Are Neutron Stars with Crystalline Color-Superconducting Cores Relevant for the LIGO Experiment?

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We estimate the maximal deformation that can be sustained by a rotating neutron star with a crystalline color-superconducting quark core. Our results suggest that current gravitational-wave data from the Laser Interferometer Gravitational-Wave Observatory have already reached the level where a detection would have been possible over a wide range of the poorly constrained QCD parameters. This leads to the nontrivial conclusion that compact objects do not contain maximally strained color crystalline cores drawn from this range of parameter space. We discuss the uncertainties associated with our simple model and how it can be improved in the future.

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Introduction.—A spinning neutron star may be an interesting gravitational-wave source provided that it is deformed in some way. Such asymmetries, colloquially referred to as "mountains", may arise in a number of different ways. The elastic crust of the star may sustain shear stresses [1,2], the size of which are limited by the breaking strain of the crustal lattice. A magnetic star in which the magnetic field is misaligned *vis-à-vis* the rotation axis may also be deformed in an interesting way [3,4]. Estimates of the possible level of neutron star asymmetry are of great current interest given the improving upper limits obtained by the Laser Interferometer Gravitational-Wave Observatory (LIGO) [5]. The best current limit set by the detectors corresponds to a deformation of $\epsilon < 7.1 \times 10^{-7}$ in PSR J2124-3358, a neutron star spinning at 202.8 Hz. This should be compared to the maximal mountain size predicted from theory. We have recently concluded that the largest crust mountain one should expect on an isolated neutron star is $\epsilon \approx 2.4 \times 10^{-6}$ [2]. This estimate is obtained assuming that the breaking strain of the crust is $\bar{\sigma}_{\rm br} = 10^{-2}$, which may be optimistic. The comparison shows that the sensitivity of the detectors have reached the level where the observations are beginning to confront the theory. Of course, we are still some way away from testing more conservative models. Recall, for example, that the crust breaking strain is usually assumed to lie in the range $\bar{\sigma}_{\rm br} = 10^{-5}$–$10^{-2}$. If the true value is at the lower end of this range, then we would still be far away from a detection. It is also important to keep in mind that the theoretical estimates are based on a maximization argument. It is not at all clear that there are physical avenues that lead to a star being deformed at this level. As a final caveat, we note that there also exist spin-down upper limits on ellipticities, where conservation of energy is used to produce a moment of inertia and distance dependent bound on ϵ. However, these are less robust than the direct upper limits and will not be our focus here.

Despite the various caveats, the existing observational data are very interesting. As shown by Owen [6], maximally strained solid quark stars could radiate at a level where a detection would have been possible. It may also be the case that strong emission could occur from an elastic phase in the neutron star core. Core deformations are, in fact, likely to be more significant than crustal ones, since the high density region provides a larger contribution to the quadrupole moment. The problem has been the lack of "quantitative" models. This may have changed recently, with the suggestion of crystalline color-superconducting phases in the deep core (see [7–9] and references therein). Our aim with this Letter is to adapt our recently developed framework for estimating the maximal crustal mountains in a neutron star [2] to consider elastic cores. This leads to a simple model for a neutron star with a crystalline core of deconfined quarks and a normal hadron fluid envelope. We compare the attainable mountains within this model to the current limits set by LIGO and ask to what extent observations are already confronting QCD modeling.

The elastic core model.—As discussed in [7,8] it may be energetically favorable for color-superconducting quark matter to form a crystalline structure. The shear modulus can be estimated as [9]

$$\mu = 3.96 \times 10^{33}\ \mathrm{erg/cm^3}\left(\frac{\Delta}{10\ \mathrm{MeV}}\right)^2\left(\frac{\mu_c}{400\ \mathrm{MeV}}\right)^2. \quad (1)$$

It depends on the gap parameter Δ and the quark chemical potential μ_c, for which the authors of [9] suggest estimated ranges of

$$\begin{aligned} 350\ \mathrm{MeV} &< \mu_c < 500\ \mathrm{MeV}, \\ 5\ \mathrm{MeV} &< \Delta < 25\ \mathrm{MeV}. \end{aligned} \quad (2)$$

There is naturally a significant uncertainty associated with these estimates. In order to keep the analysis simple, we will consider the shear modulus to be constant in the core and dependent only on the combination $(\Delta\mu_c)^2$, a parameter on which one may then be able to place constraints from observations. In reality, the various parameters are expected to be (weakly) density dependent, but it does not seem relevant to try to account for this in a first study.

