PROBLEM SHEET 1

Questions 1b,4,6,7,9,11 will form part of the first assessed problem sheet. The deadline for this assessed problem sheet is **Wednesday 31st January at 17:00**.

Question 1. Let $s, t \in \mathbb{N}$. Prove the following statement

(a) $R(1, t) = 1$.

(b) $R(2, t) = t$. **[3]**

(c) If $R(s, t)$ exists, then $R(t, s) = R(s, t)$.

[Although these statements may seem obvious, it is important to give a formal and logical argument. Please remember to prove both the upper and lower bounds.]

Question 2. Let $n \geq 6 = R(3, 3)$ be even. Find a 2-colouring of the edges of K_n with more red than blue edges but without a red K_3.
[Hint: Divide the vertex set of K_n into two sets A and B and use this partition to colour the edges of K_n.]

Question 3. Prove that for all $k, t \in \mathbb{N}$ with $k \geq 2$, $R_k(t, \ldots, t) \leq k^{k(t-1)+1}$.
[Hint: See the proof of Proposition 1.7.]

Question 4. For $s, t \geq 2$, give an explicit edge-colouring showing that $R(s, t) \geq (s-1)(t-1)$ (i.e. without using Theorem 1.13 or random colouring). Please justify your answer. **[4]**
[Remark: One can generalise the solution to prove the following statement. For $k_1, k_2, s_1, \ldots, s_{k_1+k_2} \in \mathbb{N}$, we have $R_{k_1+k_2}(s_1, s_2, \ldots, s_{k_1+k_2}) \geq (R_{k_1}(s_1, s_2, \ldots, s_{k_1}) - 1)(R_{k_2}(s_{k_1+1}, s_{k_1+2}, \ldots, s_{k_1+k_2}) - 1)$.] For $k_1, k_2, s_1, \ldots, s_{k_1+k_2} \in \mathbb{N}$, we have $R_{k_1+k_2}(s_1, s_2, \ldots, s_{k_1+k_2}) \geq (R_{k_1}(s_1, s_2, \ldots, s_{k_1}) - 1)(R_{k_2}(s_{k_1+1}, s_{k_1+2}, \ldots, s_{k_1+k_2}) - 1)$.]

Question 5. By extending the construction used to show $R(3, 4) > 8$, show that $R(3, t+1) \geq 3t$ for $t \geq 2$.

Question 6. Prove that for any $n, s, t \in \mathbb{N}$, if there exists a real number $0 < p < 1$ for which $\binom{n}{s}p^{\binom{s}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$, then $R(s, t) > n$. **[6]**
[Hint: Mimic the probabilistic proof of Theorem 1.13, but edges are not coloured red or blue with the same probability.]

Question 7. Let $n = R^{(3)}(m, m)$. Let V be a set of n points in the Euclidean plane such that no three of which on a line and no two have the same x-coordinates. Prove that V contains m points that form a convex m-gon.
You may assume that if $W = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\} \subseteq \mathbb{R}^2$ satisfies $x_1 < x_2 < x_3 < x_4$ and $\frac{y_2-y_1}{x_2-x_1} < \frac{y_3-y_2}{x_3-x_2} < \frac{y_4-y_3}{x_4-x_3}$, then W forms a convex 4-gon. **[5]**
[Remark: the assumption that no two points of S having the same x-coordinate can be achieved if we allow to rotate the plane.]

Question 8.

(a) For every $n \in \mathbb{N}$, find a 2-colouring of $[2n+1]$ such that the largest colour class does not contain a solution to the equation $x + y = z$, i.e. find a 2-colouring with (say) more red than blue elements of $[2n+1]$ for which there is no red solution to $x + y = z$.

(b) Find a 2-colouring of $\mathbb{N}$ for which there does not exist a monochromatic solution to the equation $x = 2y$.

Question 9. Prove that for every $r \in \mathbb{N}$, there exists an integer n such that any r-colouring of the elements of $\{2, 3, \ldots, n\}$ yields a monochromatic solution to the equation $xy = z$. **[4]**
[Hint: ignore all of $\{2, 3, \ldots, n\}$ except for a well-chosen subset $S \subseteq \{2, 3, \ldots, n\}$ within which you can describe multiplication in terms of addition.]

Question 10. Prove that, for each $k \in \mathbb{N}$, there exists an $n = n(k) \in \mathbb{N}$ such that the following statement holds. Whenever you colour all non-empty subsets of $[n]$ with k colours, then there exist three distinct subsets $X, Y, Z \subset [n]$ of the same colour such that $X \cup Y = Z$ and $X \cap Y = \emptyset$.
[Hint: Consider a complete graph K_n with $V(K_n) = [n]$, where each edge ij with $i < j$ can be viewed as the set $[j] \setminus [i] = \{i + 1, i + 2, \ldots, j\}$.]

Question 11. Let x, y and z be positive integers and let $n > xyz$ be an integer. Prove that any sequence of n (not necessarily distinct) real numbers contains an increasing subsequence of length $x + 1$ or a decreasing subsequence of length $y + 1$ or a constant subsequence of length $z + 1$. **[3]**

Question 12. Find a 2-colouring of $\mathbb{N}$ for which there is no infinite monochromatic arithmetic progression. That is, there should be no $a, d \in \mathbb{N}$ such that $\{a, a + d, a + 2d, a + 3d, \ldots\}$ are all the same colour. Please also justify why such colouring works.
[Once you wrote your solution, please check whether your justification fails for $a = 10^{10}$ and $d = 2$, say.]

Conjecture 13 (Erdős and Sós 1980)**.** $R(3, n + 1) - R(3, n) \to \infty$ as $n \to \infty$.