident with $5$ edges each of which is coloured by colour red or blue. At least 3 of the edges incident with $x$ are red or at least 3 are blue by the pigeonholes principle (Proposition 1.1). Assume without loss of generality that there are at least 3 red edges incident with $x$ (the complementary case, where there are at least 3 blue edges incident with $x$ can be handled similarly). Let $y_1, y_2, y_3$ be vertices of $K_6$ such that $xy_i$ is red for all $i \in [3]$. If the edge $y_iy_j$ is red for some $i, j \in [3]$, then $x, y_i, y_j$ form a red $K_3$. Otherwise, $y_1y_2, y_1y_3$ and $y_2y_3$ are all blue and thus $y_1, y_2, y_3$ form a blue $K_3$. We conclude that, in either case, there is a monochromatic $K_3$ and thus $R(3, 3) \leq 6$.

Since $R(3, 3) \leq 6$ and $R(3, 3) > 5$ it follows that $R(3, 3) = 6$. $\square$

The Ramsey numbers $R(s, t)$ are named after the British mathematician Frank Plumpton Ramsey. It is not a priori clear that Ramsey numbers are well-defined (that is, that they exist for every $s, t \in \mathbb{N}$). Next, we study some basic property of $R(s, t)$. In particular, $R(1, t)$ and $R(2, t)$ exist for all $t \in \mathbb{N}$.

**Proposition 1.4.** *For all $s, t \in \mathbb{N}$,*

(i) $R(1, t) = 1$.
(ii) $R(2, t) = t$.
(iii) *If $R(s, t)$ exists, then $R(t, s) = R(s, t)$.*

**Proof.** Exercises (see Problem Sheet 1). $\square$

In the next theorem, we show that $R(s, t)$ exists for all $s, t \in \mathbb{N}$. This was proved by Ramsey in 1930. The proof of this fact which we present below is due to Erdős and Szekeres; it yields a general method for deriving explicit upper bounds for these numbers.

**Theorem 1.5** (Erdős and Szekeres). *For all $s, t \in \mathbb{N}$, $R(s, t)$ exists. Moreover, if $s, t \geq 2$, then $R(s, t) \leq R(s - 1, t) + R(s, t - 1)$.*

**Proof.** We proceed by induction on $s + t$. If $\min\{s, t\} \leq 2$, then $R(s, t)$ exists by Proposition 1.4. We can thus assume that $s, t \geq 3$ and, in particular, $s + t \geq 6$. It follows by the induction hypothesis that both $R(s - 1, t)$ and $R(s, t - 1)$ exist. Let $p := R(s - 1, t)$, $q := R(s, t - 1)$ and $n := p + q$.

Consider an arbitrary red/blue-edge-colouring of $K_n$. Let $x$ be an arbitrary vertex of $K_n$. Since $x$ is incident with $n - 1 = (p - 1) + (q - 1) + 1$ edges, each of which is coloured either red or blue, it follows by the pigeonholes principle (Proposition 1.1) that there are at least $p$ red edges or at least $q$ blue edges which are incident with $x$. Assume without loss of generality that there are at least $p$ red edges incident with $x$ (the complementary case can be handled similarly). Let $y_1, \ldots, y_p$ be vertices of $K_n$ such that $xy_i$ is red for every $i \in [p]$. Since $p \geq R(s-1,t)$ and every edge of $\{y_iy_j : 1 \leq i < j \leq p\}$ is coloured either red or blue, it follows that this colouring yields either a red $K_{s-1}$ or a blue $K_t$ or both. In the latter case we are done. In the former case, adding $x$ to the red $K_{s-1}$ yields a red $K_s$. $\square$

**Corollary 1.6.** *For every $s,t \in \mathbb{N}$, $R(s,t) \leq \binom{s+t-2}{s-1}$. In particular $R(t,t) \leq \binom{2t-2}{t-1} < 2^{2t-2}$.*

**Proof.** We proceed by induction on $s+t$. Since $R(s,t) = R(t,s)$ holds for every $s,t \in \mathbb{N}$ by Proposition 1.4, we can assume without loss of generality that $s \leq t$. If $s \leq 2$, then the inequality holds by Proposition 1.4. We can thus assume that $t \geq s \geq 3$. Assume by induction that $R(s,t) \leq \binom{s+t-2}{s-1}$ holds for all positive integers $s$ and $t$ satisfying $s+t < n$. Let $t \geq s \geq 3$ be integers satisfying $s+t = n$. It follows by Theorem 1.5 that

$$\begin{aligned} R(s,t) &\leq R(s-1,t) + R(s,t-1) \\ &\leq \binom{(s-1)+t-2}{(s-1)-1} + \binom{s+(t-1)-2}{s-1} \\ &= \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}, \end{aligned}$$

where the second inequality holds by the induction hypothesis and the last equality holds by the binomial identity $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$. This proves the first part of the corollary, and in particular shows that $R(t,t) \leq \binom{2t-2}{t-1}$; the latter quantity is at most $2^{2t-2}$ by the identity $\sum_{i=0}^{m} \binom{m}{i} = 2^m$. $\square$

Next, we give a more direct proof of a slightly weaker upper bound.

**Proposition 1.7.** *For all $t \in \mathbb{N}$, $R(t) \leq 2^{2t-1}$ holds.*

**Proof.** Let $n = 2^{2t-1}$. Fix some red/blue-edge-colouring of $K_n$. We will prove that this colouring must yield a monochromatic copy of $K_t$. Let $x_1$ be an arbitrary vertex of $K_n$. There are $n-1 = 2^{2t-2} + 2^{2t-2} - 1$ edges incident with $x_1$. It follows that at least $2^{2t-2}$ of these edges must be of the same colour (all red or all blue). Let $N_1$ be an arbitrary set of $2^{2t-2}$ vertices such that all edges of $\{x_1v : v \in N_1\}$ have the same colour. Mark $x_1$ with this colour (that is, if $x_1v$ is red for every $v \in N_1$, then mark $x_1$ red and otherwise mark it blue). Let $x_2$ be an arbitrary vertex of $N_1$. There are $|N_1| - 1 = 2^{2t-3} + 2^{2t-3} - 1$ edges $x_2y$, where $y \in N_1$. It follows that at least $2^{2t-3}$ of these edges must be of the same colour. Let $N_2 \subseteq N_1$ be an arbitrary set of $2^{2t-3}$ vertices such that all edges of $\{x_2v : v \in N_2\}$ have the same colour. Mark $x_2$ with this colour. Repeating this process $2t-1$ times (this is possible since $n = 2^{2t-1}$ and we keep half the vertices in each step), we obtain a sequence of vertices $x_1, x_2, \ldots, x_{2t-1}$ such that

**(i)** $x_i$ is marked either red or blue for every $i \in [2t-1]$, and
**(ii)** for every $1 \leq i < j \leq 2t-1$, the vertex $x_i$ is marked by the colour of the edge $x_ix_j$.

It follows by (i) and by the pigeonhole principle that there exists a monochromatic (all red or all blue) subsequence $x_{i_1}, x_{i_2}, \ldots, x_{i_t}$ of $x_1, x_2, \ldots, x_{2t-1}$. It follows by (ii) that these vertices form a monochromatic copy of $K_t$. $\square$

Note that the proof of Theorem 1.5 (and Corollary 1.6) requires Proposition 1.4, but the proof of Proposition 1.7 does not.

## 1.2 Ramsey numbers for more than two colours

**Definition 1.8.** Let $k, s_1, \ldots, s_k \in \mathbb{N}$. The *Ramsey Number of* $s_1, \ldots, s_k$, denoted by $R_k(s_1, \ldots, s_k)$, is the smallest $n \in \mathbb{N}$ such that for any edge-colouring of $K_n$ with colours $c_1, \ldots, c_k$, there exists some $i \in [k]$ for which there exists a monochromatic copy of $K_{s_i}$ of colour $c_i$.

We will prove that these numbers are well-defined.

**Theorem 1.9.** *For all* $k, s_1, \ldots, s_k \in \mathbb{N}$*,* $R_k(s_1, \ldots, s_k)$ *exists.*

**Example 1.10.** When colouring with only one colour, we clearly have $R_1(s_1) = s_1$. Indeed, colouring $K_n$ with one colour yields a monochromatic $K_n$ which contains a monochromatic $K_{s_1}$ if and only if $n \geq s_1$. When colouring with two colours we get the previously defined Ramsey number $R(s_1, s_2)$ which is the same as $R_2(s_1, s_2)$ in the new notation. We already proved that these numbers exist.

**Proof of Theorem 1.9.** We proceed by induction on the number of colours $k$. As noted in Example 1.10, the assertion of the theorem holds for $k \leq 2$. Assume that the assertion of the theorem holds for $k-1$, that is, that $R_{k-1}(t_1, \ldots, t_{k-1})$ exists for all $t_1, \ldots, t_{k-1} \in \mathbb{N}$. Let $s_1, \ldots, s_k \in \mathbb{N}$ be arbitrary, let $m = R_{k-1}(s_1, \ldots, s_{k-1})$ and let $n = R(m, s_k)$. Note that $m$ is well-defined by the induction hypothesis and that $n$ is well-defined since the assertion of the theorem holds for $k = 2$. Consider an arbitrary edge-colouring $c$ of $K_n$ with colours $c_1, \ldots, c_k$. Define a red/blue-edge-colouring $\tilde{c}$ of $K_n$ by colouring an edge $e$ blue if $c(e) = c_k$ and red otherwise. Since $n \geq R(m, s_k)$, it follows that this colouring must yield a red $K_m$ or a blue $K_{s_k}$. In the latter case, it follows by the definition of $\tilde{c}$ that the blue $K_{s_k}$ is monochromatic under $c$ with colour $c_k$. Similarly, in the former case the red $K_m$ are edge-coloured with colours $c_1, \ldots, c_{k-1}$ under $c$. Since $m \geq R_{k-1}(s_1, \ldots, s_{k-1})$, it follows that there must exist some $i \in [k-1]$ for which there exists a copy of $K_{s_i}$ all of whose edges are coloured with the colour $c_i$. $\square$

**Remark 1.11.** This argument is sometime called '*colour blindness*'.

## 1.3 Small Ramsey numbers

The following table shows some known values of $R(s,t)$. The $ij$th entry is the value of $R(i,j)$ (if it is not known exactly, then the best current lower and upper bounds are given).

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 |
| 3 | 1 | 3 | 6 | 9 | 14 |
| 4 | 1 | 4 | 9 | 18 | 25 |
| 5 | 1 | 5 | 14 | 25 | 43-48 |

We have already determined the values of $R(1,t)$, $R(2,t)$ and $R(3,3)$. Next, we evaluated $R(3,4)$.

