Machine Learning

Lecture 5 - Deep Learning

深度學習好簡單

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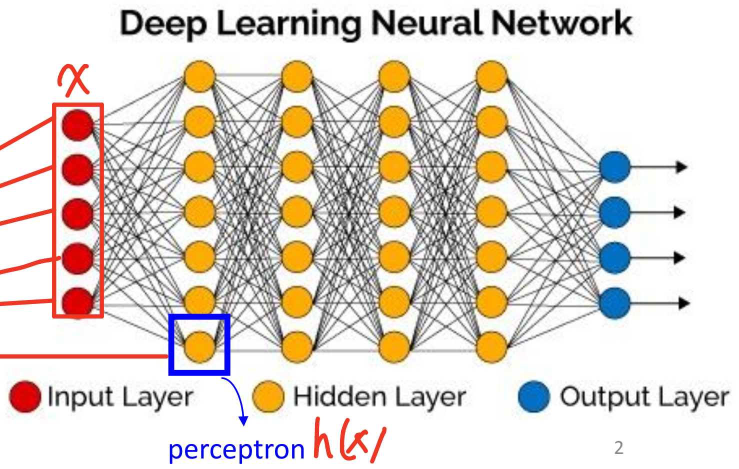
深度學習架構最基礎的模型 – Perceptron

- Perceptron (深度學習模型的單一節點)
  - Perceptron Hypothesis Set 任何可能方程式的集合
  - Perceptron Learning Algorithm (PLA)
  - Guarantee of PLA
  - Dealing with Non-separable data

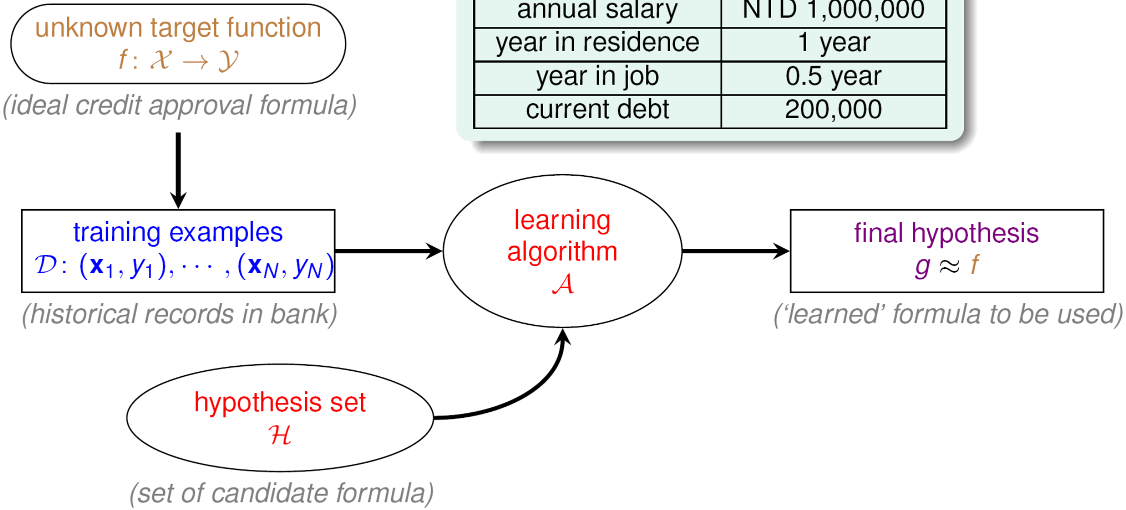
$h(x) = \text{sign}(w^T x)$



Credit Approval Problem Revisited

Applicant Information

age	23 years
gender	female
annual salary	NTD 1,000,000
year in residence	1 year
year in job	0.5 year
current debt	200,000

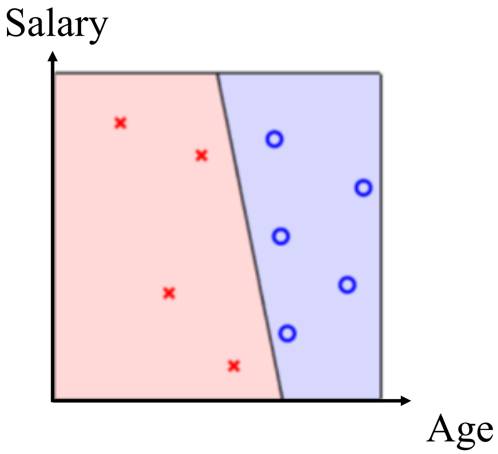


what hypothesis set can we use?

