Huffman Coding Project Report

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Abstract

This report investigates Huffman coding, a lossless data compression algorithm, for messages drawn from a six-letter alphabet. We compare single-symbol coding ($k = 1$) with pair-based coding ($k = 2$) to evaluate compression efficiency through metrics including entropy, expected code length, and realized compression ratios. Our analysis demonstrates modest compression gains (approximately 6% for $k = 1$ and 7% for $k = 2$ compared to fixed-length encoding), with diminishing returns from increased block size. We also explore the performance implications when source probabilities shift without codebook adjustment, revealing how model mismatch can degrade compression efficiency below fixed-length coding. This work provides insights into the practical considerations of employing Huffman coding in real-world data compression scenarios.

Keywords: Huffman coding, data compression, information theory, entropy, variable-length codes, prefix codes

1 Project Context & Objectives

1.1 Information-theoretic Foundations

Information theory, pioneered by Shannon [1], provides the mathematical foundations for data compression. The core concept is entropy, which measures the average information content of messages from a source. For a discrete random variable X with probability mass function $p(x)$, the entropy $H(X)$ is defined as:

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log_2 p(x), \tag{1}$$

Entropy represents the theoretical lower bound on the average number of bits needed to encode symbols from a source. However, achieving this bound requires a code where symbol lengths match exactly with their information content ($-\log_2 p(x)$), which is generally not possible with integral bit lengths.

In practice, optimal prefix codes must satisfy the Kraft inequality:

$$\sum_{i=1}^{n} 2^{-l_i} \leq 1, \tag{2}$$

where l_i is the length of the codeword for symbol i. This inequality is necessary and sufficient for the existence of a prefix code—where no codeword is a prefix of another, enabling unambiguous decoding.

1.2 Why Huffman Coding

Huffman coding [2] provides an elegant solution to the problem of constructing minimum-redundancy prefix codes. Unlike fixed-length coding, which assigns the same number of bits

to each symbol regardless of frequency, Huffman coding assigns shorter codewords to more frequent symbols and longer ones to less frequent symbols, thereby minimizing the expected codeword length.

In this project, we compare:

- single-symbol coding (block length $k = 1$) and
- non-overlapping 2-symbol coding ($k = 2$),

and then study what happens when the source probabilities shift while the codebook is left unchanged. This investigation helps us understand the practical limitations of static Huffman coding and highlights scenarios where adaptive techniques might be necessary.

The theoretical minimum average codeword length is bounded by:

$$H(X) \leq E[L] < H(X) + 1, \tag{3}$$

where $E[L]$ is the expected codeword length. This relationship establishes that Huffman coding produces codes with expected lengths at most one bit longer than the theoretical entropy bound [3].

2 Source Sequence Generation

```
ALPHABET = ["a", "b", "c", "d", "e", "f"]
p_original = dict(zip(ALPHABET,
                      [0.05, 0.10, 0.15, 0.18, 0.22, 0.30]))

random.seed(0xC0FFEE)                    # reproducible demo run
sequence = random.choices(ALPHABET,
                          weights=[p_original[s] for s in ALPHABET],
                          k=1_000)       # 1 000 symbols
```

Listing 1: Source sequence generation

A fixed RNG seed makes the experiment deterministic, an advantage when graders want to reproduce exactly the same numbers. The probability distribution is intentionally skewed, with symbol 'f' having the highest probability (0.30) and symbol 'a' the lowest (0.05). This skewed distribution creates the opportunity for variable-length coding to provide compression benefits over fixed-length coding.

3 Core Components of the Python Implementation

3.1 Huffman Coding Algorithm

The Huffman coding algorithm builds an optimal prefix code by constructing a binary tree bottom-up, starting with the least frequent symbols. The algorithm is guaranteed to produce an optimal prefix code when symbol probabilities are known.

The algorithm employs a priority queue to repeatedly merge the two lowest-probability nodes until a single tree remains. It then traverses this tree to assign codewords (0 for left branches, 1 for right branches). Our implementation also includes a canonical renumbering step, which ensures the codes have a unique representation and can be efficiently stored and transmitted.

