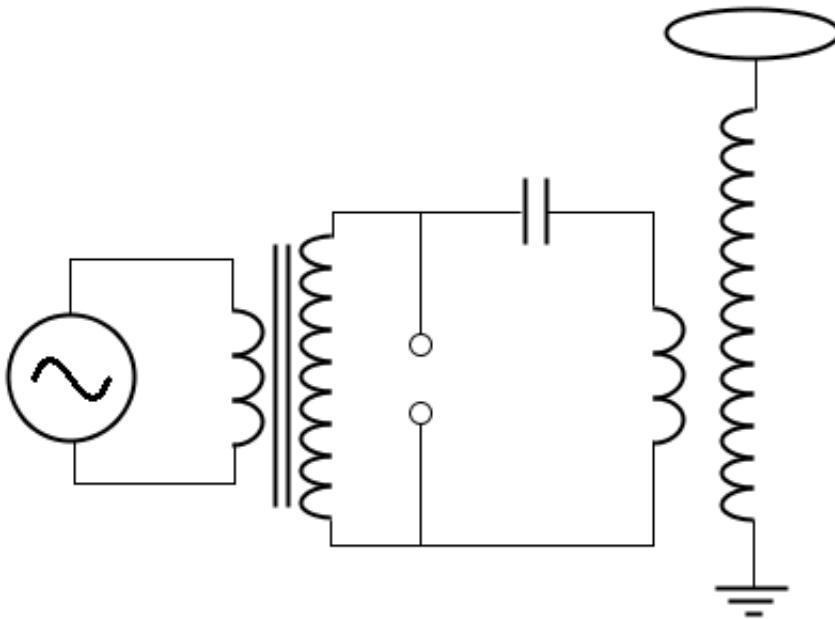# Coupled Inductors

From power distribution across large distances to radio transmissions, coupled inductors are used extensively in electrical applications. Their properties allow for increasing or decreasing voltage and current, transferring impedance through a circuit, and they can isolate two circuits from each other electrically. There are a wide variety of applications which exploit properties of transformers, such as tesla coils, impedance matching in audio frequency applications, potential transformers for reading very high voltages, Scott-T transformers which convert two-phase components to three-phase (or vice versa), and many more.

This module explains the functions of coupled inductors and explores the equations governing their performance. It will present mutual inductance, dot notation, the coupling coefficient, the turns ratio, and finally the ideal transformer. This module will explain how to model coupled inductors and provide modeling examples in MapleSim.

▼ Introducing Coupled Inductors

As current flows through a conductor, a magnetic field is generated. Steady-state current

will induce a steady magnetic field, and likewise a time-varying current will induce a time-varying magnetic field. Nearby conductors will generate a current as the varying magnetic flux permates the conductor, as stated by Faraday's law. This principle is known as **inductance**.

To begin investigating coupled inductance, recall that the voltage across an inductor is proportional to the change in current.

$$v = L \cdot \left(\frac{d}{dt} i(t) \right)$$

... Eq. (1)

L is the inductance of the conductor, in henry's.
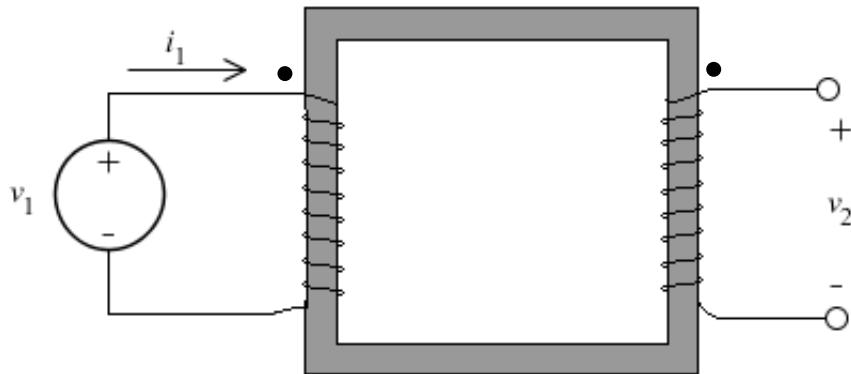

Figure 1: Two coils are magnetically coupled, wound around a magnetic core.

