Further Data Structures

- The progress so far
  - We understand the notion of an abstract data type.
  - Saw some fundamental operations as well as advanced operations on arrays, stacks, and queues
  - Saw a dynamic data structure, the linked list, and its applications.
- This week we will
  - focus on improving the performance of the find operation
  - Propose data structures for an efficient find.

Motivation

- Consider a high-level programming language such as C/C++.
- They need a compiler to translate the program into assembly language and machine instructions.
- In that process, there are several processes and several checks.
- One of them is to check for variable names, types, etc.
- Ensure also that no duplicate names appear within the same scope.

Motivation

- Let us consider this duplicate variable names problem.
- As we encounter a new variable declaration,
  - verify that in the same scope there are no other declarations with the same name.
  - If this is not a duplicate, need to store this name to check future declarations.
  - Once a scope is complete, can delete names from this scope.

Motivation

- Let us consider a few alternatives first.
- Start with using an array.
- Store names in an array, as they appear.

Using an Array

- To insert a new variable name
  - add it to the end of the array
- To check if the new is a duplicate
  - search in the array
  - called linear search
  - Too costly at O(n) when there are n names presently.

Using an Array

- To insert a new variable name
  - add it to the end of the array
- To check if the new is a duplicate
  - search in the array
  - called linear search
  - Too costly at O(n) when there are n names presently.
- Can we keep the array sorted
  - Then can use binary search to check for a name
  - But, insertion becomes difficult.
  - Why?

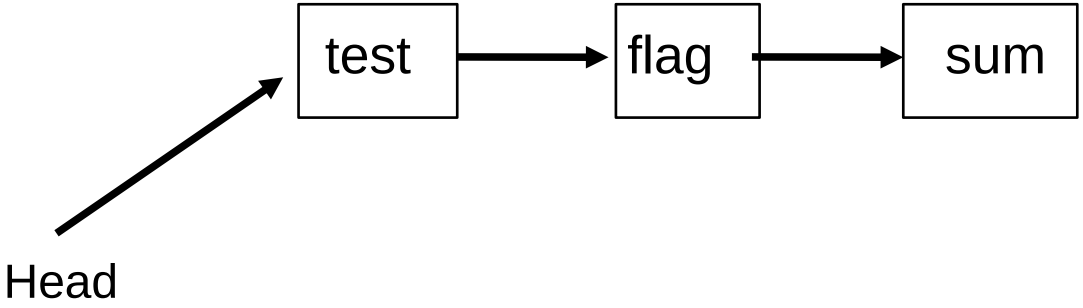
Using an Array

- So the time for ((insert, search) is:
  - (O(1), O(n)) when no sorted order
  - (O(n), O(log n)) when in sorted order.

Using a Linked List

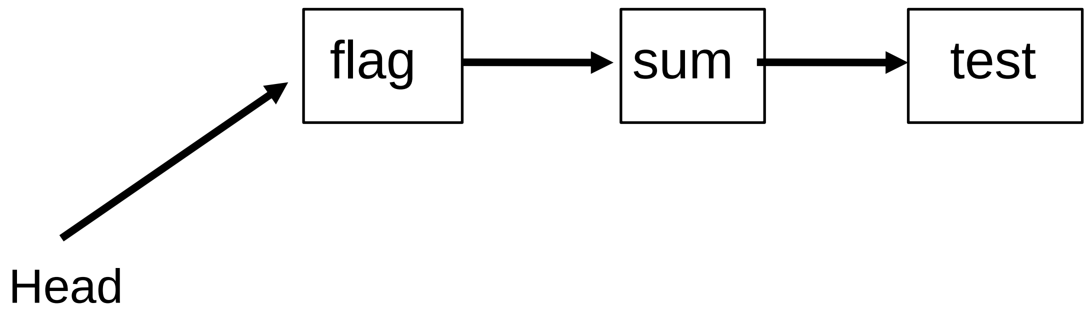
- A linked list removes the drawback that the size cannot grow dynamically.
- How would we use a linked list?