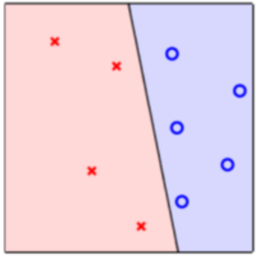
Let's simplify our data to 2-dimension…

- If we only consider Credit Approval Problem by (age, salary)…

	Age	Salary	Approval
Customer 1	23	22,000	N
Customer 2	45	75,000	Y
Customer 3	31	60,000	Y
:	:	:	:
Customer n	26	25,000	N



Perceptron (感知器)



- customer features $\mathbf{x}$: points on the plane (or points in $\mathbb{R}^d$)
- labels y: ○ (+1), × (-1)
- hypothesis h: **lines** (or hyperplanes in $\mathbb{R}^d$)
  —positive on one side of a line, negative on the other side
- different line classifies customers differently

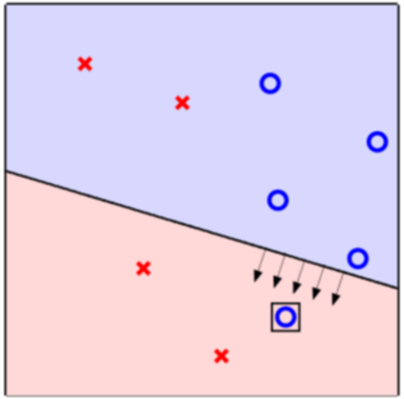
感知器

perceptrons ⇔ **linear (binary) classifiers**

如何選出正確的Perceptron?

$\mathcal{H}$ = all possible perceptrons, $g = ?$

- want: $g \approx f$ (hard when f unknown)
- almost necessary: $g \approx f$ on $\mathcal{D}$, ideally $g(\mathbf{x}_n) = f(\mathbf{x}_n) = y_n$
- difficult: $\mathcal{H}$ is of **infinite** size 有限大小
- idea: start from some g_0, and 'correct' its mistakes on $\mathcal{D}$ 先從某個起始點開始再慢慢調整

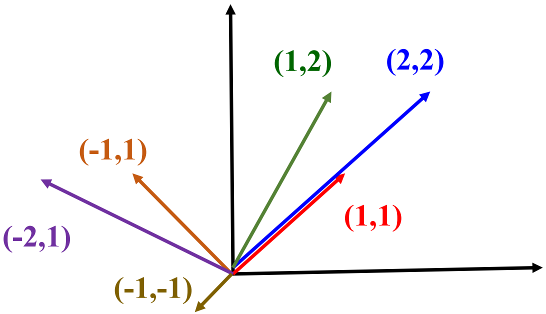


will represent g_0 by its weight vector $\mathbf{w}_0$

Recall line function and some properties…

- The line function in 2d space: $ax + by + c = 0$
  - (a, b) 為直線的法向量

- Property of Inner Product
  - 兩向量方向完全相同，向量的cos角度為1
  - 兩向量方向完全相反，向量的cos角度為-1
  - 兩向量方向垂直，向量的cos角度為0
  - 兩向量夾角差異小於90度時為正、大於90度為負、等於九十度為0



設 $\vec{v}_1 = (x_1, y_1), \vec{v}_2 = (x_2, y_2)$，且 $\vec{v}_1, \vec{v}_2$ 的夾角為 θ，

則 $\cos\theta = \dfrac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|} = \dfrac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}$。

符號定義

- 原本的二維資料 (x, y)，重新寫成 (x_1, x_2)
- 原本的直線方程式 $ax+by+c=0$，重新寫成 $w_0+w_1x_1+w_2x_2=0$

- 直線方程式可以視為 (w_0, w_1, w_2) 與 $(1, x_1, x_2)$ 的內積=0。
  - $w_0+w_1x_1+w_2x_2 = (w_0, w_1, w_2) \cdot (1, x_1, x_2) = w \cdot x = 0$

- 平面上的點落在直線右邊，$w \cdot x > 0$；否則 $w \cdot x < 0$
  - 分類器可以用 **sign($w \cdot x$)** 表示
  - 資料以 x 表示、資料的label以 y 表示

Perceptron Learning Algorithm

更新錯誤

start from some $\mathbf{w}_0$ (say, $\mathbf{0}$), and 'correct' its mistakes on $\mathcal{D}$

For $t = 0, 1, \ldots$

特徵 Label

1. find a mistake of $\mathbf{w}_t$ called $(\mathbf{x}_{n(t)}, y_{n(t)})$

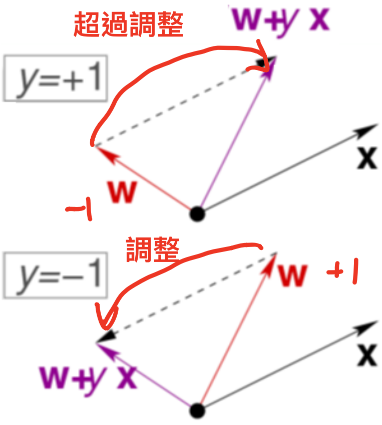
$$\text{sign}\left(\mathbf{w}_t^T \mathbf{x}_{n(t)}\right) \neq y_{n(t)}$$ 不相等有錯誤