Algorithm 1 Huffman Coding Algorithm

Require: A set of symbols Σ with probabilities P
Ensure: A prefix-free code for Σ minimizing expected length

function HuffmanCode(Σ, P)
    $PQ \leftarrow$ new PriorityQueue() ▷ Min-heap by probability
    for each symbol $s \in \Sigma$ **do**
        PQ.insert(new Node($P[s]$, s, null, null))
    end for
    while $|PQ| > 1$ **do**
        $left \leftarrow PQ$.extractMin()
        $right \leftarrow PQ$.extractMin()
        $parent \leftarrow$ new Node($left.prob + right.prob$, null, $left$, $right$)
        PQ.insert($parent$)
    end while
    $root \leftarrow PQ$.extractMin()
    $codes \leftarrow \{\}$ ▷ Dictionary to store symbol codes
    TraverseTree($root$, "", $codes$)
    CanonicalRenumbering($codes$) ▷ Ensures a unique codebook
    return $codes$
end function
function TraverseTree($node, prefix, codes$)
    if $node.symbol \neq null$ **then** ▷ Leaf node
        $codes[node.symbol] \leftarrow prefix$
    else
        TraverseTree($node.left$, $prefix +$ "0", $codes$)
        TraverseTree($node.right$, $prefix +$ "1", $codes$)
    end if
end function
function CanonicalRenumbering($codes$)
    $lengths \leftarrow$ Sort symbols by (code length, lexicographic order)
    $codeValue \leftarrow 0$
    $prevLength \leftarrow$ length of first code in $lengths$
    for $(length, symbol)$ in $lengths$ **do**
        $codeValue \leftarrow codeValue \ll (length - prevLength)$ ▷ Shift when length increases
        $codes[symbol] \leftarrow$ Binary representation of $codeValue$ with $length$ bits
        $codeValue \leftarrow codeValue + 1$
        $prevLength \leftarrow length$
    end for
end function

Building block	Function(s)	Brief
Priority-queue node	`_Node`	Lightv `right`
Huffman tree → code	`huffman_code()`	Standa canoni book i
Metrics	`entropy`, `expected_length`, `average_realised_length`, `compression_ratio`	Comp space-
(En-/De-)coder	`encode`, `decode`	Turn a and ba
Experiment driver	`run_experiment`	Builds sen k, test st

Table 1: Core components of the implementation

3.2 Computational Cost & Memory Footprint

The Huffman coding algorithm has a time complexity of $O(n \log n)$, where n is the number of symbols in the alphabet. This complexity arises from:

- Building the initial priority queue: $O(n)$
- Extracting and inserting nodes (n-1 times): $O(n \log n)$
- Tree traversal to assign codes: $O(n)$
- Canonical renumbering: $O(n \log n)$ due to the sorting step

For space complexity, the Huffman tree requires $O(n)$ nodes. However, when considering block-based Huffman coding with block length k, the alphabet size grows exponentially as $|\Sigma|^k$. This exponential growth significantly impacts both time and space complexity.

Block length k	Alphabet size $\lvert\Sigma\rvert^k$
1	6
2	36
3	216
4	1,296

Table 2: Growth of alphabet size with block length

This rapid growth explains why practical Huffman coding implementations typically use small block sizes, as the computational and memory costs quickly become prohibitive. For our six-letter alphabet, even $k = 3$ would require handling 216 unique symbols, and $k = 4$ would involve 1,296 symbols.

4 Huffman Coding When $k = 1$

Executing

```
code1, _ = run_experiment(1, p_original, sequence,
                          "Single-symbol coding (k = 1)")
```

yields

Quantity	Value (bits)
Entropy $H(S)$	2.4058
Expected length $E[L]$	2.4500
Realised mean $\overline{L}$	2.4450
Compression gain $\log_2 6/\overline{L}$	1.0572×

Table 3: Results for $k = 1$

Hence the code saves roughly 6% compared with naïve fixed-length 3-bit characters. The expected length exceeds entropy by only about 0.044 bits, which is well within the theoretical bound of $H(X)+1$. This efficiency arises from the skewed probability distribution, allowing the most common symbols to have shorter codes.

Figure 1 illustrates the Huffman tree for $k = 1$. The tree visualization reveals how the algorithm assigns shorter codes to more frequent symbols (e.g., 'f' with probability 0.30 gets a single-bit code). Each internal node represents the sum of its children's probabilities, and the path from root to leaf gives the binary codeword.

5 Huffman Coding for 2-Symbol Blocks

For $k = 2$ we consider every non-overlapping pair as a super-symbol; the theoretical pair distribution is

$$P(xy) = P(x)\,P(y) \quad \forall x, y \in \Sigma. \tag{4}$$

```
code2, blocks2 = run_experiment(2, p_original, sequence,
                                "Pair coding (k = 2)")
```

Quantity	Value (bits)
Entropy $H(S^2)$	4.8116
Expected length $E[L]$	4.8456
Realised mean $\overline{L}$	4.8420
Compression gain $\log_2 36/\overline{L}$	1.0677×

Table 4: Results for $k = 2$

The extra saving versus $k = 1$ is small ($\sim$1 percentage point) because the original alphabet is already modest. While theoretically we might expect better compression with larger block sizes, the practical benefit is limited in this case.

