Parcial #2 Señales y Sistemas

1) $x(t) = |6 sen(3t + \pi/4)|^2$ con $t \in [-\pi, \pi]$

Tras un pequeño analisis de la estructura de la ecuación, se puede asumir que la funcion no es Periodica debido a que tiene un desfase de $\pi/4$.

$$x(t) = |6 sen(3t + \pi/4)|^2 = 36 sen^2(6t + \pi/2)$$

$$x(t) = 36\left(\frac{1}{2} - \frac{\cos(6t+\pi/2)}{2}\right) = 18 - 18\cos(6t + \pi/2)$$

$$x(t) = 18 - 18[(\cos(6t)\cos(\pi/2)) - sen(6t)\,Sen(\pi/2)]$$

$$x(t) = 18 + 18\,sen(6t)$$

Serie trigonometrica de fourier.

$$x(t) = a_0 + \sum_{n=-N}^{N} a_n \cos(nWot) + b_n\,sen(nWot)$$

Empieza a_0:

$$a_0 = \frac{1}{t_f - t_i}\int_{t_i}^{t_f} x(t)\,dt$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (18 + 18 sen(6t))\,dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} 18\,dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} 18 sen(6t)\,dt$$

$$a_0 = \frac{18}{2\pi}\int_{-\pi}^{\pi} dt + \frac{18}{2\pi}\int_{-\pi}^{\pi} sen(6t)\,dt = \frac{18}{2\pi}\, t\Big|_{-\pi}^{\pi} + \frac{18}{2\pi}\left(-\frac{1}{6}\cos(6t)\right)\Big|_{-\pi}^{\pi}$$

$$a_0 = \frac{18(2\pi)}{2\pi} - \frac{18}{12\pi}\left(\cos(6\pi) - \cos(-6\pi)\right)$$

$$a_0 = 18$$

$$a_n = \frac{2}{t_f - t_i} \int_{t_i}^{t_f} x(t) \cos(n w_0 t)\, dt$$

$$a_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} (18 + 18 \,\text{sen}(6t)) \cos(n w_0 t)\, dt$$

$$a_n = \frac{18}{\pi} \int_{-\pi}^{\pi} \cos(n w_0 t) + \frac{18}{\pi} \int_{-\pi}^{\pi} \text{sen}(6t) \cos(n w_0 t)\, dt$$

Recordando la identidad trigonométrica

$$\text{sen}(\theta) \cos(\beta) = \frac{1}{2} [\text{sen}(\theta + \beta) + \text{sen}(\theta - \beta)]$$

$$a_n = \frac{18}{\pi} \int_{-\pi}^{\pi} \cos(n w_0 t)\, dt + \frac{18}{2\pi} \int_{-\pi}^{\pi} \text{sen}(6t + n w_0 t)\, dt$$

$$+ \frac{18}{2\pi} \int_{-\pi}^{\pi} \text{sen}(6t - w_0 n t)\, dt$$

$$a_n = \frac{18}{\pi n} \text{sen}(nt) \Big|_{-\pi}^{\pi} + \frac{18}{2\pi} \left(-\frac{1}{6+n} \cos(6t + nt) \right) \Big|_{-\pi}^{\pi}$$

$$+ \frac{18}{2\pi} \left(-\frac{1}{6-n} \cos(6t - nt) \right) \Big|_{-\pi}^{\pi}$$

$$a_n = \frac{18}{\pi n} (\text{sen}(n\pi) - \text{sen}(-n\pi)) - \frac{18}{2\pi(6+n)} (\cos(6\pi + n\pi) - \cos(-6\pi - n\pi))$$

$$- \frac{18}{2\pi(6-n)} (\cos(6\pi - n\pi) - \cos(-6\pi + n\pi))$$

$$a_n = 0$$

Para b_n

$$b_n = \frac{2}{t_f - t_i}\int_{t_i}^{t_f} x(t)\, \mathrm{sen}(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} (18 + 18\,\mathrm{sen}(6t))(\mathrm{sen}(n\omega_0 t))\, dt$$

$$b_n = \frac{18}{\pi}\int_{-\pi}^{\pi} \mathrm{sen}(n\omega_0 t)\, dt + \frac{18}{\pi}\int_{-\pi}^{\pi} \mathrm{sen}(6t)\,\mathrm{sen}(n\omega_0 t)\, dt$$