Using a Linked List



- Option 1
  - Insert names at the beginning of the list.
  - search would need to scan the entire list.
  - Time for these operations is (O(1), O(n))

Using a Linked List

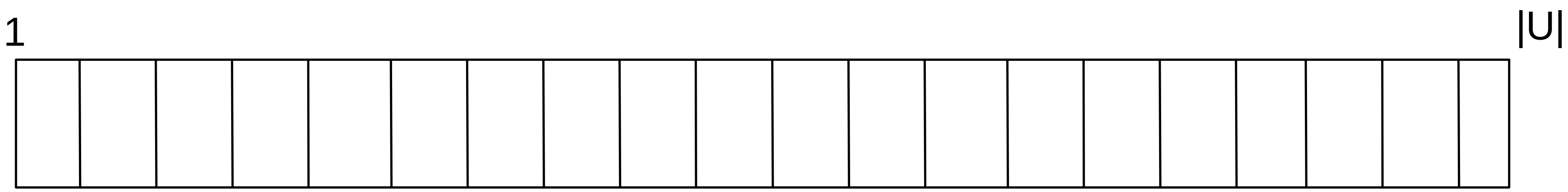


- Option 2
  - Insert names in sorted order
  - still, search would need to scan the entire list.
  - Time for these operations is (O(n), O(n))

Another Solution

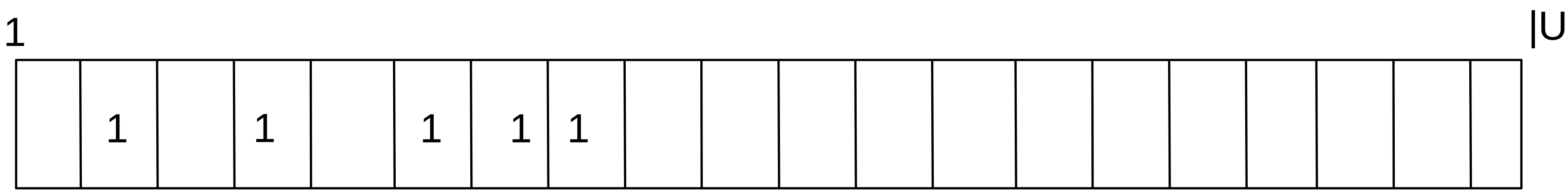
- A radically different solution for now.
- Imagine that we consider integers as the names for now.
- Let us formalize.
  - Let U be the set of all possible values. Called the universe.
  - Let K be the set of keys, a subset of U that is being used currently.

A Different Solution



- Imagine an array A of size |U|.
- Array A will have only 0 and 1 values.
- Insert an element k can be translated to setting A[k] to 1.
- Checking if k is already present would be to see if A[k] is 1 already.

A Different Solution



- Example: The following operations starting from an empty array have the effect as shown in the Figure.
  - insert(4), insert(8), insert(7), insert(2), insert(4), insert(6)
  - Empty cells assumed to contain a 0.
- Time for operations insert and search is (O(1), O(1))

A Different Solution

- has very good operation efficiency (++)
- But, can be very wasteful on space (---)
  - Imagine using such a solution for our original problem.
  - Number of valid variable names > 26^8. Why?
  - Number of variables in a typical program is about 100.
  - So, we use only 100 cells of the array of size > 26^8.
- Are there efficient solutions so that insert, search time are both small?
  - Ideally O(1) for each operation.

A New Data Structure

- The drawback of the previous solution is that a lot of space is reserved a-priori irrespective of usage.
- Our new solution will use a space only proportional to the usage.
- Still will be based on arrays.
- Called a hash table. Details follow.

Hash Table

- Consider an array T of size |T|.
  - T is called a hash table
- Consider a function h that maps elements in U to the set {0, 1, ..., |T|-1}.
  - h is called a hash function.
- Can use the function h to map elements to indices.
  - Details follow.

Hash Table

- Now U can be any set, not just integers.
- The function h can map its input to an integer in the appropriate range.
- As an example, h("test") = 12.
- We will still however use integers for our setting.

