Statistics and Graphs

CSW

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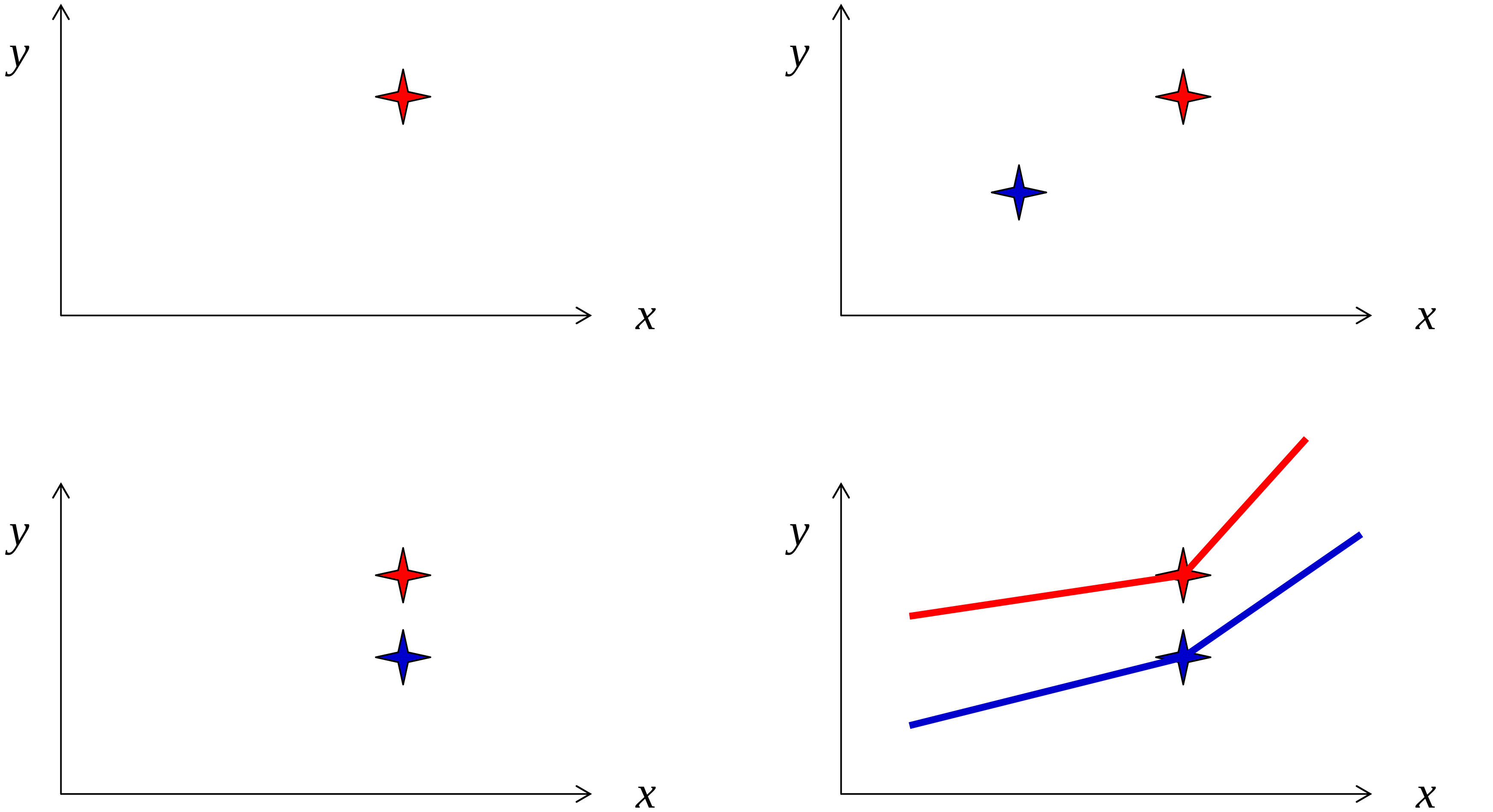
recap

- Null hypothesis H_0 is assumed true, unless ***rejected*** by the experimental data

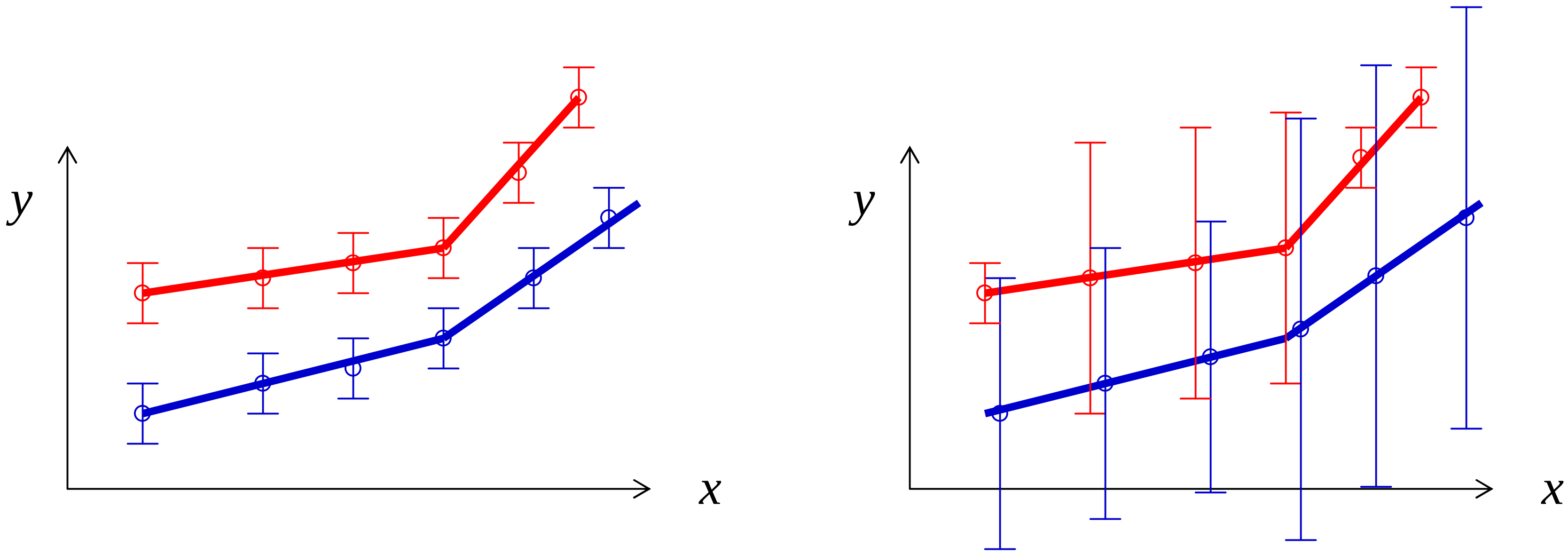
- H_0 = "all swans are white"
  - rejected by a counter-example
  - I see a black swan, and reject H_0
- H_0 = "no effect on readability"
  - rejected by *statistically significant* evidence against
  - I see a certain degree of improvement, and reject H_0
    - reject only at a given *confidence level*
  - how ***much*** improvement do I need?
    - how ***much*** confidence can I have?

what do these tell you?



it depends…

- everything varies with the data sample somehow
- is the difference *statistically significant*?



statistics

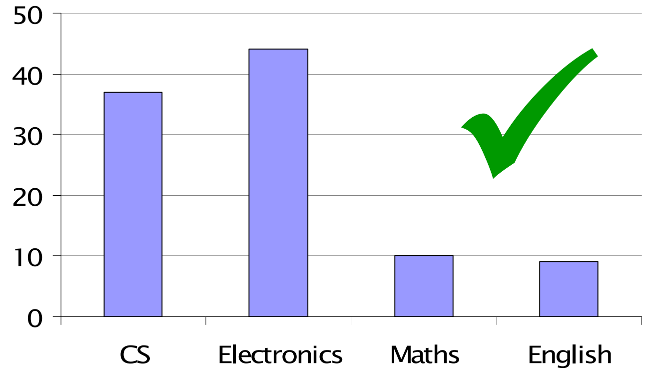
- a **statistic** is a summary of the data, a value that captures some characteristic of the data
  - **mode / ...**
  - **median / quartiles / …**
  - **mean / standard deviation / skew / …**