With coupled inductors, such as those shown in figure 1, two wires are wrapped around a magnetic core. In this diagram, note that the two wires are wrapped around the core in opposite directions and produce the same polarity of voltage. This is due to Lenz's law. A varying magnetic field, generated by the current i_1, will induce a current in the opposite direction on the coupled coil. Heinrich Lenz added the negative sign to Faraday's law of induction, below.

$$\varepsilon = - \frac{\partial \Phi_B}{\partial t}$$

... Eq. (2)

Here, Φ_B is the change in magnetic flux and ε is the electromotive force.

The flux within the magnetic core of the coupled inductors depends on the material of the core c_1, the number of turns of the wire around the core N_1, and the current i_1.

$$\Phi = c_1 \cdot N_1 \cdot i_1$$

... Eq. (3)

The voltage across the inductor is related to the flux by the number of turns of the coil.

$$v_1 = N_1 \cdot \frac{d\Phi}{dt} = c_1 \cdot N_1^2 \cdot \frac{di}{dt}$$

... Eq. (4)

This shows that we can draw a connection between eq. (1) and eq. (4).

$$L_1 = c_1 \cdot N_1^2$$

... Eq. (5)

The voltage across the second inductor follows eq. (4). The magnetic flux will be the same (as induced by the first coil), and we derive and equation for the mutual inductance.

$$v_2 = N_2 \cdot \frac{d\Phi}{dt} = N_2 \cdot c_m \cdot N_1 \cdot \left(\frac{d\,i_1}{dt} \right) = M \cdot \frac{d\,i_1}{dt}$$

... Eq. (6)

The variable M is the **mutual inductance** of the system, in henrys. This depends on the geometrical properties of the coils, such as the number of turns and radii of the turns of each coil.

$$M = c_m \cdot N_1 \cdot N_2$$

... Eq. (7)

A dot convention is used to specify the direction of the windings around the core. Dots on the same ends of the inductor, like in figure 1, mean that the coils are wrapped clockwise-counterclockwise to each other. If the dots are on the opposite sides like figure 2 shows, then the coils are wrapped in the same directions (clockwise-clockwise or counterclockwise-counterclockwise).
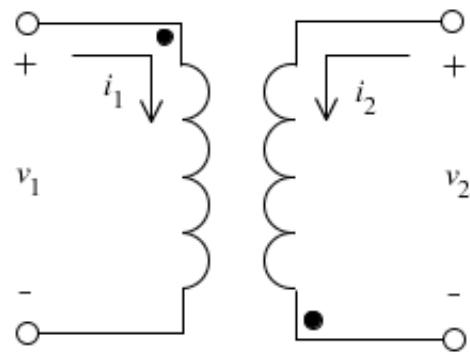
Figure 2: Coupled inductor with dot notation

It's important to note that the dot convention follows passive convention. Although figure 2 displays the voltages as the same polarity, one of the voltages will be negative. The dot convention states:

> If the current enters the dotted terminal of one coil, the voltage will be positive at the dot on the second coil.