Example of a Hash Table

	0									9
T :			22		34	65	76	97		

- Let U = {1, 2, ..., 100}.
- Let K = {34, 65, 22, 76, 97}.
- Let h(k) = k mod 10. So, |T| = 10.
- Key 22 to be stored in cell h(22) = 22 mod 10 = 2.
- Key 76 to be stored in cell h(76) = 76 mod 10 = 6.

Implementation of Operations

- Let us consider implementing operations insert, delete, and find.

```
operation insert(k)
begin
    j = h(k)
    T(j) = k;
end;
```

```
operation delete(k)
begin
    j = h(k)
    T(j) = -1;
end;
```

```
operation find(k)
begin
    j = h(k)
    if T(j) == k then
        return found
    else return not found
end;
```

Operations

- Let us consider the runtime of these operations.
- All operations run in $O(1)$ time.
  - Provided, certain conditions hold.
  - What are these conditions?
- Note the similarity to the array based solution (Solution 3)
  - Instead of accessing cell k, we now access cell $h(k)$.
  - But, instead of using a space of $|U|$, we use a space of $|T|$.

A Small? Problem

	0									9
T :		91	82			25	76			

- Suppose U = {1, 2, ..., 100} as earlier.
- Suppose h(k) = k mod 10, as earlier, with |T| = 10.
- Suppose K = { 25, 76, 82, 91, 65}.
- The figure above shows the contents of T after inserting 91.
- Where should 65 be inserted?

A Small? Problem

- Notice that 65 is different from 25. So should store both.
- But, each cell of the array T can store only one element of U.
- How do we resolve this?

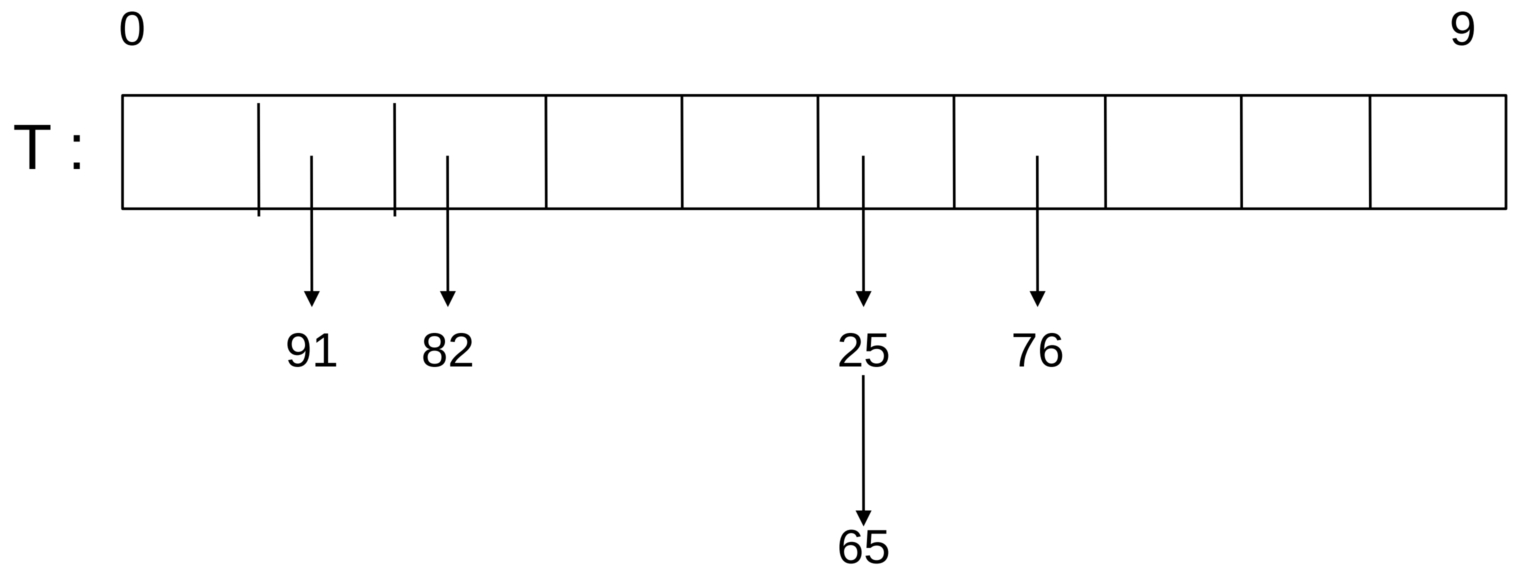
A Collision

- The situation is termed as a collision.
- Can it be avoided?
  - Not completely.
  - Notice that h maps elements of U to a range of size |T|.
  - If |U| > |T|, cannot avoid collisions.
- Can they be minimized?
  - Certainly.
  - Choosing h() carefully can minimize collisions.
  - Some guidelines to choose h() are known.

