Stochastic Applications in Engineering Design

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I. Introduction

Stochastic Processes, can be defined as any process that describes the evolution of a random phenomenon. The stochastic process phenomenon has occupied a special niche with an increasing role in various fields of Science and Engineering. Stochastic modelling, has facilitated significant contributions to a wide variety of Engineering applications such as Information Theory and Communication Systems [1].

This paper attempts to demonstrate the concepts of stochastic processes using principles of probability to illustrate their physical ramifications on, and for realisation of communication systems. The aim is to present a probabilistic model that describes the process of digital encoding and transmission through the Additive White Gaussian Noise (AWGN) channel and an analysis of probability error in this system.

An overview of the basic principles of probability theory that characterise the random processes that encompass a communication system is presented. A probabilistic model of a communication system is developed. This is followed by the presentation of concepts and stochastic application tools used in the analysis of traditional communication systems.

II. System Overview

A generalised communication system is presented by Figure 1. The parts of interest in the development of the probabilistic model is the source, channel, noise and destination as illustrated.

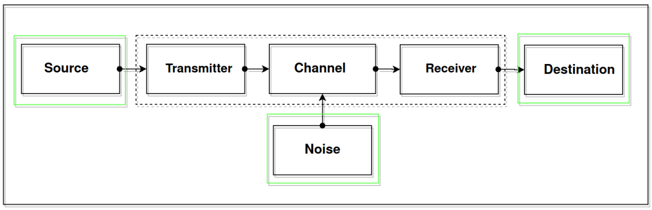

Fig. 1. Communication System Model Overview

The information that is transmitted, is modelled as a random process. In probability theory this can be analysed as a random experiment. Information that is transmitted through a channel is subjected to various typed of interference's, which when decoded, inevitably contains errors. The aim is to model these errors and determine their probability. This forms the appraisal of the systems performance through the analysis of stochastic processes.

III. Probability theory

The most fundamental concept of any stochastic probabilistic model is the concept of a randomness. The encoding of information in a binary communication system is modelled as a random process. Encoding is the process of converting information to a construct that facilitates further processing such as transmission [2].

A random experiment is defined as any process who's outcome cannot be predicted with certainty, but can be predicted nonetheless with a certain probability [3]. In this context for example, the probability of the encoding process is a random process with a binary, discrete, countably finite sample space, where a sample space is defined as all the possible outcomes of the observational experiment. The sample space of this experiment is simply defined as the probability of the event that information is either encoded to a 0 or 1. In the encoding process, a sequence of binary bits are assigned to different quantization values. The source in the communication system is characterised by a *Bernoulli trial experiment* that is modelled as a *Bernoulli random variable* [2].

The *Pulse Code Modulation* scheme is the most common coding scheme. The quantization values in a uniform PCM are taken as the midpoints of the quantization regions. These regions are defined as quantization values partitioned into a disjoint subset, such that the error is a uniformly distributed random variable. The quantization noise is defined as the expected error, or distortion. The distortion is defined as the *squared error distortion* and the expected distortion is given by:

$$D = E[X - Q(X)]^2 \tag{1}$$

where X is random variable representing the sampled information and $Q(X)$ is the familiar error function. A quantitative description of the performance of this process is defined by the signal-to-quantization noise ratio ($SQNR$). This is simply summarised as the ratio of the expected signal to the expected noise as illustrated by

$$SQNR = \frac{E[X^2]}{E[X - Q(X)]^2} \tag{2}$$

This facilitates a quantitative description of the systems expected performance by measuring the probability of the random processes.

The noise in a transmission channel is described by a *wide-sense stationary* random Gaussian process. This is a process with zero-mean and auto-correlation Gaussian functions being independent of time [3]. The PSD of each sample experiment can model the performance of the encoding process.
Information with non-uniform statistics is processed by a non-uniform PCM. This system embodies complex procedures such as a compander and reconstruction filtering [4]. The embodied processes are beyond the scope of this analysis, however these processes are characterised by functions of random variables and are therefore governed by probabilistic law of a jointly Gaussian process. It proves sufficient and efficient to model these processes as an LTI system with the output also being a random process defined on the original probability space.