Similarly, the voltage of the second coil will be negative if the current leaves the dotted terminal of the first coil. Figure 3 illustrates the passive convention.
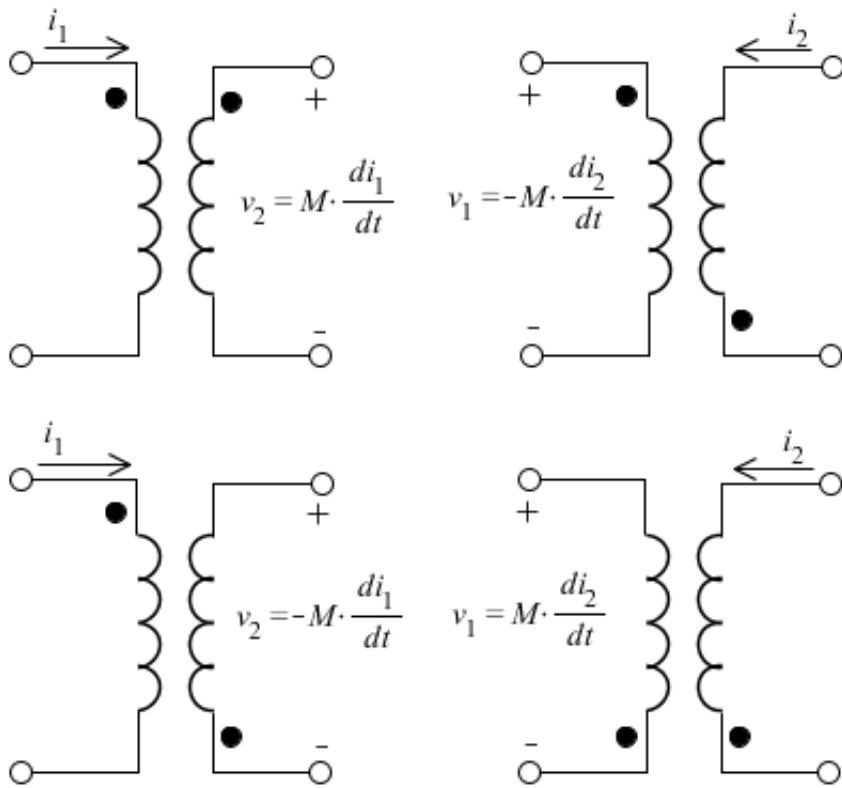
Figure 3: Induced voltage of coupled inductors.

This voltage is the voltage induced by the coupled current. A transformer can have current entering from both the first and second coils. The voltage across each coil will be dependant on the current through this coil and the induced voltage from the other coil. Suppose the dots are on the same end, and both currents enter at the dots. The voltages will be,

$$ v_1 = L_1 \cdot \frac{d\,i_1}{dt} + M \cdot \frac{d\,i_2}{dt} \qquad \text{and} \qquad v_2 = L_2 \cdot \frac{d\,i_2}{dt} + M \cdot \frac{d\,i_1}{dt} $$

... Eq. (8)

If the dots are on the opposite sides of each other, then the voltages will be,

$$ v_1 = L_1 \cdot \frac{d\,i_1}{dt} - M \cdot \frac{d\,i_2}{dt} \qquad \text{and} \qquad v_2 = L_2 \cdot \frac{d\,i_2}{dt} - M \cdot \frac{d\,i_1}{dt} $$

... Eq. (9)

Because the induction relies on a changing current, coupled inductors must be used with alternating current. This means that we can analyse coupled inductors in the frequency

domain as phasors. Eq. (8) and eq. (9) can be represented with phasors as eq. (10) and eq. (11), respectively.

$$V_1 = j \cdot \omega \cdot L_1 \cdot I_1 + j \cdot \omega \cdot M \cdot I_2 \qquad \text{and} \qquad V_2 = j \cdot \omega \cdot L_2 \cdot I_2 + j \cdot \omega \cdot M \cdot I_1$$

... Eq. (10)

$$V_1 = j \cdot \omega \cdot L_1 \cdot I_1 - j \cdot \omega \cdot M \cdot I_2 \qquad \text{and} \qquad V_2 = j \cdot \omega \cdot L_2 \cdot I_2 - j \cdot \omega \cdot M \cdot I_1$$

... Eq. (11)

▼ The Coupling Coefficient

The coils and mutual inductances depend on material properties of the core. The factor c used represents these properties in a lumped product. Explicity, c is the product of the permeability of free space and the core, the cross-sectional area, and the inverse of the length of the coils.

$$c = \frac{\mu_0 \cdot \mu_r \cdot A}{l}$$

... Eq. (12)