Collision Resolution

- Despite careful efforts, it is very likely that collisions exist.
- We should have a way to handle them properly.
- Such techniques are called collision resolution techniques.
- We shall study some of those techniques.

Collision Resolution Techniques



- Can treat each cell of the table T as a pointer to a list.
- The list at cell k contains all those elements that have a hash value of k.
- Example above.

Collision Resolution Technique

- Notice how 25 and 65 are placed at the same index, 5.
- Why should 65 come after 25?
  - No reason. Several options possible.
  - Keep at the beginning of the linked list.
  - Keep at the end of the linked list.
  - Keep the linked list in sorted order.
  - Just like insertion in linked list, each has its own applications.

Collision Resolution

- The above technique is called chaining.
- Names comes from the fact that elements with the same hash value are chained together in a linked list.
- Let us see how operations should now be implemented.
  - Assuming that insert is at the front of the list.

Operations in Chaining

```
Operation insert(k)
begin
    j = T(k);
    temp = new  node;
    temp->data = k
    if T[j] == NULL then
        temp->next = NULL;
        T[j] = temp;
    else
        temp ->next = T[j];
        T[j] = temp;
end
```

```
Operation Delete(k)
begin
    j = T(k);
    while T[j] != NULL
        temp = T[j];
        if temp->data != k
            prev = temp;
            temp = temp->next;
    end-while
        prev->next = temp->next;
end
```

Operations in Chaining

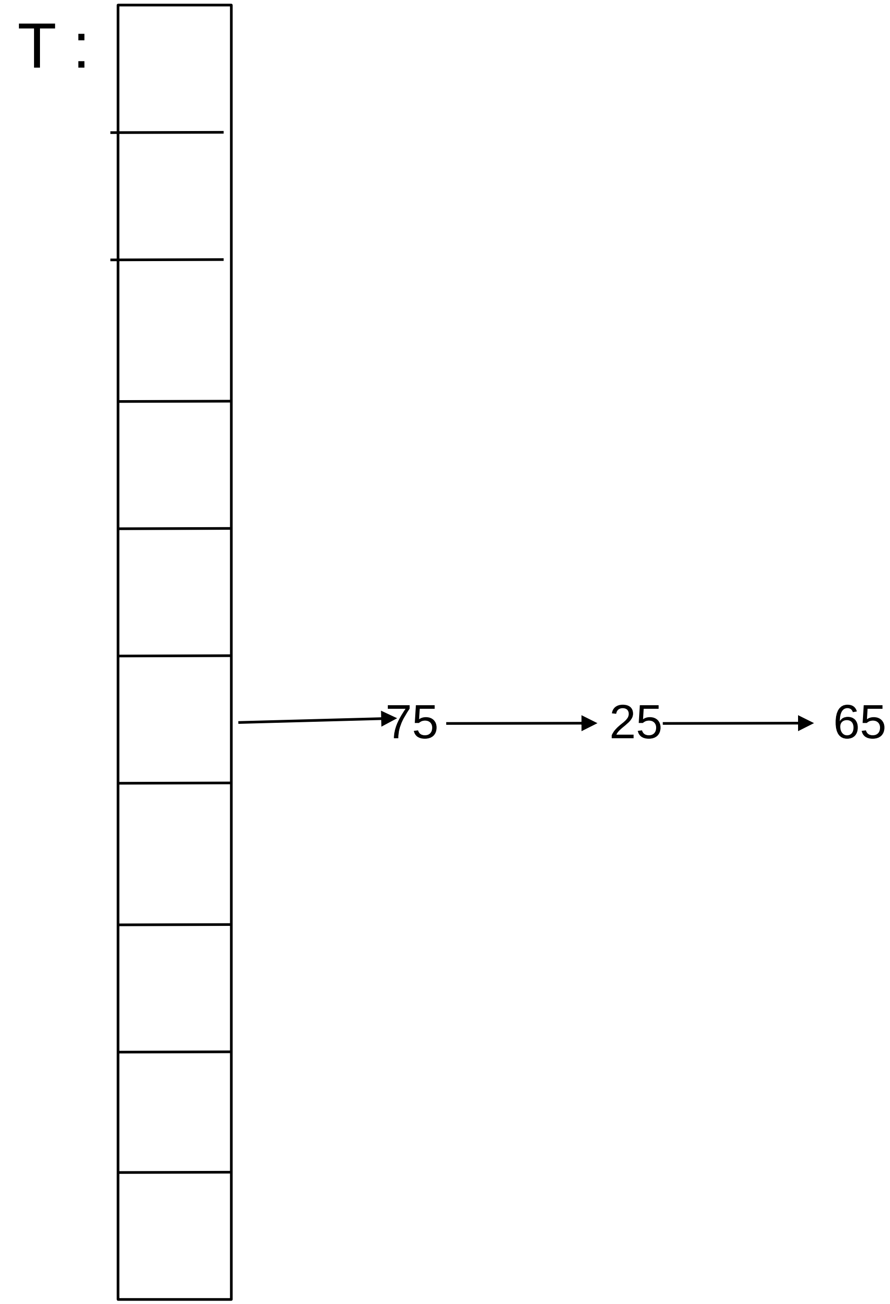
- Operation Find can be similarly implemented.