We consider a simple model of a neutron star with an elastic core of deconfined quarks and a fluid exterior. The equation of state for the exterior is taken to be an $n = 1$ polytrope, while the core is described by an incompressible fluid. This is a reasonable first approximation for a compact star with a deconfined quark core. The particular advantage of this simple core-mantle model is that the Newtonian hydrostatic equilibrium equations

$$\nabla_a p = -\rho \nabla_a \Phi, \tag{3}$$

where p is the pressure, Φ the gravitational potential, and ρ the density, can be solved analytically. We match the solutions for the two regions at a given transition density ρ_c and impose the constraints that the total mass of the star is $M = 1.4 M_\odot$ and the radius is $R = 10$ km. These constraints are obviously not necessary, but they restrict the available parameter space. This way one can construct a sequence of stellar models by varying the transition density, which we take to be in the range $3 < \rho_c/\rho_n < 8$ (where ρ_n is the nuclear saturation density). Two examples of the resulting density distributions are shown in Fig. 1 together with the radius of the solid core as a function of the transition density.

In order to calculate the maximum ellipticity we extend the formalism developed in [2]. In essence, we perturb the background configuration assuming that the perturbations have a $Y_{22}(\theta, \varphi)$ angular dependence. Deviations from sphericity build up strain in the core. It breaks when $\bar{\sigma} = \bar{\sigma}_{\rm br}$ at some point. Here, $\bar{\sigma}$ is the modulus of the strain tensor as defined in [2]. As we have already mentioned, the breaking strain $\bar{\sigma}_{\rm br}$ is highly uncertain. To make progress we will assume that it lies in the usual range considered for neutron stars; i.e., we take $10^{-5} < \bar{\sigma}_{\rm br} < 10^{-2}$.

To maximize the ellipticity we thus assume the maximum strain in the core and solve the hydrostatic equilibrium equations, which now include elastic terms due to the deformations:

$$\nabla^a t_{ab} = \rho \nabla_b \Phi + \nabla_b p, \tag{4}$$

where we have defined the stress tensor

$$t_{ab} = \mu(\nabla_a \xi_b + \nabla_b \xi_a - \tfrac{2}{3}\delta_{ab} \nabla^c \xi_c), \tag{5}$$

with μ the shear modulus, which we consider constant, and ξ^a the components of the displacement vector for the crystalline phase. As boundary conditions we require regularity at the center of the star and continuity of the components of the traction both at the surface and at the interface between the solid and the fluid, where we also impose that the deformed shapes of the star maximize the strain in the core, as described in [2].

Having solved for the equilibrium shape of the perturbed star we can calculate the ellipticity

$$\epsilon = \frac{I_{xx} - I_{yy}}{I_{zz}}, \tag{6}$$

which, for an observer on the z axis, is tied to the gravitational-wave strain amplitude by

$$h = \frac{16\pi^2 G}{c^4} \frac{\epsilon I_{zz} \nu^2}{r}, \tag{7}$$

where ν is the neutron star spin frequency, I_{zz} its principal moment of inertia, and r the distance from Earth [5]. In order to facilitate a direction comparison with the ellipticities discussed in [5], we fix $I_{zz} = 10^{45}$ g cm^2 rather than calculating it for each of our stellar models. This is just a convention, since it is clear from (6) and (7) that I_{zz} actually does not affect the final expression for the gravitational-wave amplitude.

Results and discussion.—The ellipticities that we calculate depend on the breaking strain $\bar{\sigma}_{\rm br}$ and the combination $(\Delta\mu_c)^2$. The results also depend on the chosen value of the transition density between the core and the crust. This is

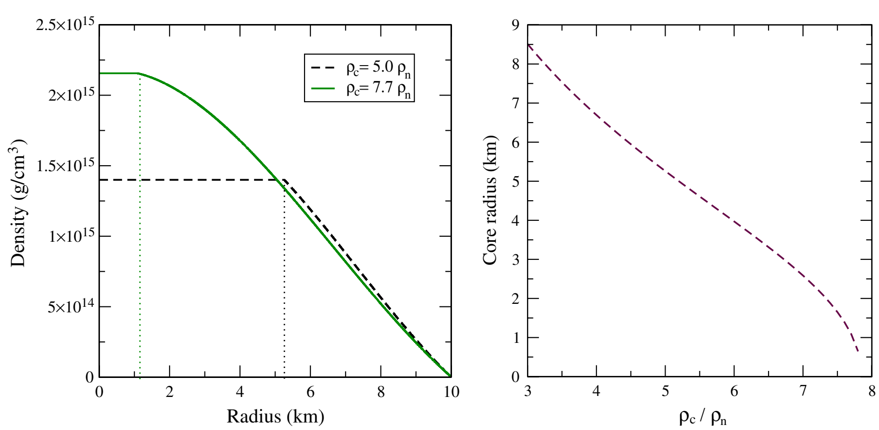

FIG. 1 (color online). The left panel shows the density profiles for two different values of the ratio ρ_c/ρ_n. One can see quite clearly that (for a fixed mass and radius of $1.4 M_\odot$ and 10 km, respectively) the core becomes significantly smaller as the transition density rises. This fact is also illustrated in the right panel, which shows the radius of the core as a function of the transition density.