- the statistics we should calculate depend on the kind of data we have

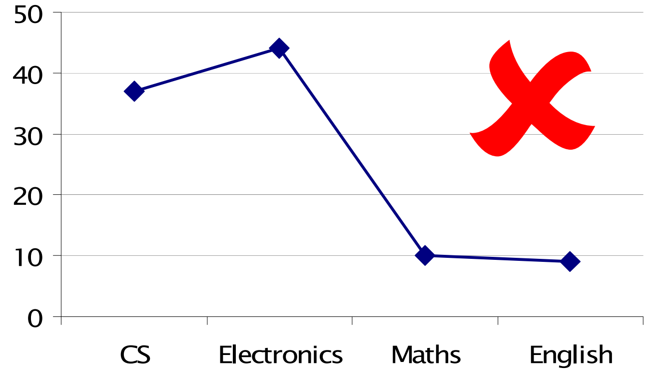
nominal / categorical data

- data grouped in ***categories***
  - on / off ; male / female ; butcher / baker / candlestickmaker ; blue / brown / green eyes ; Toyota / Ford / Volvo / other ; single / married / divorced / widowed ; …
- operations: equality of category
- statistics: **mode**
  - it ***makes no sense*** to “join the dots” in a graph

CS	37
Electronics	44
Maths	10
English	9



axis labels in any order

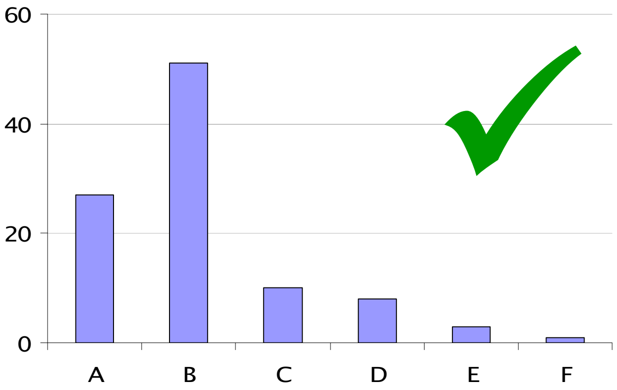


this *makes no sense* !!

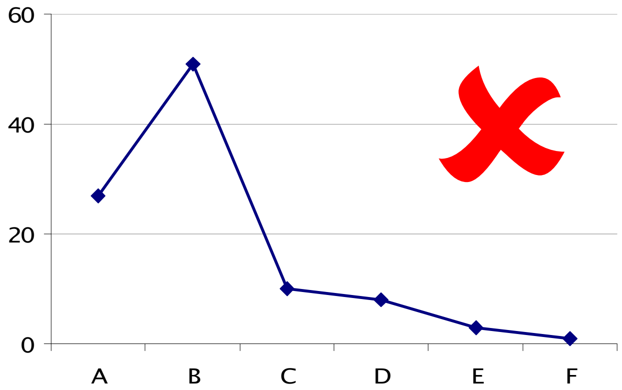
ordinal / ranked data

- data can be ranked: $a < b < c$
  - very poor < poor < satisfactory < good < very good < excellent ; Unclassified < Confidential < Secret < Top Secret ; …
- operations: comparison of ranks
  - numbers (integers) are often used to ***encode*** rankings, but it ***makes no sense*** to "calculate" with these numbers
    - eg, don't add them to find a mean, or "join the dots" in a graph
    - the encoding chosen could ***just as well*** be letters
- statistics: **median**, **quartiles** (spread)

A	27
B	51
C	10
D	8
E	3
F	1



axis labels in rank order



this *makes no sense* !!

numerical data

- interval data
  - numerical, with an ***arbitrary zero*** point
    - eg, temperature in ^oC ; dates ; …
    - be careful with arbitrary zero points:
      - "the temperature rose by 1 ^oC (33 ^oF)" [newspaper report]
  - operations: comparison of values, arithmetic: a + b, a – b
  - statistics: **mean**, **standard deviation** (spread), **skew**, ...

- ratio data
  - numerical, with an ***absolute zero*** point
    - eg, temperature in K ; length ; mass ; lifetime ; …
  - operations: comparison of values; arithmetic : $a \pm b$, a / b
  - statistics: **mean**, **standard deviation** (spread), **skew**, ...

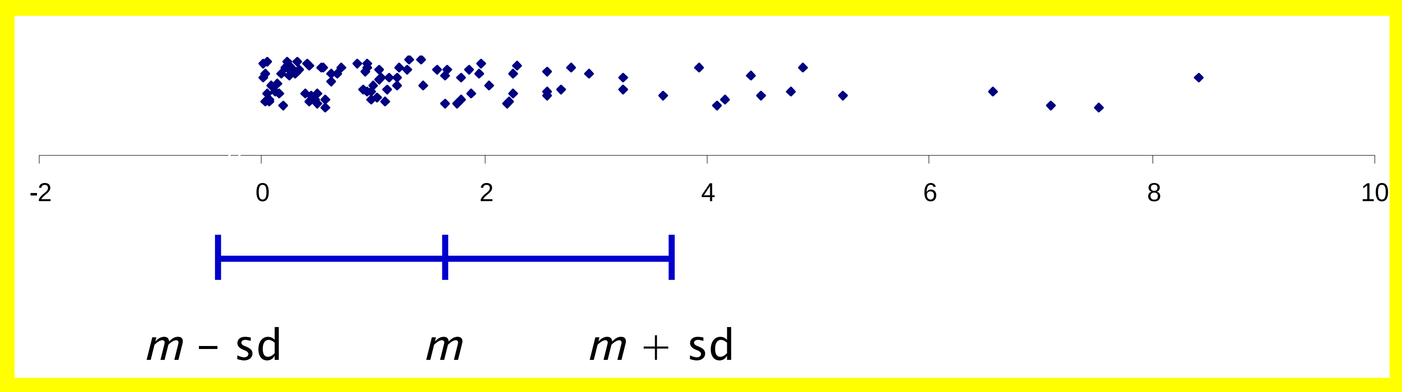
mean / standard deviation

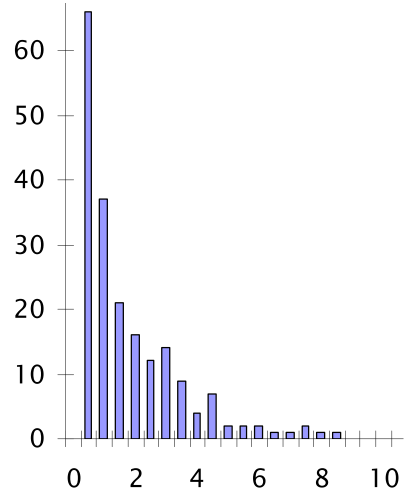
- sample mean : $$m = \frac{1}{n}\sum_{i=1}^{n} x_i$$
  - item with average ***value***
    - a measure of centrality
  - mean {**-30**, 1, 2, 3, 4} = -4 ; mean {**0**, 1, 2, 3, 4} = 2

- sample standard deviation $$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(m - x_i\right)^2}$$
    - note the $n - 1$
    - because derived from the estimated ***sample*** mean
      - it would be *n* if you could use the *population* mean
        - but you rarely know what that is
  - a measure of the spread of ***values***
  - sd {**-30**, 1, 2, 3, 4} = 14.6 ; sd {**0**, 1, 2, 3, 4} = 1.6

mean / standard deviation : example

- example: **positive** variable: mean = 1.69; s.d. = 2.14





- *majority* of the data is < m ; no recognition of skew
  - more than half the population has less than the mean value
    - more than half the population is "below average"!
      - "most people have more than the average number of legs"
- worse, m – sd is ***outside the possible range of the data*** !!