Analysis of the Operations

- How to analyze the advantage of the solution?
- Consider a hash table using chaining to resolve collisions.
- What is the runtime of insert and search?

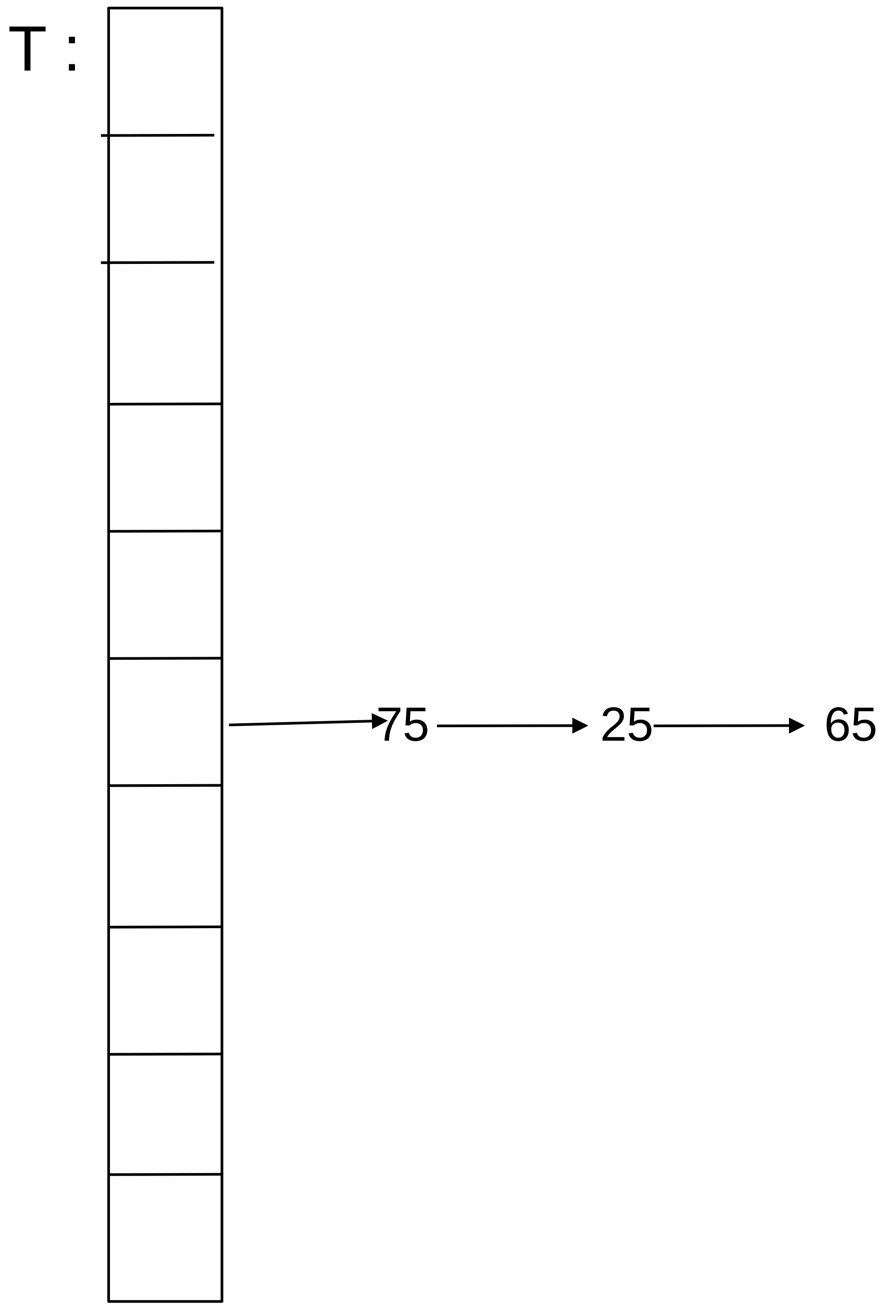
Analysis of Operations

- If we have a very bad hash function, h, then all values may map to the same index.
- The list at that index will have all |K| = n elements.



Analysis of Operations

- If we have a very bad hash function, h, then all values may map to the same index.
- The list at that index will have all |K| = n elements.
- Now, find takes O(n) time, while insert can still be done in O(1).
- What have we achieved?



Analysis of Operations

- The above approach is too pessimistic.
- Need better analysis to analyze an average scenario.
  - Average scenario?
  - Depends on the setting. In our setting, can imagine that K can be a random subset of U.
  - Still, very difficult to do average case analysis.
  - Too many assumptions, too many unknowns.

Analysis of Operations

- Still, it is observed in practice that chaining based hash table works very well.
- What is the theoretical basis for this behaviour?
- We can attempt to do this using a new parameter.
- Define λ as the load factor of a hash table.

$$\boxed{\lambda = |K| / |T|}$$

Analysis Using λ

- How does λ help in our analysis?
- Assume the following
  - The hash function has the property that it hashes each element of U to an element of T with equal probability.
  - In other words, h is a uniform hash function.
  - Does such a function exist?

Analysis Using λ

- With the above assumption on h
  - The expected number of elements in K that hash to a given cell j = $|K|/|T| = \lambda$.