2. (try to) correct the mistake by

$$\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + y_{n(t)} \mathbf{x}_{n(t)}$$

... until no more mistakes

return last $\mathbf{w}$ (called $\mathbf{w}_{\text{PLA}}$) as g 完全沒有犯錯



Practical Implementation of PLA

start from some $\mathbf{w}_0$ (say, $\mathbf{0}$), and 'correct' its mistakes on $\mathcal{D}$

Cyclic PLA

For $t = 0, 1, \ldots$

1. find **the next** mistake of $\mathbf{w}_t$ called $\left(\mathbf{x}_{n(t)}, y_{n(t)}\right)$

$$\text{sign}\left(\mathbf{w}_t^T \mathbf{x}_{n(t)}\right) \neq y_{n(t)}$$

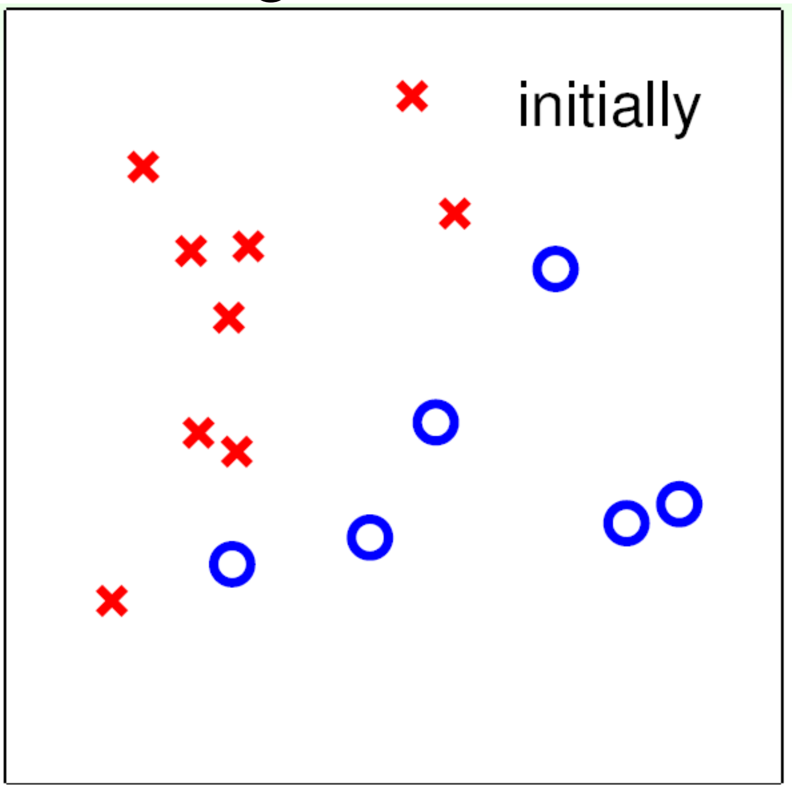
2. correct the mistake by

$$\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + y_{n(t)} \mathbf{x}_{n(t)}$$

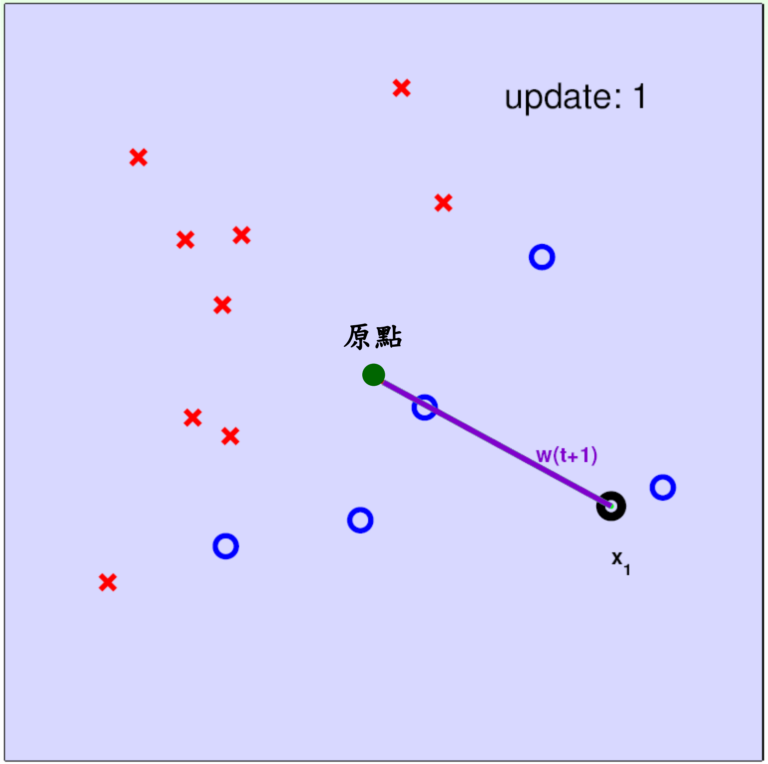
. . . until **a full cycle of not encountering mistakes** 沒有錯誤就會結束