median / quartiles : nonparametric

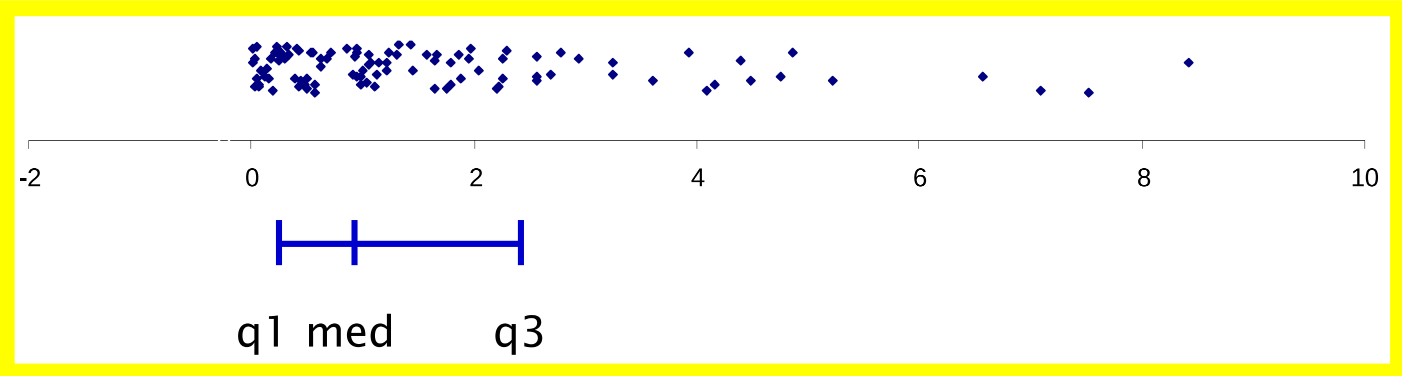
- median : item with average ***rank***
  - a **nonparametric** measure of ***centrality***

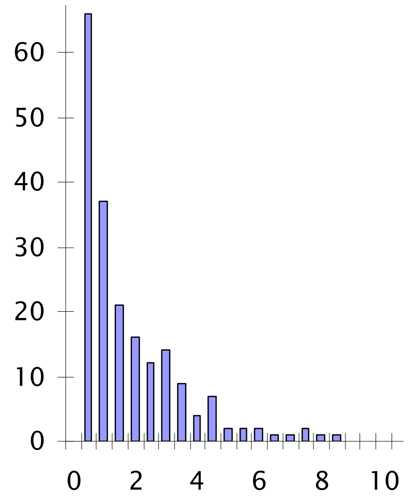
- rank the items in order, and pick the middle one
  - median {-30, 1, **2**, 3, 4} = 2 ; median {0, 1, **2**, 3, 4} = 2
    - less affected by outliers (rank, not value, is what's important)

- quartiles (25^{th} and 75^{th} percentiles) are a **nonparametric** measure of ***spread***
  - difference between quartiles can indicate **skew**
  - median = **50^{th} percentile**

median / quartiles : example

- example (same data as before): median = 0.98, 1st q = 0.34, 3rd q = 2.32

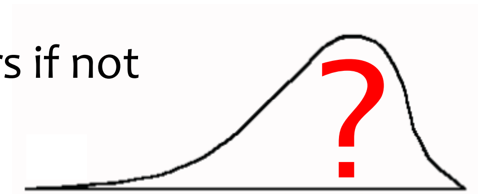


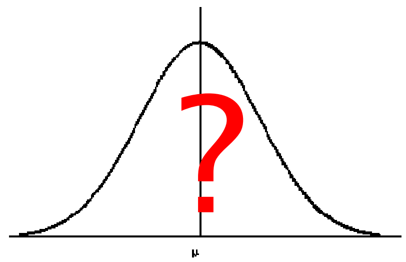


- *half* of the data is < median (by definition)
- quartiles here also indicate skew
  - and quartiles are ***inside*** the data range

normal distributions aren't normal

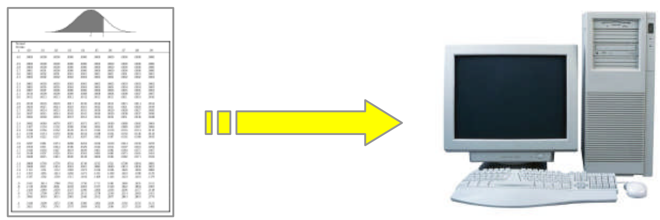
- even with ***numerical*** data, it often makes more sense to use median/quartiles than mean/sd
- non-normal distributions are affected by "outliers"
- long "tails" dominate the mean, and make the standard deviation misleading
- it is (usually **incorrectly**!) assumed that the underlying distribution is normal
  - most of the distributions we come across are not normal
  - most well-known statistical tests ***require*** normal distribution for them to work
    - and can give ***very*** wrong answers if not
  - **nonparametric** ones do not





why assume normality?

- normal distributions *are* relatively common
    - Central Limit Theorem
  - but : long tailed distributions, power laws, skewed data, etc, are ***also*** common
- statistics was an area of study well before computers
  - some very elegant analytic results for normal distributions
    - very little you can do analytically without *some* assumptions
      - there are *some* results that are *independent* of the distribution
  - calculate with tabulated results
    - table depends on distribution



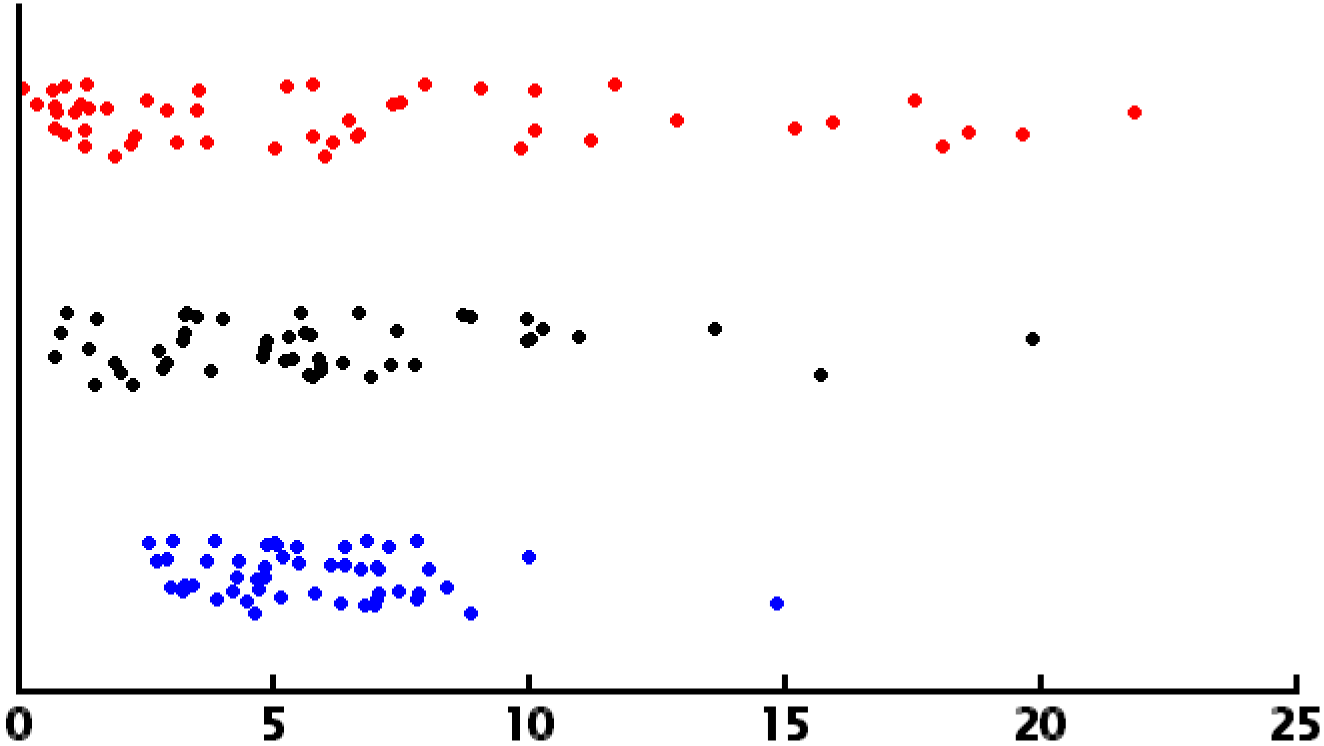
- we now have computers!
  - analyse the ***actual* distributions** rather than ***approximations***

the rise of nonparametric statistics

- nonparametric tests should be your ***default choice***
  - if not, you have to ***demonstrate*** that your distribution is normal
    - there are statistical tests…
- technically "less powerful"
  - you need more samples to get a significance level
    - but it's not *that* much more
      - eg 100 for the rank-sum test *v* 95 for the *t*-test [Siegel 1988]
    - and the criticism is only relevant for normal distributions
  - **not a problem** with typical computer simulation sample sizes!

testing samples

- these three samples are all drawn from ***different*** populations

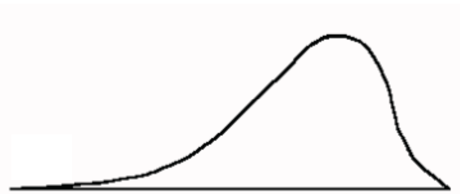


  - how might you tell?

are two distributions different?