**Proposition 1.12.** $R(3,4) = 9$.

Note that using Theorem 1.5 and Propositions 1.3 and 1.4, we would only get that $R(3,4) \leq R(2,4) + R(3,3) = 4 + 6 = 10$ (not 9).

**Proof.** Starting with the upper bound, we will prove that in any red/blue-edge-colouring of $K_9$ there is a red $K_3$ or a blue $K_4$. Consider an arbitrary red/blue-edge-colouring of $K_9$. We distinguish between the following three cases:

**Case 1:** There exists a vertex $x$ of $K_9$, which is incident with at least 4 red edges.
Let $y_1, y_2, y_3, y_4$ be vertices of $K_9$ such that $xy_r$ is red for every $r \in [4]$. If there exist some $1 \leq i < j \leq 4$ such that the edge $y_iy_j$ is red, then $y_i, y_j$ and $x$ form a red $K_3$. Otherwise, all edges of $\{y_iy_j : 1 \leq i < j \leq 4\}$ are blue and thus $y_1, y_2, y_3$ and $y_4$ form a blue $K_4$.

**Case 2:** There exists a vertex $x$ of $K_9$, which is incident with at least 6 blue edges.
Let $z_1, \dots, z_6$ be vertices of $K_9$ such that $xz_r$ is blue for every $r \in [6]$. Since $R(3,3) = 6$ by Proposition 1.3, it follows that there exist $1 \leq a < b < c \leq 6$ such that $z_a, z_b$ and $z_c$ form a monochromatic $K_3$. If this $K_3$ is red, then we are done. Otherwise, this $K_3$ is blue and thus $z_a, z_b, z_c$ and $x$ form a blue $K_4$.

**Case 3:** Every vertex of $K_9$ is incident with exactly 3 red edges and 5 blue edges.
We wish to count the total number of red edges. Since every vertex is incident with exactly 3 red edges and since every edge has exactly 2 endpoints, the number of red edges is exactly $(9 \cdot 3)/2 = 13.5$. This is clearly a contradiction. We conclude that Case 3 cannot occur.

We conclude that, in either case, there is a red $K_3$ or a blue $K_4$ and thus $R(3,4) \leq 9$.

Next, we prove the lower bound by presenting a red/blue-edge-colouring of $K_8$ with no red $K_3$ and no blue $K_4$. Let $1, 2, \dots, 8$ denote the vertices of $K_8$. For every distinct $i, j \in [8]$, colour the edge $ij$ blue if $i - j \equiv \pm 1$ or $\pm 2 \mod 8$, colour the edge $ij$ red othwerwise. See Figure 1.2.

We now show that there is no red $K_3$ or blue $K_4$. Suppose the contrary that there is a red $K_3$. Without loss of generality, we may assume that the red $K_3$ contains $1$. Then, the remaining two vertices must be in $\{4,5,6\}$, which are joined by blue edges, a contradiction. Suppose that there is a blue $K_4$. Without loss of generality, we may assume that the blue $K_4$ contains $1$. Note that $V(K_4) \subseteq \{1,2,3,7,8\}$. However such $K_4$ contains at most one vertex from $\{2,7\}$ and one from $\{3,8\}$. This implies that $K_4$ contains at most 3 vertices, a contradiction. We conclude that the given colouring yields no red $K_3$ and no blue $K_4$ entailing $R(3,4) > 8$.

Since $R(3,4) \leq 9$ and $R(3,4) > 8$ it follows that $R(3,4) = 9$.

□

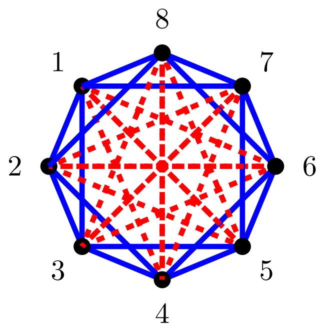


Figure 1.2: A red/blue-edge colouring of $K_8$ without a red $K_3$ or a blue $K_4$.

Note that using Theorem 1.5 and Proposition 1.12, we deduce that $R(4) \leq R(3,4) + R(4,3) = 9 + 9 = 18$. To see that $R(4) > 17$ (and hence $R(4) = 18$), we consider a red/blue-edge-colouring of $K_{17}$ with vertex set $[17]$. Imagine the vertices are sitting on a unit circle. For every $1 \leq i < j \leq 17$, we colour the edge $ij$ blue if $i$ and $j$ are distance $1, 2, 4, 8$ apart, and red otherwise. Equivalently, colour the edge $ij$ blue if $j - i \in \{1, 2, 4, 8, 9, 13, 15, 16\}$ and red otherwise. (Please check that this edge-coloured $K_{17}$ does not contain a monochromatic $K_4$.)

## 1.4 Lower bounds for Ramsey numbers

**Theorem 1.13** (Erdős 1947). *Let $t, n \in \mathbb{N}$. If $\binom{n}{t}2^{1-\binom{t}{2}} < 1$, then $R(t) > n$.*

For small values of $t$, one could usually prove an inequality of the form $R(t) > n$ by exhibiting a specific red/blue-edge-colouring of $K_n$ not containing a monochromatic copy of $K_t$. For large values of $t$ this approach becomes problematic. A different approach would be to prove that such a colouring exists without actually finding it. We present two proofs of Theorem 1.13, one based on counting and the other on probability (though in fact these are just two ways to look at the same proof).

**First proof of Theorem 1.13 (Random colouring).** Colour the edges of $K_n$ red and blue randomly (and independently). That is, for every edge $e \in E(K_n)$ flip a fair coin (i.e. the probability of "heads" is $\frac{1}{2}$ and the probability of "tails" is $\frac{1}{2}$) where all coin flips are mutually independent (i.e. the result of a coin flip does not depend on the results of any other coin flips). If the result of the coin flip for an edge $e$ is heads, then colour $e$ red; otherwise, colour $e$ blue. Let $X$ be the number of monochromatic copies of $K_t$ in this colouring (so $X$ depends on the random colouring). Then $X$ is a random variable whose value must be a non-negative integer. There are $\binom{n}{t}$ sets $T$ of $t$ vertices of $K_n$, and for each of these sets $T$ the probability that every edge within $T$ is red is $(\frac{1}{2})^{\binom{t}{2}}$, since there are $\binom{t}{2}$ edges within $T$, each of which has probability $\frac{1}{2}$ of being red, independently of each other edge. Similarly the probability that every edge within $T$ is blue is $(\frac{1}{2})^{\binom{t}{2}}$, so the probability that $T$ induces a monochromatic copy of $K_t$ is $2(\frac{1}{2})^{\binom{t}{2}} = 2^{1-\binom{t}{2}}$. Since $X$ is the number of sets $T$ which induce a monochromatic copy of $K_t$, we deduce that $\mathbb{E}(X) = \binom{n}{t}2^{1-\binom{t}{2}} < 1$ (the inequality is our hypothesis). Since $X$ must be a non-negative integer,

$$1 > \mathbb{E}(X) = \sum_{i \in \mathbb{N}} i\mathbb{P}(X = i) \geq \sum_{i \in \mathbb{N}} \mathbb{P}(X = i) = \mathbb{P}(X \geq 1).$$

Hence, $\mathbb{P}(X = 0) > 0$, that is, that with positive probability our random colouring of $K_n$ has no monochromatic copy of $K_t$. So there must exist some edge-colouring of $K_n$ with no monochromatic copy of $K_t$, which implies that $R(t) > n$. □

**Second proof of Theorem 1.13 (Counting).** A red/blue-edge-colouring of $K_n$ is called *good* if it yields no monochromatic copy of $K_t$ and is called *bad* otherwise. The number of red/blue-edge-colourings of $K_n$ is clearly $2^{\binom{n}{2}}$. We wish to bound from above the number of bad colourings. Each bad colouring contains a monochromatic $K_t$. Hence, we first fix $t$ vertices which will induce such a $K_t$ (there are $\binom{n}{t}$ ways to do so), then decide whether *monochromatic* means red or blue (there are $2$ ways to do so) and finally colour every other edge either red or blue arbitrarily (there are $2^{\binom{n}{2}-\binom{t}{2}}$ ways to do so). It follows that there are at most $\binom{n}{t} \cdot 2 \cdot 2^{\binom{n}{2}-\binom{t}{2}}$ bad colourings. Since $\binom{n}{t}2^{1-\binom{t}{2}} < 1$ by assumption, it follows that

$$\binom{n}{t} \cdot 2 \cdot 2^{\binom{n}{2}-\binom{t}{2}} = 2^{\binom{n}{2}} \left( \binom{n}{t} 2^{1-\binom{t}{2}} \right) < 2^{\binom{n}{2}}.$$

Hence, the number of bad colourings is strictly smaller than the number of all red/blue-edge-colourings and thus there must exist a good colouring. □

**Corollary 1.14.** *For every $t \geq 4$, $R(t) > \lfloor 2^{t/2} \rfloor$ holds.*

**Proof.** Let $n = \lfloor 2^{t/2} \rfloor$; by Theorem 1.13 it suffices to prove that $\binom{n}{t} 2^{1-\binom{t}{2}} < 1$. We have

$$\begin{aligned}
\binom{n}{t} 2^{1-\binom{t}{2}} &= \frac{n(n-1)\cdot \ldots \cdot (n-t+1)}{t!} 2^{1-\frac{t(t-1)}{2}} \leq \frac{n^t}{t!} 2^{1-\frac{t^2-t}{2}} \\
&\leq \frac{(2^{t/2})^t \cdot 2^{1-t^2/2+t/2}}{t!} = \frac{2^{t^2/2+1-t^2/2+t/2}}{t!} = \frac{2^{1+t/2}}{t!} \\
&\leq \frac{2^{t-1}}{t!} = \prod_{i=2}^{t} \frac{2}{i} < 1 \,,
\end{aligned}$$

where the third inequality holds since $t \geq 4$ implying $t - 1 \geq t/2 + 1$. $\square$

## 1.5 Ramsey theory for sets

**Definition 1.15.** Given $r \in \mathbb{N}$ and a set $X$ denote $X^{(r)}$ the set of all $r$-element subsets of $X$. Given distinct $i, j, k, \ldots$, we write $ij$ for $\{i, j\}$ and $ijk$ for $\{i, j, k\}$, and so on.

Note that $X^{(r)}$ can be also written as $\binom{X}{r}$. For example, $[4]^{(2)} = \{12, 13, 14, 23, 24, 34\}$. Note that if $|X| < r$, then $X^{(r)}$ is empty. Let $G$ be a graph with vertex set $V$. Note that $E(G) \subseteq V^{(2)}$.

