Problem 1

Let $\mathbf{f} = [f_1, f_2, \dots, f_n]^T$ be a vector of size $n \times 1$, and let $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r] \in \mathbb{R}^{n\times r}$, where $r < n$ and the columns of matrix $\mathbf{U}$ are independent vectors. Let $\mathbf{p} = [p_1, p_2, \dots, p_{r'}]^T$ be the vector containing indices, where $r \leq r' \leq n$. Using MATLAB indexing, $\mathbf{f}(\mathbf{p}) = [f_{p_1}, f_{p_2}, \dots, f_{p_{r'}}]^T$ selects a subset of elements from the vector $\mathbf{f}$ corresponding to the indices in $\mathbf{p}$. Let $\mathbf{P} = \mathbf{I}(:, \mathbf{p}) \in \mathbb{R}^{n\times r'}$ denote the *indexing matrix* where $\mathbf{I}$ is the identity matrix of size $n \times n$. The matrix $\mathbf{I}(:, \mathbf{p})$ selects r' columns of the identity matrix with index $\mathbf{p}$. It is easy to verify that $\mathbf{P}^T\mathbf{f} \equiv \mathbf{f}(\mathbf{p})$ and $\mathbf{P}^T\mathbf{U} \equiv \mathbf{U}(\mathbf{p}, :)$. In Lecture 3, we formulated projection ($\mathcal{P}_p$), regression ($\mathcal{P}_r$), interpolation ($\mathcal{P}_i$) matrices as in the following:

$$\mathcal{P}_p = \mathbf{U}(\mathbf{U}^T\mathbf{U})^{-1}\mathbf{U}^T, \tag{1}$$
$$\mathcal{P}_r = \mathbf{U}(\mathbf{P}^T\mathbf{U})^{\dagger}\mathbf{U}^T\mathbf{P}^T, \tag{2}$$
$$\mathcal{P}_i = \mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1}\mathbf{U}^T\mathbf{P}^T, \tag{3}$$

where $(\ )^{\dagger}$ denotes the Moore–Penrose pseudoinverse of a matrix, i.e., $\mathbf{A}^{\dagger} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$.

1. Show that if $\mathbf{U}$ is an orthonormal matrix, $\mathcal{P}_p = \mathbf{U}\mathbf{U}^T$.
2. Show that when $r' = r$, $\mathcal{P}_r = \mathcal{P}_i$.
3. Show that when $r' = n$, $\mathcal{P}_r = \mathcal{P}_p$.
4. Show that $\mathcal{P}_p$ is a projector.
5. Show that $\mathcal{P}_r$ is a projector.
6. Show that $\mathcal{P}_i$ is a projector.
7. Show that $\mathcal{P}_p$ is an orthogonal projector.
8. Show that $\mathcal{P}_r$ is an oblique projector when $r' < n$.
9. Show that $\mathcal{P}_i$ is an oblique projector.

Problem 2

Adopt the notation introduced in Problem 1 for this problem as well. Consider the vectors $\mathbf{f}$ and $\mathbf{U}$ given by:

$$\mathbf{f} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \qquad \mathbf{U} = \left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right].$$

Therefore, in this problem, the *ambient* dimension is $n = 3$ and the dimension of the subspace is $r = 2$.

1. Consider $\mathbf{p} = [1, 3]$.
   (a) Compute the interpolatory projector $\mathcal{P}_i$ and compute $\hat{\mathbf{f}} = \mathcal{P}_i\mathbf{f}$.
   (b) Confirm that $\hat{\mathbf{f}}(\mathbf{p}) = \mathbf{f}(\mathbf{p})$.
   (c) Compute the error of interpolatory projector: $\mathbf{e} = \mathbf{f} - \hat{\mathbf{f}}$.
   (d) Confirm that $\mathcal{P}_i$ is an oblique projector by showing that $\mathbf{e}$ and $\hat{\mathbf{f}}$ are not orthogonal to each other. Compute the angle between $\hat{\mathbf{f}}$ and $\mathbf{e}$.
   (e) Compare the magnitude of the error $\|\mathbf{e}\|$ and compare the value of this error with that obtained from the orthogonal projection of $\mathbf{f}$ onto $\mathbf{U}$. For the orthogonal projection, you use your results from Homework 2, Problem 2, Part 6.