- null Hypothesis H_0 : samples X and Y are drawn from the *same* population
  - that is, they have the same statistical properties: same medians, same quartiles, …

- **WARNING!** most statistical tests that you might come across (eg, various *t*-tests) ***require*** that the underlying distribution be normal for them to work
  - but it usually isn't !



- when the underlying distribution is non-normal, or even unknown, use **nonparametric** tests
  - the default choice, as they make fewest assumptions

are two medians different?

- use the nonparametric **rank-sum** test (aka Wilcoxon test, aka Mann-Whitney U test) to calculate the probability ***p*** that two samples are drawn from populations with the same ***medians***
  - H_0 : samples X and Y have the same medians
  - if they don't, then their distributions are different

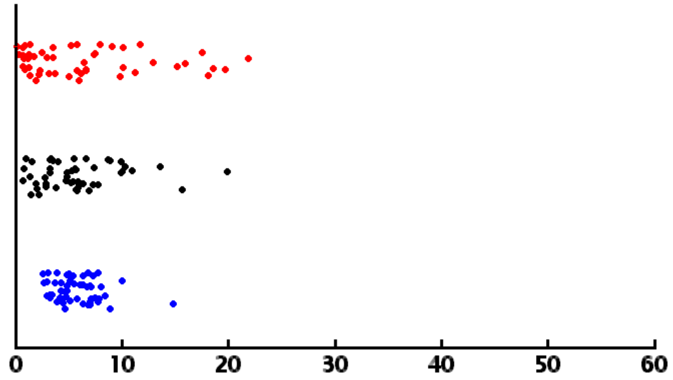
- calculate in Matlab:

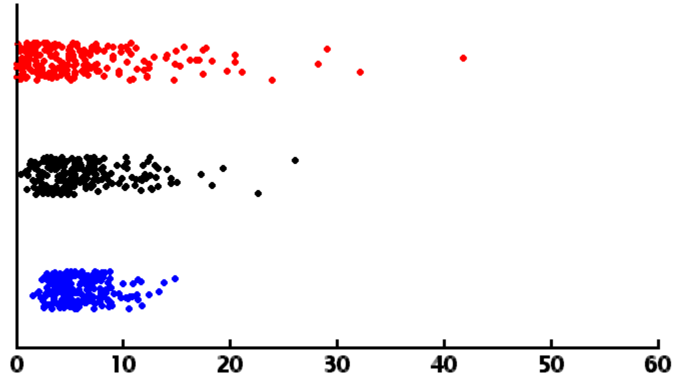
```
p = ranksum(X,Y,'alpha',0.05);
```

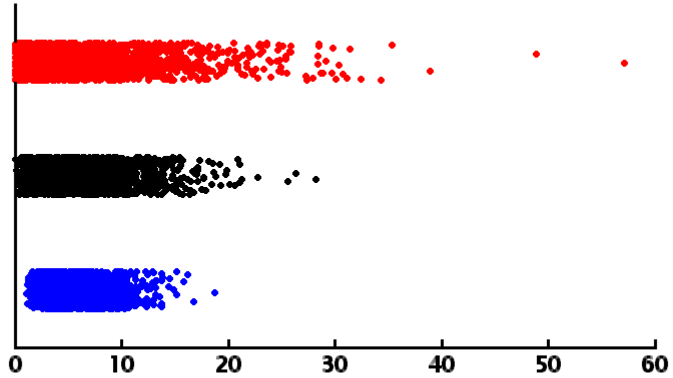
- if ***p* < 0.05**, can ***reject*** H_0 at the 95% confidence level
  - because the medians are different
  - remember, if ***p* > 0.05**, this **does not mean** that we *accept* H_0

example, rank-sum test (diff)

N = 50	rank sum *p*
blue/black	0.395
black/red	0.697
blue/red	0.316
N = 200	
blue/black	0.191
black/red	**0.030**
blue/red	**0.002**
N = 2000	
blue/black	**0.000**
black/red	**0.000**
blue/red	**0.000**







are two distributions different?

- if we can't demonstrate that the ***medians*** are different, use the nonparametric **Kolmogorov-Smirnoff** test (aka KS test) to calculate the probability ***p*** that two samples are drawn from populations with the same distributions
    - H_0 : samples X and Y have the same distributions

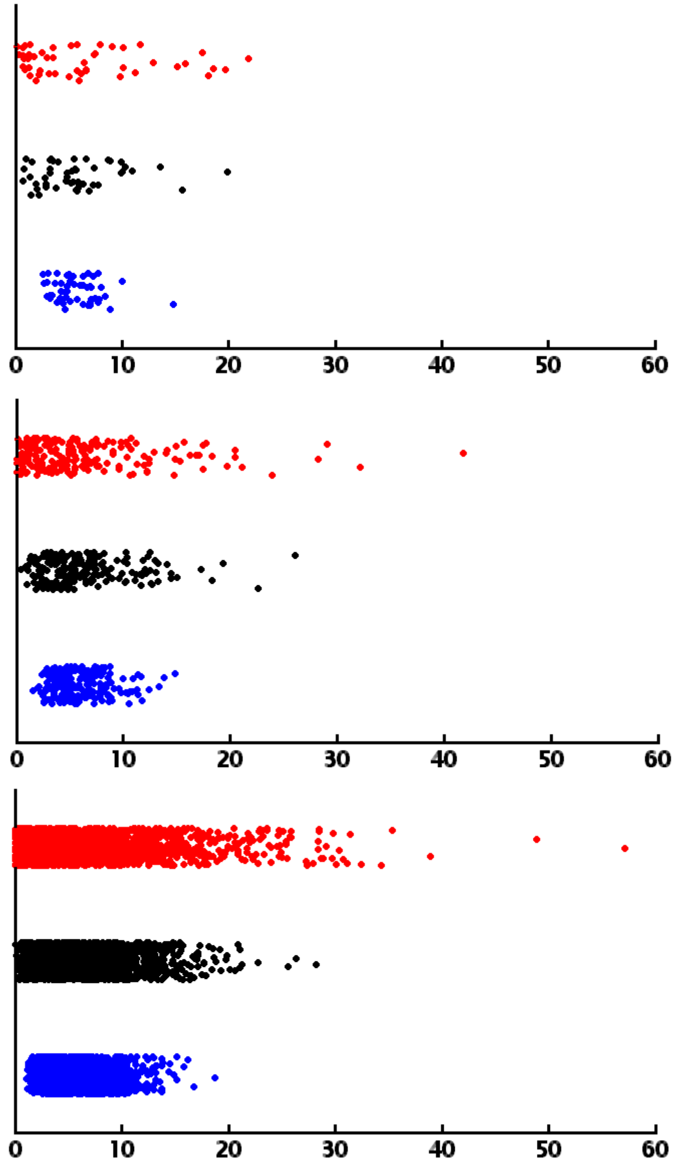
- calculate in Matlab:

```
p = ktest2(X,Y,'alpha',0.05);
```