A *white process* denotes a random process with a flat PSD. The most common type of interference in communication system is *white noise* induced by thermal noise [2]. The auto-correlation function for a white process resembles the PSD of thermal noise. Where

$$R_n(\tau) = \frac{N_o}{2}\delta(\tau) \tag{3}$$

and

$$S_n(f) = \frac{N_o}{2} \tag{4}$$

is the auto-correlation and noise power respectively [3]. This is enough to conclude that the white noise is modelled by the auto-correlation function for a white process, noise is thus modelled and characterised by a Gaussian random process with a Gaussian probability density function. This results in the definition of a channel with Additive White Gaussian Noise. The AWGN is one of the most common mathematical model for modern communication channels. The following section briefly describes the transmission of the digital information over an AWGN channel.
A geometric representation of the data to be transmitted is developed through a binary modulation scheme [2]. This results in a random vector defined by joint Gaussian probability and multivariate random variable. Such a representation greatly simplifies the analysis of AWGN channel performance.

Various schemes of signal demodulation exist, these are realised through correlation or matched filters. The demodulation scheme employed depends on the statistical characteristics of the data to be transmitted. The properties of the matched filter are briefly summarised by Eq 5 from the perspective of the random process.

$$(\frac{S}{N})_o = \frac{y_s^2(T)}{E[y_s^2(T)]} \tag{5}$$

where $E[y_s^2(T)]$ describes the variance of the noise. As expected, the PSD depends on variance of the noise.
The matched filter, as the name implies, is an LTI system who's impulse response is matched to the received signal to maximise the SNR. The output SNR is illustrated by Eq 6.

$$(\frac{S}{N})_o = \frac{2\epsilon}{E[N_o]} \tag{6}$$

where ϵ is the energy of the sample. The detector describes the optimum decision rule used to recover the sent data and minimise the probability of error. Using the fact that the noise components are statistically independent, zero-mean multivariate Gaussian variables with variance $\sigma^2 = \frac{N_o}{2}$ the probability of error in the received data is simply characterised by the vector difference (or Euclidean distance) between 2 signal points. From the theory of stochastic processes, this difference is also a Gaussian random variable with the probability density function as described by Eq 7 [3].

$$f(x) = \frac{1}{\sqrt{2\pi N_o}} \mathrm{e}^{-\frac{(x-\sqrt{\epsilon})^2}{2N_o}} \tag{7}$$

The average probability of error is illustrated by Eq 8

$$P = Q(\sqrt{\frac{d_{12}^2}{2N_o}}) \tag{8}$$

where d_{12} is the distance between the received bits and $\frac{N_o}{2}$ is the noise PSD or variance. This distance is equivalent to the energy per bit of noise.

The binary information sequence is subdivided into blocks of symbols where each symbol is represented by $M = 2^k$ bits. The received signal is a linear combination of the output of the correlators or matched-filter of the demodulator as illustrated by Figure 2.

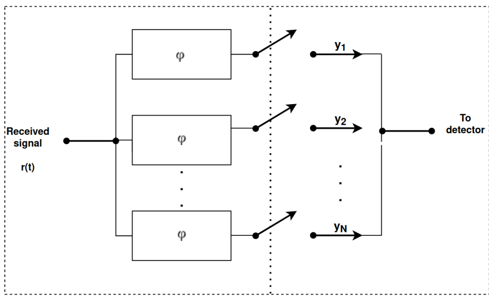

Fig. 2. Matched-filter of an M-ary demodulator

The output of the demodulator is now defined as

$$y = s_m + n \tag{9}$$

where s_m is a vector representation equivalent to the transmitted signal and **n** is an N-dimensional random vector with statistically independent variables. As previously mentioned the output of the correlator is characterised by the Gaussian random process and is in actual fact a zero-mean uncorrelated Gaussian random variable with variance equal to the PSD. The PDF of a sample is illustrated by Figure 3.