**Question.** Does every red-blue colouring of $[n]^{(r)}$ contains a subset $Y$ of $[n]$ such that every element of $Y^{(r)}$ has the same colour and $|Y| = t$ (providing $n$ is large enough)?

When $r = 1$, the answer is yes providing that $n \geq 2t-1$. When $r = 2$, the question is equivalent to "Does $R(t, t)$ exist?", which is true by Theorem 1.5. Hence, the question is asking whether 'Ramsey number' exists when we colour $r$-element subsets instead of two-element subsets. We write $Y^{(r)}$ is red to mean that every element of $Y^{(r)}$ is red.

**Definition 1.16.** For $r, s, t \in \mathbb{N}$, the *Ramsey number* $R^{(r)}(s, t)$ is the smallest $n \in \mathbb{N}$ such that any red/blue-colouring of $[n]^{(r)}$ yields an $s$-element subset $S$ satisfying $S^{(r)}$ is red or a $t$-element subset $T$ satisfying $T^{(r)}$ is blue.

We now show that $R^{(r)}(s, t)$ exists. Note that $R^{(2)}(s, t) = R(s, t)$.

**Theorem 1.17.** *For any $s, t \geq r \geq 1$, the Ramsey number $R^{(r)}(s, t)$ is finite and satisfies*

$$R^{(r)}(s,t) \leq R^{(r-1)}\left(R^{(r)}(s-1,t), R^{(r)}(s,t-1)\right) + 1.$$

**Proof.** We proceed by induction on $r$ and on $s + t$.

When $r = 1$ and $r = 2$, the theorem holds by the Pigeonhole principle, Proposition 1.1, and Theorem 1.5, respectively. It is easy see that $R^{(r)}(r, t) = t$ and $R^{(r)}(s, r) = s$ (exercise). We can assume that $s, t > r \geq 3$. Let

$$p = R^{(r)}(s-1,t),\ q = R^{(r)}(s,t-1) \text{ and } n = R^{(r-1)}(p,q) + 1,$$

which exist by our induction hypothesis.

Consider an arbitrary red/blue-colouring $c$ of $[n]^{(r)}$. [1] Define a red/blue-colouring $c^*$ of $[n-1]^{(r-1)}$ such that $c^*(U) = c(U \cup \{n\})$ for all $U \in [n-1]^{(r-1)}$. Since $n - 1 = R^{(r-1)}(p, q)$, there exists a $p$-element subset $P$ satisfying $P^{(r-1)}$ is red or a $q$-element subset $Q$ satisfying $Q^{(r-1)}$ is blue under the colouring $c^*$. Suppose without loss of generality, the former holds (the complementary case can be handled similarly) and that $P = [p]$. Recall that $p = R^{(r)}(s-1, t)$. Hence under $c$ (which is colouring of $r$-element subsets), there exists an $(s-1)$-element subset $S'$ satisfying $S^{(r)}$ is red or a $t$-element subset $T$ satisfying $T^{(r)}$ blue. In the latter case, we are done. In the former case, set $S = S' \cup \{n\}$ and observe that $S^{(r)}$ is red as $c(U \cup \{n\}) = c^*(U)$ is red for every $U \in S'^{(r-1)} \subseteq [p]^{(r-1)}$. □

Note that we can also generalise $R^{(r)}(s, t)$ for more colours, i.e. $R^{(r)}(s_1, \ldots, s_k)$.

### 1.5.1 Application of Ramsey numbers for sets

**Theorem 1.18** (Erdős-Szekeres, 1935). *Let $m \geq 4$. Then there exists $n \in \mathbb{N}$ such that any set of $n$ points in the Euclidean plane, no three of which on a line, contains $m$ points that form a convex $m$-gon.*

To deduce theorem we need the following two geometric facts.

[1] Normally, one can fix an arbitrary element of $[n]$. Here we just say that the arbitrary element is $n$.

**Fact.** (1) Among five points in the plane, no three on the same line, there are always four points in a convex position.

(2) If any four of $m$ points are in a convex position, then they are all in a convex position.

**Proof of Theorem 1.18.** Given $m$, let $n = R^{(4)}(5, m)$ and consider any set of $n$ points in the Euclidean plane, no three of which on a line. Let $V$ be the set of points. For each $U \in V^{(4)}$, we colour $U$ blue if the points of $U$ are in a convex positions and red otherwise. By Fact (1), there does not exist $S \subseteq V$ such that $|S| = 5$ and $S^{(4)}$ is red. Therefore, there must exists $T \subseteq V$ such that $T = m$ and $T^{(4)}$ is blue. Fact (2) implies that $T$ form a convex $m$-gon. $\square$

## 1.6 Ramsey theory for numbers

Consider the numbers $1, 2, \ldots, n$. We now colour each number with colours, say red and blue. Is there any subsequence in the number with colour red (or blue)? For example, does the red number contains $x, y, z$ such that $x + y = z$?

**Theorem 1.19** (Schur 1916)**.** *For all $r \in \mathbb{N}$, there exists an integer $n = n(r)$ such that in every $r$-colouring of $[n]$ there exist integers $x, y, z \in [n]$ all coloured with the same colour such that $x + y = z$.*

**Proof.** Given $r$, set $n = R_r(3, \ldots, 3)$ and let $c$ be an arbitrary $r$-colouring of $[n]$. Define an $r$-edge-colouring $c'$ of $K_n$ with vertex set $[n]$ as follows: for every $1 \leq i < j \leq n$ let $c'(ij) = c(j - i)$. Since $n \geq R_r(3, \ldots, 3)$, it follows by the definition of $R_r(3, \ldots, 3)$ that there exists a monochromatic triangle, that is, there are vertices $1 \leq i < j < k \leq n$ such that $c'(ij) = c'(jk) = c'(ik)$. Let $x = j - i$, $y = k - j$ and $z = k - i$. It follows by the definition of $c'$ that $c(x) = c(y) = c(z)$. Moreover, $x, y, z \in [n]$ and $x + y = (j - i) + (k - j) = k - i = z$. □

**Lemma 1.20.** *For all $s, t \in \mathbb{N}$, any sequence $a_1, \ldots, a_n$ of $n > st$ distinct real numbers contains an increasing subsequence of length $s + 1$ or a decreasing subsequence of length $t + 1$.*

Note that Lemma 1.20 is tight. Indeed, for every positive integers $s$ and $t$, the sequence

$$t, t-1, \ldots, 1, 2t, 2t-1, \ldots, t+1, \ldots, st, st-1, \ldots, (s-1)t+1$$

consists of exactly $st$ elements, its longest increasing subsequence has length $s$ and its longest decreasing subsequence has length $t$.

We will present two different proofs of Lemma 1.20.

**First proof of Lemma 1.20..** Consider the following algorithm for arranging the $a_i$'s in piles. Put $a_1$ in pile $1$. Assume we have already arranged elements $a_1, \ldots, a_{i-1}$ in piles $1, \ldots, r$ and now wish to add $a_i$. Let $j \in [r]$ be the smallest index (if it exists) for which $a_i$ is strictly larger than the element which is currently at the top of pile $j$. Put $a_i$ at the top of pile $j$. If no such index $j$ exists, then put $a_i$ in a new pile labeled $r + 1$. Continue until all the $a_i$'s are arranged in piles.

Note that, at any point during the algorithm, the elements in each pile form an increasing (from bottom to top) subsequence of $a_1, \ldots, a_n$. Hence, if some pile contains at least $s + 1$ elements, then we are done. Assume then that every pile contains at most $s$ elements. Since $n > st$ it follows by the pigeonhole principle that there must exist at least $t + 1$ piles. Let $a_{i_{t+1}}$ be the element at the top of pile $t + 1$. When we put $a_{i_{t+1}}$ at the top of this pile there was some element $a_{i_t} > a_{i_{t+1}}$ which, at that moment, was at the top of pile $t$ (otherwise we would not have put $a_{i_{t+1}}$ in pile $t + 1$). Clearly $i_t < i_{t+1}$. Similarly, when we put $a_{i_t}$ at the top of pile $t$ there must have been some element $a_{i_{t-1}} > a_{i_t}$ which, at that moment, was at the top of pile $t-1$ (otherwise we would not have put $a_{i_t}$ in pile $t$). Clearly $i_{t-1} < i_t$. Continuing in this way we obtain a decreasing sequence $a_1, a_2, \ldots, a_{i_{t+1}}$ such that $i_1 < i_2 < \ldots < i_{t+1}$. Therefore, $a_{i_1}, a_{i_2}, \ldots, a_{i_{t+1}}$ is a decreasing subsequence of $a_1, \ldots, a_n$. □

**Second proof of Lemma 1.20..** Define a function $f : [n] \to \mathbb{N} \times \mathbb{N}$ as follows: $f(i) = (p, q)$ where $p$ is the length of a longest increasing subsequence of $a_1, \ldots, a_n$ which starts at $a_i$ and $q$ is the length of a longest decreasing subsequence of $a_1, \ldots, a_n$ which ends at $a_i$.

We claim that $f$ is injective. Indeed, fix some $1 \leq i < j \leq n$, let $f(i) = (p, q)$ and $f(j) = (k, \ell)$. We will prove that $(p, q) \neq (k, \ell)$. Assume first that $a_i < a_j$. Let $a_{j_1}, \ldots, a_{j_k}$ be a longest increasing subsequence starting at $a_j$ (so $j_1 = j$). Then $a_i, a_{j_1}, \ldots, a_{j_k}$ is an increasing subsequence starting at $a_i$ of length $k+1$. It follows that $p \geq k+1 > k$. Assume then that $a_i > a_j$. Let $a_{i_1}, \ldots, a_{i_q}$ be a longest decreasing subsequence ending at $a_i$, then $a_{i_1}, \ldots, a_{i_q}, a_j$ is a decreasing subsequence ending at $a_j$ of length $q+1$. It follows that $\ell > q$. We conclude that $f$ is injective and thus $|\{f(i) : i \in [n]\}| = n > st$.

Since $f \colon [n] \to \mathbb{N} \times \mathbb{N}$, it follows by the pigeonhole principle that there exists some $i \in [n]$ such that $f(i) = (p, q)$ with $p > s$ or $q > t$. In particular, there exists an increasing subsequence of $a_1, \ldots, a_n$ of length $s+1$ or a decreasing subsequence of $a_1, \ldots, a_n$ of length $t+1$. $\square$

## 1.7 The Van der Waerden Thereom

An arithmetic progression of length $m$ is a sequence $(a, a+d, \dots, a+(m-1)d)$ for some $a, d \in \mathbb{N}$.