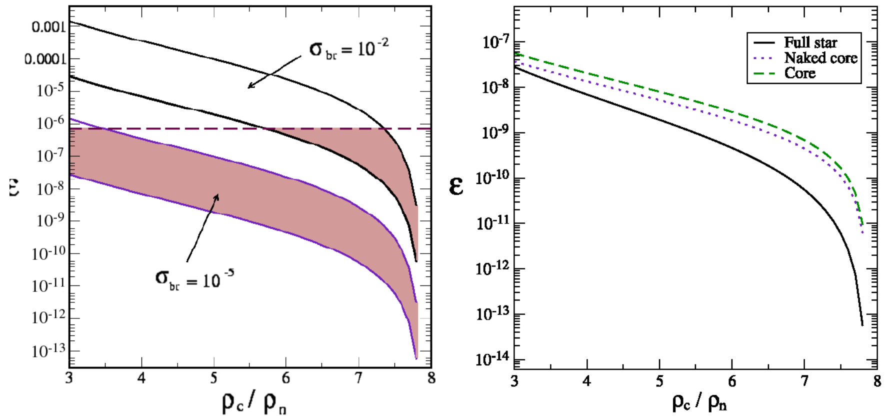

FIG. 2 (color online). In the left panel, the ellipticities obtained from our analysis are shown as a function of the core transition density ρ_c/ρ_n, where ρ_n is the nuclear saturation density. We consider two breaking strains, a maximum value $\bar{\sigma}_{\rm br} = 10^{-2}$, and a minimum of $\bar{\sigma}_{\rm br} = 10^{-5}$. For each of these values we examine the region between the maximum and the minimum shear modulus μ, obtained for the range of parameters in [9], cf. (2). The shaded region indicates the region permitted by LIGO observations and the horizontal line is the current LIGO upper limit of $\epsilon = 7.1 \times 10^{-7}$ for PSR J2124-3358 [5]. If the breaking strain is close to the maximum the observations are already confronting the theory, showing that J2124-3358 at least does not contain such a maximally strained core. If the breaking strain is close to the minimum, however, no such conclusion can be drawn. In the right panel we compare the ellipticity obtained by considering the full star, only the core of the full star and just the naked core, with no fluid around it. We take the breaking strain to be $\bar{\sigma}_{\rm br} = 10^{-5}$. As one can see, the results for the two cores do not differ significantly, while the ellipticity of the core plus fluid star is smaller, more so as the core size decreases at higher transition densities.

obvious, as one would expect a more significant elastic core to be able to sustain larger deformations.

Our main results are illustrated in the left panel of Fig. 2, where we show the obtained ellipticities for both the maximum breaking strain usually considered for a neutron star, $\bar{\sigma}_{\rm br} = 10^{-2}$, and a more conservative value, $\bar{\sigma}_{\rm br} = 10^{-5}$. For each of these values we consider the region between the maximum and the minimum value of the shear modulus μ for the range of parameters suggested in [9], cf. (2). We compare these ellipticities to the best current upper limit set by LIGO using the $S3/S4$ science runs, which is $\epsilon = 7.1 \times 10^{-7}$ for PSR J2124-3358 [5]. It is quite clear that, if the maximum breaking strain applies and the star is maximally deformed, then LIGO would have made a detection. Such a detection would have implied, cf. Fig. 2, that the transition to a quark core would have to take place below $\rho \approx 7.5\rho_n$. The fact that no detection was made rules out such a scenario. Of course, this nondetection does not say whether it is the QCD model that is at fault (i.e., there is no such high density core) or whether the QCD core exists but simply is not significantly strained.

These results are obviously interesting, especially since the sensitivity of gravitational-wave searches are set to improve. Looking ahead, the soon to be completed $S5$ science run will provide approximately 1 year's worth of data at initial design sensitivity. The $S3/S4$ data spanned approximately one month with a noise floor at near twice design sensitivity, so an upper bound from the $S5$ run will give a maximum ellipticity a factor of $2\sqrt{12}$ tighter than for $S3/S4$ (the upper bound scales linearly with the noise floor and as the inverse square root of the observation duration). For pulsar J2124-3358 this would correspond to a maximum ellipticity of approximately 1×10^{-7}. This could allow detections even if the breaking strain were at the lower end of the range we have considered. For instance, if $\bar{\sigma}_{\rm br} = 10^{-5}$ then detection would be possible if the crust-core transition occurs at a density no greater than $\rho \approx 5\rho_n$, assuming parameters at the upper end of the range given in (2). Looking further ahead, 1 year of data from Advanced LIGO would yield a further improvement of a factor of 10 in sensitivity, corresponding to $\epsilon = 1 \times 10^{-8}$ for J2124-3358. A detection would then be possible providing the crust-core transition density lies below $\rho \approx 3\rho_n$ regardless of where the shear modulus and breaking strain parameters lie in their respective likely ranges. A nondetection (i.e., upper bound) would demonstrate that the star in question does not contain such a maximally strained core with the appropriate transition density.