next can follow naïve cycle $(1, \cdots, N)$
or precomputed random cycle

Seeing is Believing

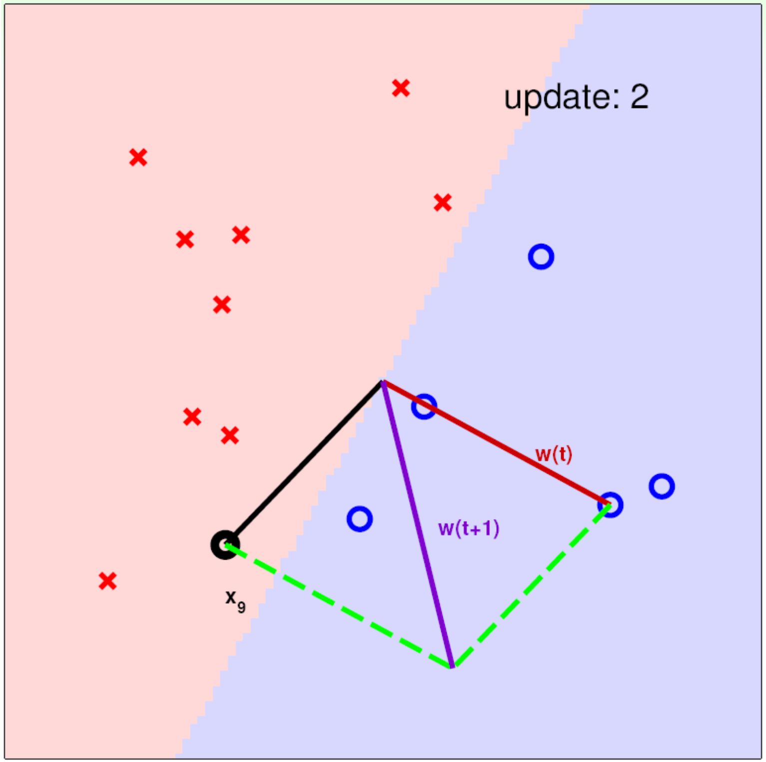


Seeing is Believing

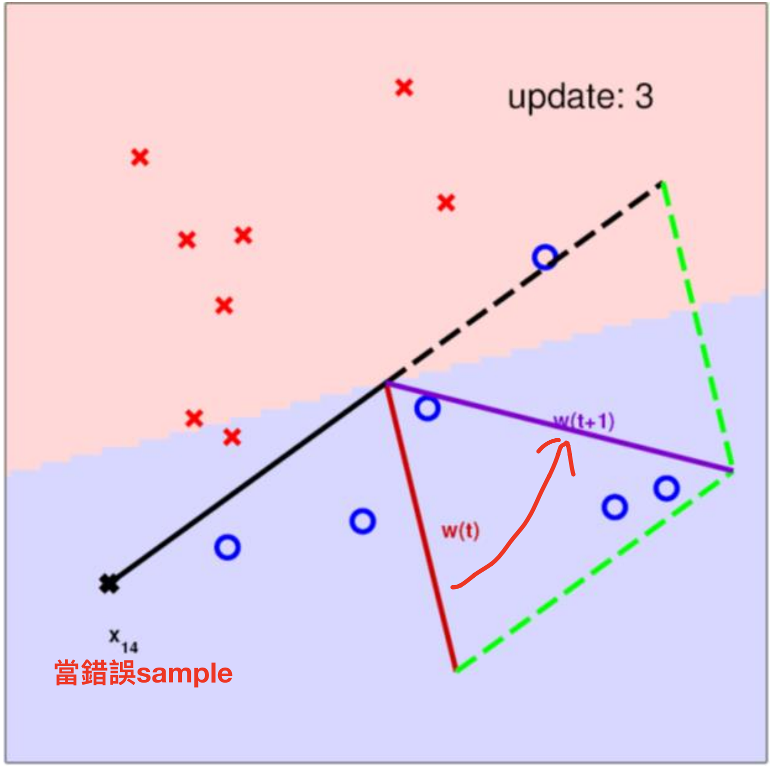


一開始沒有任何線，任取一點當錯誤點

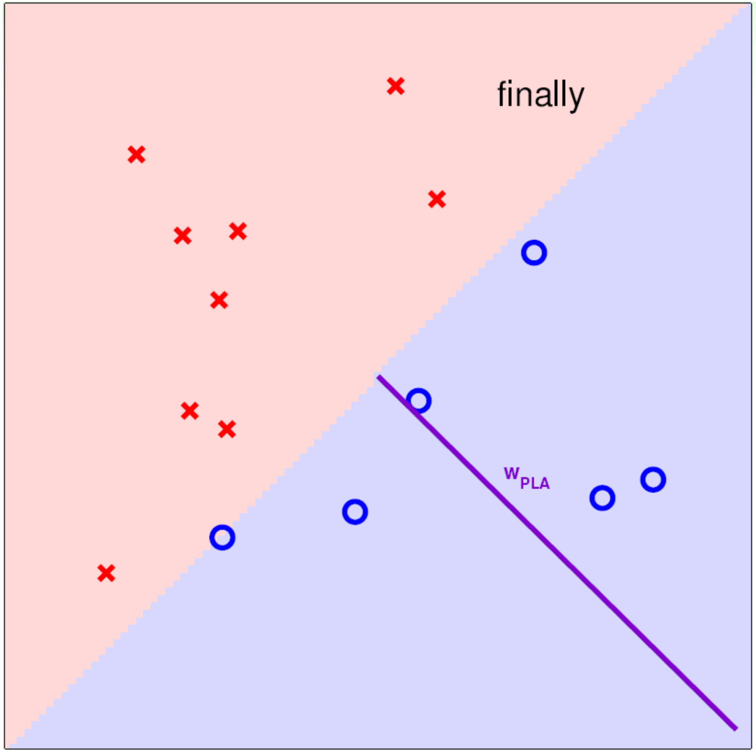
Seeing is Believing



Seeing is Believing



Seeing is Believing



Fun Time

Let's try to think about why PLA may work.

Let $n = n(t)$, according to the rule of PLA below, which formula is true?

做錯

$$\text{sign}\left(\mathbf{w}_t^T \mathbf{x}_n\right) \neq y_n, \quad \boxed{\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + y_n \mathbf{x}_n}$$

下標

區間

1. $\mathbf{w}_{t+1}^T \mathbf{x}_n = y_n$ 01

上輪錯誤 這輪正確

2. $\text{sign}(\mathbf{w}_{t+1}^T \mathbf{x}_n) = y_n$

-1轉+1 這輪會大於上輪

向量內積

3. $\underline{y_n \mathbf{w}_{t+1}^T \mathbf{x}_n \geq y_n \mathbf{w}_t^T \mathbf{x}_n}$
4. $y_n \mathbf{w}_{t+1}^T \mathbf{x}_n < y_n \mathbf{w}_t^T \mathbf{x}_n$

Reference Answer: 3

Simply multiply the second part of the rule by $y_n \mathbf{x}_n$. The result shows that **the rule somewhat 'tries to correct the mistake.'**

嘗試修正錯誤

How About High-dimensional Data?

age	23 years
annual salary	NTD 1,000,000
year in job	0.5 year
current debt	200,000 小較好

- For $\mathbf{x} = (x_1, x_2, \cdots, x_d)$ 'features of customer', compute a weighted 'score' and

$$\text{approve credit if} \quad \sum_{i=1}^{d} w_i x_i > \text{threshold} \quad \text{越大越好}$$