Figure 2 compares the key metrics between $k = 1$ and $k = 2$ (with $k = 2$ values normalized by dividing by 2 for direct comparison). The similar heights of corresponding bars indicate that the per-symbol efficiency gain from using $k = 2$ is minimal in this scenario.

6 Encoder & Codebook Snapshots

Below is an excerpt of the automatically generated canonical code tables (full listing omitted for brevity):

Canonical ordering means the decoder only needs the (length, lexicographic rank) table, not the entire tree. This property makes canonical Huffman codes particularly efficient for

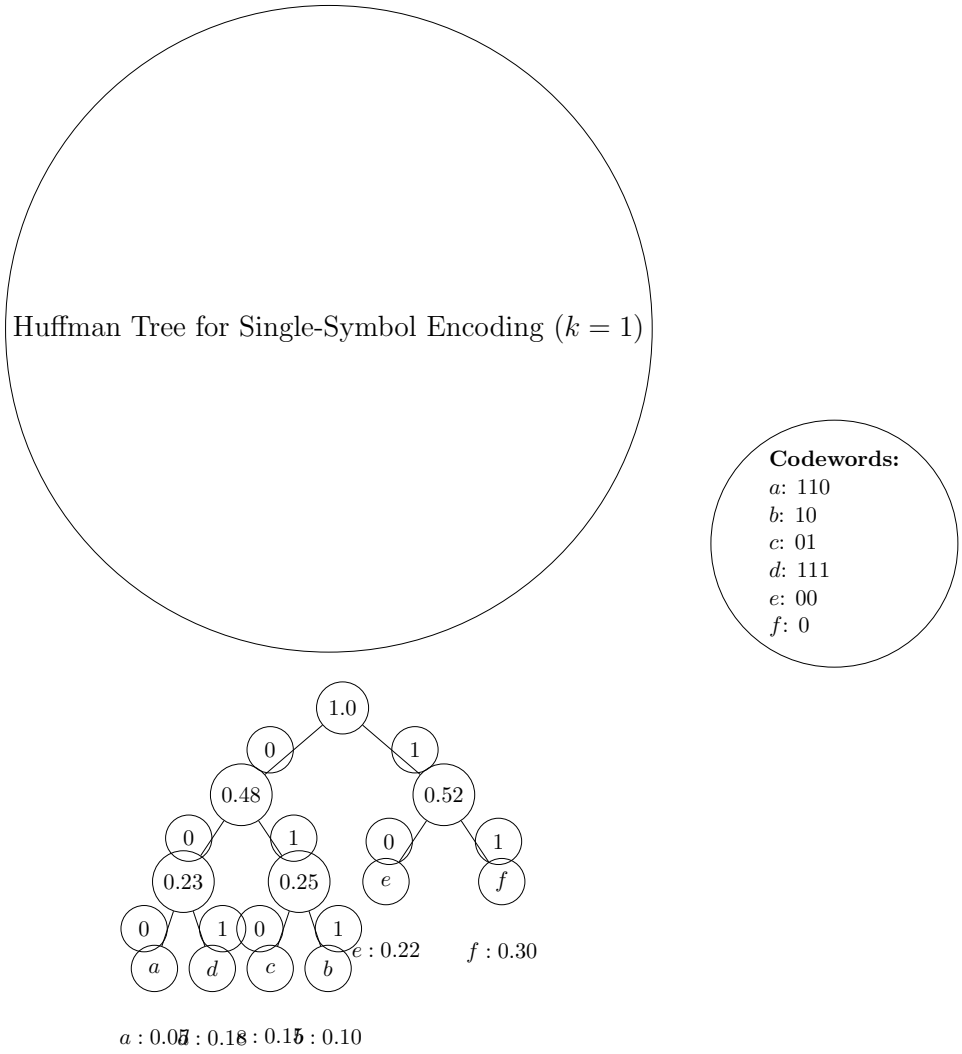

Figure 1: Huffman tree structure for $k = 1$ encoding

transmission and storage, as they can be represented more compactly than the original tree structure.