- if p < 0.05, can ***reject*** H_0 at the 95% confidence level
    - because the distributions are different in some way (maybe medians, maybe quartiles, maybe some other way, …)

example, KS test (diff)

N = 50	KS ***p***
blue/black	0.241
black/red	0.155
blue/red	**0.001**
N = 200	
blue/black	**0.008**
black/red	**0.001**
blue/red	**0.000**
N = 2000	
blue/black	**0.000**
black/red	**0.000**
blue/red	**0.000**



going further

- the rank-sum test and KS test are used to test if two samples have different distributions
  - eg, whether the results from lots of runs of one algorithm are statistically significantly different from the results from lots of runs of an alternative algorithm
- there are other tests available for other situations
  - particularly **paired data** tests, eg, sets of (before, after) data, to test whether a change has had a statistically significant effect
- use a test that is appropriate for your null hypothesis
- decide your null hypothesis and how you are going to test it ***before*** you generate the data
  - to make sure you generate the right kind of data!

the null hypothesis revisited

- "there is really no good reason to expect the null hypothesis to be true in any population"

  [Bakan 1967]

- if your algorithm *is* different from another, you can almost certainly detect this difference, *if you try hard enough*
  - the bigger the sample size, the better the statistical significance
  - mice are expensive; computers are cheap!
    - statistical tests are devised to demonstrate differences using *small* sample sizes (tens or less)
    - computer simulations routinely use **enormous** sample sizes (thousands...)

"significant" ≠ "important"

- the bigger the sample size, the better the statistical significance – that must be good?
  - not necessarily: can nearly ***always*** get a ***statistically significant*** result just by having a big enough sample size !
    - with *enough* samples, can distinguish two distributions …
  - but it might not be an ***important*** difference
    - … but the two distributions might still be very *very* similar …

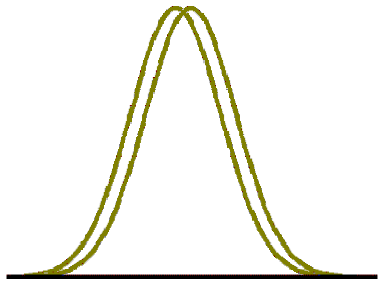
  - the old algorithm has a success rate of 52.38%
  - whereas *my* algorithm's success rate is 52.41%
    - with improvement significant at the **99.9%** confidence level
    - so why aren't you impressed by my result?
    - … because it's (probably) a very small ***effect***
  - happens easily when experimental runs are "cheap"

effect size

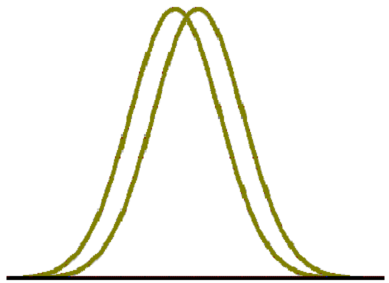
- measure of ***importance***
  (“scientific significance”)

  a small effect

  - the data’s *spread* is very much bigger than the difference in the medians
  - any “improvement” in the median is washed out by the data’s spread

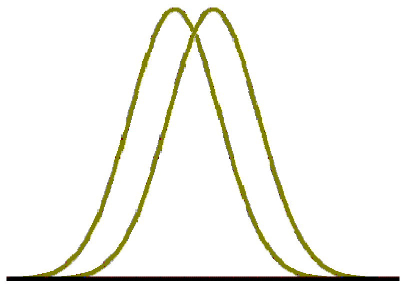


  a medium effect



  a big effect

  - *may* be worth getting excited about



effect size : “A” measure

[Vargha & Delaney 2000]

- calculate in Matlab:

```
[p,h,st] = ranksum(X,Y,'alpha',0.05);
N = size(X,1);
M = size(Y,1);

A = (st.ranksum/N – (N+1)/2)/M;
```

  - notice that you can do this and the rank-sum test *in one go* !

- measure of ***importance*** (“***scientific*** significance”)
  - A lies between 0 and 1; if $A < 0.5$, use $1 - A$ in the test

0.5 : no effect (same medians)

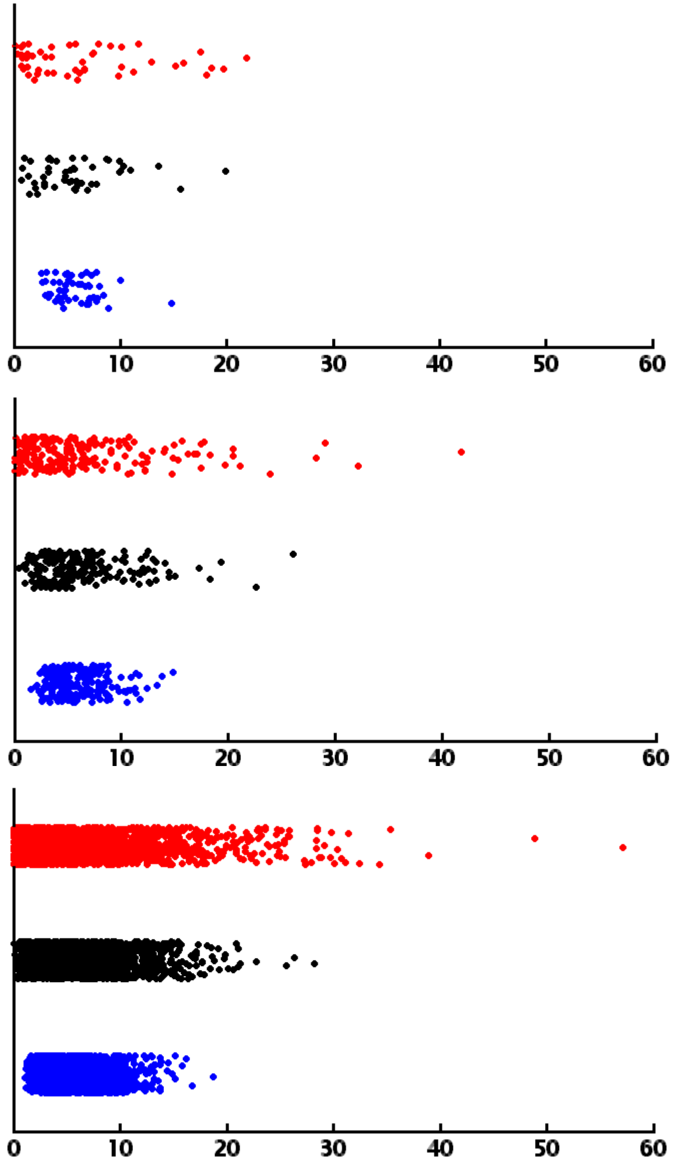
0.56 : a small effect

0.64 : a medium effect

0.71 : a big effect

more significant, but always *small* effect

N = 50	rank sum	KS ***p***	*A*
blue/black	0.395	0.241	0.550
black/red	0.697	0.155	0.523
blue/red	0.316	**0.001**	0.558
N = 200			
blue/black	0.191	**0.008**	0.538
black/red	**0.030**	**0.001**	0.563
blue/red	**0.002**	**0.000**	0.591
N= 2000			
blue/black	**0.000**	**0.000**	0.546
black/red	**0.000**	**0.000**	0.570
blue/red	**0.000**	**0.000**	0.607



Graphing your Data

is your picture worth a kilo-word?

