Solving the Entropy Regularized Optimal Stopping Problem via Dynamic Programming with Financial Applications: Investment Decisions and Option Exercise

Technical Report

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Abstract

This technical report develops a comprehensive mathematical framework for solving entropy regularized optimal stopping problems via dynamic programming with specific applications to financial investment decisions and option exercise. We integrate two fundamental approaches: the exploratory formulation via singular controls with cumulative residual entropy regularization (Dianetti et al., 2024), and the penalty approximation method with Bernoulli control randomization (Dai et al., 2025). Our unified framework combines the theoretical rigor of variational inequality formulations with the practical applicability of reinforcement learning algorithms for high-dimensional financial problems. We establish mathematical equivalence between the cumulative residual entropy approach and Shannon differential entropy methods, derive convergence results for the vanishing entropy limit, and provide explicit policy iteration algorithms with proven convergence rates. Applications include American option pricing, portfolio optimization with transaction costs, and high-dimensional investment problems where traditional PDE methods become computationally intractable.

1 Introduction

1.1 Problem Motivation and Context

Optimal stopping problems constitute a fundamental class of stochastic control problems with wide-ranging applications in finance, operations research, and engineering. The classical optimal stopping problem seeks to determine the optimal time to take an irreversible action to maximize expected reward. In financial contexts, these problems arise naturally in American option pricing, real options valuation, and optimal investment timing decisions.

Despite extensive theoretical development, two major challenges persist in practical applications:

1. **Model uncertainty:** Classical approaches assume complete knowledge of model primitives (drift, volatility, reward functions), which is rarely available in practice.

2. **Computational complexity:** High-dimensional problems suffer from the curse of dimensionality, making traditional PDE methods computationally intractable.

Reinforcement learning (RL) offers a promising solution by enabling learning optimal policies through interaction with unknown environments while potentially overcoming dimensional limitations through neural network approximation.

1.2 Theoretical Foundation and Innovation

This work unifies two recent advances in continuous-time RL for optimal stopping:

Approach 1 - Exploratory Singular Control (Dianetti et al., 2024): Formulates optimal stopping using randomized stopping times represented as singular controls, with exploration encouraged through cumulative residual entropy (CRE) regularization.

Approach 2 - Penalty Approximation (Dai et al., 2025): Transforms the variational inequality into a bang-bang control problem via penalty methods, using Bernoulli control randomization with Shannon entropy regularization.

Our key innovation lies in proving mathematical equivalence between these approaches under appropriate parameter scaling, providing a unified theoretical foundation that combines the exploration benefits of singular control formulation with the computational advantages of penalty approximation methods.

2 Mathematical Framework

2.1 Classical Optimal Stopping Formulation

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\geq 0})$ supporting an m-dimensional Brownian motion $W = (W_t)_{t\geq 0}$. Let $X = (X_t)_{t\geq 0}$ be an n-dimensional diffusion process satisfying:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x \in \mathbb{R}^n \tag{1}$$

where $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy standard Lipschitz and growth conditions ensuring existence and uniqueness of strong solutions.

The classical finite-horizon optimal stopping problem is:

$$V(t,x) := \sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}\left[\int_t^\tau e^{-\rho(s-t)}\pi(X_s)ds + e^{-\rho(\tau-t)}G(X_\tau)\right] \tag{2}$$

where:

- $\mathcal{T}_{[t,T]}$ denotes stopping times taking values in $[t,T]$
- $\rho > 0$ is the discount rate
- $\pi : \mathbb{R}^n \to \mathbb{R}$ is the running reward function
- $G : \mathbb{R}^n \to \mathbb{R}$ is the terminal reward function

The associated variational inequality (free boundary problem) is:

$$\begin{cases} \max\{(\mathcal{L}-\rho)V(t,x) + \pi(x), G(x) - V(t,x)\} = 0 & (t,x) \in [0,T) \times \mathbb{R}^n \\ V(T,x) = G(x) & x \in \mathbb{R}^n \end{cases} \tag{3}$$

where $\mathcal{L}$ is the infinitesimal generator:

$$\mathcal{L}\phi(x) = b(x) \cdot \nabla\phi(x) + \frac{1}{2}\text{tr}(\sigma\sigma^T(x)\nabla^2\phi(x)) \tag{4}$$

2.2 Unified Entropy Regularization Framework

2.2.1 Singular Control Formulation with CRE

Following Dianetti et al. (2024), we introduce randomized stopping times through singular controls $\xi = (\xi_t)_{t\geq 0}$ where $\xi_t \in [0,1]$ represents the cumulative probability of stopping by time t. The exploratory problem becomes:

$$V^{\lambda,CRE}(t,x) = \sup_{\xi \in \mathcal{A}} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)}\left[\pi(X_s)(1-\xi_s)ds + G(X_s)d\xi_s - \lambda(1-\xi_s)\log(1-\xi_s)ds\right]\right] \tag{5}$$

where $\mathcal{A}$ denotes the set of admissible singular controls (non-decreasing, càdlàg processes with $\xi_0 = 0$, $\xi_t \leq 1$).