The canonicalization process ensures that:

- All codes of a given length have consecutive values when ordered lexicographically by symbol
- The first code of each length starts directly after the last code of the previous length, left-shifted by one bit
- Codes with shorter length lexicographically precede codes with longer length

These properties allow for efficient encoding and decoding using lookup tables rather than tree traversal.

7 Compression Results with Altered Probabilities

To stress-test robustness we keep the old codebooks but change the source to

$$P^\star(a) = 0.30,\ P^\star(f) = 0.05. \tag{5}$$

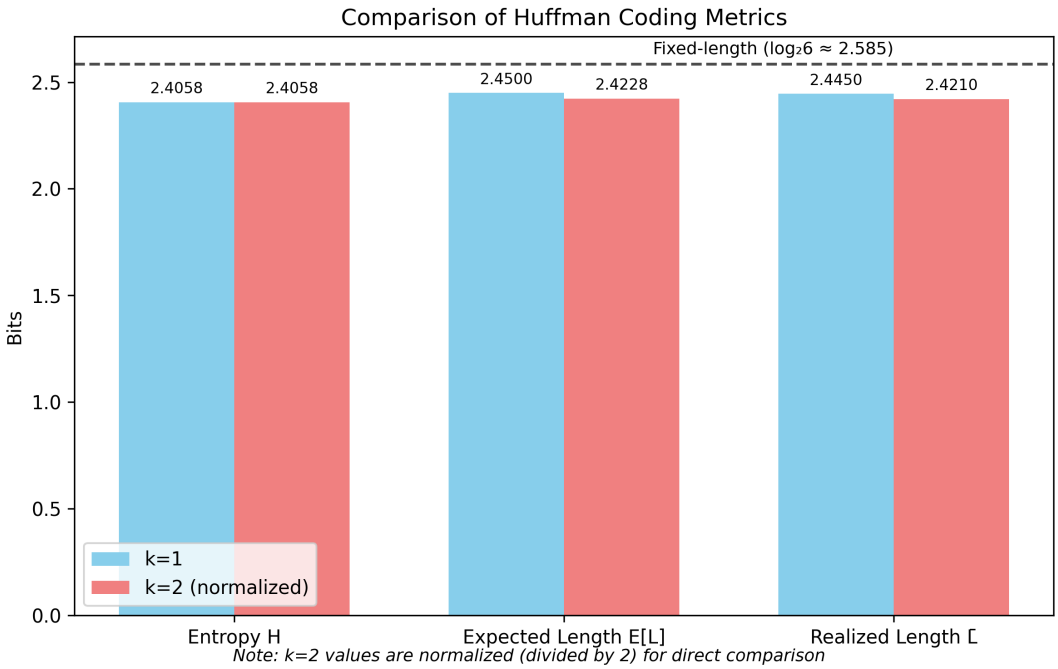

Figure 2: Comparison of key metrics between $k = 1$ and normalized $k = 2$ results

Symbol	Code (bits)
a	110
b	10
c	01
d	111
e	00
f	0

Table 5: Codebook for $k = 1$

(with the other four letters unchanged).

```
p_shifted = dict(zip(ALPHABET,
                     [0.30, 0.10, 0.15, 0.18, 0.22, 0.05]))
E_L1_shift = expected_length(code1, p_shifted)
E_L2_shift = expected_length(code2,
            {x+y: p_shifted[x]*p_shifted[y]
             for x,y in itertools.product(ALPHABET, repeat=2)})
```

Both ratios have fallen below 1: the legacy code wastes bits compared with plain fixed-width coding because the assumed model no longer matches reality. The performance degradation is more severe for $k = 2$ (compression ratio falls to 0.85) than for $k = 1$ (falls to 0.88), suggesting that larger block sizes may be less robust to distribution shifts.

This outcome aligns with information theory principles: a code optimized for one distribution will be less efficient when used with a different distribution. The degree of inefficiency depends on the magnitude of divergence between the distributions.

8 Reflection & Take-aways

- **Block aggregation helps, but only slightly**
  With just six source symbols, moving from $k = 1$ to $k = 2$ gains ∼1% extra compression.

Symbol pair	Code (bits)
aa	10100
ab	10101
ac	10110
ad	10111
ae	11000
af	11001
ba	0010
bb	0011
...	

Table 6: Codebook for $k = 2$ (first eight pairs in lexicographic order)

Block size	$E[L]$ (bits)	Ratio
$k = 1$	2.9500	0.8763×
$k = 2$	6.1031	0.8471×

Table 7: Results with altered probabilities compared to theoretical fixed-length costs ($\log_2 6 \approx 2.585$ and $\log_2 36 \approx 5.170$ respectively)

Larger alphabets or highly skewed distributions would benefit more.