**Definition 1.21.** For any $r, m \in \mathbb{N}$, the *Van der Waerden number of $r$ and $m$*, denoted by $W(r, m)$, is the smallest integer $n$ such that whenever $[n]$ are coloured with $r$ colours there is a monochromatic arithmetic progression of length $m$.

**Theorem 1.22** (Van der Waerden 1927)**.** *For all $r, m \in \mathbb{N}$, $W(r, m)$ exists.*

Theorem 1.22 is proved by induction on the length $m$ of the desired arithmetic progression. More specifically, it follows from a more detailed statement, for which we need the following definition.

**Definition 1.23.** Let $A_1, \dots, A_k$ be arithmetic progressions each of length $m$, where $A_i = (a_i, a_i + d_i, \dots, a_i + (m-1)d_i)$ for every $i \in [k]$. We say that $A_1, \dots, A_k$ are *focused at $f$* if $f = a_i + md_i$ for every $i \in [k]$.

**Lemma 1.24.** *For any $r, m, k \in \mathbb{N}$ with $k \in [r]$, there exists an integer $n = n(r, m, k)$ such that in any $r$-colouring of $[n]$ there is either a monochromatic arithmetic progression of length $m$ or a set of $k$ focused arithmetic progressions $A_1, \dots, A_k$ of length $m-1$, each of which is monochromatic in a different colour, and whose focus $f$ is at most $n$ (or both).*

We now deduce Theorem 1.22 using Lemma 1.24.

**Proof of Theorem 1.22.** Given any $r \in \mathbb{N}$, let $n = n(r, m, r)$, which exists by Lemma 1.24, and consider any $r$-colouring of $[n]$. By definition of $n$ we can then find either a monochromatic arithmetic progression of length $m$ or a collection of $r$ monochromatic focussed arithmetic progressions $A_1, \dots, A_r$ of length $m-1$ and different colour whose focus $f$ is at most $n$. Since there are only $r$ colours, in the latter case the focus $f$ must be the same colour as some $A_j$, and together these form a monochromatic arithmetic progression of length $m$. So in any case such a progression exists. $\square$

We will only prove Lemma 1.24 for the case when $m = 3$ (and so we will in fact only prove Van der Waerden's theorem in the case $m = 3$).

**Remark 1.25.** Adding $t \in \mathbb{Z}$ to every element of an arithmetic progression yields a new arithmetic progression of the same length. In particular, the assertion of Lemma 1.24 holds for every set of $n$ consecutive integers, that is, instead of colouring $[n]$ we can colour $\{t+1, \dots, t+n\}$ for some $t \in \mathbb{Z}$.

**Proof of Lemma 1.24 for $m = 3$.** Fix any $r \in \mathbb{N}$; we show by induction on $k$ that $n(r, 3, k)$ exists for any $k \in [r]$. For the base case $k = 1$, note that whenever $[r+1]$ is $r$-coloured there are two elements of the same colour; these form a monochromatic arithmetic progression of length two, say $(a, a+d)$. The focus of this arithmetic progression is $a + 2d \leq 2r + 1$, and so we find that $n(r, 3, 1)$ exists and is at most $2r + 1$.

Now suppose that $n(r, 3, k)$ exists for some $1 \leq k < r$, let $n = n(r, 3, k)$ and $N = n(r^n + 1)$. Consider an $r$-colouring of the set $[N]$. Partition $[N]$ into $r^n + 1$ blocks $B_i$ of consecutive integers each of length $n$, that is, $B_i = \{(i-1)n + 1, \dots, in\}$ for every $i \in [r^n + 1]$. Since $|B_i| = n$ for every $i \in [r^n + 1]$, there are $r^n$ possible ways to $r$-colour each block. Since there are $r^n + 1$ blocks, it thus follows by the pigeonhole principle that two blocks are coloured precisely the same; that is, there exist positive integers $s$ and $t$ such that $s + t \leq r^n + 1$ and $p$ and $p + nt$ have the same colour for every $p \in B_s$.

By our choice of $n$, the block $B_s$ either contains a monochromatic arithmetic progression of length $3$ or $k$ monochromatic focused arithmetic progressions $A_1, \dots, A_k$ of length $2$, each in a different colour, whose focus is also in $B_s$ (or both). So either the colouring of $[N]$ contains a monochromatic arithmetic progression of length $3$, or the focus $f$ of $A_1, \dots, A_k$ has a different colour than each $A_j$ (or both). For each $j \in [k]$, let $A_j = (a_j, a_j + d_j)$; then $f = a_1 + 2d_1 = \ldots = a_{r-1} + 2d_{r-1}$. It follows by the choice of $s$ and $t$ that $f, f + nt$ have the same colour, and also that, for any $j \in [k]$, the numbers $a_j, a_j + d_j, a_j + nt, a_j + d_j + nt$ have the same colour. For every $j \in [k]$, let $A'_j = (a_j, a_j + d_j + nt)$ and let $A'_{k+1} = (f, f + nt)$. Note that $A'_1, \ldots, A'_k, A'_{k+1}$ are $k + 1$ monochromatic arithmetic progressions of length $2$, each in a different colour. Moreover, they are focused at $f' = f + 2nt = a_j + 2d_j + 2nt$. Since $f'$ is at most $2N$ and our $r$-colouring of $[N]$ was arbitrary, it follows that in any $r$-colouring of $[2N]$ there is either a monochromatic arithmetic progression of length $3$ or a collection of $k + 1$ monochromatic arithmetic progressions of length $2$, each in a different colour, whose focus is at most $2N$. That is, $n(r, 3, k+1)$ exists and is at most $2N$.

We therefore conclude by induction that $n(r, 3, k)$ exists for any $k \in [r]$. $\square$

# Chapter 2

# Positional Games

**Definition 2.1.** A *positional game* $(X, \mathcal{F})$ is a two-player perfect information game in which

- $X$ is a (usually finite) set, called the *board*,
- $\mathcal{F}$ is a family of finite subsets of $X$, called the *winning sets*, and
- the two players take turns to claim previously-unclaimed elements of the board.

The player who claims first will be called Player 1 and the other player will be called Player 2. The types of positional game which we shall study are strong games, maker-breaker games (the latter are sometimes called *weak games*) and breaker-maker games

- In a *strong game* $(X, \mathcal{F})$, the player who first claims all elements of a winning set $A \in \mathcal{F}$ is the winner. If neither player succeeds in doing so the game is drawn.
- In a *maker-breaker game* $(X, \mathcal{F})$, the Player 1 wins if he or she succeeds in claiming all elements of some winning set $A \in \mathcal{F}$, otherwise the Player 2 wins. In particular, it is not possible to draw a maker-breaker game.
- In a *breaker-maker game* $(X, \mathcal{F})$, the Player 2 wins if he or she succeeds in claiming all elements of some winning set $A \in \mathcal{F}$, otherwise the Player 1 wins.

In a maker-breaker game, we often write *maker* and *breaker* for Player 1 and Player 2, respectively.

We will sometimes call Player 1 Red, and the Player 2 Blue; it may help to imagine each player colouring their chosen element on each move (so, for instance, in a strong game each player is trying to colour a winning set in his or her own colour). The following games are examples of positional games.

1. $3 \times 3$ *tic-tac-toe* (noughts and crosses) can be described as a strong game $(X, \mathcal{F})$, where $X = \{1, 2, 3, \ldots, 9\}$ and

$$\mathcal{F} = \{\{1,2,3\}, \{4,5,6\}, \{7,8,9\}, \{1,4,7\}, \{2,5,8\}, \{3,6,9\}, \{1,5,9\}, \{3,5,7\}\}.$$

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

The game of $n \times n$ *tic-tac-toe* is defined similarly as the strong game where the board is an $n \times n$ grid and the winning sets are the $n$ columns, the $n$ rows and the 2 diagonals.

2. Similarly, $n \times n$ *maker-breaker tic-tac-toe* is the maker-breaker game with the same board and winning sets (but the game is different because the players' objectives are different).
3. The *Ramsey Game* RG$(n, t)$ is the strong game $(X, \mathcal{F})$ where $X = E(K_n)$, the edge set of the complete graph on $n$ vertices, and $\mathcal{F}$ is the family of all sets of edges which form a copy of $K_t$ in $K_n$.

4. The game of HEX[1] is a maker-breaker game $(X, \mathcal{F})$ in which the board $X$ consists of hexagonal spaces and the winning sets are all crossing paths between the sides in the colour of Player 1. In order to see that this is in fact a maker-breaker game, one has to prove that it is impossible for both players to cross (intuitively clear, but the actual proof requires some topology) and that, by the end of the game, one player will surely cross (David Gale 1979).

[1] See e.g. `http://en.wikipedia.org/wiki/Hex_(board_game)`

## 2.1 Strategies

Given some game, we would like to know whether it ends in a draw, a win for Player 1, or a win for Player 2, assuming that both players make the best moves possible. We also want to know how to achieve a win or draw. In order to frame these questions sensibly, we need the notion of a strategy. Intuitively, a strategy is a rule book which tells us what move to play next in any position. For simplicity, we will only give a rigorous definition of a strategy for positional games.

**Definition 2.2.** A *strategy* $\mathcal{S}$ for Player 1 in a positional game $(X, \mathcal{F})$ is a function which, given any move number $t \geq 1$, any set $R \subseteq X$ of size $t-1$ consisting of all board elements claimed so far by Player 1 and any set $B \subseteq X \setminus R$ of size $t-1$ consisting of all board elements claimed so far by Player 2, returns a free board element $\mathcal{S}(R, B) \in X \setminus (R \cup B)$ which Player 1 should choose next.

Similarly, a strategy $\mathcal{S}$ for Player 2 is a function which, given any move number $t \geq 1$, any set $R \subseteq X$ of size $t$ consisting of all board elements claimed so far by Player 1 and any set $B \subseteq X \setminus R$ of size $t-1$ consisting of all board elements claimed so far by Player 2, returns a free board element $\mathcal{S}(R, B) \in X \setminus (R \cup B)$ which Player 2 should choose next.

We say that a player is *following* a strategy $\mathcal{S}$ if he or she always makes the move returned by this function. A strategy $\mathcal{S}$ is a *winning strategy* for some player in a game if, by following $\mathcal{S}$, this player wins regardless of the strategy of the other player. A strategy $\mathcal{S}$ is a *drawing strategy* for some player in a game if, by following $\mathcal{S}$, this player either draws or wins regardless of the strategy of the other player.

**Remark 2.3.** Any winning strategy is also a drawing strategy. The inverse implication is not true in general, but is true for maker-breaker games, since draws are not possible.

