ICTS Summer Lectures on Stochastic Gravitational Wave Background

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1 Energy Density in Gravitational Waves

Stochastic Gravitational Wave Background (SGWB) is an incoherent superposition of many GW sources. It could be cosmological: for example, vacuum fluctuations from the early universe could produce a stochastic background. It could also be astrophysical: for example, adding contributions from all binary black hole coalescences in the universe would produce a stochastic background. While this background is expected to be permanent (i.e. not transient), it is not expected to have a predictable waveform. However, different models predict different power spectra of the stochastic background, and possible different distributions across the sky and different polarizations.

We typically describe the SGWB in terms of the normalized energy density of GWs:

$$\Omega_{GW} = \frac{1}{\rho_{c,0}} \frac{d\rho_{GW}}{d \ln f} \tag{1}$$

where we think of $d\rho_{GW}$ as the energy density in the frequency band between f and $f+df$, and $\rho_{c,0}$ is the critical energy density needed to close the universe

$$\rho_{c,0} = \frac{3H_0^2 c^2}{8\pi G}, \tag{2}$$

where H_0 is the Hubble constant, G is Newton's constant and c is the speed of light. Note that this is a similar (but not identical) definition to the normalized energy densities we discussed in the context of cosmology. Further, the energy density in gravitational waves (GWs) is related to the average of the square of the time derivative of the metric (where the average is assumed to be computed over at least several wavelengths):

$$\rho_{\mathrm{GW}} = \frac{c^2}{32\pi G}\langle \dot{h}_{ab}\dot{h}^{ab}\rangle \tag{3}$$

Since this definition involves GW frequencies, it is helpful to convert our metric perturbation into the frequency domain. Specifically, we write the following decomposition:

$$h_{ab}(t,\vec{x}) = \sum_A \int_{-\infty}^{\infty} df \int d\hat{\Omega} h_A(f,\hat{\Omega}) e^{2\pi i f(t-\hat{\Omega}\cdot\vec{x}/c)} e^A_{ab}(\hat{\Omega}) \tag{4}$$

This is effectively a Fourier transform in both time and space, coupled with the decomposition into the plus and cross polarizations. The unit vector $\hat{\Omega}$ is a direction on the 2-D sphere (sky), described by two angles (θ,ϕ), from which the GW is arriving. We can therefore write the wavevector as $\vec{k} = 2\pi f\hat{\Omega}/c$. The amplitudes (at a given frequency, from a given direction in the sky) obey $h_A(f,\hat{\Omega}) = h^*_A(-f,\hat{\Omega})$, which is a consequence of the fact that h_{ab} is real. The polarization tensors have to be defined relative to the wave propagation direction, which is $\hat{\Omega}$. Specifically, we can define them as follows (see Allen-Romano paper):

$$e^+_{ab} = \hat{m}_a\hat{m}_b - \hat{n}_a\hat{n}_b \tag{5}$$
$$e^\times_{ab} = \hat{m}_a\hat{n}_b - \hat{n}_a\hat{m}_b \tag{6}$$
$$\hat{\Omega} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{7}$$
$$\hat{m} = \sin\phi\hat{x} - \cos\phi\hat{y} \tag{8}$$
$$\hat{n} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z} \tag{9}$$

At this point, we have to make some assumptions about the SGWB. It is common to assume that different polarizations are uncorrelated, and that different frequencies are uncorrelated - while this is certainly not required, most if not all SGWB models obey these assumptions. It is also typical to assume that the SGWB is isotropic so that different sky directions are not correlated - there certainly are cases where this assumption breaks down (for example, galactic SGWB will not be isotropic). We will proceed with these assumptions and write

$$\langle h^*_A(f,\hat{\Omega}) h_{A'}(f',\hat{\Omega}')\rangle = \delta_{AA'}\delta(f-f')\delta^2(\hat{\Omega},\hat{\Omega}')H(f) \tag{10}$$
$$\delta(\hat{\Omega},\hat{\Omega}') = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta') \tag{11}$$

where $H(f)$ can be thought of as the GW strain power spectrum in frequency. We will further assume that the SGWB is gaussian, so that $\langle h_A(f,\hat{\Omega})\rangle = 0$. We can then proceed with the calculation of the SGWB spectrum:

$$
\begin{aligned}
\Omega_{GW} &= \frac{1}{\rho_{c,0}}\frac{d\rho_{GW}}{d\ln f} && (12)\\
&= \frac{f}{\rho_{c,0}}\frac{c^2}{32\pi G}\frac{d}{df}\langle \dot{h}_{ab}\dot{h}^{ab}\rangle && (13)\\
&= \frac{2c^2 f}{32\pi G\rho_{c,0}}\sum_{AA'}\int df' d\hat{\Omega}d\hat{\Omega}'\langle h_A^*(f,\hat{\Omega})h_{A'}(f',\hat{\Omega}')\rangle \\
&\quad (-2\pi i f)e^{-2\pi i f(t-\hat{\Omega}\cdot\vec{x}/c)}2\pi i f' e^{2\pi i f'(t-\hat{\Omega}'\cdot\vec{x}/c)}e_{ab}^{A}(\hat{\Omega})e_{A'}^{ab}(\hat{\Omega}') && (14)\\
&= \frac{c^2 f\pi}{4G\rho_{c,0}}\sum_{AA'}\int df' d\hat{\Omega}d\hat{\Omega}'\delta_{AA'}\delta(f-f')\delta^2(\hat{\Omega},\hat{\Omega}')H(f) \\
&\quad f f' e^{-2\pi i f(t-\hat{\Omega}\cdot\vec{x}/c)}e^{2\pi i f'(t-\hat{\Omega}'\cdot\vec{x}/c)}e_{ab}^{A}(\hat{\Omega})e_{A'}^{ab}(\hat{\Omega}') && (15)\\
&= \frac{c^2 f\pi}{4G\rho_{c,0}}\sum_{A}\int d\hat{\Omega}H(f) \\
&\quad f^2 e^{-2\pi i f(t-\hat{\Omega}\cdot\vec{x}/c)}e^{2\pi i f(t-\hat{\Omega}\cdot\vec{x}/c)}e_{ab}^{A}(\hat{\Omega})e_{A}^{ab}(\hat{\Omega}) && (16)\\
&= \frac{4c^2 f 4\pi^2}{4G\rho_{c,0}}f^2 H(f) && (17)\\
&= \frac{4\pi^2 c^2}{G}\frac{8\pi G}{3H_0^2 c^2}f^3 H(f) && (18)\\
&= \frac{32\pi^3}{3H_0^2}f^3 H(f) && (19)
\end{aligned}
$$

In the third step, the factor of 2 in the numerator comes from the fact that the integral over f (which is annulled by the derivative wrt f) has the range between $-\infty$ and $+\infty$, while in line 3 we only consider positive frequencies (so $\int_{-\infty}^{+\infty} = 2\int_0^{\infty}$). So the SGWB energy density is proportional to the strain power spectrum, but note the f^3 factor which effectively weighs different frequency bins of the power spectrum. Because of this f^3 factor, detectors operating at lower frequencies (with the same strain sensitivity) would have better sensitivity to Ω_{GW}.

2 Astrophysical Isotropic SGWB

There are many models of astrophysical SGWB, obtained by integrating the contributions of astrophysical sources (such as binary black hole systems) across the entire universe. To perform this integral, it is useful to use the concept of the energy flux, i.e. energy travelling through some area A over some time t:

$$F = \frac{E}{At} \tag{20}$$

$$\frac{F}{c} = \frac{E}{Atc} = \frac{E}{V} = \rho \tag{21}$$

Then we can go back to the definition of the SGWB energy density spectrum and rewrite it in terms of the flux (per unit frequency):

$$\Omega_{GW}(f) = \frac{1}{\rho_{c,0}} \frac{d\rho_{GW}}{d\ln f} = \frac{f}{\rho_{c,0}} \frac{d\rho_{GW}}{df} = \frac{f}{\rho_{c,0} c} \frac{dF}{df} \tag{22}$$