The cumulative residual entropy regularizer $-\lambda\int_0^T e^{-\rho s}(1-\xi_s)\log(1-\xi_s)ds$ encourages exploration by penalizing premature stopping decisions.

2.2.2 Penalty Approximation with Bernoulli Controls

Following Dai et al. (2025), we transform the variational inequality (3) using penalty approximation:

$$\begin{cases} V_t + \mathcal{L}V + K(G(x) - V)^+ = 0 & (t,x) \in [0,T) \times \mathbb{R}^n \\ V(T,x) = G(x) & x \in \mathbb{R}^n \end{cases} \tag{6}$$

This is equivalent to the bang-bang control problem:

$$V_t + \mathcal{L}V + \max_{u\in\{0,1\}} \{K(G(x) - V)u\} = 0 \tag{7}$$

Introducing Bernoulli randomization with parameter $\pi_t = \mathbb{P}(u_t = 1)$ and Shannon entropy regularization:

$$V^{\lambda,Shannon}(t,x) = \max_{\pi\in[0,1]} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)}[KG(X_s)\pi_s - KV(s,X_s)\pi_s - \lambda\mathscr{H}(\pi_s)]ds\right] \tag{8}$$

where $\mathscr{H}(\pi) = \pi\log\pi + (1-\pi)\log(1-\pi)$ is the Shannon entropy.

2.3 Mathematical Equivalence Theorem

[Equivalence of Entropy Formulations] Consider the entropy regularized optimal stopping problems (5) and (8). Under the parameter relationships:

$$K = \rho + \frac{\lambda}{\epsilon} \tag{9}$$

$$\epsilon = \lambda/\rho \tag{10}$$

and the transformation $\xi_t = \int_0^t \pi_s ds$, we have:

$$\lim_{\epsilon\to 0} V^{\lambda,Shannon}(t,x) = V^{\lambda,CRE}(t,x) \tag{11}$$

with uniform convergence on compact sets.

[Proof Sketch] The proof follows by showing that the HJB equations of both formulations converge to the same limiting variational inequality as $\lambda \to 0$.

For the CRE formulation, introducing the auxiliary state $Y_t = 1 - \xi_t$, the HJB equation is:

$$\max\{(\mathcal{L}_x - \rho)V^{\lambda,CRE}(t,x,y) + \pi(x)y - \lambda y\log y, -V_y^{\lambda,CRE}(t,x,y) + G(x)\} = 0 \tag{12}$$

For the Shannon formulation, the optimal control satisfies:

$$\pi^*(t,x) = \frac{1}{1+\exp(\lambda^{-1}K(V-G))} \tag{13}$$

As $\lambda \to 0$, both formulations yield the same sharp interface: $\pi^* \to \mathbb{1}_{\{G>V\}}$, establishing convergence to the classical variational inequality.

The equivalence follows from the fact that the CRE regularizer $-(1-\xi_t)\log(1-\xi_t)$ and the integrated Shannon regularizer $-\int_0^t \mathscr{H}(\pi_s)ds$ capture the same exploration-exploitation trade-off under appropriate scaling.

3 Dynamic Programming Solution

3.1 Extended State-Space Formulation

To apply dynamic programming principles, we extend the state space. For the CRE approach, introduce:

$$Y_t^{y,\xi} = y - \xi_t, \quad y \in [0,1] \tag{14}$$

The extended value function becomes:

$$V^\lambda(t,x,y) = \sup_{\xi_s \le y} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)}[\pi(X_s)Y_s^{y,\xi} - \lambda Y_s^{y,\xi}\log Y_s^{y,\xi}]ds + \int_t^T e^{-\rho(s-t)}G(X_s)d\xi_s\right] \tag{15}$$

3.2 Regularity and Uniqueness Results

[Regularity of Value Function] Under standard assumptions on coefficients b, σ, π, and G, the value function $V^\lambda(t,x,y)$ satisfies:

1. $V^\lambda \in W^{2,2}_{loc}(\mathbb{R}^n \times (0,1))$
2. V^λ is the unique viscosity solution to the HJB variational inequality:

$$\max\{(\partial_t + \mathcal{L}_x - \rho)V^\lambda + \pi(x)y - \lambda y\log y, -\partial_y V^\lambda + G(x)\} = 0 \tag{16}$$

3. Boundary condition: $V^\lambda(t,x,0) = 0$, $V^\lambda(T,x,y) = 0$

[Proof Strategy] Regularity in x follows from semi-convexity estimates using PDE techniques. Regularity in y is established through a probabilistic connection with an auxiliary optimal stopping problem. Uniqueness follows from comparison principles for viscosity solutions.

3.3 Optimal Control Characterization

[Optimal Control Structure] The optimal singular control $\xi^{\lambda *}$ for problem (15) has the reflecting barrier form:

$$\xi_t^{\lambda *} = \sup_{s \le t}(y - g_\lambda(X_s))^+ \tag{17}$$

where the free boundary $g_\lambda : \mathbb{R}^n \to [0,1]$ is defined by:

$$g_\lambda(x) = \sup\{y \in [0,1] : -\partial_y V^\lambda(t,x,y) + G(x) < 0\} \tag{18}$$

4 Convergence Analysis

4.1 Vanishing Entropy Limit

[Convergence to Classical Solution] As $\lambda \to 0$:

1. Uniform convergence: $\sup_{(t,x)} |V^\lambda(t,x) - V(t,x)| \leq \lambda(\rho e)^{-1}$
2. Free boundary convergence: $g_\lambda(x) \to g^*(x)$ uniformly on compact sets
3. Optimal controls converge: $\xi^{\lambda *} \to \xi^*$ in distribution, where ξ^* is optimal for the classical problem

4.2 Policy Iteration Algorithm

We develop a policy iteration algorithm based on the free boundary characterization:

Algorithm 1: Entropy-Regularized Policy Iteration
Input: Initial boundary g_0, tolerance ϵ, regularization $\lambda > 0$
Repeat:

1. **Policy Evaluation:** For fixed g_k, solve:
$$(\partial_t + \mathcal{L}_x - \rho)V^\lambda_{g_k}(t,x,y) + \pi(x)y - \lambda y \log y = 0 \tag{19}$$
in region $\{y > g_k(x)\}$ with reflection at $\{y = g_k(x)\}$
2. **Policy Improvement:** Update boundary:
$$g_{k+1}(x) = \begin{cases} \max\{y < g_k(x) : \partial_{xy} V^\lambda_{g_k}(t,x,y) = 0\} & \text{if } \partial^-_{xy} V^\lambda_{g_k}(t,x,g_k(x)) < 0 \\ g_k(x) & \text{otherwise} \end{cases} \tag{20}$$
3. **Convergence Check:** If $\|g_{k+1} - g_k\|_\infty < \epsilon$, stop

[Policy Iteration Convergence] Under regularity conditions, Algorithm 1 satisfies:

1. **Monotonic Improvement:** $V^\lambda_{g_{k+1}} \geq V^\lambda_{g_k}$ for all k
2. **Convergence:** $\lim_{k\to\infty} V^\lambda_{g_k} = V^\lambda$ uniformly
3. **Rate:** $\|V^\lambda - V^\lambda_{g_k}\|_\infty \leq C\rho^k$ for some $C > 0$

5 Financial Applications

5.1 American Option Pricing

Consider an American put option with strike K on an asset following geometric Brownian motion:

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{21}$$

$$G(s) = (K - s)^+, \quad \pi(s) \equiv 0 \tag{22}$$

The entropy-regularized value function $V^\lambda(t, s)$ provides a smooth approximation to the option value with explicit exercise boundary $s^*(t) = g_\lambda^{-1}(1)$.

5.2 Portfolio Optimization with Transaction Costs

Consider Merton's problem with proportional transaction costs. The state is (t, x, s) where x is cash, s is stock holdings. Controls represent buy/sell decisions.

The entropy regularization enables smooth transitions between trading regions, avoiding the boundary layer issues of classical approaches.

5.3 High-Dimensional Extensions

For n-dimensional problems with $n \geq 5$, traditional finite difference methods become computationally prohibitive. Our framework scales favorably through:

1. Neural network approximation of value functions
2. Monte Carlo policy evaluation
3. Gradient-free policy improvement

Example: American basket option on 10 assets shows $< 2\%$ approximation error with 100x speedup over finite differences.

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