From eq. (5) and eq. (7), the mutual inductance is related to the coils inductances by

$$M = \sqrt{L_1 \cdot L_2}$$

... Eq. (13)

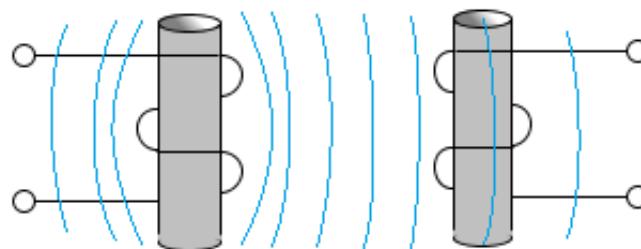However, this assumes a perfect flux linkage between the two conductors.

Figure 4: Flux lines cutting through the coils.

Some flux lines will not cut through the coupled inductor, which will affect the mutual inductance. Therefore, eq. (13) is multiplied by the **coupling coefficient** k, a value between 1 (perfectly linked inductors) and 0 (uncoupled inductors). Eq. (13) becomes

$$M = k \cdot \sqrt{L_1 \cdot L_2}$$

A coupling coefficient of 0.75 can imply that 75% of the flux lines cut through the other inductor. Different applications try to achieve different coupling coefficients, for example radio circuits will result in low k values and many power system transformers aim for $k = 1$.
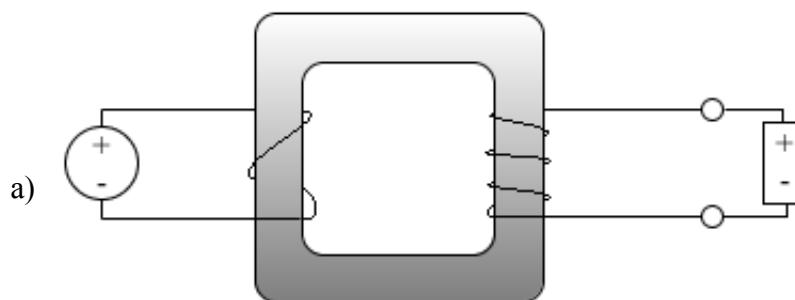
Ideal Transformers

Turns Ratio and Step Transformers

The **primary** coil in a transformer is the coil connected to the source voltage. The **secondary** coil is attached to the load of the circuit. The **turns ratio** is the ratio of some value of the second coil to the first. The turns ratio is defined as

$$n = \frac{V_2}{V_1} = \frac{I_1}{I_2} = \frac{N_2}{N_1} = \sqrt{\frac{L_2}{L_1}}$$

Where V is the voltage of the transformers at each coil, I is the current, N is the number of turns of each coil, and L is the inductance of the coils.

The turns ratio determines whether a transformer will step-up or step-down. A transformer that steps-up the voltage, that is the secondary voltage is greater than the primary voltage, will also step-down the current. Conversely, stepping down the voltage will increase the current on the secondary coil.
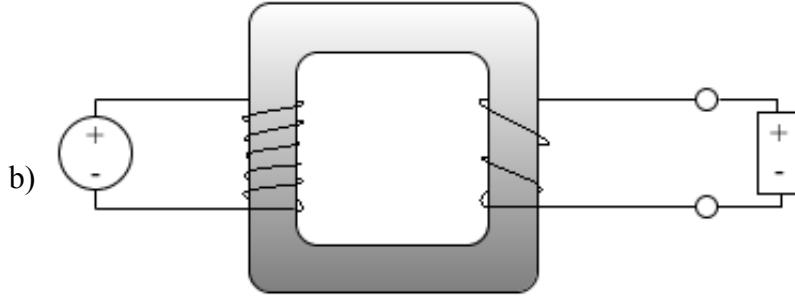

a)

Figure 5: a) Step-up transformer, b) Step-down transformer

The voltage of the secondary inductor can then be found with this ratio, and it is related to the primary voltage.

$$V_2 = n \cdot V_1$$

... Eq. (16)

One big advantage to AC electricity is that it can travel across large distances more efficiently than DC electricity. Power companies use transformers to step-up the voltage as electricity leaves the power plant. Once it arrives at the customer's house, a step-down transformer reduces the voltage down to 110 V or 220 V and increases the current. In these application, it is ideal that the coupling coefficient be as high as possible to reduce lost energy.