  - Can show the above using elementary random variables.
  - The above observation motivates the definition of λ.
- Now, how do the runtime of insert and search depend on λ.

Analysis Using λ

- For insert, since we insert at the beginning of the list, it is always $O(1)$.

Analysis Using λ

- For insert, since we insert at the beginning of the list, it is always O(1).
- For search
  - Each chain is about λ elements long.
  - If the element being searched for, say k, is not present,
    - the entire list will be searched for.
    - This is called as an unsuccessful search.

Analysis Using λ

- For insert, since we insert at the beginning of the list, it is always O(1).
- For search
  - Each chain is about λ elements long.
  - If the element being searched for, say k, is not present,
    - the entire list will be searched for.
    - This is called as an unsuccessful search.
  - If the element being searched for is present,
    - We can stop the search once the element is found.
    - Typically, this happens at the middle of the list.
    - So need to search about λ/2 elements on average.
    - This is called as a successful search.

Other Approaches to Resolve Collisions

- Problems with the chaining approach:
  - use of pointers. Can become difficult to program and verify.
  - Cannot predict the state of the hash table easily. Can keep on adding whereas a rehashing may be better.
  - Slightly more space than |T|.
- Some of these problems can be addressed.

Open Addressing

- The technique is called open addressing.
- No additional space is used apart from the table T.
- So how to address a collision?
  - Try alternative cells in the table.
  - Details follow.