On the other hand, the flux (per unit frequency as observed on Earth) can be written as an integral of fluxes arriving from all sources in the universe:

$$\frac{dF}{df} = \int_0^\infty dz \frac{R(z)}{4\pi d_L^2(z)} \frac{1}{1+z} \frac{dE_{GW}}{df}\bigg|_{f(1+z)} \tag{23}$$

where $d_L(z)$ is the luminosity distance for a given redshift z, $R(z)$ is the rate of GW sources per unit redshift as observed on Earth, and dE/df is the energy spectrum emitted by a single source (evaluated in the frame of the source, so before the redshift). The factor of $(1+z)$ accounts for the redshift of gravitons by the time they reach Earth, hence reducing the energy they carry. The rate of sources can further be cast in terms of the rate of sources per comoving volume $R_V(z)$:

$$
\begin{aligned}
R(z) &= \frac{R_V(z)}{1+z}\frac{dV}{dz} && (24)\\
dV &= (1+z)^3 dV_p = (1+z)^3 4\pi r^2 dr = (1+z)4\pi d_L^2 dr && (25)\\
dr &= \frac{c}{H_0}\frac{dz}{\sqrt{\Omega_{M,0}(1+z)^3+\Omega_{\Lambda,0}}} && (26)\\
R(z) &= \frac{4\pi d_L^2 c}{H_0}\frac{R_V(z)}{1+z}\frac{(1+z)}{\sqrt{\Omega_{M,0}(1+z)^3+\Omega_{\Lambda,0}}} && (27)\\
&= \frac{4\pi d_L^2 c}{H_0}\frac{R_V(z)}{\sqrt{\Omega_{M,0}(1+z)^3+\Omega_{\Lambda,0}}} && (28)\\
\Omega_{GW}(f) &= \frac{f}{\rho_{c,0}c}\int_0^\infty dz \frac{4\pi d_L^2 c}{H_0}\frac{R_V(z)}{\sqrt{\Omega_{M,0}(1+z)^3+\Omega_{\Lambda,0}}}\frac{1}{4\pi d_L^2(z)}\frac{1}{1+z}\left.\frac{dE_{GW}}{df}\right|_{f(1+z)} && (29)\\
&= \frac{f}{\rho_{c,0}H_0}\int_0^\infty dz \frac{R_V(z)}{(1+z)\sqrt{\Omega_{M,0}(1+z)^3+\Omega_{\Lambda,0}}}\left.\frac{dE_{GW}}{df}\right|_{f(1+z)} && (30)
\end{aligned}
$$

In the first line, the factor of $(1+z)$ accounts for the redshifting of the rate of sources: $R(z)$ is measured in local time, while $R_V(z)$ is measured in the source frame time. In the second line we defined the comoving volume dV in terms of the proper volume dV_p, and we used the relationship between the luminosity and proper distance: $d_L = (1+z)r$. In the third line we recall the proper distance evaluated in terms of cosmological parameters, as we have seen in the first part of the course. In the last line we combine all terms to compute the SGWB energy spectrum.

2.1 Compact Binary Coalescences

In order to proceed further we need to choose a model, since this will specify both dE/df and $R_V(z)$. Let us consider the background due to compact binary coalescences (binary black holes (BBH), binary neutron stars (BNS) and black-hole-neutron-star (BHNS) systems). The weak-field approximation of the energy spectrum emitted by a single binary in the inspiral phase is:

$$\frac{dE}{df} = \frac{\pi^{2/3}G^{2/3}\mathcal{M}^{5/3}}{3}f^{-1/3} \tag{31}$$

where $\mathcal{M}$ is the chirp mass of the binary system, related to the individual component masses as $\mathcal{M} = (m_1 m_2)^{3/5}/(m_1 + m_2)^{1/5}$. We conclude that the inspiral part of the binary coalescence waveform leads to $dE/df \sim f^{-1/3}$ and $\Omega_{GW}(f) \sim f^{2/3}$. Of course, adding the merger and ringdown contributions would modify the SGWB spectrum, but it turns out the effect is not significant in the frequency band of interest to LIGO/Virgo—the merger/ringdown signal is associated with the final stage of the coalescence, and therefore contributes the most at frequencies near the last stable orbit, which is typically > 100 Hz. As we will see below, SGWB searches draw most of their sensitivity from lower frequencies. Further, for the BNS or BHNS cases the merger and ringdown are additionally complicated by the presence of matter, and are therefore usually not included in the model. For the BBH case, the merger/ringdown contributions can be found in [3].

Note that the Eq. 31 assumes a single value of the chirp mass, i.e. it assumes that all binaries have the same chirp mass. This can be further developed by writing

$$\frac{dE}{df} = \frac{\pi^{2/3} G^{2/3}}{3} f^{-1/3} \int d\mathcal{M} \mathcal{M}^{5/3} P_c(\mathcal{M}) \tag{32}$$

where we now average the chirp mass over some distribution $P_c(\mathcal{M})$ that can be extracted from the catalog of observed (nearby) compact binaries by LIGO/Virgo. Or, equivalently, one can implement a double integral over the distribution of two component masses. This is indeed the approach taken in the recent LIGO/Virgo papers [4, 5].

As for the rate of these systems, if we assume that the black holes are of stellar origin, then the merger rate will be related to the star formation rate (SFR) $R_*(z)$, which gives the rate of formation of stars in units of mass per comoving volume per time. SFR has been extensively studied by many authors and their estimates are converging at least for redshifts below ~ 6. For higher redshifts there is much uncertainty (e.g. in terms of the population-3 stars etc), but the contribution of such distant systems to the overall $\Omega_{\rm GW}$ is relatively small. The functional form used in recent LIGO/Virgo paper is [6]

$$R_*(z) = \nu \frac{a \exp\left(b(z - z_m)\right)}{a - b + b \exp\left(a(z - z_m)\right)} \tag{33}$$

where $\nu = 0.146$ $M_\odot/\mathrm{yr}/\mathrm{Mpc}^3, a = 2.80, b = 2.46, z_m = 1.72$, but there are also possible variations on this, for example to account for the metallicity evolution which impacts the rate of black hole formation. Using SFR directly is not sufficient because there is a time delay between formation of a binary (which is the time when the star formation rate applies) and its merger time (which is when the binary produces most of its GW signal). This delay could be significant, and it is at the moment not fully understood. To account for this, we can write

$$R_V(z) \;=\; \lambda \int_{t_{min}}^{t_{max}} dt_d R_*(t(z)-t_d)P(t_d) \qquad (34)$$

That is, the merger rate $R_V(z)$ is evaluated by averaging over some time-delay distribution $P(t_d)$, and the star formation rate is evaluated at the formation time (that takes the time delay between formation and merger into account). The time delay distribution is not well understood, but it is typical to assume $P(t) \sim 1/t$ and that $t_{min} = 50$ Myr for BBH and 20 Myr for BNS systems. The parameter λ in Eq. 34 denotes the mass fraction of the stars that end up in compact binary systems. This parameter can be treated as a free parameter, or could be fixed to match the locally observed merger rate based on the LIGO/Virgo observations.

Another complication has to do with the fact that massive black holes are formed preferentially in low-metallicity environments, and the metallicity also evolves with redshift. To account for this, one can introduce an additional complexity in the model—for example, the O2 LIGO-Virgo analysis [4] assumed for binary systems with at least one black hole more massive than $30M_\odot$ that the star formation rate would be reweighed by the fraction of stars with metallicities smaller than 1/2 of the solar metallicity.

We then have all the ingredients to compute $\Omega_{GW}(f)$ - see slides for recent results.

2.2 Stellar Core Collapse

Stellar core collapse is the process that takes place at the end of a star's life, when the fusion processes cease. If the star is more massive than the Sun, it will collapse to a neutron star or a black hole. The process of the collapse is not yet fully understood, and it depends on the presence of magnetic field, heat transfer, neutrino flows and other physical process. However, the collapse is likely asymmetric, implying that large masses are moved at high velocities, i.e. GW production is likely.

Multiple mechanisms for GW production are possible, including a quasi-periodic signal generated during the post-shock convection phase, hot-bubble convection and the standing accretion shock instability (SASI), anisotropic neutrino emission, and the ringdown of the potentially newly formed black hole. Full three-dimensional simulations that include a complete set of relevant physical processes have become possible only recently, and their GW predictions vary. Further, the dependency of the gravitational-wave signal on stellar progenitor properties, such as mass or spin, is not well known, and the rate of core collapse events is similarly uncertain.