**Proposition 2.4.** *If $\mathcal{S}$ is a winning strategy for Breaker in the maker-breaker game $(X, \mathcal{F})$, then $\mathcal{S}$ is a drawing strategy for Player 2 in the strong game $(X, \mathcal{F})$.*

**Proof.** The fact that $\mathcal{S}$ is a winning strategy for Breaker in the maker-breaker game $(X, \mathcal{F})$ means that if Breaker follows the moves given by $\mathcal{S}$, he will win regardless of Maker's strategy. That is, Maker cannot claim all the elements of any winning set $A \in \mathcal{F}$, regardless of what moves Maker plays. It follows that in the strong game $(X, \mathcal{F})$, provided Player 2 follows the moves given by $\mathcal{S}$, Player 1 cannot win, and therefore Player 2 must either draw or win. So $\mathcal{S}$ is a drawing strategy for Player 2 in the strong game $(X, \mathcal{F})$. $\square$

The following theorem, from classical Game Theory, shows that, for games from a certain natural class, there are only three possible outcomes.

**Theorem 2.5** (Zermelo 1913)**.** *Let $\mathcal{G}$ be a finite[2], perfect information two-player game with no chance moves, then exactly one of the following three statements is true.*

1. *Player 1 has a winning strategy for $\mathcal{G}$.*
2. *Player 2 has a winning strategy for $\mathcal{G}$.*
3. *Both players have drawing strategies for $\mathcal{G}$.*

In the first case above we say that $\mathcal{G}$ is a *Player 1's win*, in the second we say that $\mathcal{G}$ is a *Player 2's win* and in the third we say that $\mathcal{G}$ is a *draw*. In fact, for strong games we can rule out the second option.

**Theorem 2.6.** *For any finite set $X$, and any family $\mathcal{F}$ of subsets of $X$, there is no winning strategy for Player 2 in the strong game $(X, \mathcal{F})$.*

[2]Here finite means that there are only finitely many possible positions; for a positional game this is equivalent to saying that the board is finite. This result also holds for many classes of infinite games.

**Proof.** Assume for the sake of contradiction that Player 2 has a winning strategy $\mathcal{S}$ for $(X, \mathcal{F})$. Using $\mathcal{S}$ we will devise a Player 1's winning strategy $\mathcal{S}'$ for $(X, \mathcal{F})$. This will contradict Theorem 2.5. In Player 1's first move, Player 1 claims an arbitrary board element $x \in X$. In every other move Player 1 follows $\mathcal{S}$, that is, Player 1 pretends that $x$ is still free and that Player 1 is 'Player 2' (and thus he/she can follow $\mathcal{S}$). Whenever $\mathcal{S}$ instructs Player 1 to claim an element which is already his/hers, Player 1 claims another arbitrary free element instead (and pretends that that element is still free). It remains to prove that this is a winning strategy for Player 1. Since $\mathcal{S}$ is a Player 2's winning strategy for $(X, \mathcal{F})$, it follows that there will be a point in the game where (ignoring the one additional board element) Player 1 will have a fully claimed $A \in \mathcal{F}$, whereas Player 2 will not have such a set. The additional board element of Player 1 will not change this fact. $\square$

This method of proof is known as a 'strategy-stealing' argument (because Player 1 'steals' the hypothetical winning strategy for Player 2). Note that the proof of Theorem 2.6 can be used to prove the following result.

**Proposition 2.7.** *Let $X$ be a finite set and $\mathcal{F}$ a family of subsets of $X$. If Maker has a winning strategy in the breaker-maker game $(X, \mathcal{F})$, then Maker also has a winning strategy in the maker-breaker game $(X, \mathcal{F})$.*

As an immediate corollary of Theorems 2.5 and 2.6 we have the following.

**Corollary 2.8.** *If $X$ is a finite set, and $\mathcal{F}$ is a family of subsets of $X$, then exactly one of the following statements is true.*

1. *Player 1 has a winning strategy for the strong game $(X, \mathcal{F})$.*
2. *Both players have drawing strategies for the strong game $(X, \mathcal{F})$.*

Using Ramsey Theory, we can sometimes eliminate the second option as well.

**Proposition 2.9.** *Let $X$ be a finite set and let $\mathcal{F}$ be a family of subsets of $X$. Suppose that every red/blue-colouring of $X$ yields a monochromatic $A \in \mathcal{F}$. Then Player 1 has a winning strategy for the strong game $(X, \mathcal{F})$. Moreover, Maker has a winning strategy for the maker-breaker game $(X, \mathcal{F})$.*

**Proof.** Assume for a contradiction that Player 1 does not have a winning strategy for $(X, \mathcal{F})$. It follows by Corollary 2.8 that both players have drawing strategies for the strong game $(X, \mathcal{F})$. If both players follow their drawing strategies, then the game must end in a draw. This means that $X$ can be coloured red and blue such that no $A \in \mathcal{F}$ is monochromatic, contrary to our assumption that no such colouring exists.

If Breaker has a winning strategy for the maker-breaker game $(X, \mathcal{F})$, then by Proposition 2.4 then Player 2 has a drawing strategy for the strong game $(X, \mathcal{F})$, a contradiction. $\square$

Let us review some of the games described in the previous section.

1. It is easy to see that $n \times n$ tic-tac-toe is a win for Player 1 if $n \leq 2$, whilst a straightforward case analysis[3] shows that it is a draw if $n = 3$.
2. On the other hand, a similar case analysis shows that the $3 \times 3$ maker-breaker tic-tac-toe game is Maker's win (see Problem Sheet 3). In particular, this shows that the converse of Proposition 2.4 does not hold in general[4].

[3] See `http://xkcd.com/832/`

[4] This fact seems quite surprising at first. The point is that in the strong game $(X, \mathcal{F})$ Player 2 can threaten to win immediately by completing some winning set $A \in \mathcal{F}$. This forces Player 1 to "waste" moves to defend against these threats, but such threats are not available for Breaker (because Maker doesn't care if Breaker completes a winning set). As an exercise, what goes wrong if we try to adapt the proof of Proposition 2.4 to prove the (false) converse?

3. If $n \geq R(t,t)$, then it follows by Proposition 2.9 that the Ramsey Game RG$(n,t)$ is a Player 1's win. For most smaller values of $n$ we do not know whether RG$(n,t)$ is a draw or a Player 1's win. Let $R_G(t)$ denote the smallest integer $n$ such that Player 1 has a winning strategy for RG$(n,t)$. Proposition 2.9 implies that $R_G(t) \leq R(t,t)$. The inverse inequality is not true in general. For example, a simple case analysis shows that $R_G(3) = 5 < 6 = R(3,3)$.
4. A strategy-stealing argument shows that Maker (Player 1) has a winning strategy for HEX. No explicit winning strategy is known for boards of side length greater than 7.

## 2.2 The Erdős-Selfridge Theorem

The Erdős-Selfridge Theorem provides a sufficient condition for Breaker to win a maker-breaker game.

**Theorem 2.10.** *Let $X$ be a set and let $\mathcal{F}$ be a family of subsets of $X$. If*

$$\sum_{A\in\mathcal{F}} 2^{-|A|} < \frac{1}{2}$$

*then Breaker has a winning strategy for the maker-breaker game $(X, \mathcal{F})$.*

Recall that Proposition 2.4 stated that a winning strategy for Breaker in a maker-breaker game $(X, \mathcal{F})$ is also a drawing strategy for Player 2 in the strong game $(X, \mathcal{F})$. Therefore, Theorem 2.10 can also be viewed as providing a sufficient condition for Player 2 to draw a strong game.

Also, note that if $\mathcal{F}$ is $n$-uniform, meaning that every $A \in \mathcal{F}$ has size $n$, then the condition in Theorem 2.10 states simply that $|\mathcal{F}| < 2^{n-1}$. This is best-possible in the sense that, for every $n \in \mathbb{N}$, there exists a set $X$ and an $n$-uniform family $\mathcal{F}$ of subsets of $X$ such that $|\mathcal{F}| = 2^{n-1}$ and Maker has a winning strategy for the maker-breaker game $(X, \mathcal{F})$. Indeed, one example is to let $X = \{w, x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}\}$ and let $\mathcal{F}$ be the family of all subsets $A$ of $X$ such that $w \in A$ and $|A \cap \{x_i, y_i\}| = 1$ for every $i \in [n-1]$. It follows from this definition that $\mathcal{F}$ is $n$-uniform and that $|\mathcal{F}| = 2^{n-1}$, so it remains only to present a winning strategy for Maker. On Maker's first move, Maker claims $w$. Then, on Maker's $i$th move for $1 < i \leq n$, he or she lets $z$ be the element claimed by Breaker in Breaker's $(i-1)$th move. Since Maker already claimed $w$, we have $z \in \{x_j, y_j\}$ for some $j \in [n-1]$, and Maker's $i$th move is to claim the other element of $\{x_j, y_j\}$ (that is, Maker claims $x_j$ if $z = y_j$; otherwise Maker claims $y_j$ if $z = x_j$). Consequently, after each move by Maker, for any $j \in [n-1]$ either both elements of $\{x_j, y_j\}$ will have been claimed, one by each player, or neither will have been claimed. Maker can therefore continue this strategy until all elements have been claimed. At the end of the game, let $M \subseteq X$ denote the set of elements claimed by Maker throughout the game; then $w \in M$ and $|M \cap \{x_i, y_i\}| = 1$ for every $i \in [n-1]$. So $M \in \mathcal{F}$, and therefore Maker wins as claimed.

**Proof of Theorem 2.10.** We define a strategy for Breaker based on a *potential function*. Indeed, consider any point in the game, let $M \subseteq X$ be the set of board elements claimed by Maker up to this point, and let $B \subseteq X$ be the set of board elements claimed by Breaker up to this point. Then we define the *potential* of the game at this point to be

$$D = \sum_{\substack{A\in\mathcal{F}\\ A\cap B=\emptyset}} 2^{-|A\setminus M|}.$$

Observe that if Maker then claims an element $x$, meaning that $x$ is added to $M$, then the effect on $D$ is to double the contribution made by each set $A \in \mathcal{F}$ with $x \in A$ and $A \cap B = \emptyset$. In particular, $D$ is at most doubled. More specifically, $D$ increases by $\phi(x) := \sum 2^{-|A\setminus M|}$, where the sum is taken over all $A \in \mathcal{F}$ such that $A \cap B = \emptyset$ and $x \in A$. If instead Breaker claims an element $y$, meaning that $y$ is added to $B$, then the effect on $D$ is to remove from the sum all terms for sets $A$ with $y \in A$ and $A \cap B = 0$. In other words, $D$ decreases by $\phi(y)$.