Our analysis suggests neutron stars containing color crystalline cores are promising candidates for gravitational-wave detection, and that in the event of no detection being made the corresponding upper limits place nontrivial constrains on the deep interiors of compact objects, showing that either the strain in the core is small or that the high density equation of state does not support such solid phases. Of course, the model we have considered is very simplistic and needs to be improved in a number of ways if we want the results to be truly quantitative. Most important would be to improve our understanding of the crystalline color superconductor. We have assumed that it behaves like ordinary elastic matter with a given shear modulus. This is a natural first assumption, but it may not be the complete picture. For example, we have not accounted for the "superfluid" nature of the core. Yet this could be an important omission. Perhaps a comparison between the crystalline color superconductor and supersolid helium [10–12] would help improve our understanding? This is an interesting possibility since supersolid helium is amenable to laboratory experiments.

Up to this point we have assumed that the nature of the phase transition at the interface between the quark core and the hadron fluid is such that there is no discontinuity in the density profile. However, there could be a (potentially sizeable) discontinuity at the interface; see, for example, Fig. 4 in [13]. To gain some insight into how this might affect our results, we have carried out a comparison of the ellipticities of three stars that can be easily modeled within our framework. First we considered a star consisting of the usual solid core surrounded by fluid, calculating $I_{xx} - I_{yy}$ (and hence the ellipticity) by integrating over the whole star in the conventional manner. Then we took the same star but integrated over the core only. Finally we took a bare core with no fluid and with the same mass and radius as the core in the first two models. The results of these calculations (as a function of core density) are shown in the right panel of Fig. 2.

Interestingly, there is little difference between the quadrupole of the bare core and the quadrupole integrated over only the core of the core plus fluid star. However, the addition of the fluid envelope serves to decrease the ellipticity, the decrease being more severe for stars with higher core-fluid transition densities. This decrease can be understood in a simple intuitive way. Unlike the solid, the fluid cannot support shear strains, so its perturbation from spherical symmetry is sourced entirely by the perturbed gravitational potential. The fluid's outer surface will therefore be close to spherical. The fluid therefore serves to "fill in" the depressions of the solid core, resulting in a partial cancellation between the $I_{xx} - I_{yy}$ contributed by the core and fluid. Clearly, this cancellation will be more substantial the closer the density of the core and fluid, the case of a smooth density transition and a bare core being extreme cases.

This leads us to conjecture that in the case of a substantial density step at the core-envelope interface, the cancellation would be far from complete, leading to an ellipticity closer to the bare core values shown in the right panel of Fig. 2. For a small core, this could lead to an order of magnitude increase in the ellipticities calculated assuming a smooth density transition, increasing further the importance of gravitational-wave observations in confronting QCD. A rigorous treatment of such a star should employ a realistic equation of state (such as that employed in [13]) which would naturally incorporate any density step.

The model we have considered is obviously simplistic in many other ways. In particular, one would want to relax the assumption that $\Delta\mu_c$ is constant. In order to make real improvements, one would (again) want to use a realistic equation of state. This would require the calculation to be carried out within general relativity. Then the determination of the background solution is straightforward, but the calculation of the mountain size would require implementing the general relativistic theory of elasticity; see, for example, [14]. To date, there have been no such calculations. Work in this direction should clearly be encouraged.

Finally, we need to improve our understanding of the breaking strain. The range of values that we have used, $10^{-5} \leq \bar{\sigma}_{\rm br} \leq 10^{-2}$, is relevant for a crust consisting of normal matter. However, the physics of the core is very different from that of the crust. There is no reason to believe that the estimates on the breaking strain for the crust should be applicable to the core. The response of the crystalline quark matter to large stresses is also uncertain. Normal matter will be predominantly brittle and break into pieces when the temperature is sufficiently far below the melting temperature and will respond by plastic flow (up to some limit) otherwise. How an elastic quark core will respond is completely unknown. Yet, for our purposes it may not matter which scenario is realized as long as the time scale for plastic flow at a given strain is longer than the observation time.

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