Adding GW contributions from all core collapse events leads to a SGWB. While there are significant uncertainties, one can model this SGWB similarly to the case of compact binaries. The rate of core collapse events will be directly proportional to the SFR (i.e. there is no time delay): $R_v(z) = \lambda_{CC} R_*(z)$. However, dE/df is far less certain because there are significant variations in the predictions from core collapse simulations. The slides include some examples of these simulations. Since it is not possible to model dE/df from first principles, one approach is to model it empirically [7]:

$$\frac{dE}{df} = \frac{G}{c^5} E_\nu^2 \langle q \rangle^2 \left(1 + \frac{f}{a}\right)^2 e^{-2f/b} \tag{35}$$

where E_ν is the energy carried away by neutrinos during the collapse, $\langle q \rangle$ is the averaged neutrino anisotropym and a, b are empirical parameters. Crocker et al. [7] studied a broad range of simulations and fitted this functional form to them, in order to find the allowed range for the parameters a and b, finding

$$5 \text{ Hz} < a < 150 \text{ Hz} \tag{36}$$

$$10 \text{ Hz} < b < 400 \text{ Hz} \tag{37}$$

$$\tag{38}$$

Pulling everything together, we arrive at the full expression for the SGWB energy density spectrum:

$$\Omega_{\rm GW}(f) = \frac{8\pi G f \xi}{3 H_0^3 c^2} \int_0^\infty dz \frac{R_*(z)}{(1+z)\sqrt{\Omega_{M,0}(1+z)^3 + \Omega_{\Lambda,0}}} \left(1 + \frac{f(1+z)}{a}\right)^2 e^{-2f(1+z)/b} \tag{39}$$

where $\xi = \lambda_{CC} E_\nu^2 \langle q \rangle^2$ captures all amplitude uncertainties into one free parameter of the model. With these assumptions, one can compare the predicted $\Omega_{\mathrm{GW}}(f)$ to the detector sensitivities to determine the regions of this empirical parameter space (ξ, a, b) that can be probed by observations. See slides for some examples.

2.3 Magnetars

Magnetars are neutron stars with a strong magnetic field, of order $B \approx 10^{15}$ Gauss. The strength of the magnetic field and its geometry, combined with the equation of state of the neutron star, can lead to asymmetry in this rotating neutron star, hence producing GWs. We can again add up contributions from all magnetars and produce a SGWB that we can model similarly to above. The rate of magnetars will be directly proportional to the SFR (i.e. there is no time delay): $R_v(z) = \lambda_m R_*(z)$, where λ_m effectively captures the fraction of star mass that ends up in magnetar objects. The energy spectrum of each magnetar is given by [8]:

$$\frac{dE}{df} = I\pi^2 f^3 \left(\frac{5c^2 R^6}{192\pi^2 G I^2} \frac{B^2}{\epsilon^2} + f^2 \right)^{-1} \tag{40}$$

where R is the radius of the magnetar, I is its moment of inertia, and ϵ is the magnetar's ellipticity. The first term in the bracket comes from the angular acceleration caused by electromagnetic radiation, while the second term is due to the acceleration caused by GW radiation.

As in the above cases, one can compare these predictions to the detector sensitivities to determine which part of the parameter space (λ_m, B, ϵ) can be probed by detectors. See slides for examples. Furthermore, different models of the equation of state in neutron stars and different geometry of the magnetic field (poloidal vs toroidal vs twisted torus) lead to different relations between the elipticity and the magnetic field. For example, the poloidal field model gives [8]:

$$\epsilon = \beta \frac{R^8 B^2}{4 G I^2} \tag{41}$$

where β is a dimensionless parameter that depends on the field geometry and equation of state. See slides for an example.

2.4 Primordial Black Hole Binaries

Another interesting possibility is that the black hole binaries observed by LIGO/VIRGO may be (at least partly) of primordial origin. That is, density fluctuations in the early universe could lead to formation of black holes that then grow over time as additional material falls into them. As such, these black hole may contribute to the dark matter problem in cosmology—e.g. these promordial black holes could have been initiated by density fluctuations in dark matter itself.

There are multiple mechanisms for production of binary systems of black holes and for production of GWs. We will consider here the mechanism in which the primordial black holes in the dark matter halo interact with each other via the emission of GWs and form binaries. The GW energy spectrum emitted by a single source (single primordial BBH) will be exactly the same as for any compact binary discussed above. However, the rate of primordial BBH systems will no longer be associated with the star formation rate, since the black holes are not created by the collapsing stars.

The cross section for the BBH capture has been computed [9]. When combined with th black hole number density in the dark matter halo, one gets the following rate of primordial BBH mergers per halo:

$$R_{\rm halo}(z) \;=\; \left(\frac{85\pi}{6\sqrt{2}}\right)^{2/7} \frac{2\pi}{3}\frac{G^2 M_{vir}^2 D(v)\lambda^2}{R_S^3 c g^2(C)}\left[1-\frac{1}{(1+C)^3}\right]. \qquad (42)$$

Here, λ is the fraction of dark matter in the form of black holes, R_S is the characteristic radius of the halo profile, M_{vir} is the mass inside the virial radius R_{vir} (defined to be the radius at which the NFW dark matter profile reaches 200 times the critical density of the universe), $C = R_{vir}/R_S$ is the concentration parameter, $g(C) = \ln(1+C) - C/(1+C)$, and $D(v)$ is the expected value of $(2v/c)^{3/7}$ for the Maxwell-Boltzmann distribution of velocity v. To calculate the merger rate, the rate per halo should be multiplied by the number of halos. This is done using the halo mass function dn/dM_{vir} which can be obtained from simulations:

$$R_V(z) \;=\; \int R_{\rm halo}(z)\frac{dn}{dM_{vir}}dM_{vir} \qquad (43)$$

One can then proceed as above the compute $\Omega_{\rm GW}$ [10]. The slides show the results of this calculation for multiple choices of the halo mass function and anchoring $R_V(0)$ to the observed rate of BBHs. Since $R_V(z)$ for the primordial BBHs is a much slower function of redshift that the SFR, the overall number of the BBH mergers in the primordial model is much smaller than that in the stellar BBH model (discussed above), and therefore the corresponding primordial BBH $\Omega_{\rm GW}$ is weaker.

3 Searching for Isotropic SGWB

3.1 Cross-Correlation Search

The cross-correlation search for SGWB is based on the formalism developed by Bruce Allen and Joe Romano [1]. The key aspect of it is that when computing the cross-correlation of two GW detectors' time series, one is effectively probing the two-point correlation of the metric, which by Eq. 3 is also related to the GW energy density. There are some differences having to do with the coupling of the metric to the GW detectors, and with the separation and orientation of two detectors, but these can be quantified. To start, let us assume that the two detector time series will have the SGWB and the detector noise contributions, i.e. we write

$$s_1(t) = n_1(t) + h_1(t) \tag{44}$$
$$s_2(t) = n_2(t) + h_2(t) \tag{45}$$

We will assume that both the noise $n_i(t)$ and the SGWB signal $h_i(t)$ are random variables obeying Gaussian distributions, and that these distributions are stationary in time. Hence,

$$\langle s_i \rangle = \langle n_i + h_i \rangle = 0. \tag{46}$$

Then, assuming that we observe over some period of time T, we can define the cross correlation estimator (following [1])

$$Y = \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' s_1(t) s_2(t') Q(t,t') \tag{47}$$

where we have added a filter $Q(t,t')$ which is yet to be determined. If we assume that the SGWB is stationary and that the detector noise is also stationary, this filter will only depend on the time difference $t-t'$, i.e. we can write $Q(t,t') = Q(t-t')$. Furthermore, we would expect that the optimal form of this filter would peak at $t = t'$—due to the geographic separation of detectors, and due to the broadband nature of the SGWB, we would expect this filter to approach zero as $|t - t'| \to \infty$, which allows us to replace one of the integral's limits:

$$Y = \int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} dt' s_1(t) s_2(t') Q(t, t') \tag{48}$$

We now switch to the frequency domain, by defining the Fourier transform as:

$$g(t) = \int_{-\infty}^{\infty} df g(f) e^{2\pi i f t} \tag{49}$$

Fourier transforming s_1, s_2 and Q then gives:

$$\begin{aligned}
Y &= \int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} dt' s_1(t) s_2(t') Q(t, t') \\
&= \int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} df s_1^*(f) e^{-2\pi i f t} \int_{-\infty}^{\infty} df' s_2(f') e^{2\pi i f' t'} \int_{-\infty}^{\infty} df'' Q(f'') e^{2\pi i f''(t-t')} \\
&= \int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \int_{-\infty}^{\infty} df'' s_1^*(f) s_2(f') Q(f'') e^{-2\pi i (f-f'')t} \int_{-\infty}^{\infty} dt' e^{2\pi i (f'-f'')t'} \\
&= \int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \int_{-\infty}^{\infty} df'' s_1^*(f) s_2(f') Q(f'') e^{-2\pi i (f-f'')t} \delta(f' - f'') \\
&= \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' s_1^*(f) s_2(f') Q(f') \int_{-T/2}^{T/2} dt e^{-2\pi i (f-f')t} \\
&= \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' s_1^*(f) s_2(f') Q(f') \delta_T(f - f')
\end{aligned} \tag{50}$$

where in the last step we define the finite-time approximation to the Dirac delta function

$$\delta_T(f - f') = \int_{-T/2}^{T/2} dt e^{-2\pi i (f-f')t} = \frac{\sin(\pi T(f - f'))}{\pi(f - f')} \tag{51}$$

