

# Path Signature for Time Series

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## Quadratic Variation

$$QV(t) = \sum_{i=0}^t (S_{i+1} - S_i)^2$$

for  $t := \{1, \dots, n\}$ ,  $QV_1, QV_2, \dots$  a running cumulative sum.  
Returns as `np.array()`

## Brownian Motion Extraction

$$B_t = \frac{(S_{t+1} - S_t)}{\sqrt{QV(t)}} = \frac{dS_t}{\sqrt{d\langle S \rangle_t}}$$

for  $t := \{1, \dots, n\}$ .

## Signature Order/Keys

Level 1 Signature:  $(1), (2)$

Level 2 Signature:  $(1), (2), (1, 1), (1, 2), (2, 1), (2, 2)$

Level 3 Signature:  $(1), (2), (1, 2), (2, 1), (2, 2), (1, 1, 1), (1, 1, 2), \dots$

for  $t := \{1, \dots, n\}$ .

## In Practice:

Level 1 Signature example (1), (2):  $t, X_t - X_s$

Level 2 Signature example (2,2):  $\frac{1}{2} (X_t^2 - X_0^2) - [X^2]_t$

Level 3 Signature example: more complicated

for  $t := \{1, \dots, n\}$ .

Formally:

Stratonovich Integral: 
$$\frac{f(X_{t_k}) + f(X_{t_{k+1}})}{2} (X_{t_{k+1}} - X_{t_k}) \approx \int_{t_k}^{t_{k+1}} f(X_s) \circ dX_s$$

## LASSO Regression

$$\begin{aligned}\text{Final Model: } \hat{S}_t &\approx \sum_I \hat{\alpha}_I \cdot \text{Sig}(X)'_{0,t} \\ &= \alpha_0 + \alpha_1 t + \alpha_2 B_t + \alpha_3 \frac{t^2}{2} + \alpha_4 \int_0^t s dB_s + \dots\end{aligned}$$