$$\text{deny credit if} \quad \sum_{i=1}^{d} w_i x_i < \text{threshold} \quad \text{越小越好}$$

- $\mathcal{Y}$: $\{+1(\textbf{good}), -1(\textbf{bad})\}$, 0 ignored—linear formula $h \in \mathcal{H}$ are

$$h(\mathbf{x}) = \text{sign}\left(\left(\sum_{i=1}^{d} w_i x_i\right) - \text{threshold}\right)$$

Vector Form of Perceptron Hypothesis

$$
\begin{aligned}
h(\mathbf{x}) &= \text{sign}\left(\left(\sum_{i=1}^{d} w_i x_i\right) - \text{threshold}\right) \\
&= \text{sign}\left(\left(\sum_{i=1}^{d} w_i x_i\right) + \underbrace{(-\text{threshold})}_{w_0} \cdot \underbrace{(+1)}_{x_0}\right) \\
&= \text{sign}\left(\sum_{i=0}^{d} w_i x_i\right) \\
&= \text{sign}\left(\mathbf{w}^T \mathbf{x}\right)
\end{aligned}
$$

學習權重 維度

向量轉至

- each 'tall' $\mathbf{w}$ represents a hypothesis h & is multiplied with 'tall' $\mathbf{x}$ —will use tall versions to simplify notation 高維度向量

Fun Time

- Consider using a perceptron to detect spam messages.
- Assume that each email is represented by the frequency of keyword occurrence, and output +1 indicates a spam. Which keywords below shall have large positive weights in a **good perceptron** for the task?

1. coffee, tea, hamburger, steak Keywords會有大的權重值
2. free, drug, fantastic, deal
3. machine, learning, statistics, textbook
4. national, Taiwan, university, courser

Some Remaining Issues of PLA

'correct' mistakes on $\mathcal{D}$ **until no mistakes**

Algorithmic: halt (with no mistake)? 是否會停下來

- naïve cyclic: ??
- random cyclic: ??
- other variant: ??