Note that $\delta_T(0) = T$, but in the limit $T \to \infty$, δ_T approaches the standard Dirac delta function. Our next task is to determine the expected value of this estimator:

$$\langle Y \rangle = \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \langle s_1^*(f) s_2(f') \rangle Q(f') \delta_T(f - f'). \tag{52}$$

If we assume that the noise in the two detectors is uncorrelated, and that the SGWB and detector noise are uncorrelated, then

$$\langle s_1^*(f)s_2(f')\rangle = \langle h_1^*(f)h_2(f')\rangle. \tag{53}$$

That is, the two-point correlation of the two detector time series reduces the two-point correlation of the SGWB signals in the two detectors. The SGWB signal observed in each detector can be written as a contraction of the GW metric and the detector response:

$$\begin{aligned} h_i(t) &= h_{ab}(t,\vec{x}_i)d^{ab}(t,\vec{x}_i) && (54) \\ d^{ab}(t,\vec{x}_i) &= \frac{1}{2}\left(\hat{X}^a\hat{X}^b - \hat{Y}^a\hat{Y}^b\right) && (55) \end{aligned}$$

where we have defined the detector response $d^{ab}(t,\vec{x}_i)$ in terms of the unit vectors $\hat{X}$ and $\hat{Y}$ along the x- and y-arms of the detector. Note that since Earth rotates, the detector response tensor will be a function of time and of position of the detector on the Earth, $\vec{x}_i$. Our next step is to express the metric as a plane-wave expansion, in frequency domain, which then gives each detector's signal as:

$$\begin{aligned} h_i(f;t) &= \sum_A \int d\hat{\Omega} h_A(f,\hat{\Omega})e^{-2\pi i f\hat{\Omega}\cdot\vec{x}_i/c}e^A_{ab}(\hat{\Omega})d^{ab}(t,\vec{x}_i) \\ &= \sum_A \int d\hat{\Omega} h_A(f,\hat{\Omega})e^{-2\pi i f\hat{\Omega}\cdot\vec{x}_i/c}F_i^A(\hat{\Omega},t) && (56) \end{aligned}$$

where the variable t now serves as an index of the time segment being analyzed (more on that later). Inserting the plane-wave expansion for both detectors into the 2-point correlation gives:

$$
\begin{aligned}
\langle h_1^*(f)h_2(f')\rangle &= \sum_{A,A'}\int d\hat{\Omega}d\hat{\Omega}'\langle h_A^*(f,\hat{\Omega})h_{A'}(f',\hat{\Omega}')\rangle \\
&\quad e^{2\pi if\hat{\Omega}\cdot\vec{x}_1/c}e^{-2\pi if'\hat{\Omega}'\cdot\vec{x}_2/c}F_1^A(\hat{\Omega},t)F_2^{A'}(\hat{\Omega}',t) \\
&= \sum_{A,A'}\int d\hat{\Omega}d\hat{\Omega}'\delta_{AA'}\delta(f-f')\delta^2(\hat{\Omega},\hat{\Omega}')H(f) \\
&\quad e^{2\pi if\hat{\Omega}\cdot\vec{x}_1/c}e^{-2\pi if'\hat{\Omega}'\cdot\vec{x}_2/c}F_1^A(\hat{\Omega},t)F_2^{A'}(\hat{\Omega}',t) \\
&= \sum_{A}\int d\hat{\Omega}\delta(f-f')H(f)e^{2\pi if\hat{\Omega}\cdot(\vec{x}_1-\vec{x}_2)/c}F_1^A(\hat{\Omega},t)F_2^A(\hat{\Omega},t) \\
&= \frac{3H_0^2}{32\pi^3f^3}\Omega_{\mathrm{GW}}(f)\delta(f-f')\sum_A\int d\hat{\Omega}e^{2\pi if\hat{\Omega}\cdot(\vec{x}_1-\vec{x}_2)/c}F_1^A(\hat{\Omega},t)F_2^A(\hat{\Omega},t) \\
&= \frac{3H_0^2}{20\pi^2f^3}\Omega_{\mathrm{GW}}(f)\delta(f-f')\gamma_{12}(f) \qquad (57)
\end{aligned}
$$

In the second step we used the assumed 2-point correlation for isotropic and stationary SGWB from Eq. 11, and in the second-to-last step we used the relationship between the GW strain power $H(f)$ and the normalized energy density $\Omega_{\mathrm{GW}}(f)$ from Eq. 19. In the last step, we defined the overlap reduction function:

$$
\gamma_{12}(f) = \frac{5}{8\pi}\sum_A\int d\hat{\Omega}e^{2\pi if\hat{\Omega}\cdot(\vec{x}_1-\vec{x}_2)/c}F_1^A(\hat{\Omega})F_2^A(\hat{\Omega}) \qquad (58)
$$

which averages the detector responses across the full sky (note, the detectors are "pointing" in different directions) and accounts for the time-delay in GW propagation between two detectors due to their different geographic locations. This is a purely geometric reduction factor, and it is dependent on the locations and relative orientations of the two detectors that are being used in the analysis. The prefactor of $5/8\pi$ is chosen so that $\gamma = 1$ for collocated and coaligned detectors. We will get back to the overlap reduction factor a bit later, for now let's press on to calculate the expected value of our estimator in Eq. 52:

$$
\begin{aligned}
\langle Y \rangle &= \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \langle s_1^*(f) s_2(f') \rangle Q(f') \delta_T(f - f') \\
&= \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \langle h_1^*(f) h_2(f') \rangle Q(f') \delta_T(f - f') \\
&= \frac{3H_0^2}{20\pi^2} \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' f^{-3} \Omega_{\mathrm{GW}}(f) \delta(f - f') \gamma_{12}(f) Q(f') \delta_T(f - f') \\
&= \frac{3H_0^2 T}{20\pi^2} \int_{-\infty}^{\infty} df \, f^{-3} \Omega_{\mathrm{GW}}(f) \gamma_{12}(f) Q(f) \qquad (59)
\end{aligned}
$$

where in the last step we integrated over f' which eliminated the Dirac delta function, forcing $f = f'$ and evaluating $\delta_T(0) = T$. Note that the final frequency integral is over both positive and negative frequencies, while in practice we deal with only positive frequencies. This issue can be resolved by forcing the frequency to be positive in the 2-point correlation, Eq. 11 (see Allen-Romano for more detail [1]).

The last result is very nice because it shows that if we cross-correlate two detectors' time series, the expected value will be directly related to $\Omega_{\mathrm{GW}}(f)$. But we have not yet determined the filter $Q(f)$ and we have not examined the uncertainty associated with the estimator Y. The variance of Y is given by:

$$
\sigma_Y^2 = \langle Y^2 \rangle - \langle Y \rangle^2 \qquad (60)
$$

We have already calculated $\langle Y \rangle$ and we saw that it is proportional to $\Omega_{\mathrm{GW}}(f)$. If we assume that the SGWB power will be much smaller than the detector noise power (which is a safe assumption), we can ignore the second term in the last equation, so we have:

$$
\begin{aligned}
\sigma_Y^2 &\approx \langle Y^2 \rangle \\
&\approx \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \\
&\quad \langle n_1^*(f) n_2(f') n_1^*(k) n_2(k') \rangle \delta_T(f - f') \delta_T(k - k') Q(f') Q(k') \\
&\approx \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \\
&\quad \langle n_1^*(f) n_1(-k) \rangle \langle n_2^*(-f') n_2(k') \rangle \delta_T(f - f') \delta_T(k - k') Q(f') Q(k') \qquad (61)
\end{aligned}
$$

where in the last line we used the fact that n_1 and n_2 are statistically independent (which implies that 4-point correlation can be written as a product of 2-point correlations) and that they are real which implies $n_i^*(f) = n_i(-f)$. Defining the noise power spectrum by

$$\langle n_i^*(f) n_i(f') \rangle \;=\; \frac{1}{2}\delta(f - f') P_i(|f|) \tag{62}$$

allows us to simplify the variance to

$$\begin{aligned}
\sigma_Y^2 \;&\approx\; \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \\
&\quad \frac{1}{2}\delta(f + k) P_1(|f|) \frac{1}{2}\delta(f' + k') P_2(|f'|) \delta_T(f - f') \delta_T(k - k') Q(f') Q(k') \\
&= \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \frac{1}{4} P_1(|f|) P_2(|f'|) \delta_T^2(f - f') Q^2(f') \\
&= \frac{T}{4} \int_{-\infty}^{\infty} df\, P_1(|f|) P_2(|f|) Q^2(f)
\end{aligned} \tag{63}$$

where in the last line we approximated one of the finite-time delta functions with an ordinary Dirac delta function, and evaluated the other finite-time delta function at $\delta_T(f - f') = \delta_T(0) = T$. We can then summarize our results as:

$$\langle Y \rangle \;=\; \frac{3H_0^2 T}{20\pi^2} \int_{-\infty}^{\infty} df\, f^{-3} \Omega_{\mathrm{GW}}(f) \gamma_{12}(f) Q(f) \tag{64}$$

$$\sigma_Y^2 \;\approx\; \frac{T}{4} \int_{-\infty}^{\infty} df\, P_1(|f|) P_2(|f|) Q^2(f) \tag{65}$$