CS	37
Electronics	44
Maths	10
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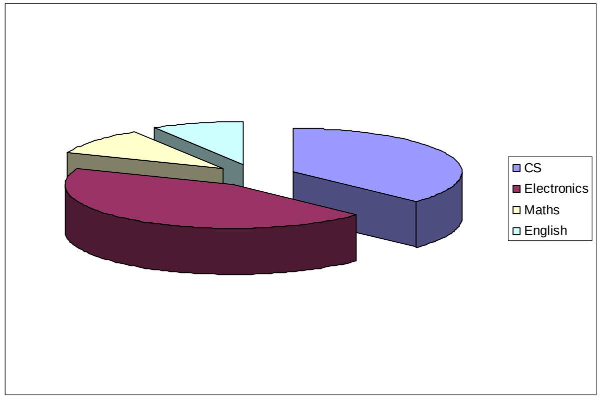
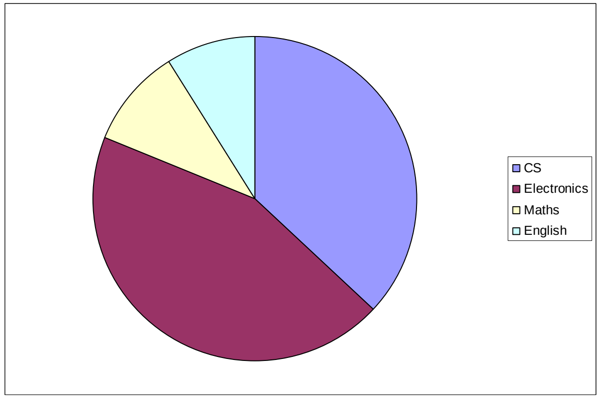
best !

for *this* data

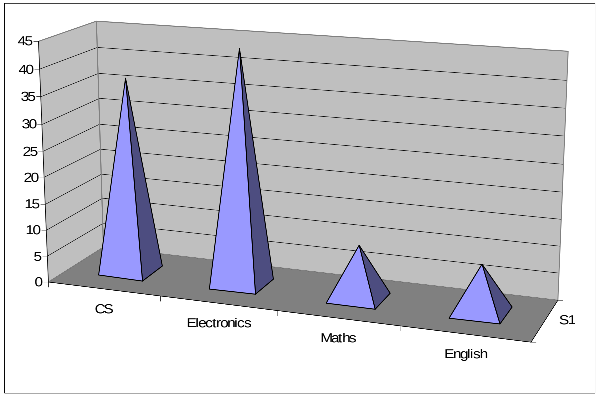
(get more data?)



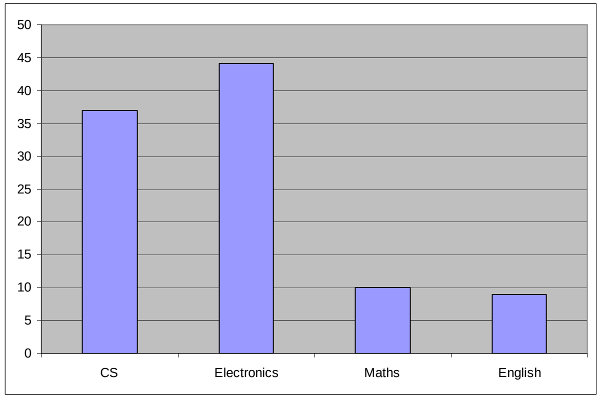
Chartjunk



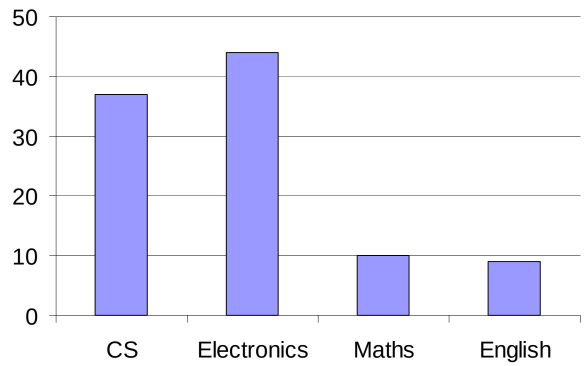
small distinctions obscured



More Chartjunk



Cluttered

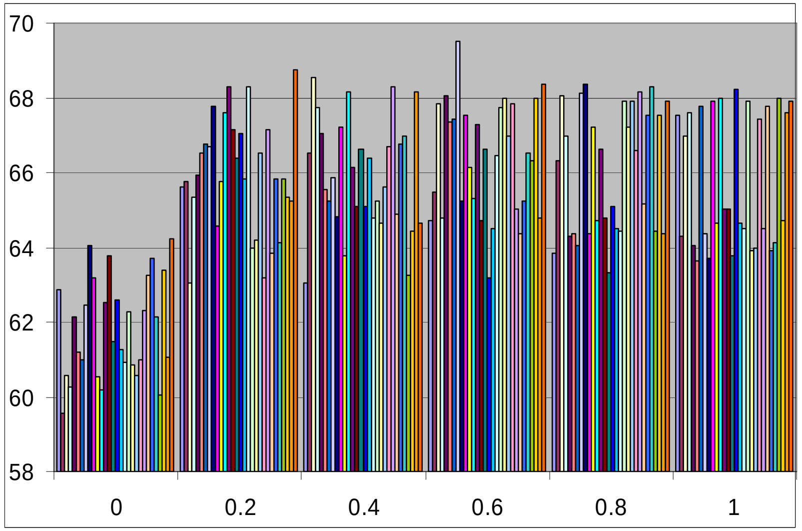


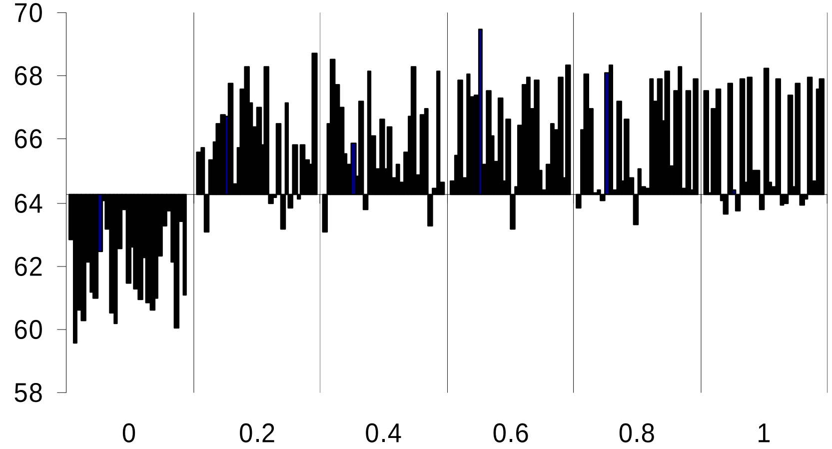
better

[Tufte. *The Visual Display of Quantitative Information*. Graphics Press, 1983]

plot your data to expose its structure

0.00	0.20	0.40	0.60	0.80	1.00
62.86	65.61	63.06	64.70	63.85	67.54
59.58	65.74	66.51	65.48	66.32	64.31
60.59	63.06	68.55	67.86	68.05	66.96
60.26	65.35	67.73	64.77	66.96	67.60
62.13	65.93	67.03	68.05	64.31	64.05
61.20	66.51	65.55	67.35	64.38	63.65
61.00	66.77	65.22	67.41	64.05	67.79
62.46	66.71	65.87	69.50	68.11	64.38
64.05	67.79	64.83	65.22	68.36	63.72
63.19	64.57	67.22	67.54	64.38	67.92
60.53	65.74	63.79	66.13	67.22	64.64
60.19	67.60	68.17	65.29	64.70	67.98
62.53	68.30	66.13	67.28	66.64	65.03
63.79	67.16	65.09	64.70	64.77	65.03
61.47	66.39	66.64	66.64	63.32	63.79
62.60	67.03	65.09	63.19	65.09	68.24
61.26	65.81	66.39	64.51	64.51	64.64
60.93	68.30	64.77	66.45	64.44	64.51
62.27	63.98	65.22	67.73	67.92	67.92
60.86	64.18	64.64	67.98	67.22	63.92
60.59	66.51	65.61	66.96	67.92	63.98
61.00	63.19	66.71	67.86	66.58	67.41
62.33	67.16	68.30	65.03	68.17	64.51
63.26	63.85	64.90	64.38	65.16	67.79
63.72	65.81	66.77	65.22	67.54	63.92
62.13	64.11	66.96	66.51	68.30	64.11
60.05	65.81	63.26	66.32	64.44	67.98
63.39	65.35	64.44	67.98	67.54	64.70
61.06	65.22	68.17	64.77	64.38	67.60
64.24	68.74	64.64	68.36	67.92	67.92