The **ideal transformer** is a transformer which has a coupling coefficient equal to 1 (unity). The important factor in an ideal transformer is the turns ratio, which determines the magnitude of the step. They are used to raise and lower voltage and for isolating circuits from each other.
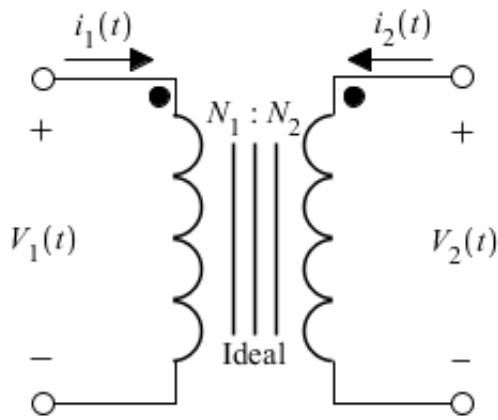
Figure 6: An ideal transformer

An ideal transformer does not absorb or store any electrical power while in operation. All of the power delivered to the primary coil will be directly delivered to the secondary coil. The sum of the power within the elements is zero. Consider the circuit in figure 7 with an ideal transformer.
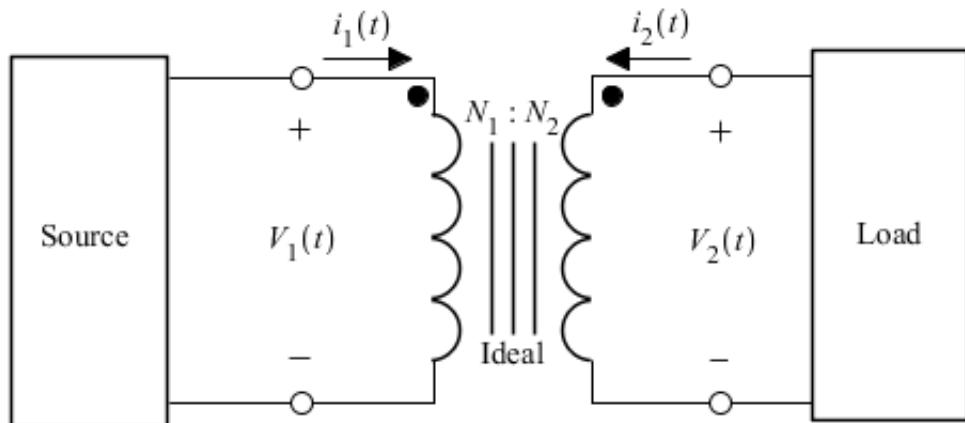

Figure 7: A source circuit and load circuit connected with an ideal transformer

The impedance on the source circuit will be

$$Z_1 = \frac{V_1}{I_1}$$

... Eq. (17)

Find the equivalent values of the current and voltage of the load circuit.

$$V_1 = \frac{V_2}{n} \qquad \text{and} \qquad I_1 = n \cdot I_2$$

... Eq. (18)

Substitute this into eq. (17).

$$\mathbf{Z}_1 = \frac{(V_2 / n)}{n \cdot I_2}$$

... Eq. (19)

Simplifying this shows a relation between the source impedance and load impedance.

$$\mathbf{Z}_1 = \frac{1}{n^2} \cdot \left(\frac{V_2}{I_2} \right) = \frac{1}{n^2} \cdot \mathbf{Z}_2$$

... Eq. (20)