For a given filter Q, we can compute the estimator Y using Eq. 52 and we can compute the variance on this estimator using Eq. 63, where the P_i's can be computed by Fourier transforming $s_i(t)$ and then squaring it (i.e. $P_i(f)\ s_i^2(f)$), which is a good approximation in the limit where the SGWB is much smaller than the detector noise. To determine the optimal form of the filter, we use the trick of Allen-Romano [1] and define the following inner product between two functions:

$$(A, B) \;=\; \int_{-\infty}^{\infty} df\, A^*(f) B(f) P_1(|f|) P_2(|f|) \tag{66}$$

In terms of this inner product, we can write

$$\langle Y \rangle = \frac{3H_0^2 T}{20\pi^2} \left(Q(f), \frac{\Omega_{\rm GW}(|f|)\gamma_{12}(f)}{|f|^3 P_1(|f|) P_2(|f|)} \right) \tag{67}$$

$$\sigma_Y^2 \approx \frac{T}{4} (Q, Q) \tag{68}$$

Finally, to maximize the signal to noise ratio we want to maximize:

$$SNR^2 = \frac{\langle Y \rangle^2}{\sigma_Y^2} = \left(\frac{3H_0^2}{10\pi^2} \right) T \frac{\left(Q(f), \frac{\Omega_{\rm GW}(|f|)\gamma_{12}(f)}{|f|^3 P_1(|f|) P_2(|f|)} \right)^2}{(Q, Q)}. \tag{69}$$

The last expression is maximized when the two "vectors" in the numerator are equal to each other (up to a scaling factor), yielding this form of the optimal filter:

$$Q(f) = \lambda \frac{\Omega_{\rm GW}(|f|)\gamma_{12}(f)}{|f|^3 P_1(|f|) P_2(|f|)}. \tag{70}$$

Inserting this back into the expression for the SNR gives:

$$SNR = \frac{3H_0^2}{10\pi^2} \sqrt{T} \left(\int_{-\infty}^{\infty} \frac{\Omega_{\rm GW}^2(|f|)\gamma_{12}^2(f)}{|f|^6 P_1(|f|) P_2(|f|)} \right)^{1/2}. \tag{71}$$

We immediately see several implications. First, the SNR increases with the square root of time—averaging longer datasets helps, but with diminishing returns. Second, due to the factors of $P_i(f)$ in the denominator, the frequencies at which the detectors are noisy are deweighed in the integral and they contribute less—this includes the very low frequencies (< 10 Hz) where seismic noise dominates, very high frequencies > 300 Hz) where the detector strain noise starts to climb, and all narrowband loud lines (power line harmonics, calibration lines etc, although many of these are notched anyways). Third, note the factor of f^6 in the denominator—this factor heavily weighs the lowest frequencies. In other words, if we can push the sensitive band of GW detectors to lower frequencies, we would very quickly gain in sensitivity to the SGWB. And fourth, note that $\Omega_{\rm GW}(f)$ shows up in the expression for the SNR and for $Q(f)$—this might be surprising at first, since we do not know a priori what is the energy density of the SGWB, nor its frequency dependence. This is therefore an assumption that must go into the analysis.

In other words, we will have to choose some template for $\Omega_{\rm GW}(f)$ to conduct the search, and the optimal filter calculated for this choice of $\Omega_{\rm GW}(f)$ is indeed the optimal filter we could choose to search for it. If we want to search for a different energy density shape, we would use a different optimal filter.

In practice, we often choose a power-law functional form for the SGWB template spectrum:

$$\Omega_{\rm GW}(f) \quad = \quad \Omega_\alpha \left(\frac{f}{f_{\rm ref}}\right)^\alpha \tag{72}$$

This functional form is sufficient for most (if not all) models of SGWB in the relatively narrow band of the terrestrial GW detectors. Furthermore, for a chosen power law index α, one can choose the scaling factor λ in Eq. 70 so that $\langle Y\rangle = \Omega_\alpha$. This can be done over the entire frequency band of the search (e.g. between $20-100$ Hz), in which case one assumes the spectral shape over the entire frequency band and therefore extracts a single parameter from the analysis (namely, Ω_α). Alternatively, one can assume $\alpha = 0$ and repeat the analysis for each individual frequency bin (which is often chosen to be 0.25 Hz or 1/32 Hz)—in this case, one is effectively measuring $Y(f)$ such that $\langle Y(f)\rangle = \Omega_{\rm GW}(f)$, which lends itself for comparisons with theoretical models of SGWB (and therefore for parameter estimation).

In particular, one can define a likelihood function of the form:

$$L(Y_i, \sigma_i|\vec{\theta}) \quad \propto \quad \exp\left(\frac{-(Y_i - \Omega_M(f_i, \vec{\theta}))^2}{2\sigma_i^2}\right) \tag{73}$$

where Y_i and σ_i are the value of the estimator and the associated uncertainty measured in the frequency bin f_i, and $\Omega_M(f_i, \vec{\theta})$ is the SGWB model evaluated at the frequency f_i for some set of parameters $\vec{\theta}$. One can then combine this with the Bayes theorem:

$$P_{post}(\vec{\theta}|Y_i, \sigma_i) \quad = \quad \frac{L(Y_i, \sigma_i|\vec{\theta})P_{prior}(\vec{\theta})}{P(Y_i, \sigma_i)} \tag{74}$$

That is, for some choice of prior distribution on the parameters $\vec{\theta}$, this procedure yields the posterior distribution of the parameters, allowing us to place constraints on parameters e.g. at 95% confidence. The simplest model we can use is the power law of Eq. 72 in which case we would be estimating the amplitude Ω_α and the corresponding power index α (see slides). But in principle Ω_M could be any SGWB model, including all of those we examined above. It could also be a combination of an SGWB model and of correlated noise spectrum (e.g. due to correlated magnetic noise between two GW detectors).

3.2 Polarized SGWB

The above analysis could be extended to study different polarizations. We consider the left- and right-handed correlators [11, 12]:

$$\langle h_{R/L}(f,\hat{\Omega}) h^*_{R/L}(f',\hat{\Omega}')\rangle = \frac{\delta(f-f')\delta^2(\hat{\Omega}-\hat{\Omega}')}{4\pi}\left[I(f) \pm V(f)\right] \tag{75}$$

where $h_L = (h_+ + ih_\times)/\sqrt{2}$, $h_R = (h_+ - ih_\times)/\sqrt{2}$, and $+$ and $\times$ are the standard plus and cross polarizations. This is the point of departure from the searches for unpolarized isotropic SGWB, which assume $V = 0$. Further note that $\langle h_R h^*_L\rangle$ vanishes due to statistical isotropy. In this notation, the normalized energy density is:

$$\Omega_{\rm GW}(f) = \frac{f}{\rho_c}\frac{d\rho_{\rm GW}}{df} = \frac{\pi f^3}{G_N \rho_c} I(f) \tag{76}$$

The standard cross-correlation estimator is now modified [12]:

$$\begin{aligned}
\langle \hat{Y}\rangle &= \int_{-\infty}^{+\infty} df \int_{-\infty}^{+\infty} df' \delta_T(f-f')\langle(s_1^*(f)s_2(f')\rangle \tilde{Q}(f') \\
&= \frac{3H_0^2 T}{10\pi^2}\int_0^\infty df \frac{\Omega'_{\rm GW}(f)\gamma_I(f)\tilde{Q}(f)}{f^3},
\end{aligned} \tag{77}$$

where

$$\begin{aligned}
\Omega'_{\rm GW}(f)\gamma_I(f) &= \Omega_{\rm GW}(f)\left[\gamma_I(f) + \Pi(f)\gamma_V(f)\right] \\
\gamma_I(f) &= \frac{5}{8\pi}\int d\hat{\Omega}(F_1^+ F_2^{+*} + F_1^\times F_2^{\times *})e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}} \\
\gamma_V(f) &= -\frac{5}{8\pi}\int d\hat{\Omega} i(F_1^+ F_2^{\times *} - F_1^\times F_2^{+*})e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}}.
\end{aligned} \tag{78}$$

The factor $\gamma_I(f)$ is the standard overlap reduction function arising from different locations and orientations of the two detectors, and $\gamma_V(f)$ is a new function, associated with the parity violating term and first computed in [11]. See slides for an example of these functions for two real detector pairs. Finally, $\Pi(f) = V(f)/I(f)$ encodes the parity violation, with maximal values $\Pi = \pm 1$ corresponding to fully right- or left-handed polarizations. Setting $\Pi = 0$ reproduces the standard unpolarized SGWB search [**?**].