Problem 3

Consider $\mathbf{x} \in \mathbb{R}^{n\times 1}$ be an equidistant discretization of $x \in [-1, 1]$. This can be achieved in MATLAB with $\mathbf{x} =$ `linspace(-1,1,n)'`. Take $n = 100$. Consider the vector $\mathbf{f}$ and the matrix $\mathbf{U}$ as in the following:

$$\mathbf{f} = \exp(-2\mathbf{x}^2) \quad \text{and} \quad \mathbf{U} = [\mathbf{1} \;\; \mathbf{x} \;\; \mathbf{x}^2 \;\; \mathbf{x}^3 \;\; \mathbf{x}^4], \tag{4}$$

where $\mathbf{1} = [1, 1, \ldots, 1]^T \in \mathbb{R}^{n\times 1}$, $\mathbf{f} \in \mathbb{R}^{n\times 1}$, and $\mathbf{U} \in \mathbb{R}^{n\times 5}$. Therefore, in this problem, the *ambient* dimension is $n = 100$ and the dimension of the subspace is $r = 5$.

1. Compute the orthogonal projection of $\mathbf{f}$ onto $\mathbf{U}$ and denote the projected vector with $\hat{\mathbf{f}}_o$. Compute the error $\mathbf{e}_o = \mathbf{f} - \hat{\mathbf{f}}_o$ and show that the error is orthogonal to $\hat{\mathbf{f}}_o$.
2. Consider the indices $\mathbf{p}_{i_1} = [1 \;\; 100 \;\; 17 \;\; 84 \;\; 50]$. Note that the order of these indices is not important. Use these indices to build an interpolatory projector $\mathcal{P}_{i_1}$ and interpolate $\mathbf{f}$ onto $\mathbf{U}$ using $\mathcal{P}_{i_1}$. Denote the interpolated function with $\hat{\mathbf{f}}_{i_1}$. Show that $\mathbf{f}(\mathbf{p}_1) = \hat{\mathbf{f}}_{i_1}(\mathbf{p}_1)$. Compute the projection error $\mathbf{e}_{i_1} = \mathbf{f} - \hat{\mathbf{f}}_{i_1}$. What is the angle between $\mathbf{e}_{i_1}$ and $\hat{\mathbf{f}}_{i_1}$? Denote the angle (in degree) with α_1.
3. Repeat all items of Part 2 with the indices $\mathbf{p}_{i_2} = [10 \;\; 20 \;\; 30 \;\; 40 \;\; 90]$. Denote the interpolated function, the projection error, and the angle with $\hat{\mathbf{f}}_{i_2}$, $\mathbf{e}_{i_2}$, and α_2, respectively.
4. Consider the indices $\mathbf{p}_r = [1 \;\; 100 \;\; 17 \;\; 84 \;\; 50 \;\; 58]$. Note that $\mathbf{p}_r$ contains all $\mathbf{p}_{i_1}$ indices as well as an extra index of 58. Use these indices to build a regression projector $\mathcal{P}_r$ and regress $\mathbf{f}$ onto $\mathbf{U}$ using $\mathcal{P}_r$. Denote the regressed function with $\hat{\mathbf{f}}_r$. Compute the projection error $\mathbf{e}_r = \mathbf{f} - \hat{\mathbf{f}}_r$. What is the angle between $\mathbf{e}_r$ and $\hat{\mathbf{f}}_r$? Denote the angle (in degree) with α_r.
5. Plot $\mathbf{f}$, and $\hat{\mathbf{f}}_o$, $\hat{\mathbf{f}}_{i_1}$, $\hat{\mathbf{f}}_{i_2}$, and $\hat{\mathbf{f}}_r$ versus $\mathbf{x}$ and show them in one figure and legend accordingly.
6. Plot the norm of the error of each of the above for parts versus the angle. To this end, first sort the angle from smallest to the largest. What conclusion can you draw from this figure?