Open Addressing

- Recall that the hash function h maps elements in U to indices in the table T.
- How can we have an alternative cell/index for a key k?
- Idea: Use more hash functions.
- Consider hash functions h_1, h_2, ... all mapping U to indices in T such that $h_i(k) = (h_{i-1}(k) + f(i)) \bmod |T|$.
  - f() is a function on N->N.

Open Addressing

- Depending on the function f() several variations are possible.
- We'll consider three choices.

Linear Probing

- Consider the case when f(i) = 1.
- When $h_0(k) = h(k)$ itself, we now have that
  - $h_1(k) = h_0(k) + 1$
  - $h_2(k) = h_1(k) + 1$
  - ...
- Suggesting that the next slots that are tried in succession starting from the slot $j = h(k)$ are
  - $j+1$
  - $j+2$
  - .

Linear Probing

- Notice that eventually all slots may be tried.
- When all slots are tried, this indicates that the table T is full.
- Also, while the table is not full, an empty cell can always be found.

Linear Probing Example

	0									9
T :				23		65				

- Consider U = {1, 2, ..., 100} as earlier.
- Let K = {23, 65, 43, 98, 64, 76}
- Let h(k) = k mod 10.
- Figure shows the table after inserting 65.
- Place the other elements

Linear Probing Example

	0									9
T :				23	43	65				

- To insert 43, realize that cell 3 is already full.
- So try cell 3+1 = 4. That cell is empty, and hence
- 43 placed in cell 4.

Linear Probing Example

T :

0									9
			23	43	65			98	

- To continue further, we can include 98 in cell 8.
- To insert 64, we see that cell 4 is occupied by 43.
- Cannot move 43 now.
  - This helps the search operation later.
- So, find a new slot for 64 as earlier.

Linear Probing Example

T :

0									9
			23	43	65	64		98	

- We realize that cell 4+1 = 5 is also full.
- Cell 4+2 =6 is empty. So, 64 stored at cell 6.
- In a similar fashion, 76 stored at cell 7.

Linear Probing

- The operation find(k) works similarly.
- Cannot declare k not found if T[h(k)] does not contain k.
- We should search at indices h(k)+1, h(k)+2, ...
- How long?
  - A good quick question.

Linear Probing

- Operations Insert and Find shown here.

```
Algorithm Insert–LinearProbing(k)  {
    v = h(k);
    if T(v) is empty then
        T(v) = k
    else
      while not done {
          v = v + 1;
          if T[v] is empty then {
             T[v] = k;  done = true;
      }
    }
}
```

```
Algorithm Find(k) {
    v = h(k);
    while not done and v ≤ |T| do {
        if T[v] = k then {
            done = true;
            return found
        }
        else v = v + 1;
    }
}
```

Linear Probing

- How to analyze the time taken?
  - More so, the average case.
- Quite difficult, but the following results are known.
  - For an insertion, the average number of probes is

$\frac{1}{2} + \frac{1}{2} (1-\lambda)^2$

  - For an unsuccessful search, the average number of probes is $\frac{1}{2} + \frac{1}{2}(1-\lambda)$.

Linear Probing

- A few issues with linear probing.
- Delete is a problem.
  - Affects searchability.
  - Normally, a virtual delete option is used.
- Another observation: The table may have areas that are more dense compared to other areas.
- Reason for this is that inserts look for empty cells linearly.
- This problem is called as primary clustering.

Linear Probing

- Primary clustering : the effect is that some operations can take more time.
  - especially those that get close to the denser area.

Quadratic Probing

- Yet another open addressing method is quadratic probing.
- As the name suggests, this uses $f(i) = i^2$.
- So, the cells tried in order are:
  - $h_0(k) = h(k)$
  - $h_1(k) = h(k) + 1$
  - $h_2(k) = h_1(k) + 4$
  - ..

Quadratic Probing Example

	0									9
T :		71			64					29

- Let U = {1, 2, ..., 100}.
- Let K = {29, 64, 71, 84, 53}
- Let h(k) = k mod 10.
- Where do 84 and 53 go?

