# Tutorial 4

## Problem 1

Define two functions $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ 0 \end{bmatrix}, \quad S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ xy \end{bmatrix}.$$

Determine whether $T$, $S$, and the composite $S \circ T$ are linear transformations.

## Problem 2

The space $C^\infty(\mathbb{R})$ is the vector space of real functions which are infinitely differentiable. Let $T : C^\infty(\mathbb{R}) \to P_3$ be the map which takes $f \in C^\infty(\mathbb{R})$ to its third order Taylor polynomial, specifically defined by

$$T(f)(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3.$$

Here, $f'$, $f''$, and $f'''$ denote the first, second, and third derivatives of $f$, respectively.
Prove that $T$ is a linear transformation.

## Problem 3

Let $C([0, 3])$ be the vector space of real functions on the interval $[0, 3]$. Let $P_3$ denote the set of real polynomials of degree 3 or less.

Define the map $T : C([0, 3]) \to P_3$ by

$$T(f)(x) = f(0) + f(1) \cdot x + f(2) \cdot x^2 + f(3) \cdot x^3.$$

Determine if $T$ is a linear transformation. If it is, determine its nullspace.

## Problem 4

For an integer $n > 0$, let $P_n$ be the vector space of polynomials of degree at most $n$. The set $B = \{1, x, x^2, \ldots, x^n\}$ is a basis of $P_n$.

Let $T : P_n \to P_{n+1}$ be the map defined by, for $f \in P_n$,

$$T(f)(x) = x \cdot f(x).$$

Prove that $T$ is a linear transformation, and find its range and nullspace.