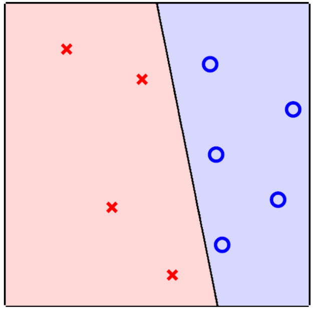
Learning: $g \approx f$? 會與原始一樣好嗎？

- on $\mathcal{D}$, if halt, yes (no mistake) 是，沒錯誤
- outside $\mathcal{D}$: ??
- if not halting: ??

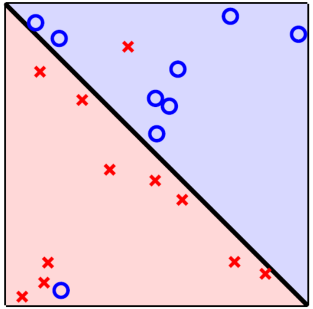
[to be shown] if (...), after 'enough' corrections,
any PLA variant halts 足夠修正後，可以停下來

Linear Separability

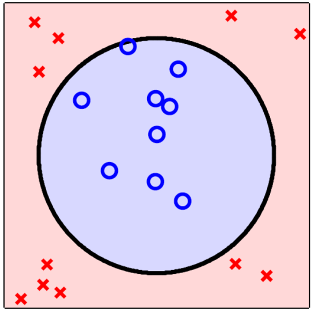
- **if** PLA halts (i.e. no more mistakes),
  (necessary condition) $\mathcal{D}$ allows some $\mathbf{w}$ to make no mistake 沒有發生任何錯誤
- call such $\mathcal{D}$ **linear separable** 線性可分割



(linear separable)



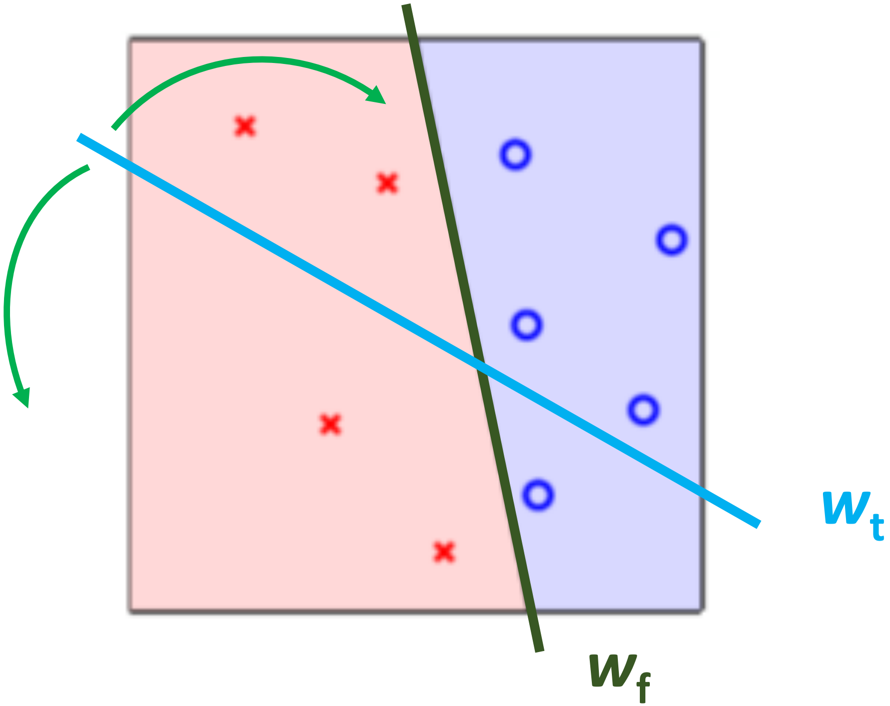
(not linear separable)



(not linear separable)

assume linear separable $\mathcal{D}$,
does PLA always **halt**? Yes

PLA找出的解，真的越來越好嗎?



PLA Fact: $\mathbf{w}_t$ Gets More Aligned with $\mathbf{w}_f$

linear separable $\mathcal{D}$ $\Leftrightarrow$ exists perfect $\mathbf{w}_f$ **such that** $y_n = \text{sign}(\mathbf{w}_f^T \mathbf{x}_n)$

一定會做對

- $\mathbf{w}_f$ perfect hence every $\mathbf{x}_n$ correctly away from line:

答案 分類

$$y_{n(t)} \mathbf{w}_f^T \mathbf{x}_{n(t)} \geq \min_n y_n \mathbf{w}_f^T \mathbf{x}_n > 0$$

- $\mathbf{w}_f^T \mathbf{w}_t \uparrow$ by updating with any $(\mathbf{x}_{n(t)}, y_{n(t)})$

前一筆 做錯與新的

$$\begin{aligned} \mathbf{w}_f^T \mathbf{w}_{t+1} &= \mathbf{w}_f^T \left(\mathbf{w}_t + y_{n(t)} \mathbf{x}_{n(t)} \right) \\ &\geq \mathbf{w}_f^T \mathbf{w}_t + \min_n y_n \mathbf{w}_f^T \mathbf{x}_n \\ &> \mathbf{w}_f^T \mathbf{w}_t + 0. \end{aligned}$$

更靠近方向

≥ 0

$\mathbf{w}_t$ appears more aligned with $\mathbf{w}_f$ after update
(really?)