Quadratic Probing Example

0 9

T :

	71			64	84				29

$h_0(84) = 4$
$h_1(84) = h_0(84) + 1$
$= 4 + 1 = 5.$

- Let U = {1, 2, ..., 100}.
- Let K = {29, 64, 71, 84, 53}
- Let h(k) = k mod 10.

Quadratic Probing Example

T :

0									9
	71			64	84			53	29

$h_0(53) = 3$
$h_1(53) = h_0(53) + 1$
$= 3 + 1 = 4.$
$h_2(53) = h_1(53) + 4$
$= 4 + 4 = 8.$

- Let U = {1, 2, ..., 100}.
- Let K = {29, 64, 71, 84, 53}
- Let h(k) = k mod 10.

The Find Operation

- How should it proceed?
- When can it stop?

Quadratic Probing

- The operations in pseudo-code.

```
Algorithm Insert(k) {
    v = h(k); i = 0;done = false;
    while not done and i < |T| do {
        v = (v + i*i) mod |T|
        if T[v] is empty then {
            T[v] = k; done = true;
        else i = i + 1;
    }
}
```

```
Algorithm Find(k) {
    v = h(k); i = 0;done = false;
    while not done do {
        v = (v + i*i) mod |T|
        if T[v] is empty then
            return not found
        if T[v] = k then {
            done = true;
            return found
        }
        else i = i + 1;
    }
}
```

Quadratic Probing

- Would all cells be probed?
  - May not be.
- But, if |T| is a prime then a new element can always be inserted if the table is half-empty.
- Proof: Idea is to show that the first |T|/2 entries probed are distinct. Here is where |T| being a prime helps.
  - Let cells j and j' are probed, so that $h(k) + j^2 = h(k) + j'^2$. Then, we have that $j^2 - j'^2 = 0 \bmod |T|$.
  - This implies that either $j - j' = 0 \bmod |T|$ or $j + j' = 0 \bmod |T|$
  - If j, j' are different, then only $j+j' = 0 \bmod |T|$.

Yet Another Mechanism – Double Hashing

- Double hashing is another mechanism.
- In this case, the function f(i) is defined as $f(i) = i.h_2(k)$ where h_2 is another hash function.
- How to choose $h_2(k)$?
- Choice to ensure that all cells will be probed eventually.

Double Hashing – An Example

- Consider U = {1, 2, ..., 100}
- Let K = {32, 54, 87, 74, 82}.
- Let h(k) = k mod 10.
- There are collisions. Let $h_2(k) = r - (k \bmod r)$ where $r = 7$.
- Place the above elements in a hash table.

Double Hashing – An Example

	0									9
T :			32		54			87		

- Let K = {32, 54, 87, 74, 82}.
- No collisions till 87.
- To insert 74
  - collision.
  - Compute (4 + (7 – (74 mod 7)) mod 10 = 0
  - So place 74 at cell 0.

Double Hashing – An Example

	0									9
T :			32	74	54			87		

- Let K = {32, 54, 87, 74, 82}.
- No collisions till 87.
- To insert 74
  - collision.
  - Compute (4 + (7 – (74 mod 7)) mod 10 = 7
  - Compute (7+2.(7 – 74 mod 7)) mod 10 = 3

Double Hashing – An Example

T :

0									9
74		32		54			87	82	

- Let K = {32, 54, 87, 74, 82}.
- To insert 82
  - collision.
  - Compute (2 + (7 – (82 mod 7)) mod 10 = 4
  - Compute (4 + 2.(7 – (82 mod 7))) mod 10 = 8
  - So, insert 82 at cell 8.

Double Hashing

- Can now write the routines for insert, search, etc.
- Guidelines to choose $h_2(k)$
  - $h_2(k)$ should never evaluate to 0.
  - ensure that all cells are probed.
- Our earlier choice works well. In general,
  - pick an r that is a prime and smaller than $|T|$
  - Define $h_2(k) = r - (k \bmod r)$.

Advanced Topics

- For most of the techniques we discussed
  - can only say that the average time is O(1), if the load factor is O(1)
- In some settings, can actually achieve a worst case O(1) time.
  - settings where there are no delete operations.
- This is called as universal hashing.