```julia
begin
    using Pkg; Pkg.add(["Distributions", "LinearAlgebra", "Plots", "PlutoUI"])
    using Distributions
    using LinearAlgebra
    using Plots
    using PlutoUI
    default(legendfontsize = 15.0, linewidth = 2.0)
end
```

```julia
# TableOfContents()
```

Problem Sheet 1

Inverse Transformation Method: Cauchy Distribution

The probability density function (pdf) of a cauchy distribution is defined

$$p(x|x_0, \gamma) = \frac{1}{\pi}\left(\frac{\gamma}{(x-x_0)^2+\gamma^2}\right)$$

where x_0 is the location of the mode and γ is a shape parameter.

Use the inverse transformation method to find a function $f(u|x_0, \gamma)$ which generates cauchy distributed samples if u is uniformly distributed over $[0, 1]$.

Fill in your answer here or on paper!

γ = 2.61

x$_0$ = 3.0

```
true_d = Distributions.Cauchy{Float64}(μ=3.0, σ=2.61)
```

We sample N uniform variables and compute the inverse transform on the uniform variables

```julia
function f(u, x₀, γ)
    ## !! CODE MISSING !! ##
    ## Fill in the function correctly transforming your uniform random variables here
    ## !! CODE MISSING !! ##
end;
```

```julia
begin
    N = 1000
```

```
    u = rand(N)
    x = f.(u, x₀, γ) # We broadcast the transformation on all u
end;
```

```
true_x = rand(true_d, N); # We also sample from the method in Distributions.jl
```

Polar Box-Muller

A computational more efficient version of the Box-Muller transformation makes use of random numbers z_1, z_2 which are uniformly distributed in the unit circle.

We can generate these by drawing z_1 and z_2 from the uniform distribution over $[-1, 1]$ and rejecting the pairs until $z_1^2 + z_2^2 \leq 1$ is true.

- **Show** that

$$y_1 = z_1 \sqrt{\frac{-2\ln(r^2)}{r^2}} \; (1)$$
$$y_2 = z_2 \sqrt{\frac{-2\ln(r^2)}{r^2}} \; (2),$$

with $r^2 = z_1^2 + z_2^2$,have the joint distribution

$$p(y_1, y_2) = \left(\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-y_1^2}{2}\right)\right)\left(\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-y_2^2}{2}\right)\right)$$

and therefore each has a standard normal distributio (Gaussian distribution with zero mean and unit variance).

Tip

Use the two first equations (1) and (2) to get a formula for r^2 depending only on y_1 and y_2. Then you can easily get the inverted function $z_{1/2}(y_1, y_2)$.

You might want to use *Mathematica* or *Maple* for computing some of the derivates. If not you can use

$$[y_1^4 - 2y_2^2 + y_1^2 y_2^2][y_2^4 - 2y_1^2 + y_1^2 y_2^2] - [y_1 y_2(2 + y_1^2 + y_2^2)]^2 = -2(y_1^2 + y_2^2)^3$$

Fill in your answers here or on paper

Rejection Sampler: General

We want to use a rejection sampler to sample from a target distribution $p(x)$. As a proposal distribution we can choose between:

- (1) $q_1(x)$, with $c_1 = 1.5$,
- (2) $q_2(x)$, with $c_2 = 2.0$,
- (3) $q_3(x)$, with $c_3 = 4.0$,

where for $i = 1, 2, 3$ the constant c_i is the smallest number which fulfills $c_i q_i(x) \geq p(x) \, \forall x \in \mathcal{R}$.

We know that on average it takes $6 \cdot 10^{-4}$, $4 \cdot 10^{-4}$ and $3 \cdot 10^{-4}$ seconds to get one sample from q_1, q_2 and q_3, respectively.