長度 還是 角度

$$\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1||\vec{v}_2| \cos\theta$$

PLA Fact: $\mathbf{w}_t$ Does Not Grow Too Fast

$\mathbf{w}_t$ changed only when mistake

預測 標準

$$\Leftrightarrow \text{sign}\left(\mathbf{w}_t^T \mathbf{x}_{n(t)}\right) \neq y_{n(t)} \Leftrightarrow \boxed{y_{n(t)} \mathbf{w}_t^T \mathbf{x}_{n(t)} \leq 0}$$

向量長度會被限制住 最長向量x

- mistake 'limits' $\|\mathbf{w}_t\|^2$ growth, even when updating with 'longest' $\mathbf{x}_n$

向量平方

$$\begin{aligned}
\|\mathbf{w}_{t+1}\|^2 &= \|\mathbf{w}_t + y_{n(t)}\mathbf{x}_{n(t)}\|^2 \\
&= \|\mathbf{w}_t\|^2 + 2y_{n(t)}\mathbf{w}_t^T\mathbf{x}_{n(t)} + \|y_{n(t)}\mathbf{x}_{n(t)}\|^2 \\
&\leq \|\mathbf{w}_t\|^2 + 0 + \|\underset{(\pm 1)}{y_{n(t)}}\mathbf{x}_{n(t)}\|^2 \\
&\leq \|\mathbf{w}_t\|^2 + \max_n \|y_n \mathbf{x}_n\|^2
\end{aligned}$$

start from $\mathbf{w}_0 = \mathbf{0}$, after T mistake corrections,

$$\frac{\mathbf{w}_f^T}{\|\mathbf{w}_f\|} \frac{\mathbf{w}_T}{\|\mathbf{w}_T\|} \geq \sqrt{T} \cdot \text{constant}$$

正規化的w_f跟w_t的內積，跟更新的次數T有一個根號的關係，隨著次數越多，兩者會越靠近

More about PLA

Guarantee

as long as linear separable and correct by mistake

越調整越正確

- inner product of $\mathbf{w}_f$ and $\mathbf{w}_t$ grows fast; length of $\mathbf{w}_t$ grows slowly
- PLA 'lines' are more and more aligned with $\mathbf{w}_f \Rightarrow$ halts

Pros

simple to implement, fast, works in any dimension d

Cons

才可停下來

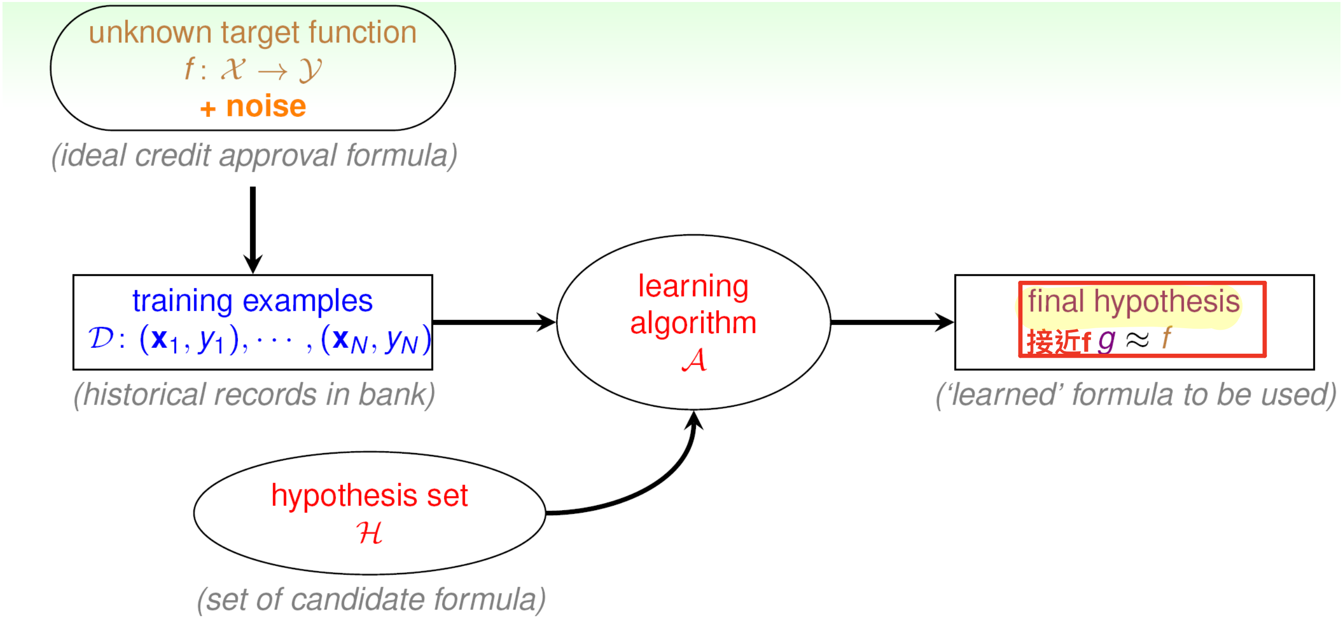
- 'assumes' linear separable $\mathcal{D}$ to halt
  —property unknown in advance (no need for PLA if we know $\mathbf{w}_f$)