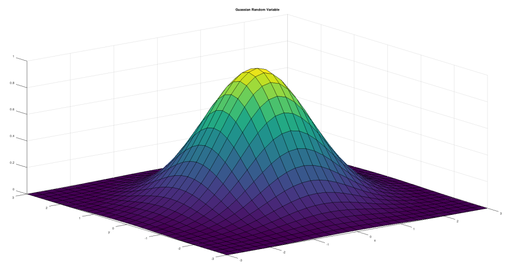

Fig. 3. PDF of signal with White Gaussian noise

The vector s_m is a point in the signal constellation and **n** is a spherical distribution of the noise vector around s_m. This represents the density of the noise around the received signal. This density is determined by the variance in the noise as previously mentioned.
The physical interpretation of the signal constellation is that the density is higher in the center, which corresponds to a high probability of detection while this density reduces with an increase in the distance from the center.

The following conclusion can be drawn from a probabilistic perspective: The higher the noise, the high the variance, the larger the spread. The larger the spread the shorter the distance between data points, the higher the probability of error of detection. The detection rule or decision criterion is determined by Baye's rule of conditional probabilities, illustrated by Eq 10 below.

$$P(s_m|y) = \frac{f(y|s_m)P(s_m)}{f(y)} \tag{10}$$

where $f(y|s_m)$ is the conditional PDF or *maximum likelihood criterion* (ML) of the observed vector given that the signal points are equiprobable (equidistant). In an AWGN channel, the ML is equivalent to finding s_m that minimises the Euclidean distance to **y**. This is referred to as minimum distance detection. The error probability is defined as the event that the receiver detects a signal other than s_m. A sufficient condition for a received message s_i to be detected, is that **y** be closer to s_i than s_m.

IV. Analysis of a Communication System

A Quadrature Amplitude Modulation (QAM) system is demonstrated and used for analysis as an application of a communication system. The signal constellation of a 16-QAM system is presented by Figure 4. A simulation of this communication system is conducted on MATLAB®. The objective of this experiment is to validate the probabilistic model developed and to verify it's performance by estimating the probability of error in the communication system. A sample of symbols representing the sequence of information is generated using a random variable. This random process

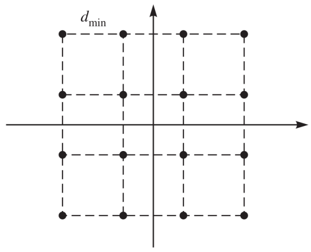

Fig. 4. QAM signal constellation

characterises the source. The sample is mapped to a 16-QAM constellation. The Gaussian random process of generating the additive white noise in applied. The error of probability is determined by comparing the detected signal points with the transmitted. This probability of symbol error is presented by Figure 5. The probability error as expected is dominated by the the $Q-function$. This forms the quantitative evaluation of performance of the communication system.

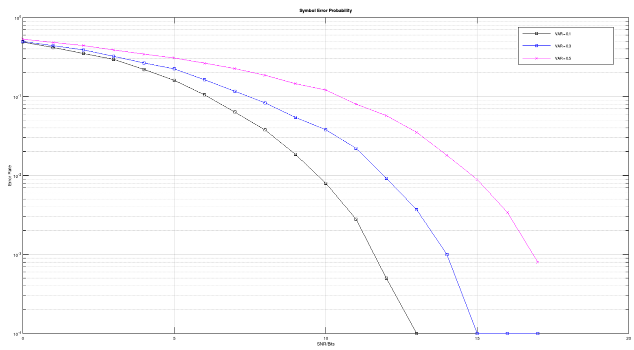

Fig. 5. Probability error as function of SNR/bit

V. Conclusion

A probabilistic model is developed, The stochastic processes of the model are analysed. The model is simulated and validated. The analysis showed how stochastic processes have physical ramifications on systems, in this case a digital communication system. The results verify that variance has a direct impact on the performance. The higher the noise variance the higher the probability error rate. This demonstrates how Stochastic processes can be used to develop systems that have reliable and consistence performance using probabilistic theory.

References

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