Let $D_i$ be the potential after Maker's $i$th move. So $D_0 < 1/2$. As observed above, any move by Maker can at most double the potential, so before Breaker's first move we have $D_1 < 1$. If Breaker's strategy guarantees that $D_i < 1$ for all $i$, then Maker will never claim all elements of any set $A \in \mathcal{F}$. Indeed, if Maker claimed all elements of some set $A^* \in \mathcal{F}$, then the contribution of $A^*$ to the potential $D$ would then be $2^0 = 1$, whilst the contribution of any other $A \in \mathcal{F}$ would be non-negative since $2^x > 0$ for any $x \in \mathbb{R}$. We would therefore have $D \geq 1$, giving a contradiction.

We can now describe Breaker's strategy as follows. Suppose that Maker has finished his or her $i$th move (with $i \geq 1$), so the current potential is $D_i$. Whenever it is Breaker's turn to move he or she should claim the element $y$ which maximises $\phi(y)$ in the current position (choosing arbitrarily if there is more than one such $y$). As described above, this reduces $D_i$ by $\phi(y)$. The element $y$ is consequently added to $B$, which may alter the values of $\phi(x)$ for other board elements $x \in X$, but crucially, for any $x \in X$ the value of $\phi(x)$ cannot increase as a result of adding $y$ to $B$. That is, for any $x \in X$, we have $\phi_{\mathrm{new}}(x) \leq \phi(x)$, where we write $\phi_{\mathrm{new}}(x)$ for the updated value after adding $y$ to $B$, and continue to write $\phi(y)$ for the value prior to adding $y$ to $B$. Following Breaker's move claiming $y$, Maker will then respond by claiming some element $x$, with the effect of increasing $D$ by $\phi_{\mathrm{new}}(x) \leq \phi(x) \leq \phi(y)$, where the second inequality holds by choice of $y$. So, regardless of the move chosen by Maker, we have

$$D_{i+1} = D_i - \phi(y) + \phi_{\mathrm{new}}(x) \leq D_i \leq D_{i-1} \leq D_1 < 1$$

as required. □

**Remark 2.11.** For a breaker-maker game, in which Breaker plays first (but the objectives are unchanged, so Breaker is still trying to prevent Maker from claiming all elements of a winning set), essentially the same proof shows that the condition in Theorem 2.10 can be relaxed to $\sum_{A\in\mathcal{F}} 2^{-|A|} < 1$.

**Example 2.12.** In $n \times n$ tic-tac-toe, there are $2n+2$ winning sets ($n$ rows, $n$ columns and 2 diagonals), each of size $n$. It follows by Theorem 2.10 that if $2n+2 < 2^{n-1}$, then Breaker has a winning strategy for $n \times n$ maker-breaker tic-tac-toe, and the second player has a drawing strategy in $n \times n$ tic-tac-toe. Note that $2n+2 < 2^{n-1}$ holds for every $n \geq 5$. As described earlier, a simple case analysis shows that Maker wins $n \times n$ maker-breaker tic-tac-toe for $n = 3$, and it is easy to see that he wins if $n = 1$ or $n = 2$. The $4 \times 4$ game is Breaker's win, but an argument other than the Erdős–Selfridge Theorem is needed.

**Example 2.13.** In the Ramsey game RG$(n, t)$, there are $\binom{n}{t}$ winning sets, each of size $\binom{t}{2}$. It follows by Theorem 2.10 that if $\binom{n}{t} < 2^{\binom{t}{2}-1}$, then the Player 2 has a drawing strategy for this game, so $R_G(t) > n$. Together with our earlier observation that $R(t,t) \geq R_G(t) > n$ (or Proposition 2.9 implies that $R(t,t) > n$), this gives another proof of Theorem 1.13.

## 2.3 Pairing strategies

One way to exhibit a drawing strategy for Player 2 in a strong game, or a winning strategy for Breaker in a maker-breaker game, is via a *pairing strategy*.

**Definition 2.14.** Fix a board $X$ and a family $\mathcal{F}$ of winning sets. Then a *pairing strategy* for Player 2 in the strong game $(X, \mathcal{F})$, or for Breaker in the maker-breaker game $(X, \mathcal{F})$, proceeds as follows.
**Preparation:** find distinct elements $x_1, \dots, x_k, y_1, \dots, y_k \in X$ with the property that for every winning set $A \in \mathcal{F}$, there exists some $j \in [k]$ such that $\{x_j, y_j\} \subseteq A$.
**Playing:** suppose it is the Player 2's $i$th move for some $i \geq 1$. Let $z \in X$ be the board element just claimed by Player 1. If $z = x_j$ for some $j \in [k]$, and $y_j$ is unclaimed, then Player 2 should claim $y_j$. If instead $z = y_j$ for some $j \in [k]$, and $x_j$ is unclaimed, then Player 2 should claim $x_j$. In any other case Player 2 may claim an arbitrary element.

By playing in the described manner, Player 2 will claim at least one element from $\{x_j, y_j\}$ for each $j \in [k]$. By choice of the elements $x_1, \dots, x_k$ and $y_1, \dots, y_k$ this ensures that Player 2 will claim at least one element from each winning set, and therefore that the Player 1 will not win. The difficulty in implementing such a strategy (which is not always possible) is to find elements $x_1, \dots, x_k$ and $y_1, \dots, y_k$ with the property described above.

**Example 2.15.** Breaker has a pairing strategy in $5 \times 5$ maker-breaker tic-tac-toe as indicated in the table below. Every two board elements which are assigned the same number form a pair. Note that the central element is not in any pair.

| | | | | |
|---|---|---|---|---|
| 11 | 1 | 8 | 1 | 12 |
| 6 | 2 | 2 | 9 | 10 |
| 3 | 7 | | 9 | 3 |
| 6 | 7 | 4 | 4 | 10 |
| 12 | 5 | 8 | 5 | 11 |

Suppose that we have a pairing strategy for $n \times n$ maker-breaker tic-tac-toe game $(X, \mathcal{F})$ for $n \geq 5$ with $X = \{a_{i,j} : i, j \in [n]\}$ and elements $x_1, \dots, x_k, y_1, \dots, y_k$. Now consider $(n+1) \times (n+1)$ maker-breaker tic-tac-toe game $(X^*, \mathcal{F}^*)$ with $X^* = \{a_{i,j} : i, j \in [n+1]\}$. The number of $A \in \mathcal{F}^*$ such that $\{x_j, y_j\} \not\subseteq A$ for all $j \in [k]$ is $3$. To be precise, such winning sets are the horizontal line on the $(n+1)$st row $\{a_{n+1,i} : i \in [n+1]\}$, the vertical line on the $(n+1)$st column $\{a_{i,n+1} : i \in [n+1]\}$ and the 'backward' diagonal line $\{a_{i,n+2-i} : i \in [n+1]\}$. Let

$$x_{k+1} = a_{n+1,1}, \qquad y_{k+1} = a_{1,n+1}, \qquad x_{k+2} = a_{n+1,2},$$
$$y_{k+2} = a_{n+1,3}, \qquad x_{k+3} = a_{2,n+1}, \qquad y_{k+3} = a_{3,n+1}.$$

Note that we have a pairing strategy for $(n+1) \times (n+1)$ maker-breaker tic-tac-toe game with elements $x_1, \dots, x_{k+3}, y_1, \dots, y_{k+3}$.

**Example 2.16.** Breaker does not have a pairing strategy in $4 \times 4$ maker-breaker tic-tac-toe. Suppose the contrary, let $x_1, \dots, x_k, y_1, \dots, y_k$ be such elements. Since $|A \cap A'| \leq 1$ for any distinct winning sets $A, A'$, each winning set $A$ contains its own unique $x_i, y_i$. That is, for each $A \in \mathcal{F}$, there exists a unique $i$ such that $x_i, y_i \in A$. This implies that $k \geq |\mathcal{F}| \geq 10$. However, $X$ has only 16 elements, a contradiction.

On the other hand, once each player has played one move, a pairing strategy becomes possible. It thus suffices to consider all possible first moves of Player 1 (denoted by X) and the appropriate response of Player 2 (denoted by O). Up to symmetry there are only 3 cases to consider.

| | | | |
|---|---|---|---|
| X | 1 | 2 | 1 |
| 3 | O | 4 | 6 |
| 3 | 4 | 7 | 7 |
| 5 | 5 | 2 | 6 |

| | | | |
|---|---|---|---|
| 1 | X | 5 | 5 |
| 1 | O | 4 | 6 |
| 7 | 4 | 3 | 7 |
| 2 | 2 | 3 | 6 |

| | | | |
|---|---|---|---|
| 5 | 6 | 1 | 1 |
| 7 | X | O | 2 |
| 4 | 4 | 5 | 2 |
| 7 | 6 | 3 | 3 |

Whilst we have only considered pairing strategies for Player 2, Player 1 may proceed similarly: Player 1 chooses a first move arbitrarily and then follows a pairing strategy (see the extremal example for the Erdős-Selfridge theorem.

## 2.4 Playing games in parallel

Finally, another way to exhibit a drawing strategy for Player 2 in a strong game, or a winning strategy for Breaker in a maker-breaker game, is by considering a game as a combination of several 'smaller games', which are played *in parallel*.

**Definition 2.17.** The $m \times m$ *$n$-in-a-row maker-breaker game* is the maker-breaker game $(X, \mathcal{F})$ where $X$ is an $m \times m$ square grid and $\mathcal{F}$ consists of all sets of $n$ consecutive grid squares in a vertical, horizontal or diagonal straight line.

Similarly, the *unrestricted-$n$-in-a-row maker-breaker game* is the maker-breaker game $(X, \mathcal{F})$ where $X = \mathbb{Z}^2$ is the 2-dimensional infinite square grid and, as before, $\mathcal{F}$ consists of all sets of $n$ consecutive grid squares in a vertical, horizontal or diagonal straight line.

Naturally, each of these games can also be played as a strong game.

Note that $n \times n$ $n$-in-a-row is exactly $n \times n$ tic-tac-toe. It is known that the unrestricted-$n$-in-a-row maker-breaker game is Maker's win[5] for $n \leq 4$ and Breaker's win for $n \geq 8$; the cases $5 \leq n \leq 7$ remain an open problem. We will prove the following weaker result.

