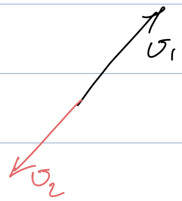
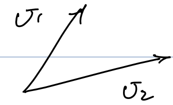
Linear Independence

v_1

v_2



$v_1 = \alpha v_2$



$\alpha_1 v_1 + \alpha_2 v_2 = \vec{0}$

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α_1, α_2: scalars

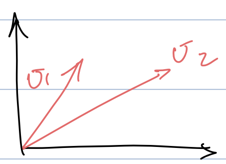
$\alpha_1 = \alpha_2 = 0$

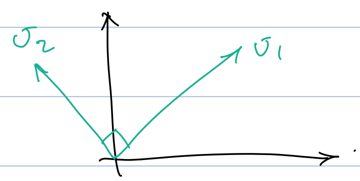
$v_1 = \dfrac{-\alpha_2}{\alpha_1} v_2$

Independence

The set of vectors $\{v_1, v_2, \dots, v_n\}$ are linearly independent if the only solution to : $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \vec{0}$ is

$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.





- If the set of vectors $A = \{v_1, v_2, \dots, v_n\}$ are orthogonal, $v_i^T v_j = 0$, $i \neq j$

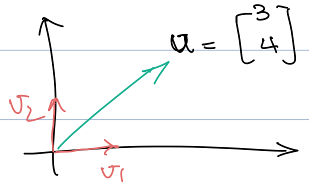
the vector $\{v_i\}_{i=1}^{n}$ are linearly independent.

Space & Span: the space spanned by vectors $\{v_1, v_2, \dots, v_n\} \equiv$ the space of vectors in the form of: $u = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

$\alpha_i \in \mathbb{R}$

Example: The space spanned by $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $\mathbb{R}^2$

$u = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3v_1 + 4v_2$



Example: The space spanned by $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is $\mathbb{R}^2$

Example: The space spanned by $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -4 \\ -8 \end{bmatrix}$ is a line:

Dimension: The dimension of the space spanned by $\{v_1, v_2, \dots, v_n\}$ is the minimum number of vectors that span the same space as $\{v_1, v_2, \dots, v_n\}$.

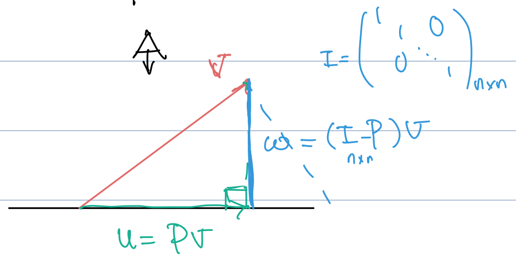
Example:

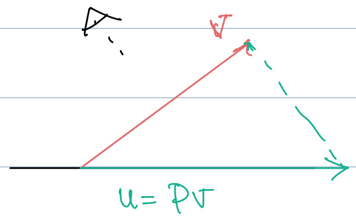
$v_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix}$

$v_3 = v_1 + v_2$

Projectors:

$u \in \mathbb{R}^n \quad u = Pv \qquad v \in \mathbb{R}^n$





$P_{n\times n} \qquad P(Pv) = (PP)v = P^2 v = Pv$

Projector. The matrix $P_{n\times n}$ is a projector if $P^2 = P$.

Complementary Projector. $P' = I - P$

$$P'^2 = P' \Rightarrow \quad (I-P)^2 \overset{?}{=} (I-P)$$

$$(I-P)(I-P) = I - P - P + P^2 = I - 2P + \underbrace{P^2}_{P} = I - 2P + P = I - P$$

Example: If $\underset{n\times r}{U} = [u_1\ u_2 \cdots u_r]_{n\times r}$ $\quad n > r$ and columns of U are a set of orthonormal vectors, then $P = UU^T$ is a projector.

$$\langle u_i, u_j \rangle = u_i^T u_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$U^T U = \begin{bmatrix} u_1^T \\ \hline u_2^T \\ \hline \vdots \\ u_r^T \end{bmatrix} \begin{bmatrix} u_1 \,\Big|\, u_2 \,\Big|\, \cdots \,\Big|\, u_r \end{bmatrix} = \begin{bmatrix} \overset{=1}{u_1^T u_1} & \overset{0}{\cancel{u_1^T u_2}} & \cdots & \overset{0}{\cancel{u_1^T u_r}} \\ \overset{0}{\cancel{u_2^T u_1}} & \overset{=1}{u_2^T u_2} & \cdots & \overset{0}{\cancel{u_2^T u_r}} \\ \vdots & \vdots & & \\ \overset{0}{\cancel{u_r^T u_1}} & \overset{0}{\cancel{u_r^T u_2}} & \cdots & \overset{=1}{u_r^T u_r} \end{bmatrix}_{r\times r} = I_{r\times r}$$

$n = 1000 \longrightarrow P = (UU^T)_{n\times n} \rightsquigarrow P^2 \overset{?}{=} P \implies P^2 = (U\underbrace{U^T)(U}_{I}U^T)$

$P^2 = UU^T = P$

$r = 5 \longrightarrow I = U^T U_{r\times r}$

Orthogonal Projectors. $\underset{n\times n}{P}$ is an orthogonal projector if $\begin{cases} P^2 = P \\ P^T = P \end{cases}$

$(Pv)^T((I-P)v) = 0 \;\ldots$

$\langle Pv, (I-P)v\rangle = 0$

Singular Value Decomposition:

$$A_{n\times m} = LU$$
$$= QR$$
$$\vdots$$
$$= U\Sigma V^T \leftarrow \text{SVD}$$

$\underset{n\times m}{A} = \underset{n\times m}{U}\,\underset{m\times m}{\Sigma}\,V^T_{m\times m}$ $\qquad n > m$

$U^T U = I_{m\times m}$ $\qquad U$: orthonormal $\leftarrow$ left singular vectors

$V^T V = I_{m\times m}$ $\qquad V$: orthonormal $\leftarrow$ right singular vector

$\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots\ \sigma_m \end{pmatrix}$ $\qquad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \geq 0 \leftarrow$ singular values