Notice that the above expressions are effectively the same as for the unpolarized background, with the only change being that $\Omega_{\rm GW}(f)$ is replaced by $\Omega'_{\rm GW}(f)$ which encodes the polarization information. Hence, the analysis can proceed as usual, with the difference taking place only in the last step, when the likelihood is evaluated in Eq. 73. Slides show an example of such an analysis for power law SGWB. Other examples have been studied for axion inflationary models [12] and for phase transitions [13].

Note that similar approach can be taken for other non-GR polarizations as well (i.e. scalar and vector polarizations), the difference in the analysis reduces to the difference in the overlap reduction function. The latest results on this can be found in [5].

4 Directional SGWB Search

The above analysis has focused on isotropic SGWB. It is possible, however, that there is anisotropy in the SGWB sky, i.e. that the GW energy density in some directions is higher than in others. It is also possible to use a single GW detector pair to measure the anisotropy. This might be surprising since one cannot triangulate the wave direction using only two detectors. However, in the case of persistent sources (which is what we assume), we can use the fact that Earth rotates to effectively make measurements with detectors in different locations.

The formalism starts off similarly to the isotropic case, but we now have to modify our 2-point correlation function:

$$\langle h^*_A(f,\hat{\Omega}) h_{A'}(f',\hat{\Omega}')\rangle \;\; = \;\; \frac{1}{4}\delta_{AA'}\delta(f-f')\delta^2(\hat{\Omega},\hat{\Omega}')\mathcal{P}(f,\hat{\Omega}). \qquad (79)$$

(The factor 1/4 is a choice, conveniently accounting for one-sided strain power.) That is, different directions on the sky are still uncorrelated, but the strain power will now be direction-dependent. It is common to assume that frequency and direction dependencies can be separated:

$$\mathcal{P}(f,\hat{\Omega}) \;\; = \;\; P(\hat{\Omega})H(f), \qquad (80)$$

where the common choice is $H(f) = (f/f_{ref})^{\beta}$. Following similar calculation to above, we have

$$\Omega_{\rm GW}(f) = \frac{2\pi^2}{3H_0^2} f^3 H(f) \int d\hat{\Omega} P(\hat{\Omega}), \tag{81}$$

which is very similar to Eq. 19, but with a different prefactor constant. The next step is to decompose the directional dependence in terms of some basis of functions on the sphere:

$$P(\hat{\Omega}) = P_\alpha e_\alpha(\hat{\Omega}), \tag{82}$$

with the summation over repeated index understood. The two choices for the basis that are commonly used are the pixel basis:

$$P(\hat{\Omega}) = P_{\hat{\Omega}'} \delta(\hat{\Omega}, \hat{\Omega}'), \tag{83}$$

and the spherical harmonics basis:

$$\begin{aligned} P(\hat{\Omega}) &= P_{lm} Y_{lm}(\hat{\Omega}) \\ P_{lm} &= \int d\hat{\Omega} P(\hat{\Omega}) Y^*_{lm}(\hat{\Omega}). \end{aligned} \tag{84}$$

We then proceed similarly to the isotropic case, calculating the cross-correlation between two detectors, but without the optimal filter in this case:

$$C_{ft} = \frac{2}{T} s_1^*(f) s_2(f) \tag{85}$$

If the noise in the two detectors in uncorrelated, then

$$\begin{aligned}
\langle C_{ft} \rangle &= \frac{2}{T}\langle s_1^*(f) s_2(f) \rangle = \frac{2}{T}\langle h_1^*(f) h_2(f) \rangle \\
&= 2\sum_{A,A'} \int d\hat{\Omega} d\hat{\Omega}' \langle h_A^*(f,\hat{\Omega}) h_{A'}(f,\hat{\Omega}') \rangle \\
&\quad e^{2\pi i f \hat{\Omega}\cdot\vec{x}_1/c} e^{-2\pi i f \hat{\Omega}'\cdot\vec{x}_2/c} F_1^A(\hat{\Omega},t) F_2^{A'}(\hat{\Omega}',t) \\
&= 2\sum_{A,A'} \int d\hat{\Omega} d\hat{\Omega}' \delta_{AA'} \delta^2(\hat{\Omega},\hat{\Omega}') H(f) P(\hat{\Omega}) \\
&\quad e^{2\pi i f \hat{\Omega}\cdot\vec{x}_1/c} e^{-2\pi i f \hat{\Omega}'\cdot\vec{x}_2/c} F_1^A(\hat{\Omega},t) F_2^{A'}(\hat{\Omega}',t) \\
&= 2\sum_{A} \int d\hat{\Omega} H(f) P(\hat{\Omega}) e^{2\pi i f \hat{\Omega}\cdot(\vec{x}_1-\vec{x}_2)/c} F_1^A(\hat{\Omega},t) F_2^A(\hat{\Omega},t) \\
&= H(f) \int d\hat{\Omega} P(\hat{\Omega}) \gamma(\hat{\Omega},f,t) \qquad (86) \\
\gamma(\hat{\Omega},f,t) &= \frac{1}{2}\sum_A e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/c} F_1^A(\hat{\Omega},t) F_2^A(\hat{\Omega},t) \qquad (87)
\end{aligned}$$

In the second line, the factor of T was cancelled by the finite delta function evaluated at $\delta_T(f-f) = T$, and in the last line we define the equivalent of the overlap reduction function which is now direction dependent. Note that again this factor is purely geometrical. If we then decompose $P(\hat{\Omega})$:

$$\begin{aligned}
\langle C_{ft} \rangle &= H(f) P_\alpha \sum_A \int d\hat{\Omega} e_\alpha(\hat{\Omega}) \gamma(\hat{\Omega},f,t) \\
&= H(f) P_\alpha \gamma_\alpha(f,t) \qquad (88) \\
\gamma_\alpha(f,t) &= \frac{1}{2}\sum_A \int d\hat{\Omega} e_\alpha(\hat{\Omega}) e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/c} F_1^A(\hat{\Omega},t) F_2^A(\hat{\Omega},t) \qquad (89)
\end{aligned}$$

Our goal is to estimate the coefficients of the expansion, P_α. In the pixel basis

$$\begin{aligned}
\gamma_{\hat{\Omega}'}(f,t) &= \frac{1}{2}\sum_A \int d\hat{\Omega} \delta(\hat{\Omega},\hat{\Omega}') e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/c} F_1^A(\hat{\Omega},t) F_2^A(\hat{\Omega},t) \\
&= \frac{1}{2}\sum_A e^{2\pi i f \hat{\Omega}'\cdot\Delta\vec{x}/c} F_1^A(\hat{\Omega}',t) F_2^A(\hat{\Omega}',t) \qquad (90)
\end{aligned}$$

and in the spherical harmonics basis

$$\gamma_{lm}(f,t) = \frac{1}{2}\sum_A \int d\hat{\Omega} Y_{lm}(\hat{\Omega}) e^{2\pi i f \hat{\Omega}\cdot\Delta\vec{x}/c} F_1^A(\hat{\Omega},t) F_2^A(\hat{\Omega},t) \qquad (91)$$

Again, γ's are geometric factors that can be calculated once, and then the expected value of the cross correlation can be expressed simply as

$$\langle C_{ft}\rangle = H(f) P_\alpha \gamma_\alpha(f,t) \qquad (92)$$

The corresponding covariance matrix is given by

$$\begin{aligned} N_{ft,f't'} &= \langle C_{ft} C^*_{f't'}\rangle - \langle C_{ft}\rangle\langle C^*_{f't'}\rangle \\ &\approx \delta_{tt'}\delta_{ff'} P_1(f,t) P_2(f,t) \end{aligned} \qquad (93)$$

where $P_i(f,t)$ are the one-sided noise power spectra, as before. To estimate P_α, we can set up a likelihood function and then attempt to maximize it. Our analysis will be done by splitting a long observation into many small segments, we can assume that the C_{ft}'s are Gaussian-distributed, so we can write

$$\begin{aligned} L &\propto \exp(-(C^*_{ft} - \langle C^*_{ft}\rangle) N^{-1}_{ft,f't'} (C_{f't'} - \langle C_{f't'}\rangle)) \\ &= \exp(-(C^*_{ft} - H(f) P^*_\alpha \gamma^*_\alpha(f,t)) \frac{1}{P_1(f,t)P_2(f,t)} (C_{ft} - H(f) P_\beta \gamma_\beta(f,t))) \end{aligned} \qquad (94)$$