Utilizando la identidad trigonometrica:

$$\mathrm{sen}(\theta)\,\mathrm{sen}(\beta) = \frac{\cos(\theta - \beta) - \cos(\theta + \beta)}{2} \qquad \text{con } \omega_0 = 1\,\mathrm{rad/s}$$

$$b_n = \frac{18}{\pi}\int_{-\pi}^{\pi} \mathrm{sen}(n\omega_0 t)\, dt + \frac{18}{2\pi}\int_{-\pi}^{\pi} \cos(6t - nt)\, dt$$

$$- \frac{18}{2\pi}\int_{-\pi}^{\pi} \cos(6t + nt)\, dt$$

$$b_n = \frac{18}{\pi}\left(-\frac{1}{n}\cos(nt)\right)\Big|_{-\pi}^{\pi} + \frac{18}{2\pi(6-n)}(\mathrm{sen}(6t - nt))\Big|_{-\pi}^{\pi}$$

$$- \frac{18}{2\pi(6+n)}(\mathrm{sen}(6t + nt)\Big|_{-\pi}^{\pi}$$

$$b_n = \frac{18}{\pi n}(\cos(n\pi) - \cos(-n\pi)) + \frac{18[\mathrm{sen}(6\pi - n\pi) - \mathrm{sen}(-6\pi + n\pi)]}{2\pi(6-n)}$$

$$- \frac{18[\mathrm{sen}(6\pi + n\pi) - \mathrm{sen}(-6\pi - n\pi)]}{2\pi(6+n)}$$

Calculo de los limites para b_n

$$\lim_{n \to 6} \frac{18(\mathrm{sen}(6\pi - n\pi) - \mathrm{sen}(-6\pi + n\pi))}{2\pi(6-n)}$$

$$18 \lim_{n \to 6} \frac{\cos((6-n)\pi)(-\pi) - \cos((-6+n)(+\pi))(\pi)}{-2\pi}$$

$$18\,\frac{\cos(0)(-\pi) - \cos(0)(\pi)}{-2\pi} = 18\left(\frac{-\pi - \pi}{-2\pi}\right) = 18$$

$$\lim_{n\to -6} -\frac{18\,(\mathrm{sen}(6\pi+n\pi)-\mathrm{sen}(-6\pi-n\pi))}{2\pi\,(6+n)}$$

$$\lim_{n\to -6} -18\left(\frac{\cos((6+n)\pi)(-\pi)-\cos((-6-n)\pi)(\pi)}{-2\pi}\right)$$

$$-18\,\frac{\cos(0)(-\pi)-\cos(0)(\pi)}{-2\pi} = -18\left(\frac{-\pi-\pi}{-2\pi}\right) = -18$$

Entonces:

$a_0 = 18$
$C_0 = a_0 = 18$
$b_n = \pm 18$

$$C_n = \frac{a_n - j b_n}{2}$$

$$C_{-6} = \frac{0-(-18j)}{2} = 9j$$

$$C_6 = \frac{0-18j}{2} = -9j$$

$$C_n = \begin{cases} 18 & n=0 \\ 9j & n=-6 \\ -9j & n=6 \\ 0 & \forall\ \{0,6,-6\} \end{cases}$$

La recontrucción de la señal por la exponencial.

$$X(t) = \sum_{n=-N}^{N} C_n e^{jnt} \qquad \text{Nota: } e^{\pm \omega j t} = \cos(\omega t) \pm j\,\mathrm{sen}(\omega t)$$

$$X(t) = C_{-6}e^{-j6t} + C_0 e^{0} + C_6 e^{j6t}$$

$$X(t) = 9j\left(\cos(6t) - j\,\mathrm{sen}(6t)\right) + 18 - 9j\left(\cos(6t) + j\,\mathrm{sen}(6t)\right)$$

$$X(t) = 18 + 9j\cos(6t) - 9j^2\mathrm{sen}(6t) - 9j\cos(6t) - 9j^2\,\mathrm{sen}(6t)$$

$$X(t) = 18 - 9j^2\,\mathrm{sen}(6t) \quad \text{con } j^2 = -1$$

$X(t) = 18 + 18\,\mathrm{sen}(6t)$ Reconstrucción por exponencial.

Reconstruccion Por trigonometrica

$$x(t) = a_0 + \sum_{n=-N}^{N} b_n \operatorname{sen}(w_0 n t)$$

$x(t) = 18 + 18\operatorname{sen}(6t) \rightarrow$ Reconstrucción Trigonométrica.