With this information, **which proposal distribution should you use and why?**

Fill in you answer here or on paper

Rejection Sampler: Rayleigh distribution

We want to use a rejection sampler to sample from a Rayleigh distribution. The pdf of a Rayleigh distribution is

$$p(x|\sigma_R) = \begin{cases} \frac{x}{\sigma_R^2} \exp\left(-\frac{x^2}{2\sigma_R^2}\right), & x \geq 0 \\ 0, & x < 0. \end{cases}$$

We want to use a Gaussian distribution as a proposal distribution and set its mean to σ_R (which is the mode of the Rayleigh distribution) and its standard deviation to σ_G.

a) Show that if $\sigma_G \geq \sigma_R$, c has to be at least

$$\sqrt{\frac{2\pi}{\exp(1)}} \frac{\sigma_G}{\sigma_R}$$

Fill in your answer here or on paper

For $\sigma_G \geq \sigma_R$ only x_1 is positive.

$$\frac{d^2c}{dx^2}(x) = \left[\left(\frac{2x(1-\frac{\sigma_G^2}{\sigma_R^2})-\sigma_R}{\sigma_G\sigma_R^2}\right) + \left(\frac{2x(\sigma_R^2-\sigma_G^2)-2\sigma_R^3}{2\sigma_G^2\sigma_R^2}\right)\left(\frac{x^2(1-\frac{\sigma_G^2}{\sigma_R^2})-x\sigma_R+\sigma_G^2}{\sigma_G\sigma_R^2}\right.\right.$$

$$\times\sqrt{2\pi}\exp\left(\frac{x^2(\sigma_R^2-\sigma_G^2)-2x\sigma_R^3+\sigma_R^4}{2\sigma_G^2\sigma_R^2}\right)$$

$$\begin{aligned}\operatorname{sign}\left(\frac{d^2c}{dx^2}(\sigma_R)\right) &= \operatorname{sign}\left(\left(\frac{2\sigma_R(1-\frac{\sigma_G^2}{\sigma_R^2})-\sigma_R}{\sigma_G\sigma_R^2}\right) + \left(\frac{2\sigma_R(\sigma_R^2-\sigma_G^2)-2\sigma_R^3}{2\sigma_G^2\sigma_R^2}\right)\left(\frac{\sigma_R^2(1-\frac{\sigma_G^2}{\sigma_R^2}}{\sigma}\right.\right. \\ &= \operatorname{sign}\left(\left(\frac{\sigma_R-\frac{\sigma_G^2}{\sigma_R}}{\sigma_G\sigma_R^2}\right)+\left(\frac{-2\sigma_G^2\sigma_R)}{2\sigma_G^2\sigma_R^2}\right)\left(\frac{\sigma_R^2-\sigma_G^2-\sigma_R^2+\sigma_G^2}{\sigma_G\sigma_R^2}\right)\right) \\ &= \operatorname{sign}\left(\frac{\sigma_R^2-\sigma_G^2}{\sigma_G\sigma_R^3}\right) \\ &= -1, \text{for } \sigma_G \geq \sigma_R.\end{aligned}$$

