• Least Squares

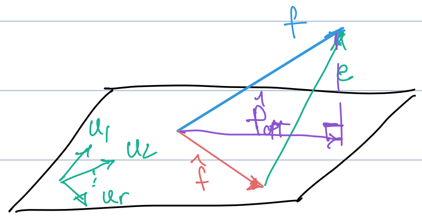
$$\begin{bmatrix} \\ f \\ \\ \end{bmatrix}_{n\times1} \simeq \begin{bmatrix} \\ u_1 \\ \\ \end{bmatrix}_{n\times1}\alpha_1 + \begin{bmatrix} \\ u_2 \\ \\ \end{bmatrix}_{n\times1}\alpha_2 + \cdots + \begin{bmatrix} \\ u_r \\ \\ \end{bmatrix}_{n\times1}\alpha_r$$

n: ambient dimension

r: the number of basis vectors

$r < n$

$\{u_i\}_{i=1}^{r}$: independent vectors



$$f \simeq \hat{f} = \sum_{i=1}^{r} \alpha_i u_i = U\alpha \qquad U = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix}_{n\times r} \qquad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \end{bmatrix}$$

$$\underset{n\times1}{f} = \underset{n\times1}{\hat{f}} + \underset{n\times1}{e} \implies e = f - U\alpha$$

$$E(\alpha) = \|e\|^2 = \|f - U\alpha\|^2 \qquad \langle e, e\rangle = \|e\|^2$$

$$= (f - U\alpha)^T (f - U\alpha) \qquad (AB)^T = B^T A^T$$

$$= f^T f - \underbrace{f^T U\alpha}_{\alpha^T U^T f} - \alpha^T U^T f + \alpha^T U^T U \alpha \qquad a\text{:scalar} \quad a^T = a$$

$$= f^T f - 2\alpha^T U^T f + \alpha^T U^T U \alpha$$

The best α:

$$\alpha^T c \qquad \alpha|_{r\times1} \qquad c|_{r\times1}$$

$$\frac{\partial E}{\partial \alpha} = \begin{bmatrix} \frac{\partial E}{\partial \alpha_1} \\ \frac{\partial E}{\partial \alpha_2} \\ \vdots \\ \frac{\partial E}{\partial \alpha_r} \end{bmatrix}$$

$$\frac{\partial(\alpha^T c)}{\partial \alpha} = \frac{\partial(\alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_r c_r)}{\partial \alpha} = \begin{bmatrix} \frac{\partial(\quad)}{\partial \alpha_1} \\ \frac{\partial(\quad)}{\partial \alpha_2} \\ \vdots \\ \frac{\partial(\quad)}{\partial \alpha_r} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{bmatrix} = c$$

$$\frac{\partial(c^T \alpha)}{\partial \alpha} = c$$

$$\frac{\partial (f^T f)}{\partial \alpha} = 0_{r\times 1} \qquad \frac{\partial(\alpha^T U^T f)}{\partial \alpha} = (U^T f)_{r\times 1} \qquad \frac{\partial(\alpha^T U^T U \alpha)}{\partial \alpha} = 2U^T U \alpha$$

$$\frac{\partial E}{\partial \alpha} = -2U^T f + 2U^T U \alpha = 0 \Rightarrow U^T U \alpha = U^T f \Rightarrow \alpha = (U^T U)^{-1} U^T f$$

$$\hat{f} = U\alpha = U(U^T U)^{-1} U^T f$$

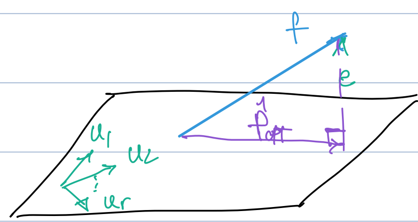
$$\underset{n\times n}{P} = \Big[\, U \,\Big]_{n\times r} \big[U^T U\big]^{-1}_{r\times r} \big[\quad U^T \quad\big]_{r\times n}$$

$$P^2 = P \rightarrow \big(U(U^TU)^{-1}\underbrace{U^T\big)\big(U}_{U^TU}(U^TU)^{-1}U^T\big) =$$

$$P^2 = U(U^TU)^{-1}U^T \quad \checkmark$$

$P^T = P$ symmetric $\rightarrow$ P is an orthogonal projector to the space spanned by U.

Galerkin Projection



$$f = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r + e$$

$$\hat{f} = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r \qquad \hat{f} \perp e \Rightarrow \langle u_i, e\rangle = 0$$

$$u_i^T e = 0 \qquad i = 1, 2, \ldots, r$$

$$u_1^T \big(f = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r + e\big)$$

$$u_2^T \big(f = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r + e\big)$$

$$u_1^T f = \alpha_1 u_1^T u_1 + \alpha_2 u_1^T u_2 + \cdots + \alpha_r u_1^T u_2$$
$$u_2^T f = \alpha_1 u_2^T u_1 + \alpha_2 u_2^T u_2 + \cdots + \alpha_r u_2^T u_2$$
$$\vdots$$
$$u_r^T f = \alpha_1 u_r^T u_1 + \alpha_2 u_r^T u_2 + \cdots + \alpha_r u_r^T u_r$$