Maximizing this likelihood (i.e. set derivative relative to P_α to zero) yields the solution of the form:

$$\hat{P}_\alpha = (\Gamma^{-1})_{\alpha\beta} X_\beta \qquad (95)$$

$$X_\beta = \sum_{tf} \gamma^*_\beta(f,t) \frac{H(f)}{P_1(f,t)P_2(f,t)} C_{ft} \qquad (96)$$

$$\Gamma_{\alpha\beta} = \sum_{tf} \gamma^*_\alpha(f,t) \frac{H^2(f)}{P_1(f,t)P_2(f,t)} \gamma_\beta(f,t) \qquad (97)$$

The array X_β is known as the "dirty" map—it encodes the map of the SGWB sky convolved with the response of the detector network. The matrix $\Gamma_{\alpha\beta}$

is known as the Fisher matrix, and it can be shown that in the weak signal approximation

$$\langle X_\alpha X^*_\beta\rangle - \langle X_\alpha\rangle\langle X^*_\beta\rangle \;\approx\; \Gamma_{\alpha\beta} \tag{98}$$

i.e. the Fisher matrix is the covariance matrix of the dirty map. Similarly, the array $\hat{P}_\alpha$ is known as the "clean" map, as the response of the detector network has been deconvolved. This map is therefore a better representation of the SGWB sky. It can also be shown that in the weak signal approximation

$$\langle \hat{P}_\alpha \hat{P}^*_\beta\rangle - \langle \hat{P}_\alpha\rangle\langle \hat{P}^*_\beta\rangle \;\approx\; \Gamma^{-1}_{\alpha\beta} \tag{99}$$

i.e. the inverse of the Fisher matrix is the covariance matrix of the clean map.

It is important to note that the analysis can be performed in any basis, and the results will be the same. Indeed there are python routines that allow conversion of maps from pixel to spherical harmonic basis and back. However, the targeted source may lead to a preference of one basis over another. For example, when searching for point sources on the sky, pixel basis is the more natural choice—one can then simply use the dirty map and the diagonal entries of the Fisher matrix as the variances at each pixel (by construction, SGWB pixels are not correlated). On the other hand, for extended sources on the sky (such as the galactic plane), the spherical harmonic basis may be a better choice—one would then prefer to use the clean map to estimate the source, along with the full inverse of the Fisher matrix as its covariance matrix.

In the spherical harmonic case, it is possible to take one step further and define

$$\hat{C}_l \;=\; \frac{1}{2l+1}\sum_{m=-l}^{l} |\hat{P}_{lm}|^2 \tag{100}$$

The C_l's are encoding the (clean) angular power spectrum of the SGWB, analogous to the angular temperature spectrum of the CMB. It is important to note, however, that the Fisher matrix might not be invertible. Each detector pair will have directions (or modes) on the sky that it is not very sensitive to. This can be seen by diagonalizing the Fisher matrix—some of the eigenvalues can be very small, implying there are directions with little power. As a consequence, the Fisher matrix is singular and has to be regularized before inversion. Regularization can be done in different ways, a common approach is to replace the lowest third of the eigenvalues with infinity and then invert. This in turn can lead to a bias in C_l's—the bias can be estimated and subtracted:

$$\hat{C}_l^{reg} = \hat{C}_l - \frac{1}{2l+1} \sum_M (\Gamma^{-1})_{lm,lm} \tag{101}$$

It should also be noted that if there are more than two detectors available, the analysis can be repeated for each detector pair and the final dirty maps and Fisher matrices can be added across all pairs. In this case, different pairs will be insensitive to different modes, implying that they can naturally regularize the overall Fisher matrix. Slides show some of the recent results from LIGO/Virgo analysis.

4.1 Angular Resolution

The question of angular resolution is a very interesting one. Namely, the nature provides some GW power at all angular scales, but we have to cut off our spherical harmonic expansion at some l (or, equivalently, we have to define a finite size of the pixels in the pixel basis). What determines the angular resolution at which we should preform the analysis? Historically, we used the diffraction limit as the argument:

$$\theta = \frac{c}{2Df}, \quad l_{\max} = \frac{\pi}{\theta} = \frac{2\pi D f}{c} \tag{102}$$

where D is the separation between two detectors. For $f = 50$ Hz, which is roughly the most sensitive frequency bin for $\alpha = 0$ analysis, this yields $l_{\max} = 3 - 4$. For higher values of α, the most sensitive frequency will be higher, implying the higher value of $l_{\max}$.

However, diffraction limit has to do with waves interacting with detectors (telescopes in EM case). This is not what happens in the SGWB measurement—rather, we are using the time-delay between two detectors to ascertain the direction of an incoming GW. Consider a toy example: a single plane waves on the surface of a lake passing by two buoys that can measure the height of water as the waves pass by. The time delay between the time-series of water heights at the two buoys indicate the direction of this single plane wave. If the buoys have zero error, the time-delay will also be measured with zero uncertainty, and therefore the direction of the plane wave can be determined with zero uncertainty as well. That is, the measurement surpasses the "diffraction limit" as defined above.

Things become more complex when we have multiple plane waves and if we don't know how many plane waves exist. In this case, we are measuring a superposition of all plane waves (as we do in the SGWB searches), and there will be partial cancellation of waves propagating in different directions. In fact, it can be shown (by Andrew Matas) that for isotropic buoy detectors operating in this situation, there is a limitation in the angular resolution that is similar to the diffraction limit defined above. If detectors are not-isotropic and rotate (which is the case with LIGO/Virgo detectors), then it is possible to surpass the diffraction limit again.

Further complication arises from the fact that optimal angular resolution will depend on what exactly we are trying to accomplish. For example, if we ask "Is there a point source in the sky map?" the optimal choice may be to use poor angular resolution, minimizing the number of parameters we need to estimate. However, if we want to know "Where on the sky is the point source?", we should repeat the analysis with a higher $l_{\max}$ allowing better angular resolution, even though the SNR of the source will decrease as we increase $l_{\max}$. And, of course, the situation is further complicated by the many possible choices of the frequency band to be used in the analysis: higher frequencies carry information about smaller angular scales. This is an active area study.

4.2 Theoretical Models of Anisotropy

Theoretical modeling of SGWB anisotropy is a relatively new area in the literature. The anisotropy is possible in both cosmological and astrophysical models. Furthermore, it is possible to look for correlations between the SGWB sky-maps and the maps of electromagnetic tracers of matter structure, such as galaxy counts and gravitational lensing.

In the case of compact binaries SGWB model, the first paper on the topic came from Cusin et al. [14, 15], revealing that the SGWB anisotropy is driven by 3 scales: (i) cosmological, in the sense of expansion of the Universe; (ii) galactic, in the sense that mass clustering introduces anisotropy; and (iii) astrophysical, taking into account the dynamics (and inspiral time) of individual binaries. The calculation is beyond the scope of these lectures, but the result can be summarized as [15]:

$$\delta\Omega_\ell(k,f) \;=\; \frac{f}{4\pi\rho_c}\int_{\eta_*}^{\eta_o} d\eta\,\mathcal{A}(\eta,f)\left[\left(b\delta_{\rm m,k}(\eta)+(b-1)3\mathcal{H}v_k(\eta)\right)j_\ell(k\Delta\eta)-2kv_k(\eta)j'_\ell(k\Delta\eta)\right]\,, \tag{103}$$

where k stands for the wavenumbers of density fluctuations, δ_m is the matter density fluctuation, η depicts the comoving distance, b captures the bias between galaxy and dark matter overdensity ($\delta_G = b\delta_m$), v is the comoving velocity field, $\mathcal{H}$ is the comoving Hubble parameter, and j_ℓ is the spherical Bessel function. The function $\mathcal{A}(\eta, f)$ is the kernel computed by Cusin et al [14, 15] by solving the evolution of density equations. The slides show what this function looks like. With these definitions, one can compute the angular power spectrum in the SGWB autocorrelation:

$$C_\ell(f) \;=\; \frac{2}{\pi}\int dk\,k^2|\delta\Omega_\ell(k,f)|^2\,, \tag{104}$$

Slides depict estimates of this angular power spectrum as well.

Cusin et al. have also computed the corresponding angular spectrum for cross-correlation of the SGWB sky with similar sky estimates of electromagnetic tracers of the matter structure, such as galaxy counts and gravitational (weak) lensing. In the case of galaxy counts, they find:

$$\begin{aligned}\Delta_\ell(k,\eta) \;&=\; \int d\eta\,W(\eta)\left[\left(b\delta_{\rm m,k}(\eta)+(b-1)3\mathcal{H}v_k(\eta)\right)j_\ell(k\Delta\eta)+k\partial_\eta\left(\frac{v_k}{\mathcal{H}}\right)v_k(\eta)j'_\ell(k\Delta\eta)\right]\,,\\ D_\ell(f) \;&\equiv\; \frac{2}{\pi}\int dk\,k^2\delta\Omega^*_\ell(k,f)\,\Delta_\ell(k)\,.\end{aligned} \tag{105}$$

Here, $W(\eta)$ is a window function used in the galaxy survey. The slides show some of their calculation results.