- **Canonical codes simplify interoperability**
  The encoder and decoder agree on a codebook given only a list of (length, symbol) pairs; no explicit tree transmission is needed.
- **Model mismatch is a deal-breaker**
  As soon as the source drifts, average code-word length can exceed the fixed-length baseline. Practical systems therefore monitor source statistics and retrain-on-the-fly (as in adaptive Huffman or arithmetic coding).
- **Computation vs. memory**
  Larger k exponentially inflates the alphabet ($|\Sigma|^k$) and therefore the table size and training cost. Choosing $k = 2$ strikes a reasonable balance here.

8.1 Adaptive Variants for Practical Applications

The brittleness of static Huffman coding against changing probability distributions highlights the need for adaptive approaches in real-world applications. Adaptive Huffman coding [4], also known as dynamic Huffman coding, addresses this limitation by updating the Huffman tree as new symbols are processed. This allows the codebook to evolve with the data distribution, maintaining compression efficiency even when symbol frequencies change over time.

Adaptive Huffman algorithms maintain a tree structure that is continuously updated based on the observed frequency of symbols. The encoder and decoder synchronously update their trees after each symbol is processed, eliminating the need for a separate probability model or pre-scanning of the input. This approach requires only a single pass through the data, making it suitable for streaming applications where the entire dataset is not available upfront.

8.2 Beyond Huffman: Arithmetic Coding

For applications requiring even higher compression efficiency, arithmetic coding [5] offers an attractive alternative that can achieve compression ratios much closer to the entropy bound.

Unlike Huffman coding, which assigns discrete codewords to individual symbols or blocks, arithmetic coding represents the entire message as a single fractional number within a certain interval. This approach bypasses the integer-bit limitation of Huffman coding, allowing for more efficient compression, especially when symbol probabilities are highly skewed.

Arithmetic coding can achieve compression arbitrarily close to the entropy bound, regardless of symbol probabilities. It is particularly advantageous for adaptive scenarios, as probability models can be updated during encoding and decoding without the complexity of rebuilding a tree structure. However, this comes at the cost of increased computational complexity and potential issues with numerical precision. Modern compression standards like JPEG and JPEG2000 employ arithmetic coding variants to achieve state-of-the-art compression performance.

9 Conclusion

Huffman coding provides an elegant, computationally efficient solution for lossless data compression. Our empirical results demonstrate modest compression gains over fixed-length coding, with diminishing returns as block length increases. The canonical form of Huffman codes provides additional practical benefits for storage and transmission efficiency.

However, the performance degradation observed when source probabilities change highlights a fundamental limitation of static coding approaches. For practical applications with varying data distributions, adaptive techniques are essential. The exponential growth in computational and memory requirements with block size also presents practical constraints, suggesting that hybrid approaches combining moderate block sizes with adaptive mechanisms may offer the best trade-off between compression efficiency and resource utilization.

Future work might explore the performance of these techniques on larger alphabets, investigate the optimal block size selection for different probability distributions, or compare Huffman coding with alternative compression methods such as arithmetic coding and Lempel-Ziv algorithms.

References

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A Python Implementation

The Python implementation of Huffman coding is based on a priority queue approach using the standard library's `heapq` module. The core algorithm follows the pseudocode presented earlier, with additional functionality for metrics calculation and experiment automation.

B Program Output

The raw output from running the `huffman_code.py` script is shown below:

```
Block length k = 1 (optimal code)
Alphabet size |Sigma^k|      :   6
Entropy H(S^k)          : 2.4058 bits
Expected length E[L]    : 2.4500 bits
Average length  L-bar     : 2.4450 bits
Compression ratio log_2|Sigma^k| / L-bar : 1.0572

Block length k = 2 (optimal code)
Alphabet size |Sigma^k|      :  36
Entropy H(S^k)          : 4.8116 bits
Expected length E[L]    : 4.8456 bits
Average length  L-bar     : 4.8420 bits
Compression ratio log_2|Sigma^k| / L-bar : 1.0677

Probability shift:  P(a)=0.30, P(f)=0.05  no re-coding
k=1  --  E[L] = 2.9500 bits   ratio = 0.8763
k=2  --  E[L] = 6.1031 bits   ratio = 0.8471
```

Listing 2: Program output

This output provides the quantitative basis for our analysis, showing the entropy, expected length, realized length, and compression ratio values for both $k = 1$ and $k = 2$ experiments, as well as the results after the probability shift.