$$\Longrightarrow \underbrace{\begin{bmatrix} u_1^T u_1 & u_1^T u_2 & \cdots & u_1^T u_r \\ u_2^T u_1 & u_2^T u_2 & \cdots & u_2^T u_r \\ & \vdots & & \\ u_r^T u_1 & u_r^T u_2 & \cdots & u_r^T u_r \end{bmatrix}}_{= U^T U} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \end{bmatrix} = \underbrace{\begin{bmatrix} u_1^T f \\ u_2^T f \\ \vdots \\ u_r^T f \end{bmatrix}}_{U^T f}$$

$$\alpha = (U^T U)^{-1} U^T f$$

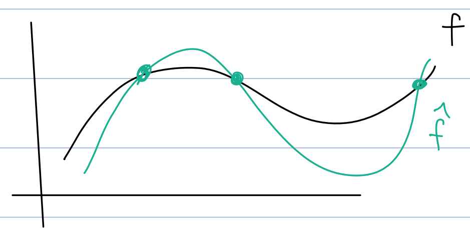
- U: is an orthonormal basis $\rightarrow U^T U = I \Rightarrow \alpha = U^T f$

$$P = U (U^T U)^{-1} U^T = U U^T$$

Interpolation:

$$\underbrace{\begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \vdots \\ \bullet \\ \bullet \end{bmatrix}_{n\times 1}}_{f} \simeq \underbrace{\begin{bmatrix} & & \\ & U & \\ & & \end{bmatrix}_{n\times r} [\alpha]_{r\times 1}}_{\hat{f}}$$

p_1, p_2, $\vdots$, p_r



$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_r \end{bmatrix}_{r\times 1} \leftarrow \text{integer index vector} \qquad f(p) = \begin{bmatrix} f_{p_1} \\ f_{p_2} \\ \vdots \\ f_{p_r} \end{bmatrix}$$

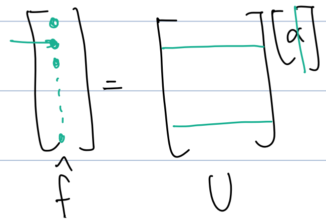
$n = 100$

$$p = \begin{bmatrix}5\\6\\9\\23\end{bmatrix} \rightsquigarrow f(p) = \begin{bmatrix}f_5\\f_6\\f_9\\f_{23}\end{bmatrix}$$

$$f(p) = \hat{f}(p) \qquad p_{r\times 1}$$

$$\underbrace{\hat{f}(p)}_{r\times 1} = \underbrace{U(p,:)}_{r\times r}\,\alpha_{r\times 1}$$

$$f(p) = \hat{f}(p) \longleftarrow \text{interpolation}$$



$$f(p)\Big|_{r\times 1} = U(p,:)_{r\times r}\,\alpha_{r\times 1}$$

$$\alpha = U^{-1}(p,:)\,f(p)$$

$$\hat{f} = U\alpha = U\,U^{-1}(p,:)\,f(p)$$

$$p = \begin{bmatrix}25\\3\\39\end{bmatrix}$$

$$\hat{f} = \varphi f$$

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$$P = \begin{bmatrix}0&0&0\\0&0&0\\ \vdots&1&0\\1&0&\vdots\\0&\vdots&1\\0&0&0\end{bmatrix}_{n\times r}$$

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$$P^T f = \begin{bmatrix}0&0&\cdots&1&0&\cdots\\0&0&1&&&\end{bmatrix}_{r\times n}\begin{bmatrix}f\end{bmatrix}_{n\times 1} = \begin{bmatrix}f_{25}\\f_3\\f_{39}\end{bmatrix} = f(p)$$

$$P^T U = U(p,:)$$

$$\hat{f} = U\alpha = U\,U^{-1}(p,:)\,f(p) = U\left(P^T U\right)^{-1} P^T f$$

projector

$$\varphi = \underbrace{U}_{n\times r}\underbrace{(P^T U)^{-1}}_{r\times r}\underbrace{P^T}_{r\times n} \rightsquigarrow \varphi^2 = \varphi$$

$$\varphi^T \neq \varphi$$

Regression

$$\hat{f}_{n\times 1} = U_{n\times r}\,[\alpha]_{r\times 1}$$

$r < r' < n$

$r \ll r' \ll n$

$$[f(p)]_{5\times 1} \simeq [U(p,:)]_{5\times 4}\,[\alpha]_{4\times 1}$$

$r = 4$

$r' = 5$

$$\alpha_{orth} = (U^T U)^{-1} U^T f$$

$$\alpha_{reg} = \underbrace{\left(U(p,:)^T U(p,:)\right)^{-1} U(p,:)^T}_{\text{Pseudo-inverse}} f(p)$$

$$A^{\dagger} = (A^T A)^{-1} A^T$$

$$f(x) \simeq \sum_{i=1}^{r} a_i u_i(x) \qquad x \in [a,b]$$

$$\left\langle f(x) = \sum_{i=1}^{r} a_i u_i(x) + e(x),\ u_j(x) \right\rangle \qquad \langle u(x), v(x)\rangle = \int_a^b u(x) v(x)\,dx$$

$$\langle e(x), u_j(x)\rangle = 0$$

$$M_{r\times r}\,\alpha_{r\times 1} = b_{r\times 1} \rightarrow M_{ij} = \langle u_i(x), u_j(x)\rangle$$

$$f(t) \simeq a_0 + \sum_{i=1}^{r} a_i \cos(it) + \sum_{i=1}^{r} b_i \sin(it)$$