PROBLEM SHEET 2

Questions 1,2,6,7a,7b will form part of the first assessed problem sheet. The deadline for this assessed problem sheet is **Wednesday 14 February at 17:00**.

Question 1. Let X be a finite set and let $\mathcal{F}$ be a family of subsets of X. Prove that if Maker has a winning strategy in the breaker-maker game $(X, \mathcal{F})$, then Maker also has a winning strategy in the maker-breaker game $(X, \mathcal{F})$. **[5]**

Show that the converse is false. That is, find $(X, \mathcal{F})$ such that Maker wins the maker-breaker game $(X, \mathcal{F})$, but Breaker wins the breaker-maker game $(X, \mathcal{F})$. **[2]**

[This question asks you to prove Proposition 2.7 and shows its converse is false.]

Question 2. Let G be a graph on t vertices and m edges. Let $R(G, G)$ be the smallest integer n such that any red/blue colouring of the edges of K_n yields a monochromatic copy of G By considering an appropriate strong game and applying the Erdős–Selfridge theorem, prove that if $t!\binom{n}{t} < 2^{m-1}$, then we have $R(G, G) > n$. **[7]**

[Hint: You may want to bound the number of copies of G in K_n from above.]

Question 3. Let X be a finite set, and let $\mathcal{F}$ be an n-uniform family of winning sets $A \subseteq X$ (that is, $|A| = n$ for any $A \in \mathcal{F}$). Prove that in the strong game $(X, \mathcal{F})$ the Player 2 has a strategy which ensures that one of the following outcomes will occur.

(i) He will win.
(ii) He will draw.
(iii) He will lose, but before losing will claim $n - 1$ elements of some $A \in \mathcal{F}$.

[Hint: You are not asked to give such a strategy explicitly, only to prove that it exists. You might want to define a new game after Player 1's first move.]

Question 4. Let $X = \mathbb{N}$ and $\mathcal{F}$ be the families of set of three consecutive numbers or three consecutive odd number. That is $\mathcal{F} = \{\{i, i+1, i+2\} : i \in \mathbb{N}\} \cup \{\{2i-1, 2i+1, 2i+3\} : i \in \mathbb{N}\}$. Determine and justify whether the strong game $(X, \mathcal{F})$ is a Player 1's win or a draw.

Question 5. Give an explicit winning strategy for the first player in 3×3 maker-breaker tic-tac-toe. (Technically, our definition of a strategy means that you should state what move to make following any sequence of moves, but in your answer you only need to give the moves to make in any position which can be reached when the first player follows the strategy you give.)

[Please ensure that you cover all possibilities. Flow diagrams are useful to aid explanations.]

Question 6. Use the Erdős-Selfridge theorem to prove that the 4×4 maker-breaker tic-tac-toe is a Breaker's win. **[4]**

[Hint: Consider applying the Erdős-Selfridge theorem after Maker claimed their first element.]

Question 7.

(a) Let $X = \{1, 2\} \times \mathbb{Z}$ and $\mathcal{F} = \{\{(1, y), (2, y), (1, y+1), (2, y+1)\} : y \in \mathbb{Z}\}$. Show that Player 2 has at least 4 pairing strategies in the strong game $(X, \mathcal{F})$. **[4]**
[In fact, there are infinity many pairing strategies for Player 2.]

(b) Let $X = \mathbb{Z}^2$ and $\mathcal{F} = \{\{(x, y), (x+1, y), (x, y+1), (x+1, y+1)\} : x, y \in \mathbb{Z}\}$. Prove that the Player 2 has a drawing strategy in the strong game $(X, \mathcal{F})$. **[3]**

(c) Let $X = \mathbb{Z}^3$ and $\mathcal{F} = \{\{(x, y, z), (x+1, y, z), (x, y+1, z), (x+1, y+1, z), (x, y, z+1), (x+1, y, z+1), (x, y+1, z+1), (x+1, y+1, z+1)\} : x, y, z \in \mathbb{Z}\}$. Prove that the Player 2 has a drawing strategy in the strong game $(X, \mathcal{F})$.

Observation: Geometrically, the winning sets for (a) and (b) are 2×2 sub-squares and for (c) are $2 \times 2 \times 2$ subcubes.
[Diagrams aids explanations, but you still need to define the pairing formally.]

Question 8. Show that every finite poset $(P, \leq)$ has a maximal element.

Question 9.
(a) Write down two antichains of size 10 in $\mathcal{P}([5])$.
(b) Write down an anitchain of size 8 in $\mathcal{P}([5])$, whose elements are not of the same size.

Question 10. Given a natural number $k \leq n/2$, let $\mathcal{A}$ be an antichain in $\mathcal{P}([n])$ which consists of sets of size at most k. Prove that $|\mathcal{A}| \leq \binom{n}{k}$.

Question 11. Suppose that $A_1, \dots, A_n$ are n distinct sets. Let $k := \lceil \sqrt{n} \rceil$. Prove that there must be k of these sets $A_{i_1}, \dots, A_{i_k}$ such that $A_{i_s} \cup A_{i_t} \neq A_{i_u}$ for all $u \neq s, t$.
[Hint: Consider the poset P whose elements are the sets $A_1, \dots, A_n$ ordered under inclusion. What can you say if P contains a chain of size k or an antichain of size k? Try to use Mirsky's or Dilworth's theorem.]

Question 12. Let n be even and $1 \leq r \leq n/2$. Let $\mathcal{A} \subseteq \mathcal{P}([n])$ be such that $\mathcal{A}$ does not contain any chain of size 3. Prove that $|\mathcal{A}| \leq \binom{n}{n/2} + \binom{n}{n/2-1}$. Moreover, show that there exists $\mathcal{A}$ such that equality holds.
[Hint: You may want to prove the corresponding LBYM-inequality.]

Question 13. Let $\mathcal{A}$ be an antichain in $\mathcal{P}([n])$ such that $\mathcal{A} \neq [n]^{(r)}$ for all $r \in [n] \cup \{0\}$. Show that there exists a chain of size $n+1$ in $\mathcal{P}([n])$, which is disjoint from $\mathcal{A}$.
[Hint: You may want to start by considering some $r \in [n-1]$ with $\mathcal{A} \cap [n]^{(r)} \neq \emptyset$ and showing that there are $A \in \mathcal{A} \cap [n]^{(r)}$ and $B \in [n]^{(r)} \setminus \mathcal{A}$ such that A and B differ precisely in one element.]