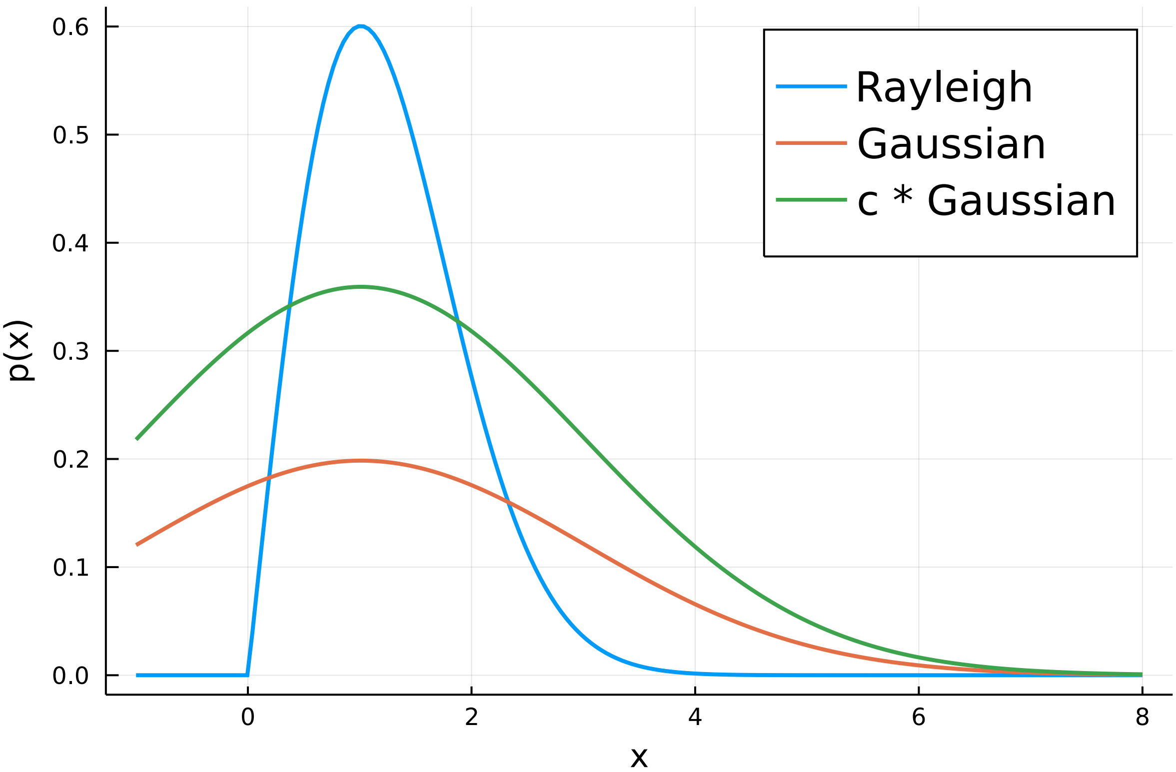
σr = 1.0 σg = 2.0

c = 1.81

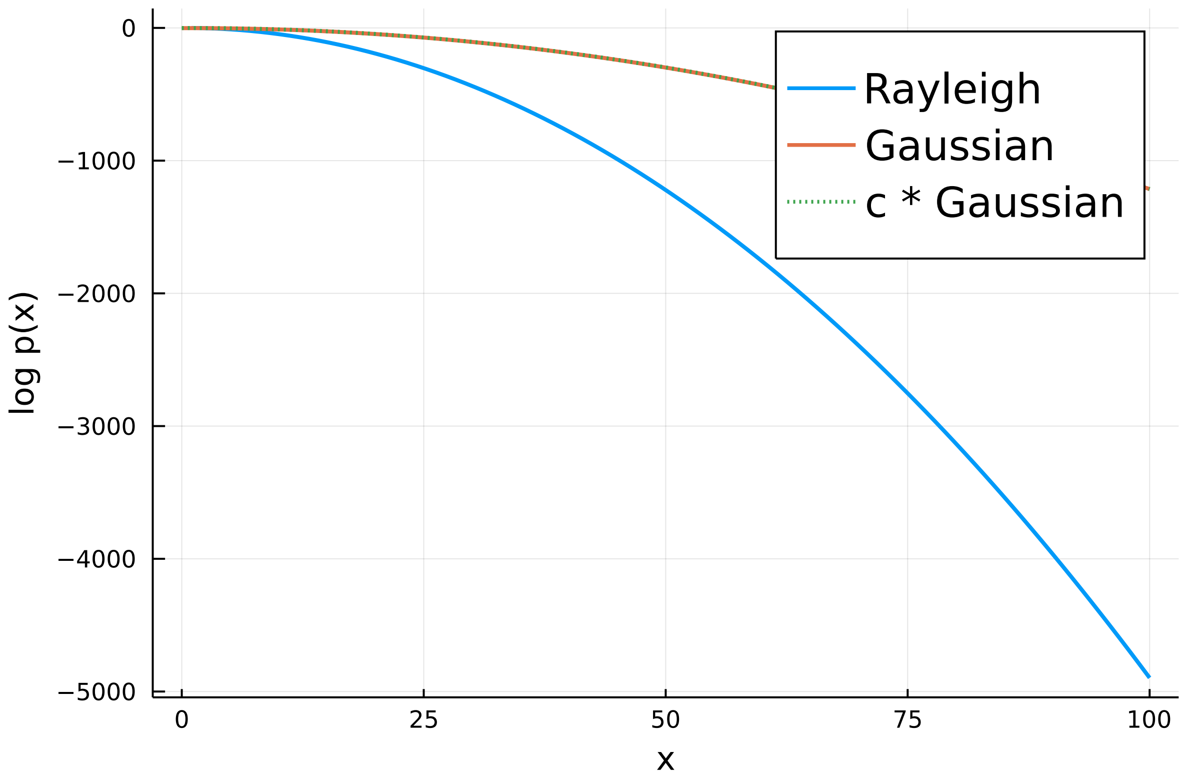
```
md"c = $(@bind c Slider(0.01:0.1:10; default=min_c, show_value=true))"
```

```
min_c = 5.0
```

```
min_c = 5.0 ## !! CODE MISSING !! ## Write here what hte minimum value of c should be
```