El espectro se grafica.

$$|C_n| = \sqrt{C_n C_n^*}$$

$$\phi_n = \tan^{-1}\left(\frac{Im\{C_n\}}{Re\{C_n\}}\right)$$

El error relativo se calcula:

$$E_r[\%] = \left[1 - \frac{1}{P_x}\sum_{n=-N}^{N}|C_n|^2\right] \times 100\%$$

$$P_x = \frac{1}{T}\int_{ti}^{tf} |x(t)|^2\, dt$$

$$P_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} |18 + 18\operatorname{sen}(6t)|^2\, dt$$

$$P_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} 18^2\, dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} 2(18)^2 \operatorname{sen}(6t)\, dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} 18^2 \operatorname{sen}^2(6t)\, dt$$

$$P_x = \frac{1}{2\pi}(18^2 t)\Big|_{-\pi}^{\pi} + \frac{2(18)^2}{12\pi}\cos(6t)\Big|_{-\pi}^{\pi} + \frac{18^2}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}(1 - \cos(12t)\, dt$$

$$P_x = \frac{1}{2\pi}(18^2 t)\Big|_{-\pi}^{\pi} + \frac{2(18)^2}{12\pi}\cos(6t)\Big|_{-\pi}^{\pi} + \frac{(18)^2}{4\pi}\left(t\Big|_{-\pi}^{\pi} - \frac{1}{12}\operatorname{sen}(12t)\Big|_{-\pi}^{\pi}\right)$$

$$P_x = \frac{162}{\pi}(2\pi) - 54(\cos(6\pi) - \cos(-6\pi)) + \frac{1}{2\pi}\left[\frac{18^2}{2}(2\pi)\right.$$

$$\left. - \frac{1}{12}[\operatorname{sen}(12\pi) - \operatorname{sen}(-12\pi)]\right]$$

$$P_x = 324 + 162 = 486$$

Entonces: $$E_r[\%] = \left[1 - \frac{|C_{-6}|^2 + |C_0|^2 + |C_6|^2}{P_x}\right] \times 100\%$$

$$|C_{-6}|^2 = |C_6|^2 = |9|^2 = 81$$

$$|C_0|^2 = |18|^2 = 324$$

$$E_r[\%] = \left[1 - \frac{81 + 324 + 81}{486}\right] 100\% = \left[1 - 1\right] 100\%$$

$$E_r[\%] = 0\%$$

Punto #2

$$C(t) = A_c \cos(2\pi F_o t)$$

$$y(t) = \left(1 + \frac{m(t)}{A_c}\right) c(t)$$

La señal modulada de la canción se puede expresar como la transformada de fourier de la siguiente manera:

$$Y(\omega) = \mathcal{F}\{y(t)\} = \mathcal{F}\left\{\left(1 + \frac{m(t)}{A_c}\right) c(t)\right\} = \mathcal{F}\{c(t)\} + \frac{1}{A_c}\mathcal{F}\{M(t)c(t)\}$$

Mediante las tablas de fourier:

$$\mathcal{F}\{A\cos(2\pi F_c t)\} = A_c \cdot \mathcal{F}\left\{\frac{e^{j2\pi F_c t} + e^{-j2\pi F_c t}}{2}\right\}$$

$$= A_c \cdot \left[\mathcal{F}\left\{\frac{e^{j2\pi F_c t}}{2}\right\} + \mathcal{F}\left\{\frac{e^{-j2\pi F_c t}}{2}\right\}\right]$$

$$= \frac{A_c}{2}\left[2\pi\,\delta(\omega - 2\pi F_c) + 2\pi\,\delta(\omega + 2\pi F_c)\right]$$

$$= A_c\pi\,\delta(\omega - 2\pi F_c) + A_c\pi\,\delta(\omega + 2\pi F_c)$$

Entonces la señal en el dominio de la frecuencia

$$C(\omega) = A_c\pi\,\delta\left[(\omega - 2\pi F_c) + (\omega + 2\pi F_c)\right]$$

Ahora para $y(t)$

$$\mathcal{F}\{y(t)\} = \mathcal{F}\left\{\frac{m(t)\,(A_c\cos(2\pi F_c t))}{A_c}\right\} = \mathcal{F}\{\cos(2\pi F_c t)\, m(t)\}$$

$$\mathcal{F}\{y(t)\} = \mathcal{F}\left\{\frac{m(t)\, e^{j2\pi F_c t}}{2}\right\} + \mathcal{F}\left\{\frac{m(t)\, e^{-j2\pi F_c t}}{2}\right\}$$

$$\mathcal{F}\{y(t)\} = \frac{m(\omega - 2\pi F_c t)}{2} + \frac{m(\omega + 2\pi F_c t)}{2}$$

$$\mathcal{F}\{y(t)\} = \frac{1}{2} m\left[(\omega - 2\pi F_c t) + (\omega + 2\pi F_c t)\right]$$

Entonces:

$$Y(\omega) = A_c \pi \; \delta \left[(\omega - 2\pi F_c t) + (\omega + 2\pi F_c t) \right] + \frac{1}{2} m \left[(\omega - 2\pi F_c t) + (\omega + 2\pi F_c t) \right]$$