This story is not complete for two reasons. First, there is a shot noise contribution having to do with the realization of the galaxy field. And second, there is an additional shot (popcorn) noise having to do with the realization of merging compact binaries during the observation time. These noise terms can be thought of biases in the angular power spectra, which have been computed [15]. For shot noise:

$$
\begin{aligned}
(N_\ell^S)^{\text{cross}} &= \frac{f}{4\pi\rho_c}\int \frac{dr}{r^2}\,\frac{1}{a^3\bar{n}_G}W(r)\mathcal{A}(r,f)\,, && (106)\\
(N_\ell^S)^{\text{Gal}} &= \int \frac{dr}{r^2}\,\frac{1}{a^3\bar{n}_G}W^2(r)\,, && (107)\\
(N_\ell^S)^{\text{GW}} &= \left(\frac{f}{4\pi\rho_c}\right)^2\int \frac{dr}{r^2}\,\frac{1}{a^3\bar{n}_G}\left(\mathcal{A}(r,f)\right)^2\,. && (108)
\end{aligned}
$$

For popcorn noise

$$
\begin{aligned}
(N_\ell^{PC})^{\text{cross}} &= (N_\ell^S)^{\text{cross}}\,, && (109)\\
(N_\ell^{PC})^{\text{GW}} &= \left(1+\frac{1}{\beta_T}\right)(N_\ell^S)^{\text{GW}} \gg (N_\ell^S)^{\text{GW}}\,, && (110)\\
(N_\ell^{PC})^{\text{Gal}} &= (N_\ell^S)^{\text{Gal}}, && (111)
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_T &= \frac{T}{a^3 n_G}\times\frac{d\mathcal{N}}{dtdV} && (112)\\
\frac{d\mathcal{N}}{dtdV} &< \frac{d\mathcal{N}}{dt_m dV}\sim 100\text{Gpc}^{-3}\text{yr}^{-1}\,. && (113)
\end{aligned}
$$

Here t_m is the comoving time of the source, i.e. $t = (1+z)t_m > t_m$. Then using a constant comoving galaxy density $a^3 n_G \sim 0.1$ Mpc^{-3}, we find $\beta_T/T < 10^{-6}$/yr implying that the popcorn noise bias is much larger in the SGWB autocorrelation case than in the SGWB-Galaxy case. Finally, we note that the SGWB autocorrelation anisotropy model has been studied by other groups of authors as well—the agreement is not yet perfect, there are differences in the predictions at the level of 1 order of magnitude [16, **?**, 18].

We briefly note that cosmological SGWB models can also result in anisotropy, and there are recent studies in this area too. For example, cosmic strings are topological defects that may have been produced in the phase transitions in the early universe. They are typically described by the string tension $G\mu$. The strings can intersect each other or themselves, in which case they can reconnect and produce loops. Cusps or kinks on these strings can produce GW radiation, the sum of which results in a SGWB. The amplitude of the SGWB depends on the density of string loops, which in turn depends on the dynamics of the cosmic strings network. Recently, [19] have studied the anisotropy in some of the currently favored cosmic string models, predicting the angular power spectrum of specific shape, although typically several orders of magnitude smaller than the isotropic (monopole) component (see slides for example).

Another example is the SGWB due to phase transitions (PT) in the early universe, which are expected to produce GWs via a variety of mechanisms (bubble collisions, turbulence etc). The PT will occur at slightly different redshift at different sky directions, due to slightly different temperatures in different regions in the early universe [20]. If the phase transition happens after the reheating phase of inflation, i.e. after the primordial density fluctuations are established, then the anisotropy in the PT SGWB will be correlated with the CMB temperature anisotropy. The level of correlation will therefore depend on the physics of inflation and of the phase transition [20].

4.3 GW-EM Correlations

We conclude this discussion by highlighting another rising area, that of angular correlations between the SGWB and electromagnetic tracers of the matter structure. We saw above that there are already theoretical predictions for this cross-correlation, and there are now the first attempts to measure this correlation using recent LIGO/Virgo data. There are multiple directions one can pursue, and we briefly mention them here.

The simplest approach is to compute the coherence between two skymaps, following [21]:

$$\Gamma = \frac{\langle \delta M_{GW} \delta M_{GC} \rangle^2}{\langle \delta M_{GW}^2 \rangle \langle \delta M_{GC}^2 \rangle} \tag{114}$$

where δM_{GW} and δM_{GC} denote the fluctuation maps for the SGWB and galaxy counts, respectively. The GC map can be extracted from the Sloan Digital Sky Survey (SDSS), which includes both photometric and spectrometric catalogs. For the GW map, [21] used the O2 LIGO data and computed the anisotropy maps in several frequency bands (resulting in maps of different angular resolution, and also of different sensitivity). To assess the significance of the computed value of Γ, a series of simulations of noise-only maps was used to understand the noise distribution of the Γ parameter. The slides highlight some of the results of this analysis, which did not detect a correlation.

Another approach is to compute the 2-point correlations in a hierarchical Bayesian inference framework, which performs the parameter estimation procedure all all LIGO/Virgo data (and not only on segments that contain clear CBC signals). This approach allows an estimate of the sky-distribution of the CBC population, which can then be correlated with the SDSS galaxy count maps. This is beyond the scope of these lectures, but see [22] for more detail and the slides for an example of a simulated angular spectrum recovery.

Yet another approach is to compute the cross-power spectrum of the SGWB and GC maps,

$$D_\ell(f) = \frac{1}{2\ell+1}\sum_m P^*_{GW,lm} a_{GC,lm}. \tag{115}$$

This quantity is closely related to the theoretically computed D_ℓ in Eq. 105. This is an active area of study, we are working on understanding the covariance matrix for these estimators and to use the measurement to constrain the theoretical model, taking into account the shot noise contribution to the angular power spectrum.

5 Tutorials

5.1 Days 1 and 2

During the first two tutorials, you will compute the SGWB frequency spectrum using python, matlab, or another platform your are comfortable with. Start with the Eq. 30, which is the general expression for $\Omega_{\rm GW}$ for astrophysical SGWB models. Apply it first to the CBC model. Start simple: assume that all binaries have the same chirp mass and that their merger rate follows exactly the star formation rate (i.e. ignore the time delay). Compare your results with the sensitivity of the upcoming LIGO/Virgo detectors, or with the future 3rd generation detectors.

Then add the time-delay, as described by Eq. 34. This will require you to figure out the connection between redshift and time, so you may have to look up some cosmology texts. Try different time delay distributions to get a sense of how significant this factor is.

Next, try to add a more complex distribution of chirp masses following Eq. 32. Play with different mass distributions to see how much this impacts the overall $\Omega_{\rm GW}(f)$.

If time allows, try to vary the star formation rate. There are multiple papers you could use to this end, but [6] gives a few examples you could try.

5.2 Day 3

Repeat the exercise from days 1 and 2 but for other astrophysical SGWB models, namely magnetars and stellar core collapse. These models are easier to code up because there is no need to worry about the time delays. However, they are far less understood because the mechanism for GW production is complex. Try to vary the parameters of the model (i.e. those that determine dE/df) and see how much the overall $\Omega_{\rm GW}$ changes. Compare your results with the sensitivity of the upcoming LIGO/Virgo detectors, or with the future 3rd generation detectors.

5.3 Day 4

Repeat the calculation leading to the estimator of the isotropic SGWB estimator. That is, re-derive Eqs. 59, 63, 70, and 71.

If time allows, make a simulation of a simple toy model:

$$s_1(t) = n_1(t) + h_1(t) \tag{116}$$

$$s_2(t) = n_2(t) + h_2(t) \tag{117}$$

$$Y = \sum_t s_1(t) s_2(t) \tag{118}$$

That is, we are computing the correlation estimator in time-domain with a delta function as the filter. Draw $n_i(t)$ from a Gaussian distribution (e.g. zero mean, unit variance). As a first step, try $h_1(t) = h_2(t) = h = const$, where $h << 1$. How long does you simulation need to be for you to detect h? Then repeat the above by let h be a random variable drawn from a Gaussian distribution.

5.4 Day 5

Repeat the derivation of the estimators in the directional SGWB search. Fill in the gaps in the lecture, such as computing X_β and $\Gamma_{\alpha\beta}$ by optimizing the likelihood function, and computing the regularization bias on C_l.

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