**Proposition 2.18.** *The unrestricted-$n$-in-a-row maker-breaker game is Breaker's win for every $n \geq 40$.*

Since there are infinitely many winning sets, we cannot apply the Erdős–Selfridge Theorem directly. The main idea of the proof is to split the infinite game into infinitely many finite games which we play in parallel. More specifically, each smaller game will be a $40 \times 40$ $14$-in-a-row maker-breaker game. Note that this game is Breaker's win by the Erdős–Selfridge Theorem. Indeed, the $40 \times 40$ $14$-in-a-row maker-breaker game has at most $4 \cdot 40^2$ winning sets, since there are $4$ ways to choose the type of line, that is, vertical, horizontal, diagonal with slope $1$ or diagonal with slope $-1$, and at most $40^2$ ways to choose the endpoint with smaller $y$ coordinate if the line is vertical and smaller $x$ coordinate otherwise. Each of these winning sets has size $14$, and so the Erdős–Selfridge criterion is satisfied since $4 \cdot 40^2 < 2^{14-1}$.

**Proof of Proposition 2.18.** It suffices to prove the result for $n = 40$. Split the board into $40 \times 40$ pairwise disjoint squares, that is, let $\mathbb{Z}^2 = \bigcup_{(x,y) \in \mathbb{Z}^2} S_{xy}$ where, for every $(x, y) \in \mathbb{Z}^2$ we have $S_{xy} = \{(40x + i, 40y + j) : i, j \in [40]\}$. Breaker will play the $40 \times 40$ $14$-in-a-row maker-breaker game on $S_{xy}$ for each $(x, y) \in \mathbb{Z}^2$, and moreover Breaker will play these games in parallel. That is, whenever Maker claims an element of $S_{xy}$ for some $(x, y) \in \mathbb{Z}^2$ (according to whatever strategy Maker is playing), Breaker will respond by claiming an element of the same square $S_{xy}$ (according to Breaker's winning strategy for the $40 \times 40$ $14$-in-a-row maker-breaker game). This is always possible as Breaker is the second player and $|S_{xy}| = 40^2$ is even. So, if we ignore all moves except for those played in the square $S_{xy}$ for some $(x, y) \in \mathbb{Z}^2$, Maker and Breaker are taking turns to choose elements of $S_{xy}$, with Breaker following a winning strategy for the $40 \times 40$ $14$-in-a-row maker-breaker game. It follows that Breaker will win each of these smaller games, meaning that Maker will never succeed in claiming 14 consecutive elements of any square $S_{xy}$. Since every winning set of the unrestricted $40$-in-a-row game intersects at most $3$ of the $40 \times 40$ boards $S_{xy}$ (at most $2$ for vertical and horizontal lines and up to $3$ for diagonal), each such winning set must contain a sub-line of length $\lceil 40/3 \rceil = 14$ in some $40 \times 40$ board $S_{xy}$. As Maker will never succeed in claiming such a subline, we conclude that Maker will never succeed in winning the unrestricted $40$-in-a-row maker-breaker game. So the strategy described above is a winning strategy for Breaker. $\square$

[5] As an exercise, try to find an explicit winning strategy for Maker.

## 2.5 Connectivity game

A multigraph $G$ is a graph, where each edge may appear mutliple times (and we also allow loops). A graph is *connected* if every pair of vertices can be joined by a path. A *spanning tree* is minimal connected graph.

Given a multigraph $G$, the *connectivity game on* $G$ is a breaker-maker game $(X, \mathcal{F})$, where $X$ is the set of edges of $G$ and $\mathcal{F}$ is the set of spanning tree of $G$.

We investigate the property of $G$ that ensure a Maker's win. Clearly, $G$ must contain a spanning tree. In fact, $G$ having two edge-disjoint spanning trees is a sufficient and necessary condition for Maker's win in the connectivity game on $G$.

**Theorem 2.19.** *If Maker wins the connectivity game on a multigraph $G$, then $G$ has two edge-disjoint spanning trees.*

**Proof.** Suppose that Maker has a winning strategy $\mathcal{S}$ in the connectivity game on $G$. By Proposition 2.7, Maker has a winning strategy $\mathcal{S}'$ if he goes first. Now let play the game, where Maker follows $\mathcal{S}$ and Breaker follows $\mathcal{S}'$. Since both $\mathcal{S}'$ and $\mathcal{S}$ are 'winning strategies' for the connectivity games, both players will each claim a winning set, which is a spanning tree. These two spanning trees are clearly edge-disjoint. $\square$

**Theorem 2.20** (Lehman)**.** *If a multigraph $G$ has two edge-disjoint spanning trees, then Maker wins the connectivity game on $G$.*

**Proof.** We proceed by induction on $|V(G)|$. This is obvious for $|V(G)| = 2$, so we may assume that $|V(G)| \geq 3$. Let $T_1$ and $T_2$ be edge-disjoint spanning trees of $G$. If Breaker claims an edge $e_1$ belonging to one of these trees, say $T_1$, then this move cuts $T_1$ into two connected parts. Since $T_2$ is a spanning tree, there exists an edge $e_2 = uv$ of $T_2$ connecting these two parts. Maker then claims the edge $e_2$. Let $G'$ be the multigraph obtained from $G - e_1 - e_2$ by contracting $e_2$, that is, identifying its endpoints $u$ and $v$ (which may create more multiple edges). Let $T_1'$ and $T_2'$ be the updated $T_1 - e_1$ and $T_2 - e_2$, respectively. Note that $T_1'$ and $T_2'$ are edge-disjoint spanning trees of $G'$ and none of its edges are claimed. Note that $|V(G')| = |V(G)| - 1$. So by induction hypothesis, Maker wins the connectivity game played on $G'$. That is, Maker claims a spanning tree $T'$ of $G'$, which together with the edge $e_2$ corresponds a spanning tree of $G$.

If Breaker claims an edge $e'$ not belonging $T_1$ or $T_2$, then Maker will just pretend that Breaker claims an edge $e_1$ belonging to $T_1$ (and remove $e'$ from $G$) and precede as above. $\square$

# Chapter 3

# Extremal Set Theory

## 3.1 Posets

**Definition 3.1.** A *partially ordered set* or *poset* is a set $P$ together with a relation '$\leq$' which is

- reflexive, that is, $\forall x \in P$ we have $x \leq x$;
- antisymmetric, that is, $\forall x, y \in P$ if $x \leq y$ and $y \leq x$ then $x = y$;
- transitive, that is, $\forall x, y, z \in P$ if $x \leq y$ and $y \leq z$ then $x \leq z$.

We say that two elements $x$ and $y$ are *comparable* if $x \leq y$ or $y \leq x$. The poset is a *total ordering* if every two elements are comparable.

**Example 3.2.** 1. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, [n]$ with the usual order.
2. $[n] \times [n]$ with '$\leq$' defined by $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$. Note this is not a total ordering as $(n, 1)$ and $(1, n)$ are not comparable.
3. the set of all divisors of $n$ under divisibility (e.g. if $n = 20$ then $1, 2, 4, 5, 10$ are the divisors, $2 \leq 4$ but $4$ and $5$ are not comparable).
4. Let $S$ be a finite set. The set of all subsets of $S$ under inclusion is a poset (see Figure 3.1).

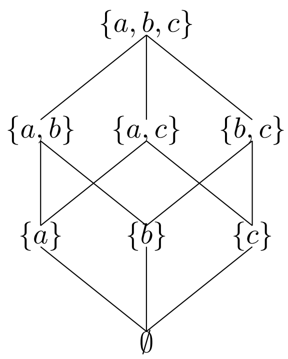


Figure 3.1: An example for $S = \{a, b, c\}$

5. The set of all proper subsets of a finite set $S$ with $|S| \geq 2$ under inclusion.

**Definition 3.3.** Let $(P, \leq)$ be a poset. We say that $x \in P$ is a *maximal element of* $P$ if there exists no $y \in P$ such that $x \leq y$ and $x \neq y$.

**Example 3.4.** 1. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, [n]$ with the usual order do not have a maximal element.
2. $[n] \times [n]$ (with '$\leq$' defined as above): the unique maximal element is $(n, n)$ .
3. the set of all divisors of $n$ under divisibility: $n$ is the only maximal element.

4. the set of all subsets of a finite set $S$ under inclusion: $S$ is the only maximal element.
5. the set of all proper subsets of a finite set $S$ with $|S| \geq 2$ under inclusion: every $(|S|-1)$-element subsets of $S$ is a maximal element.

**Fact 3.5.** *Every finite poset has a maximal element.*

**Proof.** Exercise. □

## 3.2 Partition into chains and antichain

**Definition 3.6.** Let $(P, \leq)$ be a poset. A subset $C \subseteq P$ is a *chain* if any two elements of $C$ are comparable. A subset $A \subseteq P$ is an *antichain* if any two distinct elements of $A$ are incomparable.

**Example 3.7.**
1. $\mathbb{N}$ itself is a chain with the usual order, and every antichain can contain only one element.
2. $[n] \times [n]$ with the above order:
   $C = \{(x, 1)|\ x \in [n]\} \cup \{(n, y)|\ y \in [n]\}$ is a chain.
   $A = \{(1, n), (2, n-1), \ldots, (n, 1)\}$ is an antichain.
3. subsets of $[4]$ under inclusion:
   $C = \{\emptyset, 1, 12, 123, 1234\}$ is a chain.
   $A = \{12, 13, 14, 23, 24, 34\}$ is an antichain.
   $A' = \{12, 13, 23, 4\}$ is also an antichain.

**Remark 3.8.** If a poset $(P, \leq)$ can be partitioned into $r$ chains (that is, there are $r$ disjoint chains $C_1, \ldots, C_r$ such that $P = C_1 \cup \ldots \cup C_r$), then the largest antichain has size at most $r$, since it meets every chain in at most one element. Similarly, if $P$ can be partitioned into $r$ antichains, then the largest chain has size at most $r$.

We will investigate the partitionings of a poset into chains/antichains more closely. Before we prove some theorems let's consider another example: the poset on $[n] \times [n]$ with the above order. Let $C_i = \{(i, x)|\ 1 \leq x \leq n\}$ for all $1 \leq i \leq n$. Then $C_1 \cup \ldots C_n$ is a partition of the poset into $n$ chains, hence, the largest antichain has at most $n$ elements. Let $A = \{(1, n), (2, n-1), \ldots, (n, 1)\}$, clearly, $A$ is an antichain with $n$ elements.

**Theorem 3.9** (Mirsky 1971)**.** *Let $(P, \leq)$ be a poset. If the largest chain in $P$ has size $r$, then $P$ can be partitioned into $r$ antichains.*

**Proof.** Let $A_i$ be the set of all $x \in P$ such that the longest chain whose greatest element is $x$ has exactly $i$ points, including $x$. Clearly, $A_i = \emptyset$ for $i > r$ since the largest chain has size $r$. Moreover, $P = A_1 \cup A_2 \cup \ldots \cup A_r$ and the sets $A_i$ are disjoint.