無法保證多久可以停下來

- not fully sure how long halting takes
  —though practically fast

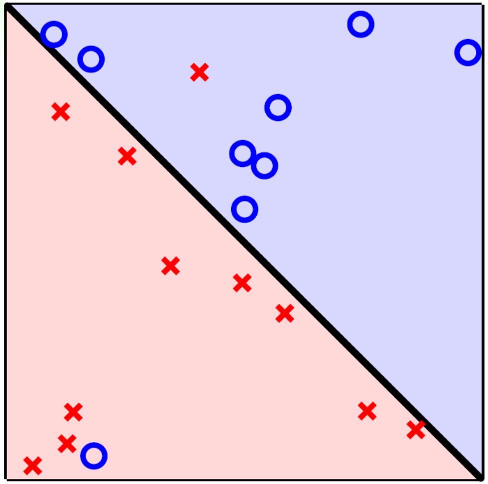
what if $\mathcal{D}$ not linear separable?

Learning with **Noisy Data**



how to at least get $g \approx f$ on **noisy** $\mathcal{D}$?

Line with Noise Tolerance



- assume 'little' noise: $y_n = f(\mathbf{x}_n)$ **usually**
- if so, $g \approx f$ on $\mathcal{D} \Leftrightarrow y_n = g(\mathbf{x}_n)$ **usually**
- how about

最小化式子

不成立為1，反之0

$$\mathbf{w}_g \leftarrow \underset{\mathbf{w}}{\operatorname{argmin}} \sum_{n=1}^{N} [\![y_n \neq \operatorname{sign}(\mathbf{w}^T \mathbf{x}_n)]\!]$$

不可微分

—**NP-hard to solve, unfortunately**

can we modify PLA to get
an 'approximately good' g?

Pocket Algorithm

口袋演算法

modify PLA algorithm (black lines) by $\boxed{\text{keeping best weights in pocket}}$

initialize pocket weights $\hat{\mathbf{w}}$

For $t = 0, 1, \cdots$

1. find a (random) mistake of $\mathbf{w}_t$ called $(\mathbf{x}_{n(t)}, y_{n(t)})$ 找錯
2. (try to) correct the mistake by 調整線

$$\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + y_{n(t)}\mathbf{x}_{n(t)}$$

3. if $\mathbf{w}_{t+1}$ makes fewer mistakes than $\hat{\mathbf{w}}$, replace $\hat{\mathbf{w}}$ by $\mathbf{w}_{t+1}$ 新的線是否有比前一筆

...until enough iterations 有更好才收起來

return $\hat{\mathbf{w}}$ (called $\mathbf{w}_{\text{POCKET}}$) as g 做錯個數變少達成分類器學習目的

a simple modification of PLA to find (somewhat) $\boxed{\text{'best' weights}}$

Fun Time

Should we use pocket or PLA?

PLA一定要

Since we do not know whether $\mathcal{D}$ is linear separable in advance, we may decide to just go with pocket instead of PLA. If $\mathcal{D}$ is actually linear separable, what's the difference between the two?

最後是

1. pocket on $\mathcal{D}$ is slower than PLA
2. pocket on $\mathcal{D}$ is faster than PLA
3. pocket on $\mathcal{D}$ returns a better g in approximating f than PLA
4. pocket on $\mathcal{D}$ returns a worse g in approximating f than PLA

Linear separable都會得到正確相同的答案

Summary

- Perceptron Hypothesis Set

  hyperplanes/linear classifiers in $\mathbb{R}^d$

- Perceptron Learning Algorithm (PLA) 有無限多條線

  correct mistakes and improve iteratively

- Guarantee of PLA 一定有確定答案

  no mistake eventually if linear separable

- Non-Separable Data 非線形區分可以使用pocket

  hold somewhat 'best' weights in pocket