Let's plot the same thing but in log-space

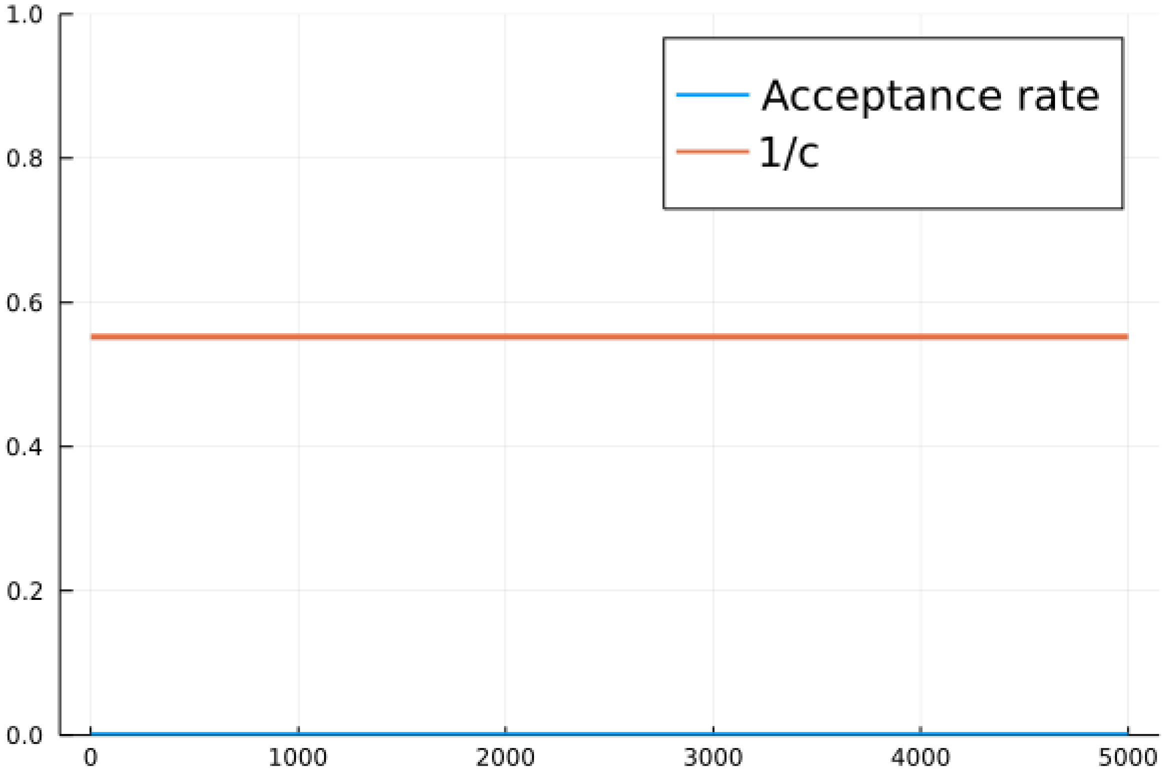


(b) It is obvious that c is minimal for $\sigma_G = \sigma_R$. Why is it not possible to use a Gaussian distribution with $\sigma_G < \sigma_R$ as a proposal?

Fill in your answer here or on paper

(c) Program a rejection sampler for a Rayleigh distribution with an arbitary σ_R and a Gaussian distribution with standard deviation and mean σ_R and the minimal c as computed in (a). Plot the acceptance rate against the number of drawn samples to show that it converges to $\frac{1}{c}$.

Solution



Acceptance rate
1/c
1.0
0.8
0.6
0.4
0.2
0.0
0
1000
2000
3000
4000
5000