We show that each $A_i$ is an antichain. Suppose not. Then let $x, y \in A_i$ be such that $x \leq y$ and $x \neq y$. Since $x \in A_i$, there exists a chain $C$ of size $i$ whose greatest element is $x$. Then $C \cup \{y\}$ is a chain whose greatest element is $y$ and $|C \cup \{y\}| = i + 1$, contradicting the fact that $y \in A_i$. □

Dilworth's theorem below is closely related to Mirsky's theorem, these theorems are also called the *dual* of each other. However, it is harder to prove.

**Theorem 3.10** (Dilworth 1950)**.** *Let $P$ be a finite poset. Suppose that the largest antichain in $P$ has size $r$. Then $P$ can be partitioned into $r$ chains.*

**Proof.** (due to Galvin, 1994) We proceed by induction on $|P|$. If $|P| = 1$ then there is nothing to show. So suppose that $|P| > 1$. Let $x$ be a maximal element of $P$.

Suppose first that the largest antichain of $P \setminus \{x\}$ has size $r - 1$. Then, by our induction hypothesis, $P \setminus \{x\}$ can be partitioned into $r - 1$ chains $C_1, \ldots, C_{r-1}$. Thus, $C_1, \ldots, C_{r-1}, \{x\}$ is a partition of $P$ into $r$ chains.

We may assume that the largest antichain of $P \setminus \{x\}$ has size $r$. By our induction hypothesis, $P \setminus \{x\}$ can be partitioned into $r$ chains $C_1, \ldots, C_r$. For each $i \in [r]$, let $a_i$ be the highest element of $C_i$ which belongs to some antichain of size $r$ in $P \setminus \{x\}$. That is,

**Claim 3.11.** *$A = \{a_1, \ldots, a_r\}$ is an antichain.*

**Proof of claim.** Suppose not that $a_i \leq a_j$ for some $i \neq j$. Let $A'$ be an antichain of size $r$ in $P \setminus \{x\}$ which contains $a_j$. Then $A'$ meets each of $C_1, \ldots, C_r$ in precisely one point. Let $\{a\} = A' \cap C_i$. By definition of $a_i$, we have that $a \leq a_i$ and so $a \leq a_j$ This contradicts the fact that $A'$ is an antichain with $a, a_j \in A'$. □

Note that $A \cup \{x\}$ is not an antichain since the largest antichain in $P$ has size $r$. So $a_i \leq x$ for some $i$. Thus $C^* = \{x\} \cup \{a \in C_i \mid a \leq a_i\}$ is a chain. By the definition of $a_i$, $P \setminus C^*$ does not contains an antichain of size $r$. Hence by our induction hypothesis, $P \setminus C^*$ can be partitioned into $r-1$ chains $C'_1, \ldots, C'_{r-1}$. Then $C'_1, \ldots, C'_{r-1}, C^*$ is a partition of $P$ into $r$ chains, as required.

□

Dilworth's theorem implies Hall's matching theorem. A matching is a set of vertex-disjoint edges. For a graph $G$ and a subset $S \subseteq V(G)$, let $N(S)$ be the *neighbourhood of* $S$, the set of vertices that are adjacent to some vertex in $S$.

**Theorem 3.12** (Hall). *Let $G$ be a bipartite graph with vertex classes $A$ and $B$ such that $|N(S)| \geq |S|$ for all $S \subseteq A$. Then $G$ contains a matching covering all of $A$.*

**Proof.** We define a poset $(P, \leq)$ such that $P = A \cup B$; $x \leq x$ for all $x \in P$; $a \leq b$ if and only $a \in A$ and $b \in N(a) \subseteq B$. It is easy to see that each of $A$ and $B$ is an antichain. We claim that every antichain has at most $|B|$ elements. Let $X$ be an antichain and $S = X \cap A$. Then $N(S) \cap X$ must be the empty set, since $X$ is an antichain. Hence

$$|X| = |X \cap A| + |X \cap B| \leq |S| + (|B| - |N(S)|) \leq |B|,$$

where we used that $|S| \leq |N(S)|$. By Dilworth's theorem (Theorem 3.10), $P$ can be partitioned into $|B|$ chains. Each of these chains has to contain exactly one point of the largest antichain $B$ and at most one point of $A$. Hence the chains of size $2$ in this partition correspond to a matching covering all of $A$. □

## 3.3 Largest antichain in $\mathcal{P}([n])$

For a finite set $S$, the *power set of* $S$ is the set of all subsets of $S$. We write $(\mathcal{P}([n]), \subseteq)$ for the poset under inclusion.

**Question.** How large can an antichain of $(\mathcal{P}([n]), \subseteq)$ be?

The set of all $r$-element subsets of $[n]$, $[n]^{(r)}$ is an antichain (for all $r = [n] \cup \{0\}$). Since $\binom{n}{r}$ is maximized for $r = \lfloor n/2 \rfloor$, $\mathcal{P}([n])$ has an antichain of size $\binom{n}{\lfloor n/2 \rfloor}$. Can we do better?

**Theorem 3.13** (Sperner's lemma 1928)**.** *The size of the largest antichain in* $(\mathcal{P}([n]), \subseteq)$ *is* $\binom{n}{\lfloor n/2 \rfloor}$.

Sperner's theorem follows for following YBLM-inqualtiy. This inequality was proved (and reproved) by Yamamoto (1954), Bollobás (1965), Lubell (1966) and Meshalkin (1963).

**Theorem 3.14** (YBLM inequality)**.** *Let* $A$ *be an antichain of* $(\mathcal{P}([n]), \subseteq)$*. Then*

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1$$

**Proof.** For a permutation $\sigma = s_1 \dots, s_n$ of $[n]$, define a chain $C_\sigma$ of length $n+1$ such that

$$C_\sigma = \{\emptyset, \{s_1\}, \{s_1, s_2\}, \dots, \{s_1, \dots, s_r\}, \dots, \{s_1, \dots, s_n\}\}.$$

Each $C_\sigma$ contains at most one member of $\mathcal{A}$ and there are $n!$ such chains.

Consider $A \in \mathcal{A}$. Note that $A$ is contained in $C_\sigma$ is equivalent to $A = \{s_1, \dots, s_{|A|}\}$. So $A$ is contained in $|A|!(n-|A|)!$ many chains $C_\sigma$, as there are $|A|!$ ways to order the elements of $A$ and $(n-|A|)!$ ways to order the elements of $[n] \setminus A$. Therefore

$$\begin{aligned} n! &\geq \sum_\sigma |\mathcal{A} \cap C_\sigma| = \sum_{A \in \mathcal{A}} \#\{C_\sigma \ni A\} = \sum_{A \in \mathcal{A}} |A|!(n-|A|)!, \\ 1 &\geq \sum_{A \in \mathcal{A}} \frac{|A|!(n-|A|)!}{n!} = \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}}, \end{aligned}$$

as required. □

We now prove Theorem 3.13 using YBLM-inequality.

**Proof of Theorem 3.13.** Let $A$ be an antichain in $(\mathcal{P}([n]), \subseteq)$. By Theorem 3.14, we have

$$1 \geq \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \geq \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{|A|}{\binom{n}{\lfloor n/2 \rfloor}}.$$

Hence $|A| \leq \binom{n}{\lfloor n/2 \rfloor}$. Note that $[n]^{\lfloor n/2 \rfloor}$ is an antichain of size $\binom{n}{\lfloor n/2 \rfloor}$. □

**Remark 3.15.** (1) Examining the proof in more detail shows that if $\mathcal{A}$ is an antichain with $|\mathcal{A}| = \binom{n}{\lfloor n/2 \rfloor}$ then $\mathcal{A} = [n]^{(n/2)}$ if $n$ is even and $\mathcal{A} = [n]^{(\lfloor n/2 \rfloor)}$ or $[n]^{(\lceil n/2 \rceil)}$.

(2) Sperner's theorem and Dilworth's theorem (Theorem 3.10) together imply that $\mathcal{P}([n])$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains.

(3) One can use Hall's theorem to show that $\mathcal{P}([n])$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains. Together with Mirsky's theorem (Theorem 3.9) to prove Sperner's theorem.

## 3.4 The Littlewood–Offord problem

Let $a_1, \ldots, a_n \in \mathbb{R} \setminus \{0\}$. Suppose that $\sum_{i\in[n]} a_i = 0$. For each $i \in [n]$, we flip the sign of $a_i$ with probability $1/2$ independently. What is the probability that $\sum_{i\in[n]} a_i$ is still zero?

**Theorem 3.16** (Erdős 1945)**.** *Let $a_1, \ldots, a_n \in \mathbb{R} \setminus \{0\}$. Let $x_1, \ldots, x_n$ be independent random variable taking value in $\{-1, +1\}$ uniformly. Then for all $z \in \mathbb{R}$,*

$$\mathbb{P}\left(\sum_{i\in[n]} x_i a_i = z\right) \leq \frac{\binom{n}{\lfloor n/2 \rfloor}}{2^n}$$

**Proof.** Without loss of generality, we may assume that each $a_i$ is positive. Fixed any $z \in \mathbb{R}$.

Given $x_1, \ldots, x_n$, define the set $S \subseteq [n]$ to be the set of $i \in [n]$ such that $x_i = 1$. Note that $S \in \mathbb{P}([n])$ choose uniformly at random. Note that

$$\sum_{i\in[n]} x_i a_i = \sum_{i\in[n]} (1 + x_i)a_i - \sum_{i\in[n]} a_i = 2\sum_{i\in S} a_i - \sum_{i\in[n]} a_i,$$

so

$$\mathbb{P}\left(\sum_{i\in[n]} x_i a_i = z\right) = \mathbb{P}\left(2\sum_{i\in S} a_i = z + \sum_{i\in[n]} a_i\right).$$

Let $\mathcal{S}$ be the set of $S \subseteq [n]$ such that $2\sum_{i\in S} a_i = z + \sum_{i\in[n]} a_i$. Recall that $a_1, \ldots, a_n$ are strictly positive, $\mathcal{S}$ is an antichain. By Sperner's lemma (Theorem 3.13), $|\mathcal{S}| \leq \binom{n}{\lfloor n/2 \rfloor}$. Therefore,

$$\mathbb{P}\left(2\sum_{i\in S} a_i = z + \sum_{i\in[n]} a_i\right) = \frac{|\mathcal{S}|}{2^n} \leq \frac{\binom{n}{\lfloor n/2 \rfloor}}{2^n}.$$

□