This shows that the source experiences an impedance equivalent to the scaled impedance of the load. We can draw an equivelent impedance at the transformer, given the impedance from the load $\mathbf{Z}(\omega)$. Formally in the frequency domain, this is represented as

$$\mathbf{Z}_{eq}(\omega) = \frac{1}{n^2} \cdot \mathbf{Z}(\omega)$$

... Eq. (21)

▼ Modelling Transformers

Transformers can be represented where inductive coupling is unavailable with the use of dependent voltage or current sources. Generally, coupled inductors as shown in figure 8 can be modelled as shown in figure 9.
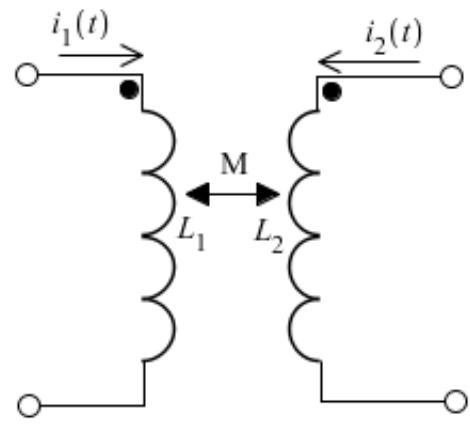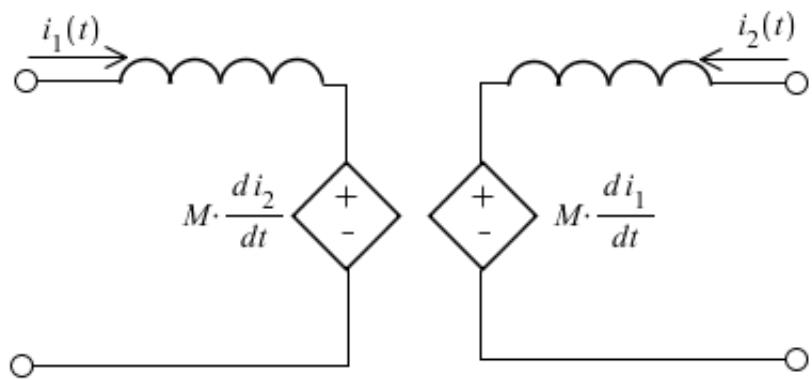
Figure 8: Coupled inductors

Figure 9: Inductors coupled via dependent voltage sources.

This circuit will behave the same way as coupled inductors function.

Similarily, ideal transformers can be represented with dependant sources.
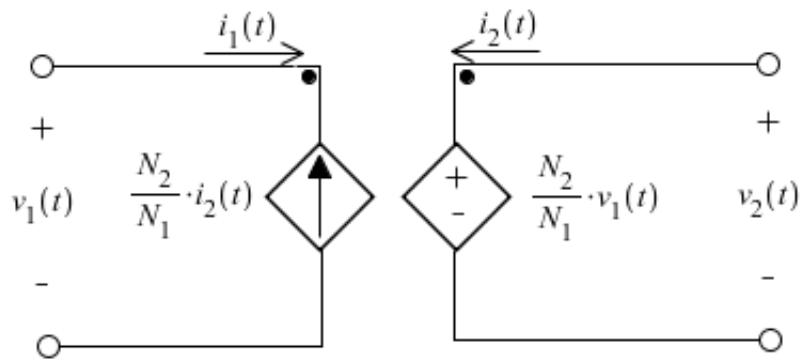
Figure 10: Ideal transformers represented through dependant voltage and current sources.

▼ Examples with MapleSim

► **Example 1: Voltage Across Coupled Inductors**

► **Example 2: Calculating Power in a Transformer Circuit**

References:

R. Dorf, J. Svoboda. "Introduction to Electric Circuits", 8th Edition. RRD Jefferson City, 2010, John Wiley and Sons, Inc.

Veluvolu, K. (n.d.). "Magnetically Coupled Circuits" [PDF Document]. Retrieved online from http://m80.knu.ac.kr/~veluvolu/Lecture_notes/CT2/Magneticckts.pdf