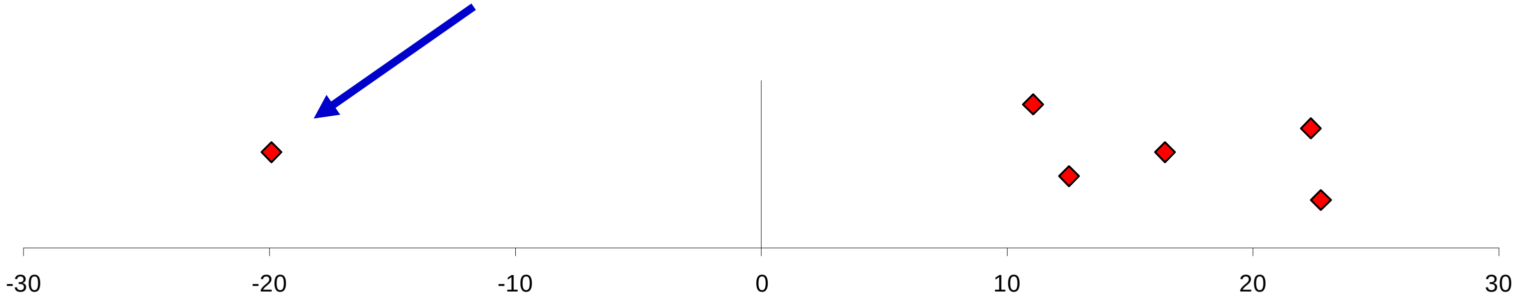




[Clark, Jacob, Stepney. Secret Agents Leave Big Footprints. *GECCO 2003*]

plot all your data, to see outliers

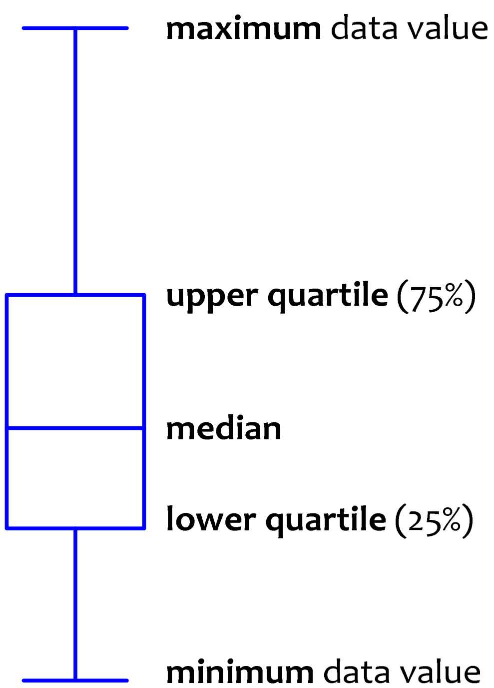
- you have some “anomalous” data



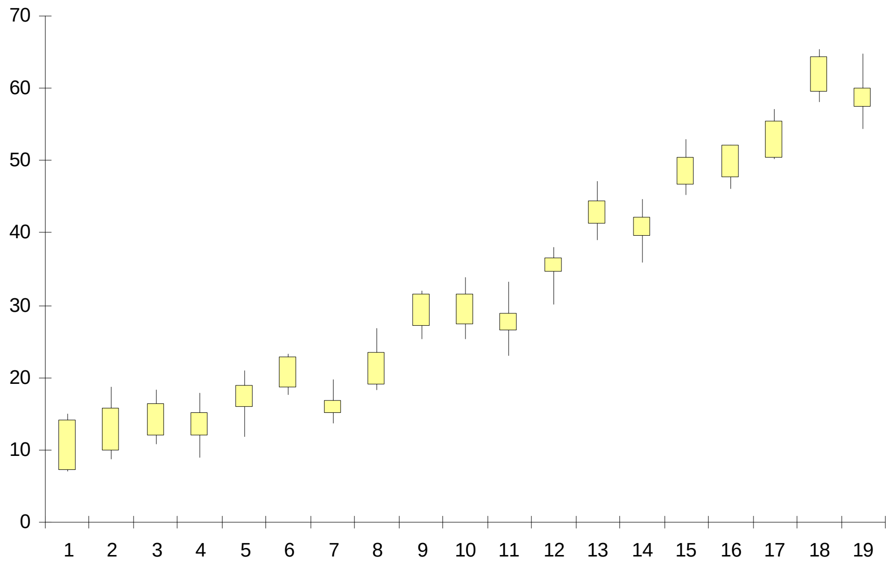
- don’t just discard it as an “outlier” – *understand* it!
  - is it just a statistical fluctuation?
    - a once-in-a-blue-moon “six sigma” outlier?
  - is it an error in the experimental design or implementation?
    - fix the problem, and rerun *all* the experiments
  - is it in interesting unexpected effect?
    - investigate it further!
    - it *might* be the basis of a new discovery

"box-and-whisker" plots

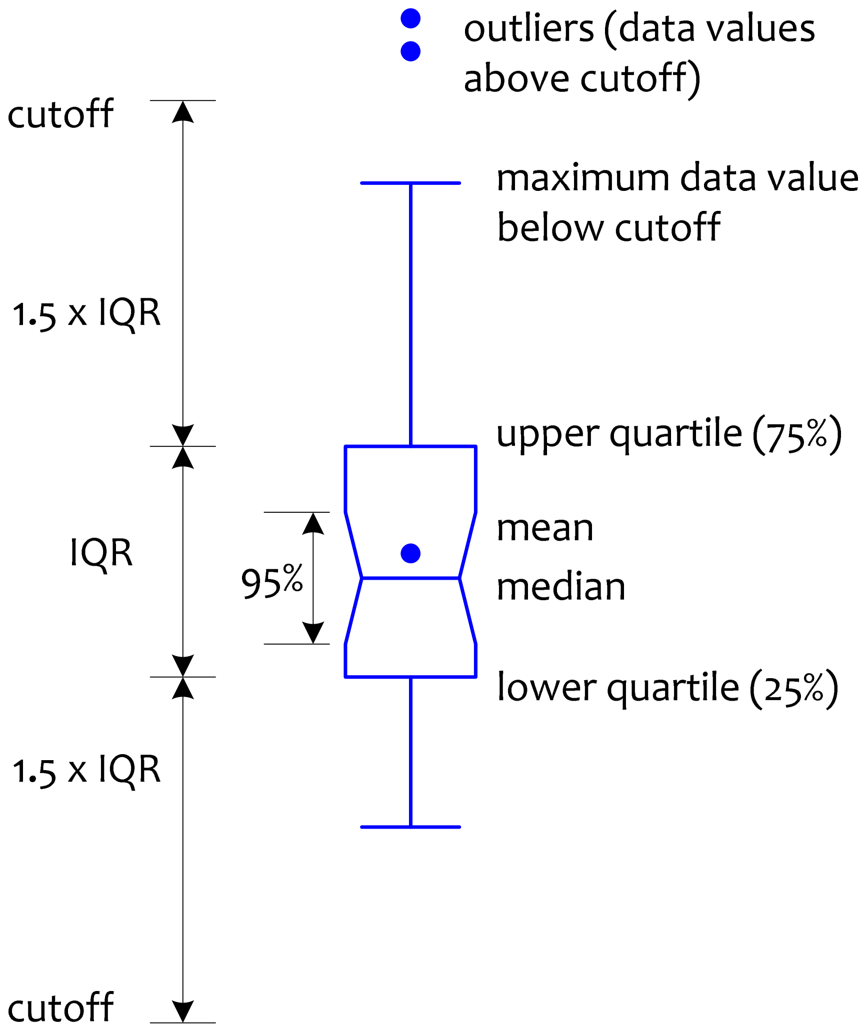
- median, quartiles
  - John W. Tukey. *Exploratory data analysis.* Addison Wesley, 1977



if you *want* to use Excel, use
Chart Wizard > Standard Types > Stock:



“deluxe” box-and-whisker plots



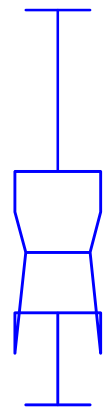
schematic plot: use a “cutoff” to highlight outliers (for ***numerical*** data)

plot *mean*, to highlight skew

“notches” at

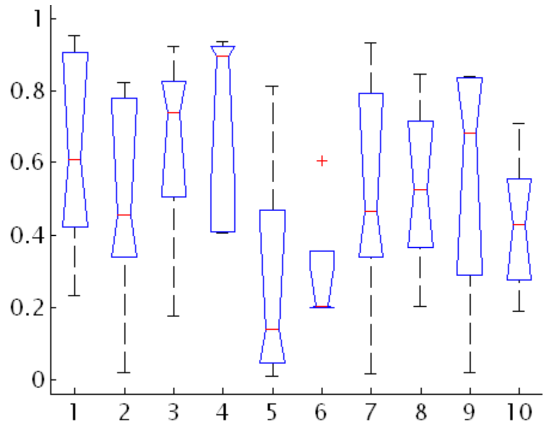
median $\pm$ 1.58(IQR / $\sqrt{n}$)

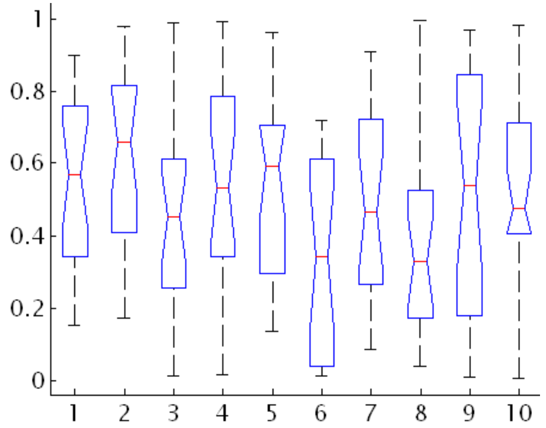
- if notches on separate bars do not overlap, ~ 95% confidence the medians are different
    - but do a proper test to check this
- small samples may have “folded back” notches outside the IQR:

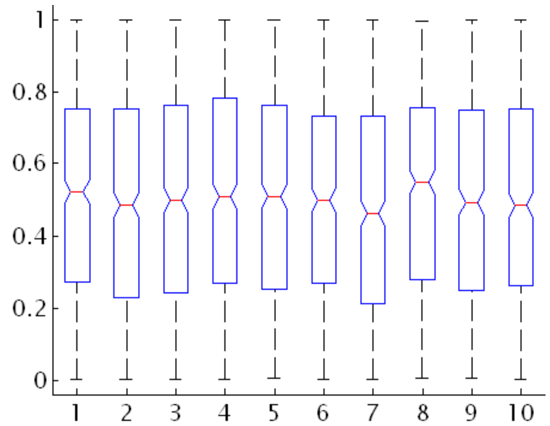


box plot examples

- uniform distribution, 10 samples
    - true median = 0.5, true IQR = 0.25-0.75
  - each of 5 elements, 20 elements, 500 elements
    - notice how the notches get smaller







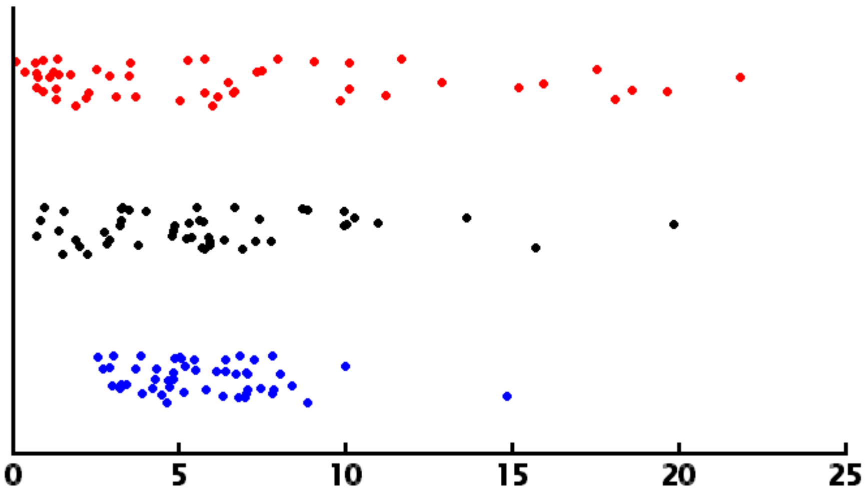

```
x = rand(5,10);
boxplot(x,'notch','on');
```

```
x = rand(20,10);
boxplot(x,'notch','on');
```

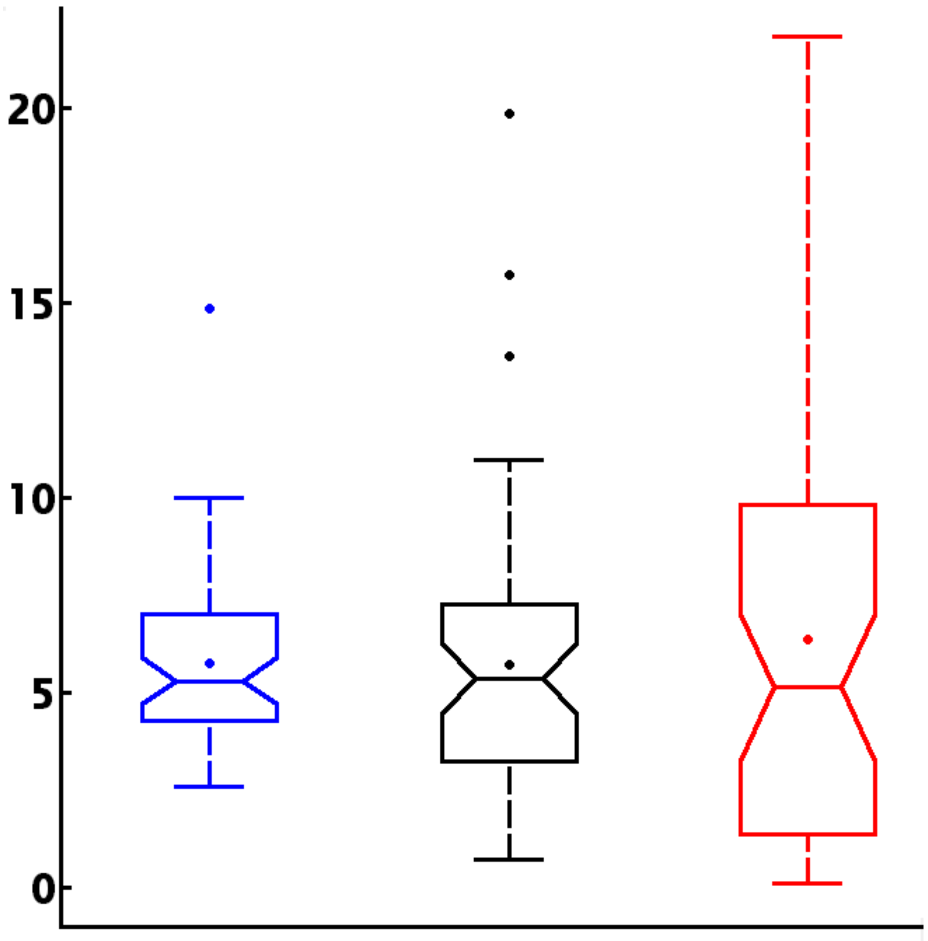
```
x = rand(500,10);
boxplot(x,'notch','on');
```

(the *entire* Matlab code that generated the random numbers, and drew the boxplot)

box plot : running example



N = 50	rank-sum	KS *p*	*A*
blue/black	0.395	0.241	0.550
black/red	0.697	0.155	0.523
blue/red	0.316	**0.001**	0.558



summary : statistics

1. present the ***data spread*** (s.d. / quartiles)
   - ***not*** **just an average** (mean / median)
2. plot your data
   - *with* **error bars / whiskers** ; *without* **chartjunk**
3. use the ***rank-sum test*** to calculate ***confidence levels for statistical significance***
   - or the KS test, if the medians are the same
4. calculate ***effect size*** (scientific significance) as well
   - and don't get excited unless the effect is ***large***

- *lots* more to experimental design, statistics and graphs