Problem 1

Show that the projector $\mathbf{P} \in \mathbb{R}^{n \times n}$ is an orthogonal projector if $\mathbf{P}^T = \mathbf{P}$.

Problem 2

Consider the vector $\mathbf{x}$ and matrix $\mathbf{U}$ given by:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \qquad \mathbf{U} = \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right].$$

1. Are the columns of matrix $\mathbf{U}$ linearly independent? Why?
2. Are the columns of matrix $\mathbf{U}$ for an orthonormal set of vectors? Why? Show your calculation.
3. Compute the orthogonal projector matrix $\mathbf{P}$ that projects any vector in $\mathbb{R}^3$ onto the span of columns of matrix $\mathbf{U}$.
4. Compute $\mathbf{P}^2$ and show that $\mathbf{P}^2 = \mathbf{P}$.
5. Compute the orthogonal projection of $\mathbf{x}$ onto $\mathbf{U}$. What is the projection error denoted by $\mathbf{e}$, where $\mathbf{e} = \mathbf{x} - \mathbf{P}\mathbf{x}$. Justify your answer.
6. Consider the space spanned by the first two columns of matrix $\mathbf{U}$ and denote that matrix with $\mathbf{U}'$:

$$\mathbf{U}' = \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right].$$

   (a) Repeat Part 3 and Part 5 with $\mathbf{U}'$ as the basis.

   (b) Show that the projection error $\mathbf{e}$ is orthogonal to the space spanned by $\mathbf{U}'$.

   (c) Show that the projection error $\mathbf{e}$ is aligned (linearly dependent) with the third column of matrix $\mathbf{U}$.

Problem 3

Assume $x \in [-1, 1]$.

1. Consider the monomials given by:

$$u_0(x) = 1$$
$$u_1(x) = x$$
$$u_2(x) = x^2$$
$$u_3(x) = x^3$$

   Are these functions orthogonal to each other? Are they orthonormal? Show your computation.

2. Consider the Legendre polynomials given by:

$$u_0(x) = \sqrt{\frac{1}{2}}$$
$$u_1(x) = \sqrt{\frac{3}{2}}x$$
$$u_2(x) = \sqrt{\frac{5}{2}} \cdot \frac{1}{2}(3x^2 - 1)$$
$$u_3(x) = \sqrt{\frac{7}{2}} \cdot \frac{1}{2}(5x^3 - 3x)$$

Are these functions orthogonal to each other? Are they orthonormal? Show your computation.